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Lower bounds for numbers of real solutions in problems of Schubert calculus

by

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1. Introduction

It is well known that the problem of finding the number of real solutions to algebraic systems is very difficult, and not many results are known; see [16]. In this paper we address the counting of real points in intersections of Schubert varieties associated with osculating flags in the Grassmannian of n-dimensional planes in a d-dimensional space. These problems are parameterized by partitions $\lambda^{(1)}, ..., \lambda^{(k)}$ and ν with at most n parts satisfying the condition $|\nu|+\sum_{i=1}^{k}|\lambda^{(i)}|=n(d-n)$, and distinct complex numbers $z_1, ..., z_k$. In this parametrization, $\lambda^{(1)}, ..., \lambda^{(k)}$ and ν are respectively paired with $z_1, ..., z_k$ and infinity.

Equivalently, we count n-dimensional real vector spaces of polynomials that have ramification points $z_1, ..., z_k$ with respective ramification conditions $\lambda^{(1)}, ..., \lambda^{(k)}$ and are spanned by polynomials of degrees $d-i-\nu_{n+1-i}$, $i=1, ..., n$, see §3 for details.

The same number is obtained by counting real monic monodromy-free Fuchsian differential operators with singular points $z_1, ..., z_k$ and infinity, exponents $\lambda_n^{(i)}, \lambda_{n-1}^{(i)}+1, ..., \lambda_1^{(i)}+n-1$ at the z_i's, $i=1, ..., k$, and exponents $\nu_n+1-d, \nu_{n-1}+2-d, ..., \nu_1+n-d$ at infinity.

The number of complex solutions to the above-mentioned algebraic systems is readily given by the Schubert calculus and equals the multiplicity of the irreducible $\mathfrak{gl}_n$-module L_μ of highest weight $\mu=(d-n-\nu_n, d-n-\nu_{n-1}, ..., d-n-\nu_1)$ in the tensor product $L_{\lambda^{(1)}}\otimes...\otimes L_{\lambda^{(k)}}$ of irreducible $\mathfrak{gl}_n$-modules of highest weights $\lambda^{(1)}, ..., \lambda^{(k)}$.

The Shapiro–Shapiro conjecture proved in [4] for $n=2$ and in [13] for all n asserts that, if all $z_1, ..., z_k$ are real, then all solutions of the Schubert problem associated with osculating flags are real. Therefore, in this case, the number of real solutions is the maximum possible.

Next we wonder how many real solutions we can guarantee in other cases. For the Schubert problem to have real solutions, the set $z_1, ..., z_k$ should be invariant under complex conjugation and the ramification conditions at complex conjugate points should be the same. This can be readily seen, for example, from the above-mentioned counting of real monic monodromy-free Fuchsian differential operators. In this case we say that the data $z_1, ..., z_k$ and $\lambda^{(1)}, ..., \lambda^{(k)}$ are invariant under complex conjugation. In general, the number of real solutions is not known, and based on extensive computer experimentation, see [8], the answer to this question should be very interesting.

Prior to this paper, there were several approaches in order to obtain lower bounds. First, one can compute the real topological degree of the Wronski map, and it gives bounds for the case when all $\lambda^{(1)}, ..., \lambda^{(k)}$ are one-box partitions, see [2]. The lower bound can be extended to the case when all partitions but one consist of one box, see [15]. While this method gives non-trivial bounds, it has several serious drawbacks—the answer does not depend on the number of real points among $z_1, ..., z_k$, does not apply to general ramification conditions, and is far from being sharp in many cases.

Another method is to consider parity conditions. It is proved in [9] that if all partitions are symmetric, the number of solutions can change only by 4. Unfortunately, this is also a very special situation and the only lower bound one can obtain this way is 2. Finally, in some cases, see [7, Theorem 7], the required spaces of polynomials can be described relatively explicitly to estimate the number of solutions. This estimate is sharp, that is, it is attained for some choice of $z_1, ..., z_k$, but it works only for very special choices of $\lambda^{(1)}, ..., \lambda^{(k)}$.

We propose one more way to attack the problem. The proof of the Shapiro–Shapiro conjecture in [12] and [13] is based on the identification of the spaces of polynomials with points of the spectrum of a remarkable family of commuting linear operators known as higher Gaudin Hamiltonians. For real $z_1, ..., z_k$, these operators are self-adjoint with respect to a positive definite Hermitian form, and hence have real eigenvalues. Eventually, this shows that the spaces of polynomials with real ramification points are real.

If some of $z_1, ..., z_k$ are not real, but the data $z_1, ..., z_k$ and $\lambda^{(1)}, ..., \lambda^{(k)}$ are invariant under complex conjugation, then the higher Gaudin Hamiltonians are self-adjoint with respect to a non-degenerate Hermitian form, but this form is indefinite. Since the number of real eigenvalues of such operators is at least the absolute value of the signature of the Hermitian form, see Lemma 6.1, this gives a lower bound for the number of real solutions to the Schubert problem in question.

We reduce the computation of the signature of the form to the computation of values of characters of products of symmetric groups on products of commuting transpositions. There is a formula for such characters, see Proposition 2.1, similar to the Frobenius formula [5]. Thus, we obtain a lower bound for all possible choices of partitions $\lambda^{(1)}, ..., \lambda^{(k)}$ and ν, and the obtained bound depends on the number of real points among $z_1, ..., z_k$, see Corollary 7.3.

We check the obtained lower bound against the available results and computer experiments, see §8. We find that our bound is sharp in many cases. For example, all available data for $n=2$ match our bound. However, our bound is not sharp in general. We hope that the bound can be improved in some cases by modifying the Hermitian form given in this paper so that higher Gaudin Hamiltonians remain self-adjoint relative to the new form.

The paper is organized as follows. We start with computations of characters of symmetric groups in §2, see Proposition 2.1. Then we prepare notation and definitions for osculating Schubert calculus in §3. We recall definitions and properties of higher Gaudin Hamiltonians in §4 and their symmetries in §5. We discuss the key facts from linear algebra about self-adjoint operators with respect to indefinite Hermitian forms in §6. In §7 we prove our main statement, see Theorem 7.2 and Corollary 7.3. In §8 we compare our bounds with known data and results.

2. Characters of the symmetric groups

The study of characters of the symmetric groups is a classical subject which goes back to Frobenius [5]. In this section we deduce a formula for characters of a product of the symmetric groups appearing in a tensor product of irreducible $\mathfrak{gl}_n$-modules.

Let S_k be the group of all permutations of a k-element set, GL_n be the group of all non-degenerate $n\times n$ matrices, and $\mathfrak{gl}_n$ be the Lie algebra of $n\times n$ matrices.

Let $\lambda=(\lambda_1, \lambda_2, ..., \lambda_n)$ be a partition with at most n parts, $\lambda_1\geqslant\lambda_2\geqslant...\geqslant\lambda_n\geqslant 0$. We use the notation $|\lambda|=\sum_{i=1}^n \lambda_i$.

For each partition λ with at most n parts, denote by L_λ the irreducible finite-dimensional $\mathfrak{gl}_n$-module of highest weight λ. We call the module corresponding to $\lambda=(1, 0, ..., 0)$ the *vector representation*.

Let

$$\Delta_n = \prod_{\substack{i,j=1\\ i>j}}^{n} (x_i - x_j) = \det(x_i^{n-j})_{i,j=1}^n \in \mathbb{C}[x_1, ..., x_n] \tag{2.1}$$

be the Vandermonde determinant. Let $S_\lambda\in\mathbb{C}[x_1, ..., x_n]$ be the Schur polynomial given by

$$S_\lambda(x_1,...,x_n)=\frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\Delta_n}. \tag{2.2}$$

The Schur polynomial is a symmetric polynomial in $x_1,...,x_n$. It is well known that the character of the module L_λ is given by the Schur polynomial:

$$S_\lambda(x_1,...,x_n)=\mathrm{tr}_{L_\lambda}X,$$

where $X=\mathrm{diag}(x_1,...,x_n)\in\mathrm{GL}_n$.

Consider the tensor product of $\mathfrak{gl}_n$-modules

$$L_{\boldsymbol{\lambda}}=L_{\lambda^{(1)}}^{\otimes k_1}\otimes L_{\lambda^{(2)}}^{\otimes k_2}\otimes...\otimes L_{\lambda^{(s)}}^{\otimes k_s} \tag{2.3}$$

and its decomposition into irreducible $\mathfrak{gl}_n$-submodules:

$$L_{\boldsymbol{\lambda}}=\bigoplus_\mu L_\mu\otimes M_{\boldsymbol{\lambda},\mu}. \tag{2.4}$$

Notice that the multiplicity space $M_{\boldsymbol{\lambda},\mu}$ is trivial unless

$$|\mu|=\sum_{i=1}^s k_i|\lambda^{(i)}|. \tag{2.5}$$

The product of symmetric groups $S_{\boldsymbol{k}}=S_{k_1}\times S_{k_2}\times...\times S_{k_s}$ acts on $L_{\boldsymbol{\lambda}}$ by permuting the corresponding tensor factors. Since the $S_{\boldsymbol{k}}$-action commutes with the $\mathfrak{gl}_n$-action, the group $S_{\boldsymbol{k}}$ acts on the multiplicity space $M_{\boldsymbol{\lambda},\mu}$ for all μ. If $s=1$ and all tensor factors are vector representations, $\lambda^{(1)}=(1,0,...,0)$, by the Schur–Weyl duality, the space $M_{\boldsymbol{\lambda},\mu}$ is the irreducible representation of S_{k_1} corresponding to the partition μ. In general, $M_{\boldsymbol{\lambda},\mu}$ is a reducible representation of $S_{\boldsymbol{k}}$.

For $\sigma=\sigma_1\times\sigma_2\times...\times\sigma_s\in S_{\boldsymbol{k}}$, $\sigma_i\in S_{k_i}$, let $\chi_{\boldsymbol{\lambda},\mu}(\sigma)=\mathrm{tr}_{M_{\boldsymbol{\lambda},\mu}}\sigma$ be the value of the character of $S_{\boldsymbol{k}}$ corresponding to the representation $M_{\boldsymbol{\lambda},\mu}$ on σ. Writing σ_i as a product of disjoint cycles, denote the number of cycles in the product by c_i and the lengths of cycles by l_{ij}, $j=1,...,c_i$. We have $l_{i,1}+...+l_{i,c_i}=k_i$.

Proposition 2.1. *The character value $\chi_{\boldsymbol{\lambda},\mu}(\sigma)$ equals the coefficient of the monomial $x_1^{\mu_1+n-1}x_2^{\mu_2+n-2}...x_n^{\mu_n}$ in the polynomial*

$$\Delta_n\cdot\prod_{i=1}^s\prod_{j=1}^{c_i}S_{\lambda^{(i)}}(x_1^{l_{ij}},...,x_n^{l_{ij}}).$$

Proof. Let V be a vector space, $P\in\mathrm{End}(V\otimes V)$ be the flip map, and $A,B\in\mathrm{End}(V)$. Then $(\mathrm{id}\otimes\mathrm{tr}_V)((A\otimes B)P)=AB\in\mathrm{End}(V)$.

Let $\sigma=(1\,2\,\dots\,l)$ be a cycle permutation and $X=\mathrm{diag}(x_1,\dots,x_n)\in\mathrm{GL}_n$. Using the presentation $\sigma=(1\,2)(2\,3)\dots(l-1\,l)$, we get

$$\mathrm{tr}_{L_\lambda^{\otimes l}}(X\times\sigma)=\mathrm{tr}_{L_\lambda}(X^l)=S_\lambda(x_1^l,\dots,x_n^l). \tag{2.6}$$

For any $\sigma\in S_{\boldsymbol{k}}$ and $X\in\mathrm{GL}_n$, formulae (2.3) and (2.6) yield

$$\mathrm{tr}_{L_{\boldsymbol{\lambda}}}(X\times\sigma)=\prod_{i=1}^{s}\prod_{j=1}^{c_i}S_{\lambda^{(i)}}(x_1^{l_{ij}},\dots,x_n^{l_{ij}}),$$

and formulae (2.4) and (2.2) give

$$\mathrm{tr}_{L_{\boldsymbol{\lambda}}}(X\times\sigma)=\sum_{\mu}\chi_{\boldsymbol{\lambda},\mu}(\sigma)S_\mu(x_1,\dots,x_n)=\frac{1}{\Delta_n}\sum_{\mu}\chi_{\boldsymbol{\lambda},\mu}(\sigma)\det(x_i^{\mu_j+n-j})_{i,j=1}^n.$$

The proposition follows. □

For the case of vector representations, that is $s=1$ and $\lambda^{(1)}=(1,0,\dots,0)$, the Schur polynomial is $S_{\lambda^{(1)}}(x_1,\dots,x_n)=x_1+x_2+\dots+x_n$ and Proposition 2.1 reduces to the famous Frobenius formula [5] for characters of irreducible representations of the symmetric group.

3. Osculating Schubert calculus

In this section we recall the problem of computing intersections of Schubert varieties corresponding to osculating flags.

Let n and d be positive integers such that $d>n$. Let V be a d-dimensional complex vector space. We realize V as the space of polynomials in a variable x of degree less than d: $V=\mathbb{C}_d[x]$. The Grassmannian $\mathrm{Gr}(n,d)$ of n-dimensional planes in V is a smooth projective variety of dimension $n(d-n)$. A point of $\mathrm{Gr}(n,d)$ is called real if the corresponding space of polynomials has a basis consisting of polynomials with real coefficients. We also call such spaces of polynomials real.

For $z\in\mathbb{C}$ we define a full flag $\mathcal{F}_\bullet(z)$ in V as follows:

$$\mathcal{F}_\bullet(z)=\{\mathcal{F}_1(z)\subset\mathcal{F}_2(z)\subset\dots\subset\mathcal{F}_{d-1}(z)\subset\mathcal{F}_d(z)=V\},$$

where $\mathcal{F}_i(z)=(x-z)^{d-i}\mathbb{C}_i[x]$ is the subspace of polynomials vanishing at z to the order at least $d-i$. Clearly, $\mathcal{F}_i(z)$ has a basis $(x-z)^{d-i},\dots,(x-z)^{d-1}$ and $\dim\mathcal{F}_i(z)=i$. We also define a full flag $\mathcal{F}_\bullet(\infty)=\{\mathcal{F}_1(\infty)\subset\mathcal{F}_2(\infty)\subset\dots\subset\mathcal{F}_{d-1}(\infty)\subset\mathcal{F}_d(\infty)=V\}$, where $\mathcal{F}_i(\infty)=$

$\mathbb{C}_i[x]$ is the subspace of polynomials of degree less than i. The subspace $\mathcal{F}_i(\infty)$ has a basis $1, x, ..., x^{i-1}$.

Given $z\in\mathbb{C}\cup\{\infty\}$ and a partition λ with at most n parts, the corresponding Schubert variety is

$$\Omega_\lambda(z) = \{W\in \mathrm{Gr}(n,d) : \dim W\cap\mathcal{F}_{d-\lambda_{n-i}-i}(z)\geqslant n-i \text{ for } i=0, ..., n-1\}.$$

The Schubert variety $\Omega_\lambda(z)\subset\mathrm{Gr}(n,d)$ has codimension $|\lambda|$. For $z\in\mathbb{C}$, the Schubert variety $\Omega_\lambda(z)$ consists of n-dimensional spaces of polynomials $W\subset V$ that have a basis $f_1, ..., f_n$ such that f_j has a root at z of order at least $\lambda_{n+1-j}+j-1$. The Schubert variety $\Omega_\lambda(\infty)$ consists of n-dimensional spaces of polynomials $W\subset V$ that have a basis $f_1, ..., f_n$ such that $\deg f_j\leqslant d-j-\nu_{n+1-j}$,

Given partitions $\lambda^{(1)}, ..., \lambda^{(k)}$ and ν with at most n parts such that

$$|\nu|+\sum_{i=1}^{k}|\lambda^{(i)}| = n(d-n), \tag{3.1}$$

and distinct complex numbers $z_1, ..., z_k$, the corresponding osculating Schubert problem asks to find the intersection of Schubert varieties

$$\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) = \bigcap_{i=1}^{k}\Omega_{\lambda^{(i)}}(z_i)\cap\Omega_\nu(\infty). \tag{3.2}$$

LEMMA 3.1. *The intersection $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ consists of n-dimensional spaces of polynomials $W\subset V$ such that*

(a) *the space W has a basis $f_{1,0}, ..., f_{n,0}$ such that* $\deg f_{j,0}=d-j-\nu_{n+1-j}$, *and*

(b) *for each $i=1, ..., k$, the space W has a basis $f_{1,i}, ..., f_{n,i}$ such that $f_{j,i}$ has a root at z_i of order exactly $\lambda^{(i)}_{n+1-j}+j-1$.*

Proof. Let $w(x)$ be the Wronski determinant of a basis of W. Clearly, $w(x)$ does not depend on a choice of basis up to a non-zero multiplicative constant. The lemma follows from equality (3.1) by comparing $\deg w(x)$ with the number of zeros of $w(x)$. □

According to Schubert calculus on Grassmannians, see [6], the set $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ is finite, and the number $m(\boldsymbol{\lambda}, \nu)$ of points in $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ counted with multiplicities equals the multiplicity of the irreducible $\mathfrak{gl}_n$-module L_μ in the tensor product $L_{\lambda^{(1)}}\otimes...\otimes L_{\lambda^{(k)}}$, where the partition μ is the complement of ν in the $n\times(d-n)$ rectangle:

$$\mu = (d-n-\nu_n, d-n-\nu_{n-1}, ..., d-n-\nu_1). \tag{3.3}$$

It is known that for generic complex numbers $z_1, ..., z_k$, all points of intersection are multiplicity-free. Moreover, for distinct real $z_1, ..., z_k$, all points of intersection are multiplicity-free as well, and all the corresponding spaces of polynomials are real, see [13]. That is, for distinct real $z_1, ..., z_k$ the osculating Schubert problem has $m(\boldsymbol{\lambda}, \nu)$ real solutions.

Let us make two pertinent remarks. First, notice that $m(\boldsymbol{\lambda}, \nu) = m(\tilde{\boldsymbol{\lambda}}, \varnothing)$, where $\tilde{\boldsymbol{\lambda}}$ is the $(k+1)$-tuple $\lambda^{(1)}, ..., \lambda^{(k)}, \nu$ and $\varnothing = (0, ..., 0)$ is the empty partition.

Second, fix partitions $\lambda^{(1)}, ..., \lambda^{(k)}$ and μ such that $|\mu| = \sum_{i=1}^{k} |\lambda^{(i)}|$, take $d \geqslant n + \mu_1$, and set

$$\nu = (d-n-\mu_n, ..., d-n-\mu_1). \tag{3.4}$$

Then the spaces of polynomials that are points of $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ do not depend on d.

4. Gaudin model

Let E_{ij}, $i, j = 1, ..., n$, be the standard basis of $\mathfrak{gl}_n$: $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$. The current Lie algebra $\mathfrak{gl}_n[t]$ is spanned by the elements $E_{ij} \otimes t^r$, $i, j = 1, ..., n$, $r \in \mathbb{Z}_{\geqslant 0}$, satisfying the relations $[E_{ij} \otimes t^r, E_{kl} \otimes t^s] = \delta_{jk} E_{il} \otimes t^{r+s} - \delta_{il} E_{kj} \otimes t^{r+s}$. We identify $\mathfrak{gl}_n$ with the subalgebra in $\mathfrak{gl}_n[t]$ by the rule $E_{ij} \mapsto E_{ij} \otimes 1$, $i, j = 1, ..., n$.

Given $z \in \mathbb{C}$, define the evaluation homomorphism $\varepsilon_z : \mathfrak{gl}_n[t] \to \mathfrak{gl}_n$, $E_{ij} \otimes t^r \mapsto E_{ij} z^r$. For a $\mathfrak{gl}_n$-module L, the evaluation $\mathfrak{gl}_n[t]$-module $L(z)$ is the pull-back of L through the evaluation homomorphism ε_z.

For $g \in \mathfrak{gl}_n$, define the formal power series in x^{-1}: $g(x) = \sum_{s=0}^{\infty} (g \otimes t^s) x^{-s-1}$. The series $g(x)$ acts in the evaluation module $L(z)$ as $g(x-z)^{-1}$.

Let ∂_x be the differentiation with respect to x. Set $X_{ij} = \delta_{ij} \partial_x - E_{ij}(x)$, $i, j = 1, ..., n$. Define the formal differential operator $\mathcal{D}$ by the rule

$$\mathcal{D} = \sum_{\sigma \in S_n} X_{\sigma(1),1} X_{\sigma(2),2} ... X_{\sigma(n),n} = \partial_x^n + \sum_{i=1}^{n} \sum_{j=i}^{\infty} B_{ij} x^{-j} \partial_x^{n-i}, \tag{4.1}$$

where the B_{ij}'s are elements of the universal enveloping algebra $U(\mathfrak{gl}_n[t])$. The operator $\mathcal{D}$ is called the *universal operator*.

The unital subalgebra of $U(\mathfrak{gl}_n[t])$ generated by B_{ij}, $i = 1, ..., n$, $j \in \mathbb{Z}_{\geqslant i}$, is called the Bethe subalgebra and denoted by $\mathcal{B}_n$. Also, $\mathcal{B}_n$ is called the algebra of higher Gaudin Hamiltonians.

PROPOSITION 4.1. ([17]) *The subalgebra $\mathcal{B}_n$ is commutative and commutes with $\mathfrak{gl}_n$.*

For partitions $\lambda^{(1)}, ..., \lambda^{(k)}$ and distinct complex numbers $z_1, ..., z_k$, consider the tensor product $L_{\boldsymbol{\lambda}}(\boldsymbol{z}) = L_{\lambda^{(1)}}(z_1) \otimes ... \otimes L_{\lambda^{(k)}}(z_k)$ of evaluation $\mathfrak{gl}_n[t]$-modules. For every $g \in \mathfrak{gl}_n$, the series $g(x)$ acts on $L_{\boldsymbol{\lambda}}(\boldsymbol{z})$ as a rational function of x.

As a $\mathfrak{gl}_n$-module, $L_{\boldsymbol{\lambda}}(\boldsymbol{z})$ does not depend on $z_1, ..., z_k$ and equals

$$L_{\boldsymbol{\lambda}} = L_{\lambda^{(1)}} \otimes ... \otimes L_{\lambda^{(k)}}.$$

Let $L_{\boldsymbol{\lambda}} = \bigoplus_\mu L_\mu \otimes M_{\boldsymbol{\lambda},\mu}$ be its decomposition into irreducible $\mathfrak{gl}_n$-submodules. Recall that the multiplicity space $M_{\boldsymbol{\lambda},\mu}$ is trivial unless

$$|\mu| = \sum_{i=1}^{k} |\lambda^{(i)}|. \tag{4.2}$$

As a subalgebra of $U(\mathfrak{gl}_n[t])$, the algebra $\mathcal{B}_n$ acts on $L_{\boldsymbol{\lambda}}(\boldsymbol{z})$. Since $\mathcal{B}_n$ commutes with $\mathfrak{gl}_n$, this action descends to the action of $\mathcal{B}_n$ on each multiplicity space $M_{\boldsymbol{\lambda},\mu}$. For $b \in \mathcal{B}_n$, denote by $b(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) \in \operatorname{End}(M_{\boldsymbol{\lambda},\mu})$ the corresponding linear operator.

Given a common eigenvector $v \in M_{\boldsymbol{\lambda},\mu}$ of the operators $b(\boldsymbol{\lambda}, \mu, \boldsymbol{z})$, we let $b(\boldsymbol{\lambda}, \mu, \boldsymbol{z}; v)$ denote the corresponding eigenvalues, and define the scalar differential operator

$$\mathcal{D}_v = \partial_x^n + \sum_{i=1}^{n} \sum_{j=i}^{\infty} B_{ij}(\boldsymbol{\lambda}, \mu, \boldsymbol{z}; v) x^{-j} \partial_x^{n-i}. \tag{4.3}$$

One can check that $\mathcal{D}_v$ is a Fuchsian differential operator with singular points at the points $z_1, ..., z_k$ and infinity. Moreover, for every $i = 1, ..., k$, the exponents of $\mathcal{D}_v$ at the point z_i are $\lambda_n^{(i)}, \lambda_{n-1}^{(i)} + 1, ..., \lambda_1^{(i)} + n - 1$, the exponents of $\mathcal{D}_v$ at infinity are $-\mu_1 + 1 - n$, $-\mu_2 + 2 - n, ..., -\mu_n$, and the kernel of $\mathcal{D}_v$ is spanned by polynomials, see [11].

Theorem 4.2 below connects Schubert calculus and the Gaudin model. Let a partition μ satisfy (4.2). Take $d \geqslant n + \mu_1$, and define the partition ν by (3.4). Let $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ be the intersection of Schubert varieties (3.2).

THEOREM 4.2. ([13]) *There is a bijective correspondence τ between common eigenvectors of the operators $b(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) \in \operatorname{End}(M_{\boldsymbol{\lambda},\mu})$, $b \in \mathcal{B}_n$, and points of $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ such that $\tau(v)$ is the kernel of the scalar differential operator $\mathcal{D}_v$. For generic $\boldsymbol{z}$, the operators $b(\boldsymbol{\lambda}, \mu, \boldsymbol{z})$ are diagonalizable and have simple joint spectrum.*

In particular, Theorem 4.2 implies that if for a common eigenvector v all eigenvalues $B_{ij}(\boldsymbol{\lambda}, \mu, \boldsymbol{z}; v)$ are real, then the point $\tau(v) \in \Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) \subset \operatorname{Gr}(n, d)$ is real, because the operator $\mathcal{D}_v$, see (4.3), has real coefficients.

Remark. Denote by $\mathcal{B}_n(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) \subset \operatorname{End}(M_{\boldsymbol{\lambda},\mu})$ the commutative subalgebra, generated by the operators $b(\boldsymbol{\lambda}, \mu, \boldsymbol{z})$, $b \in \mathcal{B}_n$. It is proved in [13] that for all $\boldsymbol{z} = (z_1, ..., z_k)$ with distinct coordinates, $\mathcal{B}_n(\boldsymbol{\lambda}, \mu, \boldsymbol{z})$ is a maximal commutative subalgebra of dimension $\dim M_{\boldsymbol{\lambda},\mu}$, and for a generic vector $w \in M_{\boldsymbol{\lambda},\mu}$, the map

$$\mathcal{B}_n(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) \longrightarrow M_{\boldsymbol{\lambda},\mu},$$
$$b(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) \longmapsto b(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) w,$$

is an isomorphism of vector spaces.

5. Shapovalov form

For any partition λ with at most n parts, the irreducible $\mathfrak{gl}_n$-module L_λ admits a positive definite Hermitian form $(\cdot,\cdot)_\lambda$ such that $(E_{ij}v,w)_\lambda=(v,E_{ji}w)_\lambda$ for any $i,j=1,...,n$ and any $v,w\in L_\lambda$. Such a form is unique up to multiplication by a positive real number. We will call this form the Shapovalov form.

For partitions $\lambda^{(1)},...,\lambda^{(k)}$ we define the positive definite Hermitian form $(\cdot,\cdot)_{\boldsymbol{\lambda}}$ on the tensor product $L_{\boldsymbol{\lambda}}=L_{\lambda^{(1)}}\otimes...\otimes L_{\lambda^{(k)}}$ as the product of Shapovalov forms on the tensor factors. For each multiplicity space $M_{\boldsymbol{\lambda},\mu}$, the form $(\cdot,\cdot)_{\boldsymbol{\lambda}}$ induces a positive definite Hermitian form $(\cdot,\cdot)_{\boldsymbol{\lambda},\mu}$ on $M_{\boldsymbol{\lambda},\mu}$.

PROPOSITION 5.1. *For any $i=1,...,n$, $j\in\mathbb{Z}_{\geqslant i}$, and any $v,w\in M_{\boldsymbol{\lambda},\mu}$,*

$$(B_{ij}(\boldsymbol{\lambda},\mu,\boldsymbol{z})v,w)_{\boldsymbol{\lambda},\mu}=(v,B_{ij}(\boldsymbol{\lambda},\mu,\bar{\boldsymbol{z}})w)_{\boldsymbol{\lambda},\mu}, \tag{5.1}$$

where the elements B_{ij} are defined by (4.1), $\bar{\boldsymbol{z}}=(\bar z_1,...,\bar z_k)$ and the bar stands for complex conjugation.

Proof. The claim follows from [10, Theorem 9.1]. □

If some of the partitions $\lambda^{(1)},...,\lambda^{(k)}$ coincide, then the operators $b(\boldsymbol{\lambda},\mu,\boldsymbol{z})$ have additional symmetry. Assume that $\lambda^{(i)}=\lambda^{(i+1)}$ for some i. Let $P_i\in\operatorname{End}(L_{\boldsymbol{\lambda}})$ be the flip of the ith and $(i+1)$-st tensor factors and $\tilde{\boldsymbol{z}}^{(i)}=(z_1,...,z_{i-1},z_{i+1},z_i,z_{i+2},...,z_k)$.

LEMMA 5.2. *For any $b\in\mathcal{B}_n$, we have $P_ib(\boldsymbol{\lambda},\mu,\boldsymbol{z})P_i=b(\boldsymbol{\lambda},\mu,\tilde{\boldsymbol{z}}^{(i)})$.*

6. Self-adjoint operators with respect to indefinite Hermitian forms

In this section we discuss some key statements from linear algebra.

Given a finite-dimensional vector space M, a linear operator $A\in\operatorname{End}M$, and a number $\alpha\in\mathbb{C}$, let $M_A(\alpha)=\ker(A-\alpha)^{\dim M}$. When $M_A(\alpha)$ is not trivial, it is the subspace of generalized eigenvectors of A with eigenvalue α.

LEMMA 6.1. *Let M be a complex finite-dimensional vector space having a non-degenerate Hermitian form of signature m, and let A be a self-adjoint operator. Let $R=\bigoplus_{\alpha\in\mathbb{R}}M_A(\alpha)$ be the subspace of generalized eigenvectors of A with real eigenvalues. Then the restriction of the Hermitian form on R is non-degenerate and has signature m. In particular*, $\dim R\geqslant|m|$.

Proof. Since A is self-adjoint, $M_A(\alpha)^\perp=\bigoplus_{\beta\neq\bar\alpha}M_A(\beta)$. In particular, if α is an eigenvalue of A that is not real, then the restriction of the Hermitian form on the subspace $M_A(\alpha)^\perp\oplus M_A(\bar\alpha)$ is non-degenerate and has zero signature. Thus, the restriction of the Hermitian form on the subspace R is non-degenerate and has signature m. □

Corollary 6.2. *Let M be a complex finite-dimensional vector space with a non-degenerate Hermitian form of signature m, and let $\mathcal{A}\subset\mathrm{End}(M)$ be a commutative subalgebra over $\mathbb{R}$, whose elements are self-adjoint operators. Let $R=\bigcap_{A\in\mathcal{A}}\bigoplus_{\alpha\in\mathbb{R}}M_A(\alpha)$. Then the restriction of the Hermitian form on R is non-degenerate and has signature m. In particular*, $\dim R\geqslant|m|$.

Proof. Let $A_1,...,A_k$ be a basis of $\mathcal{A}$. Clearly,

$$R=\bigcap_{i=1}^{k}\bigoplus_{\alpha\in\mathbb{R}}M_{A_i}(\alpha).$$

Let $M_1=\bigoplus_{\alpha\in\mathbb{R}}M_{A_1}(\alpha)$. The subspace M_1 is $\mathcal{A}$-invariant and the restriction of the Hermitian form on M_1 is non-degenerate and has signature m by Lemma 6.1. The corollary follows by induction. □

In fact, Lemma 6.1 can be strengthened.

Lemma 6.3. ([14]) *Under the assumption of Lemma 6.1, the operator A has at least m linearly independent eigenvectors with real eigenvalues*: $\dim\bigoplus_{\alpha\in\mathbb{R}}\ker(A-\alpha)\geqslant m$.

Contrary to the case of a positive definite Hermitian form, Lemma 6.3 does not extend to a pair of commuting self-adjoint operators. A counterexample is given by the multiplication operators in the ring $\mathbb{C}[x,y]/(x^2=y^2,xy=0)$ with the usual Grothendieck residue form. Explicitly, we have a 4-dimensional commutative real unital algebra of linear operators in $\mathbb{C}^4$ generated by two matrices

$$x=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\quad\text{and}\quad y=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\1&0&0&0\end{pmatrix},$$

that satisfy the relations $x^2=y^2$ and $x^3=y^3=xy=yx=0$. In particular, both x and y have the only eigenvalue that equals zero: $M=M_x(0)=M_y(0)$. Clearly,

$$\dim\ker x=\dim\ker y=2\quad\text{and}\quad\dim(\ker x\cap\ker y)=1.$$

The Hermitian form is given by the matrix

$$J=\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}.$$

It is non-degenerate and has signature 2. Since $x^t J = J\bar{x}$ and $y^t J = J\bar{y}$, the operators x and y are self-adjoint and commuting, but have only one common eigenvector.

The given counterexample is minimal. If in addition to the assumption of Corollary 6.2, for each character $\varrho: \mathcal{A} \to \mathbb{C}$ we have $\dim \bigcap_{A\in\mathcal{A}} M_A(\varrho(A)) < 4$, then there are at least m linearly independent common eigenvectors of the elements of $\mathcal{A}$ with real eigenvalues:

$$\dim \bigcap_{A\in\mathcal{A}} \bigoplus_{\alpha\in\mathbb{R}} \ker(A-\alpha) \geqslant m.$$

7. The lower bound

In this section we prove our main theorem—the lower bound for the number of real solutions to osculating Schubert problems, see Theorem 7.2 and Corollary 7.3.

Recall the notation from §3. For positive integers n and d such that $d>n$ we consider the Grassmannian of $\mathrm{Gr}(n,d)$ of n-dimensional planes in the space $\mathbb{C}_d[x]$ of polynomials of degree less than d. Recall that $W\in\mathrm{Gr}(n,d)$ is called real if it has a basis consisting of polynomials with real coefficients.

Given partitions $\lambda^{(1)}, ..., \lambda^{(k)}$ and ν with at most n parts satisfying (3.1), and distinct complex numbers $z_1, ..., z_k$, denote by $d(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ the number of real points counted with multiplicities in the intersection of Schubert varieties $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) \subset \mathrm{Gr}(n,d)$. Clearly, $d(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) = 0$ unless the set $\{z_1, ..., z_k\}$ is invariant under complex conjugation and $\lambda^{(i)} = \lambda^{(j)}$ whenever $z_i = \bar{z}_j$. In what follows we denote by c the number of complex conjugate pairs in the set $\{z_1, ..., z_k\}$ and without loss of generality assume that $z_1 = \bar{z}_2, ..., z_{2c-1} = \bar{z}_{2c}$ while $z_{2c+1}, ..., z_k$ are real. We will also always assume that $\lambda^{(1)} = \lambda^{(2)}, ..., \lambda^{(2c-1)} = \lambda^{(2c)}$.

For the sake of clarity, let us emphasize that by *generic* we always mean *on a non-empty Zariski-open subset of* $\mathbb{C}^k$. Recall that for any $\boldsymbol{\lambda}, \nu$ and generic complex $\boldsymbol{z}$, the intersection of Schubert varieties is transversal, that is, all points of $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$ are multiplicity-free. The same holds true under the reality condition on $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ imposed above for any c.

Let $L_{\boldsymbol{\lambda}} = L_{\lambda^{(1)}} \otimes ... \otimes L_{\lambda^{(k)}}$ be the tensor product of irreducible $\mathfrak{gl}_n$-modules and let $M_{\boldsymbol{\lambda},\mu}$ be the multiplicity space of L_μ in $L_{\boldsymbol{\lambda}}$, see §4. Since $\lambda^{(2i-1)} = \lambda^{(2i)}$ for $i=1, ..., c$, the flip P_{2i-1} of the $(2i-1)$-st and $2i$-th tensor factors of $L_{\boldsymbol{\lambda}}$ commutes with the $\mathfrak{gl}_n$-action and thus acts on $M_{\boldsymbol{\lambda},\mu}$. Denote by $P_{\boldsymbol{\lambda},\mu,c} \in \mathrm{End}(M_{\boldsymbol{\lambda},\mu})$ the action of the product $P_1 P_3 ... P_{2c-1}$ on $M_{\boldsymbol{\lambda},\mu}$.

The operator $P_{\boldsymbol{\lambda},\mu,c}$ is self-adjoint relative to the Hermitian form $(\cdot,\cdot)_{\boldsymbol{\lambda},\mu}$ on $M_{\boldsymbol{\lambda},\mu}$ given in §5. Define a new Hermitian form $(\cdot,\cdot)_{\boldsymbol{\lambda},\mu,c}$ on $M_{\boldsymbol{\lambda},\mu}$ by the rule: for any

$v, w \in M_{\boldsymbol{\lambda},\mu}$,

$$(v, w)_{\boldsymbol{\lambda},\mu,c} = (P_{\boldsymbol{\lambda},\mu,c} v, w)_{\boldsymbol{\lambda},\mu}.$$

Denote by $q(\boldsymbol{\lambda}, \mu, c)$ the signature of the form $(\cdot, \cdot)_{\boldsymbol{\lambda},\mu,c}$.

PROPOSITION 7.1. *The signature $q(\boldsymbol{\lambda}, \mu, c)$ equals the coefficients of the monomial $x_1^{\mu_1+n-1} x_2^{\mu_2+n-2} \ldots x_n^{\mu_n}$ in the polynomial*

$$\Delta_n \cdot \prod_{i=1}^{c} S_{\lambda^{(2i)}}(x_1^2, \ldots, x_n^2) \prod_{j=2c+1}^{k} S_{\lambda^{(j)}}(x_1, \ldots, x_n).$$

Here Δ_n is the Vandermonde determinant (2.1) *and S_λ are the Schur polynomials* (2.2).

Proof. Since $P_{\boldsymbol{\lambda},\mu,c}^2 = 1$, we have $q(\boldsymbol{\lambda}, \mu, c) = \operatorname{tr}_{M_{\boldsymbol{\lambda},\mu}} P_{\boldsymbol{\lambda},\mu,c}$, and the claim follows from Proposition 2.1. □

THEOREM 7.2. *We have $d(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) \geqslant |q(\boldsymbol{\lambda}, \mu, c)|$, where μ is the complement of ν in the $n \times (d-n)$ rectangle, $\mu = (d-n-\nu_n, d-n-\nu_{n-1}, \ldots, d-n-\nu_1)$; cf.* (3.3).

Proof. By Proposition 5.1 and Lemma 5.2, the operators $B_{ij}(\boldsymbol{\lambda}, \mu, \boldsymbol{z}) \in \operatorname{End}(M_{\boldsymbol{\lambda},\mu})$ are self-adjoint relative to the form $(\cdot, \cdot)_{\boldsymbol{\lambda},\mu}^P$. By Corollary 6.2,

$$\dim \Bigl(\bigcap_{i,j} \bigoplus_{\alpha \in \mathbb{R}} M_{B_{ij}(\boldsymbol{\lambda},\mu,\boldsymbol{z})}(\alpha) \Bigr) \geqslant |q(\boldsymbol{\lambda}, \mu, c)|.$$

By Theorem 4.2, for any $\boldsymbol{\lambda}$, ν and generic complex $\boldsymbol{z}$ the operators $B_{ij}(\boldsymbol{\lambda}, \mu, \boldsymbol{z})$ are diagonalizable. The same holds true under the reality condition on $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ imposed in this section for any c. Thus, for generic $\boldsymbol{z}$, the operators $B_{ij}(\boldsymbol{\lambda}, \mu, \boldsymbol{z})$ have at least $|q(\boldsymbol{\lambda}, \mu, c)|$ common eigenvectors with real eigenvalues, which provides $|q(\boldsymbol{\lambda}, \mu, c)|$ distinct real points in $\Omega(\boldsymbol{\lambda}, \nu, \boldsymbol{z})$. Hence, $d(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) \geqslant |q(\boldsymbol{\lambda}, \mu, c)|$ for generic $\boldsymbol{z}$, and therefore, for any $\boldsymbol{z}$, due to counting with multiplicities. □

COROLLARY 7.3. *We have $d(\boldsymbol{\lambda}, \nu, \boldsymbol{z}) \geqslant |a(\boldsymbol{\lambda}, \nu, c)|$, where $a(\boldsymbol{\lambda}, \nu, c)$ is the coefficient of the monomial $x_1^{d-1-\nu_n} x_2^{d-2-\nu_{n-1}} \ldots x_n^{d-n-\nu_1}$ in the polynomial*

$$\Delta_n \cdot \prod_{i=1}^{c} S_{\lambda^{(2i)}}(x_1^2, \ldots, x_n^2) \prod_{j=2c+1}^{k} S_{\lambda^{(j)}}(x_1, \ldots, x_n).$$

Here Δ_n is the Vandermonde determinant (2.1) *and S_λ are the Schur polynomials* (2.2).

Proof. The claim follows from Theorem 7.2 and Proposition 7.1. □

Recall that the total number of points in $\Omega(\boldsymbol{\lambda},\nu,\boldsymbol{z})$ equals $\dim M_{\boldsymbol{\lambda},\mu}=q(\boldsymbol{\lambda},\mu,0)$. It is proved in [13] that for real $z_1,...,z_k$ all points in $\Omega(\boldsymbol{\lambda},\nu,\boldsymbol{z})$ are real and multiplicity-free. The proof of Theorem 7.2 here is a modification of the reasoning used in [13].

Let $\tilde{\boldsymbol{\lambda}}$ be the $(k+1)$-tuple $\lambda^{(1)},...,\lambda^{(k)},\nu$ and $\delta=(d-n,...,d-n)$ be the rectangular partition with n rows. There is a natural isomorphism of the multiplicity spaces $M_{\boldsymbol{\lambda},\mu}$ and $M_{\tilde{\boldsymbol{\lambda}},\delta}$ that is consistent with the forms $(\cdot,\cdot)_{\boldsymbol{\lambda},\mu}$ and $(\cdot,\cdot)_{\tilde{\boldsymbol{\lambda}},\delta}$ and intertwines the operators $P_{\boldsymbol{\lambda},\mu,c}$ and $P_{\tilde{\boldsymbol{\lambda}},\delta,c}$. Therefore, $q(\boldsymbol{\lambda},\mu,c)=q(\tilde{\boldsymbol{\lambda}},\delta,c)$ and $a(\boldsymbol{\lambda},\nu,c)=a(\tilde{\boldsymbol{\lambda}},\varnothing,c)$, where $\varnothing=(0,...,0)$ is the empty partition.

The corresponding statement in the osculating Schubert calculus is as follows. Let F be a Möbius transformation mapping the real line to the real line and such that $\infty\notin\{F(z_1),...,F(z_k),F(\infty)\}$. Set $\tilde{\boldsymbol{z}}=(F(z_1),...,F(z_k),F(\infty))$. Then F defines an isomorphism of $\Omega(\boldsymbol{\lambda},\nu,\boldsymbol{z})$ and $\Omega(\tilde{\boldsymbol{\lambda}},\varnothing,\tilde{\boldsymbol{z}})$ that maps real points to real points, and $d(\boldsymbol{\lambda},\nu,\boldsymbol{z})=d(\tilde{\boldsymbol{\lambda}},\varnothing,\tilde{\boldsymbol{z}})$.

Consider the transposed partitions $(\lambda^{(1)})',...,(\lambda^{(k)})',\nu'$, and treat them as partitions with at most $d-n$ parts, adding extra zero parts if necessary. Denote by $\boldsymbol{\lambda}'$ be the k-tuple $(\lambda^{(1)})',...,(\lambda^{(k)})'$. By the Lagrangian involution for the osculating Schubert problems, see [9, §4], the intersections of Schubert varieties $\Omega(\boldsymbol{\lambda},\nu,\boldsymbol{z})\subset\mathrm{Gr}(n,d)$ and $\Omega(\boldsymbol{\lambda}',\nu',\boldsymbol{z})\subset\mathrm{Gr}(d-n,d)$ are isomorphic by taking the orthogonal complements in $\mathbb{C}_d[x]$ relative to the following bilinear form: $\langle x^p/p!,x^q/q!\rangle=(-1)^p\delta_{p+q,d-1}$, $p=0,...,d-1$. In particular, $d(\boldsymbol{\lambda},\nu,c)=d(\boldsymbol{\lambda}',\nu',c)$.

On the other hand, define the multiplicity space $M_{\boldsymbol{\lambda}',\mu'}$ using the Lie algebra $\mathfrak{gl}_{d-n}$. There is a natural isomorphism of the spaces $M_{\boldsymbol{\lambda},\mu}$ and $M_{\boldsymbol{\lambda}',\mu'}$ that is consistent with the forms $(\cdot,\cdot)_{\boldsymbol{\lambda},\mu}$ and $(\cdot,\cdot)_{\boldsymbol{\lambda}',\mu'}$ and intertwines the operators $P_{\boldsymbol{\lambda},\mu,c}$ and $(-1)^m P_{\boldsymbol{\lambda}',\mu',c}$, where $m=\sum_{i=1}^{c}|\lambda^{(2i)}|$. Therefore,

$$q(\boldsymbol{\lambda},\mu,c)=(-1)^m q(\boldsymbol{\lambda}',\mu',c)\quad\text{and}\quad a(\boldsymbol{\lambda},\nu,c)=(-1)^m a(\boldsymbol{\lambda}',\nu',c).$$

8. Comparison with the available results and data

In this section we will compare the lower bound for the number of real solutions of the osculating Schubert problem provided by Corollary 7.3 against other available data.

Recall that we consider the Grassmannian $\mathrm{Gr}(n,d)$, so the partitions $\lambda^{(1)},...,\lambda^{(k)}$ and ν have at most n non-zero parts and satisfy (3.1).

We discuss bounds that are independent of $z_1,...,z_k$ and say that a bound is sharp if it is attained for some values of $z_1,...,z_k$. We assume that the set $\{z_1,...,z_k\}$ is invariant under complex conjugation and $\lambda^{(i)}=\lambda^{(j)}$ whenever $z_i=\bar{z}_j$. The number of complex conjugate pairs in $\{z_1,...,z_k\}$ is denoted by c.

To save writing, we will indicate only non-zero parts in partitions and omit zeros. We call the osculating Schubert problem for the case of $\lambda^{(1)}=\ldots=\lambda^{(k)}=(1)$ and arbitrary ν, the *vector Schubert problem*.

The topological degree of a real Wronski map gives a lower bound for the number of real solutions for the vector Schubert problem. This degree was computed in [2] and extended in [15] to the case of $\lambda^{(1)}=\ldots=\lambda^{(k-1)}=(1)$ and arbitrary $\lambda^{(k)}$ and ν. The result is given in terms of the sign-imbalance of the skew Young diagram $\mu/\lambda^{(k)}$, where the partition μ is the complement of ν in the $n\times(d-n)$ rectangle, see (3.3). For $\lambda^{(k)}=(1)$ and empty ν, so that $k=n(d-n)$, the sign-imbalance was computed in [18]. The results is 0 for even d and

$$\frac{\left(\frac{1}{2}n(d-n)\right)!}{\left(\frac{1}{2}(d-1)\right)!}\prod_{i=1}^{n-1}\frac{i!(d-n-i)!}{(d-2n+2i)!\left(\frac{1}{2}(d-2n-1)+i\right)!} \tag{8.1}$$

for odd d. Unlike Corollary 7.3, this bound is independent of the number of complex conjugate pairs among $z_1, \ldots, z_k$.

This bound is not sharp for the case $n=3$, $d=6$, $k=9$, as shown in [9]. It is proved there that the problem has at least two real solutions. For this case, Corollary 7.3 gives lower bounds $a=42, 0, 2, 0, 6$ for $c=0, 1, 2, 3, 4$ respectively. Thus our bound is not sharp for $c=1, 3$, but, according to the computer data, see [8], it is sharp for $c=0, 2, 4$.

On the other hand, for the case of $n=3$, $d=8$, $k=15$, the topological bound of [2] gives zero, the results of [9] are not applicable, and Corollary 7.3 yields

$$a=6006, 858, 198, 42, 6, 10, 10, 70 \quad \text{for } c=0, 1, 2, 3, 4, 5, 6, 7, \text{ respectively.}$$

In particular, it shows that the real Wronski map $\mathrm{Gr}^{\mathbb{R}}(3,8)\to\mathbb{RP}^{15}$, which sends 3-dimensional subspaces of $\mathbb{R}_8[x]$ to their Wronski determinants, is surjective; see [3] for discussion of surjectivity of real Wronski maps.

In another example, $n=3$, $d=9$, $k=18$, the topological bound (8.1) is 12, and Corollary 7.3 gives

$$a=87516, 15444, 3432, 792, 180, 60, 0, 0, 140, 420 \quad \text{for } c=0, \ldots, 9, \text{ respectively.}$$

Thus the topological bound is better for $c=6, 7$, while Corollary 7.3 wins in the other cases.

For the case $n=2$, $k=2(d-2)$, $c=d-3$, the bounds of (8.1) and Corollary 7.3 coincide: both equal zero for even d and $(2s)!/s!(s+1)!$ for odd $d=2s+3$. The bounds are known to be sharp in this case and the examples attaining the bounds can be obtained for the points $z_1, \ldots, z_k$ located on a circle, see [1].

A large amount of computer generated data is available at [8], and we have tested the bound given by Corollary 7.3 against them. It coincides with the computer prediction in amazingly many cases. For example, out of eleven computer generated tables presented in [7], the bound given by Corollary 7.3 is sharp in all the cases except for the second row of Table 5 corresponding to the vector Schubert problem with $k{=}7$, $\nu{=}(3,3,3)$, for $\mathrm{Gr}(4,8)$. In this case, Corollary 7.3 gives the bounds $a{=}20,0,4,0$ for $c{=}0,1,2,3$, and the computer data are $20,8,4,0$, indicating a possible deficiency for $c{=}1$.

Also, for the case of $n{=}2$, there are sixty computer generated bounds with nineteen of them being non-zero. All of them match the bounds given by Corollary 7.3.

Call the osculating Schubert problem *symmetric* if $\lambda^{(i)}{=}(\lambda^{(i)})'$ for all $i{=}1,...,k$, and $\nu{=}\nu'$. In this case, the numbers of real solutions for different c are often congruent modulo four, see [9]. Since the number of real solutions for $c{=}0$ is known, it gives under some additional assumptions a lower bound of two for the number of real solutions whenever the number of complex solutions is not divisible by four. It seems that many, though not all, discrepancies we found between the bound given by Corollary 7.3 and the computer data happen in symmetric problems. For example, the remark at the end of §7 shows that $a(\boldsymbol{\lambda},\nu,c){=}0$ for the symmetric Schubert problem if $\sum_{i=1}^{c}|\lambda_{2i}|$ is odd, but in some of those cases the zero bound is not sharp according to the computer generated data.

Finally, consider the vector Schubert problem with $\nu{=}(k{-}n,...,k{-}n)$ having $n{-}1$ non-zero parts, for the Grassmannian $\mathrm{Gr}(n,k{+}1)$. The number of real solutions of this problem for given $z_1,...,z_k$ has been found in [7] and is given by the coefficient $r(k,n,s)$ of the monomial $x^{k-n}y^{n-1}$ in the polynomial $(x{+}y)^{k-1-2s}(x^2{+}y^2)^s$, where $k{-}1{-}2s$ is the number of real roots of the polynomial

$$g(u)=\frac{d}{du}\prod_{i=1}^{k}(u-z_i).$$

It is easy to check that $r(k,n,s{-}1)\geqslant r(k,n,s)$ if $1\leqslant s<\frac{1}{2}k$. By Rolle's theorem, $s\leqslant c$ if $2c<k$, and $s\leqslant c{-}1$ if $2c{=}k$. Thus, either $r(k,n,c)$ or $r(k,n,c{-}1)$ gives the lower bound for the number of real solutions of the Schubert problem in question, depending on whether $2c<k$ or $2c{=}k$. These lower bounds are sharp because the equalities $s{=}c$ for $2c<k$ and $s{=}c{-}1$ for $2c{=}k$ are attained, as the following examples show.

Example. Let $s{=}c$ and $2c<k$. For sufficiently small real ε, the polynomial

$$\prod_{i=1}^{c}(u^2+1-\varepsilon^i)\prod_{j=1}^{k-2c}(u-\varepsilon^j)$$

has exactly $k{-}2c$ real roots and its derivative has exactly $k{-}1{-}2c$ real roots.

Example. Let $s=c-1$ and $2c=k$. The polynomial $(x^2+1)^c$ has no real roots and its derivative has exactly one real root.

For $n=3$, $k=14$, and $c=0, ..., 7$, the sharp lower bounds respectively equal $78, 56, 38, 24, 14, 8, 6, 6$, while the bounds given by Corollary 7.3 are $78, 54, 34, 18, 6, 2, 6, 6$. Similarly, for $n=4$, $k=11$, and $c=0, ..., 5$, the sharp lower bounds are $120, 64, 32, 16, 8, 0$ versus the bounds $120, 48, 8, 8, 8, 0$ given by Corollary 7.3.

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