



Local Hodge theory of Soergel bimodules

by

GEORDIE WILLIAMSON

*University of Sydney
Sydney, NSW, Australia*

Dedicated to Ben and Yeppe.

1. Introduction

In this paper we show the local hard Lefschetz theorem for Soergel bimodules, as conjectured by Soergel and Fiebig. A key new ingredient is the study of intersection forms and a proof of local Hodge–Riemann bilinear relations. These properties are interesting in themselves, as a further example of the remarkable Hodge theoretic structure present in Soergel bimodules (even when no geometry is obviously present). It is also important because, by work of Soergel and Kübel, it may be used to give an algebraic proof of the Jantzen conjectures on the Jantzen filtration on Verma modules. (The first proof of the Jantzen conjectures was given by Beilinson and Bernstein [2].)

In geometric situations Soergel bimodules may be obtained as the equivariant intersection cohomology of Schubert varieties. In this setting the local hard Lefschetz theorem and local Hodge–Riemann relations for Soergel bimodules follow from the hard Lefschetz theorem and Hodge–Riemann relations for equivariant intersection cohomology, applied to a punctured standard affine neighbourhood of a torus fixed point. Hence the results of this paper may be seen as a translation and proof of these Hodge theoretic statements into the algebra of Soergel bimodules.

This paper is a sequel to [11] by Ben Elias and the author. The main ideas for the proofs are already contained in [11]. This paper, like [11], draws much motivation from de Cataldo and Migliorini’s Hodge theoretic proof of the decomposition theorem [8], [9].

1.1. The fundamental example

We start by recalling the geometric setting that led Soergel and Fiebig to the local hard Lefschetz conjecture. It is based on [4, Chapter 14], where Bernstein and Lunts call this setting the “fundamental example”. For us the name is very appropriate: although all the proofs of this paper are algebraic, all the motivation comes from the fundamental example.

Assume that \mathbb{C}^* acts linearly on \mathbb{C}^n with positive weights (i.e. $\lim_{z \rightarrow 0} z \cdot v = 0$ for all $v \in \mathbb{C}^n$). Let $X \subset \mathbb{C}^n$ denote a closed \mathbb{C}^* -stable subvariety. Let $H_{\mathbb{C}^*}^*(pt; \mathbb{R})$ denote the \mathbb{C}^* -equivariant cohomology of a point, which we identify with $\mathbb{R}[z]$, where z “is” the first Chern class (of degree 2). When we come to discuss Soergel bimodules the choice of coefficients in the real numbers will be important. When discussing the fundamental example we could take coefficients in any field of characteristic zero. To simplify notation we take coefficients in the real numbers throughout.

Let IH^* (resp. $\mathrm{IH}_{\mathbb{C}^*}^*$, $\mathrm{IH}_{\mathbb{C}^*,c}^*$) denote (resp. equivariant, compactly supported) intersection cohomology. A basic fact is that we have a short exact sequence of graded $\mathbb{R}[z]$ -modules

$$0 \longrightarrow \mathrm{IH}_{\mathbb{C}^*,c}^*(X) \longrightarrow \mathrm{IH}_{\mathbb{C}^*}^*(X) \longrightarrow \mathrm{IH}^{*+1}(\dot{X}/\mathbb{C}^*) \longrightarrow 0, \tag{1.1}$$

where $\dot{X} := X \setminus \{0\}$ and $\mathrm{IH}^*(\dot{X}/\mathbb{C}^*)$ is a graded $H_{\mathbb{C}^*}^*(pt) = \mathbb{R}[z]$ module via the identification

$$\mathrm{IH}^{*+1}(\dot{X}/\mathbb{C}^*) = \mathrm{IH}_{\mathbb{C}^*}^*(\dot{X}),$$

which holds because \mathbb{C}^* acts on \dot{X} with finite stabilisers.

The above sequence is obtained by taking equivariant hypercohomology of the standard (“Gysin”) distinguished triangle for the equivariant intersection cohomology sheaf on X with respect to the decomposition $X = \{0\} \sqcup \dot{X}$. The first and second terms of (1.1) can be identified with the hypercohomology of the costalk and stalk of the intersection cohomology sheaf at $0 \in X$ respectively. The resulting long exact sequence yields the short exact sequence (1.1) by purity, which ensures that all connecting homomorphisms are zero.

To lighten notation we set $M^! := \mathrm{IH}_{\mathbb{C}^*,c}^*(X)$, $M := \mathrm{IH}_{\mathbb{C}^*}^*(X)$ and $H := \mathrm{IH}^*(\dot{X}/\mathbb{C}^*)$ so that our sequence takes the form

$$0 \longrightarrow M^! \longrightarrow M \longrightarrow H[1] \longrightarrow 0. \tag{1.2}$$

($H[1]$ denotes a degree shift: $H[1]^i = H^{i+1}$.) Important ingredients in the fundamental example are the following facts about the sequence (1.2):

- (1) $M^!$ (resp. M) is a finitely generated free $\mathbb{R}[z]$ -module (as follows from purity) generated in degrees > 0 (resp. < 0) (a consequence of the degree bounds on the stalks and costalks of intersection cohomology complexes).

(2) For all $i \geq 0$ multiplication by z^i induces an isomorphism

$$z^i: H^{-i} \xrightarrow{\sim} H^i.$$

(Indeed, the operator of multiplication by our generator $z \in H_{\mathbb{C}^*}^2(pt)$ on

$$\mathrm{IH}_{\mathbb{C}^*}^*(\dot{X}) = \mathrm{IH}^*(\dot{X}/\mathbb{C}^*)[-1]$$

may be identified, up to a non-zero scalar, with the action of the Chern class of the closed embedding $\dot{X}/\mathbb{C}^* \hookrightarrow (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^* = \mathbb{P}$, into a weighted projective space. Now the result follows by the hard Lefschetz theorem for intersection cohomology.)

(3) As the intersection cohomology of a projective variety, H is equipped with a non-degenerate graded intersection pairing

$$\langle \cdot, \cdot \rangle: H \times H \rightarrow \mathbb{R}.$$

Moreover, for each $i \geq 0$, the ‘‘Lefschetz’’ form on H^{-i} given by $(h, h') := \langle h, z^i h' \rangle$ (non-degenerate by (2)) induces a Hermitian form on $H^{-i} \otimes_{\mathbb{R}} \mathbb{C}$ whose signature is governed by the Hodge–Riemann bilinear relations.

This paper is concerned with establishing algebraic analogues of (2) and (3) in the setting of Soergel bimodules. The bimodule analogue of (1) is Soergel’s conjecture, which was established in [11].

1.2. Results

Let (W, S) denote a Coxeter system. Let \mathfrak{h} denote the reflection faithful representation of (W, S) over \mathbb{R} described in §3.2, and let $\{\alpha_s\} \subset \mathfrak{h}^*$ and $\{\alpha_s^\vee\} \subset \mathfrak{h}$ denote the simple roots and coroots (see §3.2). Let R denote the symmetric algebra on \mathfrak{h}^* with $\deg \mathfrak{h}^* = 2$. Let \mathcal{B} denote the category of Soergel bimodules (see §6.2). For any $y \in W$, let $B(y)$ denote the indecomposable self-dual Soergel bimodule parameterised by y .

Let B denote a Soergel bimodule and fix $x \in W$. Define $B_x^!$ (resp. B_x) to be the largest submodule (resp. largest quotient) of B on which we have the relation $b \cdot r = x(r) \cdot b$ for all $r \in R$. Then $B_x^!$ and B_x are free left R -modules. If B is indecomposable and self-dual then $B_x^!$ (resp. B_x) is generated in degrees > 0 (resp. < 0) and their graded ranks are given by Kazhdan–Lusztig polynomials (Soergel’s conjecture). Inclusion followed by projection gives a canonical map

$$i_x: B_x^! \hookrightarrow B \twoheadrightarrow B_x.$$

Moreover i_x is an isomorphism over Q , the localisation of R at all roots.

Any $\zeta^\vee \in \mathfrak{h}$ yields a specialisation $R \rightarrow \mathbb{R}[z]$ given on degree-2 elements by $\alpha \mapsto \langle \alpha, \zeta^\vee \rangle z$ (‘‘restriction to the line $\mathbb{R}\zeta^\vee \subset \mathfrak{h}$ ’’).

THEOREM 1.1. (Local hard Lefschetz) *Suppose that $\varrho^\vee \in \mathfrak{h}$ is dominant (that is, $\langle \alpha_s, \varrho^\vee \rangle > 0$ for all $s \in S$) and that B is indecomposable and self-dual. Define $H[1]$ as the cokernel of the inclusion:*

$$0 \longrightarrow \mathbb{R}[z] \otimes_R B_x^! \xrightarrow{i_x} \mathbb{R}[z] \otimes_R B_x \longrightarrow H[1] \longrightarrow 0.$$

Then H satisfies the hard Lefschetz theorem: multiplication by z^i yields an isomorphism $H^{-i} \rightarrow H^i$ for all $i \geq 0$.

This result was conjectured by Soergel [28, Bemerkung 7.2] and Fiebig [13, Conjecture 6.2], motivated (as we will explain below) by the fundamental example applied to the link of a singularity in a Schubert variety. In fact they conjectured the theorem to hold for any $\zeta^\vee \in \mathfrak{h}$ such that $\langle \alpha, \zeta^\vee \rangle \neq 0$ for any root α . The conjecture is false in this generality. We will explain below that for Weyl groups its failure is related to the failure of semi-simplicity of the layers of the Jantzen filtration associated with certain non-dominant regular deformation directions.

The local hard Lefschetz theorem is the only geometric ingredient in Fiebig’s bound for the exceptional characteristics occurring in Lusztig’s conjecture (see [15, §1.2]). Using the above theorem one can deduce the results of [15] without recourse to geometry.

We now discuss the Hodge–Riemann bilinear relations. Suppose that B is indecomposable and self-dual. Then B carries an intersection form⁽¹⁾

$$\langle \cdot, \cdot \rangle_B : B \times B \rightarrow R$$

which is graded, symmetric and non-degenerate. (This is the analogue of the equivariant intersection pairing in equivariant cohomology.)

Restricting $\langle \cdot, \cdot \rangle_B$ to $B_x^! \subset B$, extending scalars to Q and using that i_x gives us a canonical identification $Q \otimes_R B_x^! = Q \otimes_R B_x$ we obtain a symmetric and R -bilinear Q -valued form

$$\langle \cdot, \cdot \rangle_B^x : B_x \times B_x \rightarrow Q.$$

This is the *local intersection form* on B_x ; it is the main object in this paper.

Let $\varrho^\vee \in \mathfrak{h}$ be dominant as above, and let $R \rightarrow \mathbb{R}[z]$ denote the corresponding specialisation. To simplify notation, set

$$N := \mathbb{R}[z] \otimes_R B_x.$$

Since $\langle \varrho^\vee, \alpha \rangle \neq 0$ for any root α , we have that ϱ^\vee also induces a specialisation $Q \rightarrow \mathbb{R}[z^{\pm 1}]$, and the local intersection form $\langle \cdot, \cdot \rangle_B^x$ induces a symmetric $\mathbb{R}[z]$ -bilinear $\mathbb{R}[z^{\pm 1}]$ -valued form $\langle \cdot, \cdot \rangle_N$ on N .

⁽¹⁾ Throughout this paper, *form* always means symmetric bilinear form.

The Hodge–Riemann bilinear relations give the signatures of the restrictions of these forms to any homogeneous component of N . For $i > 0$ set

$$P^{-i} := (\deg_{<-i} N)^\perp \cap N^{-i}.$$

(Here $\deg_{<i} N$ denotes the submodule of N generated by all elements of degree $< i$.) Then the hard Lefschetz theorem implies that we have a decomposition

$$N = \bigoplus_{i>0} \mathbb{R}[z] \otimes_{\mathbb{R}} P^{-i},$$

which is orthogonal with respect to $\langle \cdot, \cdot \rangle_N$. Let \min denote the minimal non-zero degree of N .

THEOREM 1.2. (Local Hodge–Riemann bilinear relations) *For any $i > 0$ the restriction of the \mathbb{R} -valued form $(n, n') := z^i \langle n, n' \rangle_N$ on N^{-i} to P^{-i} is $(-1)^{\ell(x)} (-1)^d$ -definite, where $d = \frac{1}{2}(-i - \min)$.*

(The module N vanishes unless i and \min are congruent modulo 2, and hence the sign makes sense.)

Let us try to explain what the Hodge–Riemann bilinear relations mean concretely for the local intersection forms $\langle \cdot, \cdot \rangle_B^x$. Fix a graded basis e_1, e_2, \dots, e_m for B_x as a left R -module, such that $\deg e_1 \leq \deg e_2 \leq \dots \leq \deg e_m$. We can think of the Gram⁽²⁾ matrix $((e_i, e_j)_B^x)_{1 \leq i, j \leq m}$ as giving us a non-degenerate symmetric form on the trivial vector bundle of rank m over $\mathfrak{h}_{\text{reg}} := \text{Spec } Q$. Moreover, this vector bundle is naturally filtered by the subspaces generated by $\{e_i\}_{\deg e_i \leq d}$. In other words, we can think of our form as a form on a filtered vector bundle. The Hodge–Riemann bilinear relations predict the signatures of the restriction of our form to all steps of the filtration over the dominant regular locus

$$\mathfrak{h}_{\text{reg}}^+ := \{\lambda^\vee \in \mathfrak{h}_{\text{reg}} : \langle \alpha_s, \lambda^\vee \rangle > 0 \text{ for all } s \in S\} \subset \mathfrak{h}_{\text{reg}}.$$

Roughly speaking the signs must alternate at each step in the filtration.

For example, if the graded rank of B_x is given by $v^{-5} + 3v^{-3} + 2v^{-1}$ and $\ell(x)$ is even, then the signs alternate as follows:

$$\begin{pmatrix} + & & & & & \\ & - & & & & \\ & & - & & & \\ & & & - & & \\ & & & & + & \\ & & & & & + \end{pmatrix}$$

⁽²⁾ Throughout this paper *Gram matrix* means the (symmetric) matrix of a (symmetric) form in some basis.

If $\ell(x)$ is odd then the signs are given by $-+ + + - -$.

Finally, there is one entry of the local intersection form which is canonical. If B is indecomposable and self-dual, then $B \cong B(y)$ for some $y \in W$, the smallest non-zero degree of $B(y)_x$ is $-\ell(y)$ and $B(y)_x^{-\ell(y)}$ is generated by an element $c_{x,y}$ which is well defined up to a non-zero scalar. Our final result calculates the pairing of this element with itself (see Theorem 6.19).

THEOREM 1.3. $\langle c_{x,y}, c_{x,y} \rangle_{B(y)}^x = \gamma e_{x,y}$ for some $\gamma \in \mathbb{R}_{>0}$.

Here $e_{x,y}$ is the “equivariant multiplicity”, a certain homogenous rational function in Q given by an explicit formula in the nil Hecke ring.

1.3. Relation to the fundamental example

Let us briefly comment on the connection between our results and the fundamental example.

Let $G \supset B \supset T$ denote a complex reductive algebraic group, a Borel subgroup and maximal torus, and let (W, S) denote its Weyl group and simple reflections. If we set X_* and X^* to be the cocharacter and character lattice of T then we can take $\mathfrak{h} := \mathbb{R} \otimes_{\mathbb{Z}} X_*$ and $\mathfrak{h}^* := \mathbb{R} \otimes_{\mathbb{Z}} X^*$. The Borel homomorphism gives us a canonical identification

$$R = S(\mathfrak{h}^*) = H_T^*(pt).$$

Given any $y \in W$ we can consider the Schubert variety $Z_y := \overline{ByB}/\overline{B} \subset G/B$. By a theorem of Soergel [27, §3.4] we may identify $B(y)$ with the equivariant intersection cohomology $\mathrm{IH}_T^*(Z_y)$. The bimodule structure comes from the fact that $\mathrm{IH}_T^*(Z_y)$ is a module over $H_T^*(G/B) = R \otimes_{R^W} R$.

If $B := B(y)$ then the R -modules $B_x^!$ and B_x can be described as the T -equivariant cohomology of the costalk and stalk of the intersection cohomology complex of Z_y at the torus fixed point $xB/B \in G/B$. Moreover, any choice of homomorphism $\gamma^\vee: \mathbb{C}^* \rightarrow T$ yields a line $\mathbb{R}\gamma^\vee \subset \mathfrak{h}$, hence a specialisation $R \rightarrow \mathbb{R}[z]$, and one may obtain the equivariant cohomology (with respect to the induced \mathbb{C}^* -action) of the stalk and costalk via extension of scalars.

Now each T -fixed point xB/B in Z_y has a unique T -stable affine neighbourhood $X_{x,y}$. We deduce from the exact sequence (1.1) that if $\gamma^\vee: \mathbb{C}^* \rightarrow T$ is such that the induced action of \mathbb{C}^* on $X_{x,y}$ is attractive, then we have

$$H = \mathrm{IH}^*(\dot{X}_{x,y}/\mathbb{C}^*),$$

where $\dot{X}_{x,y} := X_{x,y} \setminus \{xB/B\}$. The hard Lefschetz and Hodge–Riemann relations now follow from the hard Lefschetz and Hodge–Riemann bilinear relations in intersection cohomology.

The need to reduce from the T -action to a \mathbb{C}^* -action to apply the fundamental example corresponds to the choice of cocharacter $\varrho^\vee \in \mathfrak{h}$ in Theorems 1.1 and 1.2. If one chooses a cocharacter $\mathbb{C}^* \rightarrow T$ such that the induced action on $X_{x,y}$ is no longer attractive but is still regular (i.e. $X_{x,y}^{\mathbb{C}^*} = xB/B$) one still has

$$H[1] = \mathrm{IH}_{\mathbb{C}^*}^*(\dot{X}_{x,y}),$$

but now there is no longer any reason why H should satisfy hard Lefschetz or the Hodge–Riemann bilinear relations, because we cannot identify H with the intersection cohomology of a projective variety. We will see below that hard Lefschetz does indeed fail for certain specialisations corresponding to regular (that is, $\langle \alpha, \gamma^\vee \rangle \neq 0$ for all roots α) non-dominant $\gamma^\vee \in \mathfrak{h}$.

1.4. The Jantzen filtration

We conclude the introduction with a discussion of how our results are connected to the Jantzen filtration and conjectures.

Let $\mathfrak{g}^\vee \supset \mathfrak{b}^\vee \supset \mathfrak{t}^\vee$ denote a complex semi-simple Lie algebra, Borel subalgebra and Cartan subalgebra. (The notation is intended to suggest that this data should be Langlands dual to that of §1.3.) Given any weight $\lambda \in (\mathfrak{t}^\vee)^*$, we can consider $\Delta(\lambda)$, the corresponding Verma module. It is generated by a highest weight vector v_λ which satisfies

$$h \cdot v_\lambda = \lambda(h)v_\lambda \quad \text{for all } h \in \mathfrak{t}^\vee.$$

Given a deformation direction $\gamma \in (\mathfrak{t}^\vee)^*$ one can consider the deformed Verma module $\Delta_{\mathbb{C}[z]}(\lambda)$, which is a $(\mathfrak{g}, \mathbb{C}[z])$ -bimodule generated by a highest weight vector v_λ satisfying

$$h \cdot v_\lambda = (\lambda(h) + z\gamma(h))v_\lambda \quad \text{for all } h \in \mathfrak{t}^\vee.$$

That is, $\Delta_{\mathbb{C}[z]}(\lambda)$ is a “deformation of $\Delta(\lambda)$ in the direction γ ”. The deformed Verma module $\Delta_{\mathbb{C}[z]}(\lambda)$ admits a unique $\mathbb{C}[z]$ -bilinear contravariant form which specialises at $z=0$ to the contravariant form on $\Delta(\lambda)$. On $\Delta_{\mathbb{C}[z]}(\lambda)$ one has a filtration by order of vanishing of the form, and if one considers the specialisation at $z=0$ one obtains the *Jantzen filtration*

$$\dots \subset J^1 \subset J^0 = \Delta(\lambda),$$

which is exhaustive if γ is regular.

The Jantzen conjectures [19, §5.17] are the statements (for deformation direction $\gamma = \varrho$, the half sum of the positive roots):

- (1) certain canonical maps (e.g. embeddings $\Delta(\mu) \hookrightarrow \Delta(\lambda)$) are strict for Jantzen filtrations (see [19, §5.17, equation (2)]);
- (2) the Jantzen filtration coincides with the socle filtration.

(In [19, §5.17] both statements are questions rather than conjectures, and (1) is given more weight than (2).) It was subsequently realised that the Jantzen conjectures have remarkable consequences: Gabber and Joseph [17] showed that (1) implies the Kazhdan–Lusztig conjectures on multiplicities of simple modules in Verma modules (in a stronger form: the Kazhdan–Lusztig polynomials give multiplicities in the layers of the Jantzen filtration). Building on the work of Gabber and Joseph, Barbasch [1] showed that (1) implies (2).

The Jantzen conjectures were proved by Beilinson and Bernstein in [2]. They prove that the Jantzen filtration corresponds under Beilinson–Bernstein localisation to the weight filtration on a standard D -module. Part (1) of the Jantzen conjectures follows from the fact that any morphism between mixed perverse sheaves strictly preserves the weight filtration. Part (2) follows via a pointwise purity argument.

1.5. The approach of Soergel and Kübel

An alternative (“Koszul dual”) proof of the Jantzen conjectures was initiated by Soergel [29] and completed by Kübel [22], [23]. Recall that, by results of Soergel (see [25]), the principal block \mathcal{O}_0 of category \mathcal{O} is equivalent to (ungraded) modules over a graded algebra $A_{\mathcal{O}}$. Soergel’s conjecture is equivalent to the fact that $A_{\mathcal{O}}$ may be chosen positively graded and semi-simple in degree zero.

If one instead considers *graded* modules over $A_{\mathcal{O}}$ then one obtains a graded enhancement of the principal block of \mathcal{O} . It is known that Verma modules are gradable; that is, the corresponding $A_{\mathcal{O}}$ -modules admit gradings. Taken together, the results of Soergel and Kübel show that the Jantzen filtration on a Verma module agrees with the degree filtration on a graded lift. Then part (1) of the Jantzen conjectures is immediate, because the canonical maps in question can be lifted to maps of graded modules. Part (2) follows because the socle, radical and degree filtrations for the graded lifts of Verma modules coincide. (Once one knows that the degree zero part of $A_{\mathcal{O}}$ is semi-simple, that $A_{\mathcal{O}}$ is generated in degrees ≤ 1 , and that the head and socle of a Verma module is simple, this follows from a simple observation about modules over graded algebras [3, Proposition 2.4.1].)

We now explain in more detail how the link between the Jantzen and grading filtra-

tions is made. Let W denote the Weyl group of $\mathfrak{g}^\vee \supset \mathfrak{t}^\vee$. For $x \in W$, denote by $\Delta(x)$ and $\nabla(x)$ the Verma and dual Verma modules of highest weight $x(\varrho) - \varrho$. Let T denote an indecomposable tilting module in \mathcal{O}_0 . Then $\Delta(x)$, $\nabla(x)$ and T all admit “deformations in the direction γ ” over $\mathbb{C}[[z]]$. (We need to pass from $\mathbb{C}[z]$ to its completion $\mathbb{C}[[z]]$ to apply idempotent lifting arguments.) We denote these deformations by $\Delta_{\mathbb{C}[[z]]}(x)$, $\nabla_{\mathbb{C}[[z]]}(x)$ and $T_{\mathbb{C}[[z]]}$. Consider the canonical pairing

$$\mathrm{Hom}(\Delta_{\mathbb{C}[[z]]}(x), T_{\mathbb{C}[[z]]}) \times \mathrm{Hom}(T_{\mathbb{C}[[z]]}, \nabla_{\mathbb{C}[[z]]}(x)) \longrightarrow \mathrm{Hom}(\Delta_{\mathbb{C}[[z]]}(x), \nabla_{\mathbb{C}[[z]]}(x))$$

which lands in $\mathrm{Hom}(\Delta_{\mathbb{C}[[z]]}(x), \nabla_{\mathbb{C}[[z]]}(x)) = \mathbb{C}[[z]]$. As in the definition of the Jantzen filtration, we may define a filtration on $\mathrm{Hom}(\Delta_{\mathbb{C}[[z]]}(x), T_{\mathbb{C}[[z]]})$ via order of vanishing. Upon specialisation at $z=0$, we obtain the *Andersen filtration*

$$\dots \subset F^{i+1} \subset F^i \subset \dots \subset F^0 = \mathrm{Hom}(\Delta(x), T),$$

which is exhaustive if γ is regular.

Because everything in sight is free over $\mathbb{C}[[z]]$ we can view the pairing defining the Andersen filtration instead as an inclusion (for γ regular)

$$\mathrm{Hom}(\Delta_{\mathbb{C}[[z]]}(x), T_{\mathbb{C}[[z]]}) \xrightarrow{\varkappa} \mathrm{Hom}(T_{\mathbb{C}[[z]]}, \nabla_{\mathbb{C}[[z]]}(x))^*$$

where $*$ means $\mathbb{C}[[z]]$ dual. Now the Andersen filtration is obtained as the specialisation at $z=0$ of the filtration:

$$\dots \subset \varkappa^{-1}(z^{i+1} \mathrm{Hom}(T_{\mathbb{C}[[z]]}, \nabla_{\mathbb{C}[[z]]}(x))^*) \subset \varkappa^{-1}(z^i \mathrm{Hom}(T_{\mathbb{C}[[z]]}, \nabla_{\mathbb{C}[[z]]}(x))^*) \subset \dots$$

In [29, §10.2], Soergel identifies the inclusion \varkappa defining the Andersen filtration with the inclusion (now over \mathbb{C} rather than \mathbb{R})

$$\mathbb{C}[[z]] \otimes_{\mathbb{R}} B_x^! \xrightarrow{\mathbb{C}[[z]] \otimes i_x} \mathbb{C}[[z]] \otimes_{\mathbb{R}} B_x$$

appearing in the statement of local hard Lefschetz. Here B is an indecomposable self-dual Soergel bimodule such that $\widehat{B} = \mathbb{V}T_{\widehat{S}}$, where \widehat{B} denotes the completion along the grading of B , \mathbb{V} is Soergel’s structure functor [29, §5.10], $T_{\widehat{S}}$ denotes an \widehat{S} -deformation of T [29, §3.5], and \widehat{S} denotes the completion along the grading of $S(\mathfrak{t}^\vee)$ [29, Theorem 8.2].

Now comes the key observation: Theorem 1.1 holds if and only if the filtration on $\mathbb{C}[z] \otimes_{\mathbb{R}} B_x^!$ given by

$$\dots \subset i_x^{-1}(z^{i+1} \otimes B_x) \subset i_x^{-1}(z^i \otimes B_x) \subset \dots \tag{1.3}$$

induces the degree filtration on $\mathbb{C} \otimes_{\mathbb{R}} B_x^!$. (A sketch: our inclusion $\mathbb{C}[z] \otimes i_x$ is isomorphic to a direct sum of inclusions $N \hookrightarrow M$ of free graded $\mathbb{C}[z]$ -modules of rank 1. Now the above

filtration induces the degree filtration on $\mathbb{C} \otimes_R N$ if and only if N and M are generated in degrees symmetric about degree zero. See the last paragraph of [29].)

In other words, local hard Lefschetz holds if and only if the Andersen and degree filtrations match under the identification

$$\mathbb{C} \otimes_R B_x^! = \text{Hom}(\Delta(x), T).$$

Finally, there is a contravariant “tilting” equivalence t on the additive category of modules with Verma flag, constructed by Soergel in [26]; it takes projective modules to tilting modules and sends $\Delta(x)$ to $\Delta(-\varrho - x(\varrho))$. This equivalence induces an isomorphism

$$t: \text{Hom}(P, \Delta(-\varrho - x(\varrho))) \xrightarrow{\sim} \text{Hom}(\Delta(x), T),$$

where $P := t^{-1}(T)$ is a projective module. Now Kübel shows (see [23, Corollary 5.8] and the discussion afterwards) that this isomorphism can be upgraded to an isomorphism of graded vector spaces, matching the (filtration induced by the) Jantzen filtration on $\Delta(-\varrho - x(\varrho))$ on the left with the Andersen filtration on the right. By the discussion above, the grading filtration on the right agrees with the Andersen filtration, and hence the grading filtration on the left agrees with the Jantzen filtration. This is enough to conclude that the grading and Jantzen filtrations agree on $\Delta(-\varrho - x(\varrho))$, which implies the Jantzen conjectures.

1.6. Dependence on the deformation direction

The statement of the local Hard Lefschetz theorem for Soergel bimodules involved the choice of a specialisation parameter $\varrho^\vee \in \mathfrak{h}$. Similarly, the definition of the Jantzen filtration involves the choice of a deformation direction $\gamma \in (\mathfrak{t}^\vee)^*$. These choices match in the proof of Soergel and Kübel. In particular, for an arbitrary (regular) specialisation $\varrho^\vee \in \mathfrak{h}$, local hard Lefschetz is equivalent to the Jantzen filtration agreeing with the degree filtration. We have already commented that unless ϱ^\vee is dominant, there is no geometric reason to expect hard Lefschetz to hold.

Using these observations we are able to answer the following fundamental question about the Jantzen filtration (which seems to have first been raised by Deodhar) in the negative.

Question 1.4. (Deodhar, [19, §5.3, Bemerkung 2]) Is the Jantzen filtration independent of the choice of non-degenerate deformation direction γ ?

Remarkably, this already fails for $\mathfrak{sl}_4(\mathbb{C})$. Using Soergel bimodules and Theorem 1.3 one can see this failure via a simple calculation in the nil Hecke ring. One can also verify

this example directly (without leaving the world of Verma modules). However here the author needs computer assistance.

1.7. Structure of the paper

This paper is structured as follows:

§2 We recall basic notation (shifts, gradings, degree filtration).

§3 We recall the structure related to Coxeter groups underlying this paper (the reflection representation \mathfrak{h} , positivity properties, nil Hecke ring).

§4 We develop some algebra around the fundamental example (hard Lefschetz, Hodge–Riemann, weak Lefschetz substitute).

§5 We develop the theory of §4 “over \mathbb{P}^1 ”. We define \mathbb{P}^1 -sheaves, study their structure and establish various conditions for their global sections to satisfy hard Lefschetz and Hodge–Riemann. Although elementary, the results of this section are the main new ingredients in this paper.

§6 We give background on Soergel bimodules, define local intersection forms and establish some formulas for induced forms. We formulate a list of properties which form the “local Hodge theory” of Soergel bimodules.

§7 We prove the main results of the paper.

§8 We outline an example in $\mathfrak{sl}_4(\mathbb{C})$ where the Jantzen filtration does not coincide with the socle filtration.

§9 Contains a list of the most important notation.

1.8. Acknowledgements

Although somewhat hidden, this paper makes heavy use of Peter Fiebig’s way of thinking about Soergel bimodules and Braden–MacPherson sheaves. I would like to thank him for many hours of patient explanations. Thanks to Roman Bezrukavnikov, Jens Carsten Jantzen and Johannes Kübel for useful discussions and to Wolfgang Soergel for many useful conversations and for providing the proof of Proposition 6.24. I would also like to thank the referee for a careful reading and many comments, which have contributed significantly to the readability and accuracy of this paper. This paper started as joint work with Ben Elias, and can be seen as a contribution to our joint long-term project of throwing some light on the Hodge theoretic shadows cast by Soergel bimodules. To perform some of the calculations in §8 the author had some help from magma [5].

The results of this paper were announced in December 2013 at the Taipei Conference on Representation Theory IV and in January 2014 in Patagonia. I would like to thank

the organisers of both conferences for the opportunity to present these results.

2. Notation

2.1. Gradings and graded ranks

Given a \mathbb{Z} -graded object (vector space, module, bimodule) $M = \bigoplus M^i$ we let $M[j]$ denote the shifted object with $M[j]^i = M^{i+j}$. We call a graded object M *even* if $M^{\text{odd}} = 0$ and *odd* if $M^{\text{even}} = 0$. We say that M is *parity* if it is either even or odd. Given graded objects M and M' , we denote by

$$\text{Hom}^\bullet(M, M') = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(M, M'[i])$$

the (graded) space of homomorphisms of all degrees.

Let R denote a polynomial ring which we view as a graded ring with all generators of degree 2. (Starting from §3, R will have a more specific meaning.) Given a graded free and finitely generated R -module M we can choose an isomorphism

$$M \cong \bigoplus R[m]^{\oplus p_m}.$$

We call $p = \sum p_{-m} v^m \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ the *graded rank* of M .

2.2. Lattices and their duals

Let Q denote a localisation of R at some multiplicatively closed set of homogeneous elements. Let M_Q denote a finitely generated graded free Q -module equipped with a non-degenerate graded symmetric form

$$\langle \cdot, \cdot \rangle : M_Q \times M_Q \longrightarrow Q.$$

(Throughout non-degenerate means that $\langle \cdot, \cdot \rangle$ induces an isomorphism $M_Q \xrightarrow{\sim} M_Q^* = \text{Hom}_Q^\bullet(M, Q)$, or alternatively that the determinant of $\langle \cdot, \cdot \rangle$ in some basis is a unit in Q .) An R -submodule $M \subset M_Q$ is a *lattice* if the natural map $Q \otimes_R M \rightarrow M_Q$ is an isomorphism. If $M \subset M_Q$ is a lattice the *dual lattice* is

$$M^* := \{m \in M_Q : \langle m, M \rangle \subset R\}.$$

Then M^* is canonically isomorphic to the dual of M in the usual sense (i.e. the natural map $M^* \xrightarrow{\sim} \text{Hom}^\bullet(M, R)$ is an isomorphism), $M^* \subset M_Q$ is a lattice and $M = (M^*)^*$. In particular:

$$\text{if } M \text{ has graded rank } p \text{ then } M^* \text{ has graded rank } \bar{p}. \tag{2.1}$$

(By definition $\bar{p}(v) = p(v^{-1})$ for a polynomial $p \in \mathbb{Z}[v^{\pm 1}]$.)

2.3. The degree filtration

Let R be as in the previous section. Let $M = \bigoplus M^j$ be a graded R -module. Set

$$\text{deg}_{\leq i} M := R \cdot \bigoplus_{j \leq i} M^j.$$

This gives the *degree filtration*

$$\dots \hookrightarrow \text{deg}_{\leq i} M \hookrightarrow \text{deg}_{\leq i+1} M \hookrightarrow \dots$$

of M by R -submodules. Any morphism of graded R -modules $f: M \rightarrow N$ preserves this filtration. Hence $\text{deg}_{\leq i}$ can be viewed as an endofunctor on the category of graded R -modules.

3. Coxeter group background

3.1. Coxeter group

Let (W, S) be a Coxeter system with length function $\ell: W \rightarrow \mathbb{Z}_{\geq 0}$ and Bruhat order \leq .

An *expression* $\underline{x} = s_1 \dots s_m$ will denote a word in S . Dropping the underline gives an element $x \in W$. An expression $\underline{x} = s_1 \dots s_m$ is *reduced* if $\ell(x) = m$. Given an expression $\underline{x} = s_1 \dots s_m$, a *subexpression* is a sequence $\underline{u} = t_1 \dots t_m$ such that $t_i \in \{s_i, \text{id}\}$ for all i . Again, dropping the underline denotes the product in W . For example, if \underline{y} is a reduced expression for y then $\{x \in W : x \leq y\} = \{u : \underline{u} \text{ is a subexpression of } \underline{y}\}$.

3.2. The reflection representation

We fix a realisation $(\mathfrak{h}, \mathfrak{h}^*, \{\alpha_s\}, \{\alpha_s^\vee\})$ of (W, S) over \mathbb{R} as in [28] and [11]. That is, \mathfrak{h} is a finite-dimensional real vector space and we have fixed linearly independent subsets

$$\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^* \quad \text{and} \quad \{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$$

such that, for all $s, t \in S$, $\langle \alpha_s, \alpha_t^\vee \rangle = -2 \cos(\pi/m_{st})$ (where m_{st} denotes the order, possibly ∞ , of $st \in W$). In addition, we assume that \mathfrak{h} is of minimal possible dimension satisfying the above two conditions. We can define an action of W on \mathfrak{h} via $s \cdot v = v - \langle \alpha_s, v \rangle \alpha_s^\vee$ for all $s \in S$. This action is (reflection) faithful [28, Proposition 2.1].

We consider the roots $\Phi := \bigcup_{w \in W} w \cdot \{\alpha_s\} \subset \mathfrak{h}^*$ and coroots $\Phi^\vee := \bigcup_{w \in W} w \cdot \{\alpha_s^\vee\} \subset \mathfrak{h}$. We write $\Phi_+ \subset \Phi$ and $\Phi_+^\vee \subset \Phi^\vee$ for the positive roots and coroots. We have $\Phi = \Phi_+ \sqcup -\Phi_+$ and $\Phi^\vee = \Phi_+^\vee \sqcup -\Phi_+^\vee$.

We write T for the reflections (i.e. conjugates of S) in W . We have bijections

$$\begin{aligned} T &\xrightarrow{\sim} \Phi_+, & \text{and} & & T &\xrightarrow{\sim} \Phi_+^\vee, \\ t &\mapsto \alpha_t, & & & t &\mapsto \alpha_t^\vee, \end{aligned}$$

such that $t(v) = v - \langle v, \alpha_t^\vee \rangle \alpha_t$ for all $v \in \mathfrak{h}^*$.

Now let $\varrho \in \mathfrak{h}^*$ be such that $\langle \varrho, \alpha_s^\vee \rangle > 0$ for all $s \in S$. (Such a ϱ exists because the set $\{\alpha_s^\vee\}$ is linearly independent.) Then we have

$$tx > x \quad \text{if and only if} \quad \langle x(\varrho), \alpha_t^\vee \rangle > 0. \tag{3.1}$$

Now fix ϱ^\vee in \mathfrak{h} such that $\langle \alpha_s, \varrho^\vee \rangle > 0$ for all $s \in S$.

Remark 3.1. The choice of ϱ and ϱ^\vee subject to the above positivity conditions is made arbitrarily and fixed throughout. This positivity property is used in a crucial way throughout this paper. We do not know if the results of [11] or this paper are valid for an arbitrary reflection faithful representation of (W, S) .

LEMMA 3.2. *Suppose that $x < w$. Then $\langle x(\varrho), \varrho^\vee \rangle > \langle w(\varrho), \varrho^\vee \rangle$.*

Remark 3.3. One can view this lemma as saying that the map

$$\begin{aligned} W &\longrightarrow \mathbb{R}, \\ w &\longmapsto -\langle w\varrho, \varrho^\vee \rangle, \end{aligned}$$

gives a refinement of the Bruhat order.

Proof. By definition of the Bruhat order we may assume without loss of generality that $w = tx > x$ for some reflection $t \in T$. Then

$$tx(\varrho) = x(\varrho) - \langle x(\varrho), \alpha_t^\vee \rangle \alpha_t.$$

Since $tx > x$, we know that $\langle x(\varrho), \alpha_t^\vee \rangle > 0$. Hence

$$\langle w(\varrho), \varrho^\vee \rangle = \langle tx(\varrho), \varrho^\vee \rangle = \langle x(\varrho), \varrho^\vee \rangle - \langle x(\varrho), \alpha_t^\vee \rangle \langle \alpha_t, \varrho^\vee \rangle < \langle x(\varrho), \varrho^\vee \rangle$$

because $\langle \alpha_t, \varrho^\vee \rangle > 0$. □

3.3. Positivity

From now on R denotes the regular functions on \mathfrak{h} , or equivalently the symmetric algebra $S(\mathfrak{h}^*)$ of \mathfrak{h}^* . We view R as a graded ring with $\deg \mathfrak{h}^* = 2$. Throughout Q denotes the localisation of R at the multiplicatively closed subset generated by Φ . In formulas,

$$Q = R \left[\frac{1}{\Phi} \right].$$

By functoriality, W acts on R and Q via graded automorphisms.

For $s \in S$ we denote by ∂_s the divided difference operator

$$\partial_s(f) = \frac{f - sf}{\alpha_s}.$$

Each ∂_s preserves R . If $\lambda \in R$ is of degree 2 then $\partial_s(\lambda) = \langle \lambda, \alpha_s^\vee \rangle$.

We let $A = \mathbb{R}[z]$ and $K = \mathbb{R}[z^{\pm 1}]$, graded with $\deg z = 2$.

The map $\lambda \mapsto \langle \lambda, \varrho^\vee \rangle z$ extends multiplicatively to a morphism of graded rings

$$\sigma: R \longrightarrow A$$

(“restriction to the line $\mathbb{R}\varrho^\vee \subset \mathfrak{h}$ ”). This map (fixed by our choice of ϱ^\vee) will play an important role below. Whenever we write $A \otimes_R (\cdot)$ we always mean that we view A as an R -module via σ .

Any homogenous element f of K is of the form az^m for some $a \in \mathbb{R}$. We will write $f > 0$, $f < 0$ and say that az^m is *positive*, *negative* etc. if a is.

3.4. The nil Hecke ring

Let Q_W denote the smash product of Q with W . That is, Q_W is a free left Q -module with basis $\{\delta_w : w \in W\}$ and multiplication determined by

$$(f\delta_x)(g\delta_y) = f(xg)\delta_{xy}.$$

Inside Q_W we consider the elements

$$D_s := \frac{1}{\alpha_s}(\delta_{\text{id}} - \delta_s) = (\delta_{\text{id}} + \delta_s) \frac{1}{\alpha_s}.$$

The elements D_s satisfy the following relations:

$$D_s^2 = 0; \tag{3.2}$$

$$\text{the } D_s \text{ satisfy the braid relations;} \tag{3.3}$$

$$D_s f = (sf)D_s + \partial_s(f) \quad \text{for all } f \in Q. \tag{3.4}$$

If $\underline{y} = st \dots u$ is a reduced expression for $y \in W$ then, by (3.3), we obtain well-defined elements

$$D_{\underline{y}} := D_s D_t \dots D_u \in Q_W.$$

We define rational functions $e_{x,y}$ for all x and y through the identity

$$D_y = \sum e_{x,y} \delta_x.$$

The rational functions $e_{x,y} \in Q$ are called *equivariant multiplicities*. They are homogenous of degree $-2\ell(y)$.

Remark 3.4. The ring Q_W acts naturally on Q via $f\delta_x \cdot g = fx(g)$. The *nil Hecke ring* [21] is defined as the subring $\{q \in Q_W : q(R) \subset R\}$. In [21, Theorem 4.6] it is shown that the nil Hecke ring is a free left (or right) R -module with basis $\{D_w : w \in W\}$. Kostant and Kumar treat Weyl groups of Kac–Moody Lie algebras and take Q to be the field of fractions of R , but their argument goes through in our setting.

Remark 3.5. For Kac–Moody groups, the $e_{x,y}$ describe the localisations at torus fixed points of the equivariant fundamental classes of Schubert varieties, hence their name. These functions were introduced by Kostant–Kumar and may be used to detect smoothness and rational smoothness [24], [7] of Schubert varieties, as well as p -smoothness [20].

If $y's = y$ and $y' < y$ then expanding $D_y = D_{y'}D_s$ one obtains

$$e_{x,y} = \frac{1}{x(\alpha_s)}(e_{x,y'} + e_{xs,y'}). \tag{3.5}$$

The following well-known proposition provides a useful characterisation of equivariant multiplicities.

PROPOSITION 3.6. *The equivariant multiplicities are characterised by the following three properties:*

- (1) *We have $e_{x,y} = 0$ unless $x \leq y$.*
- (2) *We have*

$$e_{y,y} = (-1)^{\ell(y)} \prod_{t \in L_T(y)} \frac{1}{\alpha_t}$$

where $L_T(y) = \{t \in T : ty < y\}$.

- (3) *Let $\underline{y} = s_1 \dots s_m$ denote a reduced expression for y . Then for all $\lambda \in \mathfrak{h}^*$ and x we have*

$$(x\lambda - y\lambda)e_{x,y} = \sum \langle s_{i+1} \dots s_m \lambda, \alpha_{s_i}^\vee \rangle e_{x,y_i},$$

where the sum on the right-hand side runs over those $1 \leq i \leq m$ such that

$$\underline{y}_i = s_1 \dots s_{i-1} s_{i+1} \dots s_m$$

is a reduced expression.

Proof. It is obvious that (1)–(3) provide an inductive recipe for the computation of $e_{x,y}$. It remains to show that the claimed properties hold. Now (1) is immediate from the definition. For (2) consider a subexpression $\underline{u} = t_1 \dots t_m$ of $\underline{y} = s_1 s_2 \dots s_m$ such that

$u=t_1 \dots t_m=y$. Then clearly $t_i=s_i$ for all i because \underline{y} is reduced. Hence

$$\begin{aligned} e_{y,y}\delta_y &= (-1)^{\ell(y)} \frac{1}{\alpha_{s_1}} \delta_{s_1} \frac{1}{\alpha_{s_2}} \delta_{s_2} \dots \frac{1}{\alpha_{s_m}} \delta_{s_m} (-1)^{\ell(y)} \prod_{i=1}^m \frac{1}{s_1 \dots s_{i-1}(\alpha_i)} \delta_y \\ &= (-1)^{\ell(y)} \prod_{t \in L_T(y)} \frac{1}{\alpha_t} \delta_y, \end{aligned}$$

and (2) follows.

For (3) we can repeatedly apply (3.4) (using that $\partial_s(\lambda)=\langle \lambda, \alpha_s^\vee \rangle$) to obtain the equality (where $\hat{}$ denotes omission)

$$D_{s_1} \dots D_{s_m} \lambda - (y\lambda) D_{s_1} \dots D_{s_m} = \sum_{i=1}^m \langle s_{i+1} \dots s_m \lambda, \alpha_{s_i}^\vee \rangle D_{s_1} \dots \widehat{D}_{s_i} \dots D_{s_m}. \quad (3.6)$$

If $s_1 \dots \hat{s}_i \dots s_m$ is not a reduced expression then $D_{s_1} \dots \widehat{D}_{s_i} \dots D_{s_m} = 0$ by (3.2) and (3.3). Now writing both sides of (3.6) in terms of the basis δ_x gives the identity in (3). \square

Recall the homomorphism $\sigma: R \rightarrow \mathbb{R}[z]$ from § 3.3. The following positivity property of equivariant multiplicities will later fix a sign ambiguity in the Hodge–Riemann bilinear relations.

COROLLARY 3.7. *If $x \leq y$ then $(-1)^{\ell(x)} \sigma(e_{x,y}) > 0$.*

Proof. We fix x and induct on $\ell(y) - \ell(x)$. The base case $x=y$ follows from Proposition 3.6(2) because all α_t appearing are positive, and hence $\sigma(\alpha_t) > 0$.

Now let $x < y$ and assume by induction that the proposition is known for all $e_{x,y'}$ with $\ell(y') < \ell(y)$. Applying Proposition 3.6(3) with $\lambda = \varrho$ (our fixed element with $\langle \varrho, \alpha_s^\vee \rangle > 0$ for all $s \in S$) and multiplying by $(-1)^{\ell(x)}$ we get the identity

$$(-1)^{\ell(x)} \sigma(x\varrho - y\varrho) \sigma(e_{x,y}) = (-1)^{\ell(x)} \sum \langle s_{i+1} \dots s_m \varrho, \alpha_{s_i}^\vee \rangle \sigma(e_{x,y_i}).$$

Now by (3.1) the $\langle s_{i+1} \dots s_m \varrho, \alpha_{s_i}^\vee \rangle$ are all strictly positive, and by Lemma 3.2, $\sigma(x\varrho - y\varrho)$ is strictly positive. Induction now gives that the right-hand side is positive and the corollary follows. \square

4. Algebra around the fundamental example

4.1. Hard Lefschetz

Recall that $A = \mathbb{R}[z]$ and $K = \mathbb{R}[z^{\pm 1}]$, graded with $\deg z = 2$.

Let N be a free finitely generated graded A -module generated in degrees ≤ 0 . We set $N_K := K \otimes_A N$ and assume that N_K is equipped with a symmetric non-degenerate graded form

$$\langle \cdot, \cdot \rangle: N_K \times N_K \longrightarrow K.$$

We say that N satisfies *hard Lefschetz* if the restriction of $\langle \cdot, \cdot \rangle$ to

$$(\deg_{\leq d} N)_K := K \otimes_A \deg_{\leq d} N$$

is non-degenerate for all d .

It is immediate that N satisfies hard Lefschetz if and only if any of the following statements holds for all $d \leq 0$:

- (1) the determinant of the Gram matrix of the restriction of $\langle \cdot, \cdot \rangle$ to $\deg_{\leq d} N$ is non-zero (\Leftrightarrow invertible in K);
- (2) the determinant of the Gram matrix of the restriction of $\langle \cdot, \cdot \rangle$ to N^d is non-zero (\Leftrightarrow invertible in K);
- (3) $\langle m, \deg_{\leq d} N \rangle = 0$ for some $m \in \deg_{\leq d} N$ implies $m = 0$;
- (4) the determinant of the Gram matrix of the form $\langle n, z^{-d} n' \rangle$ on $\deg_{\leq d} N$ is non-zero (\Leftrightarrow invertible in K).

Remark 4.1. Condition (4) probably seems like a strange reformulation at this point. We have included it here, because it is this condition that will generalise to \mathbb{P}^1 -sheaves in the next section.

Because N is generated in degrees ≤ 0 , the dual lattice $N^! \subset N_K$ is generated in degrees ≥ 0 and hence $N^! \subset N$. Set

$$H := N / (zN^!).$$

The following lemma (whose proof is an exercise) explains the terminology:

LEMMA 4.2. N satisfies hard Lefschetz if and only if for all $d \geq 0$ multiplication by $z^d: H^{-d} \rightarrow H^d$ is an isomorphism.

4.2. Hodge–Riemann

Let N be as in the previous section and assume that N satisfies hard Lefschetz. For $d \leq 0$ define the *primitive subspaces*:

$$P^d := N^d \cap (\deg_{\leq d-1} N)^\perp \subset N^d.$$

The following is an easy application of Gram-Schmidt orthogonalisation.

LEMMA 4.3. *We have an orthogonal decomposition $N = \bigoplus_{d \leq 0} A \cdot P^d$.*

The following explains the “primitive” terminology.

LEMMA 4.4. *We have the decomposition (as A -modules)*

$$H = \bigoplus_{d \leq 0} A/(z^{-d+1}) \otimes P^d.$$

The restriction of our form to N^d for $d \leq 0$ takes values in $K^{2d} = \mathbb{R}z^d$ for degree reasons. Hence the Lefschetz form $(n, n') \mapsto z^{-d} \langle n, n' \rangle$ on N^d takes values in \mathbb{R} .

We say that N satisfies HR (short for “satisfies the Hodge–Riemann bilinear relations”) if:

- (1) N is parity (i.e. N vanishes in either odd or even degree);
- (2) if \min denotes the minimal non-zero degree of N then there exists $\varepsilon \in \{\pm 1\}$ such that, for all $d = \min + 2i \leq 0$, the Lefschetz form $z^{-d} \langle p, p' \rangle$ on P^d is $\varepsilon(-1)^i$ -definite.

LEMMA 4.5. *Suppose that N satisfies HR and that $N = N' \oplus N''$ is an orthogonal decomposition. Then N' and N'' satisfy HR.*

Proof. Fix $d \leq 0$ and $p \in P^d$. Then we can write $p = p' + p''$ with $p' \in N'$ and $p'' \in N''$. Because p is primitive and N' and N'' are orthogonal

$$0 = \langle p, \deg_{<d} N \rangle = \langle p', \deg_{<d} N' \rangle = \langle p', \deg_{<d} N \rangle,$$

and hence $p' \in P^d$. Similarly, $p'' \in P^d$. Hence our decomposition $N = N' \oplus N''$ induces a refinement of the decomposition in Lemma 4.3 and the result follows, because the restriction of a definite form to a subspace is definite of the same sign. \square

Suppose that N is parity and generated in degrees ≤ 0 , and that \min denotes its minimal non-zero degree. Then we can write its graded dimension as $v^{\min} f(v^2)$ for some $f \in \mathbb{Z}_{\geq 0}[v]$.

LEMMA 4.6. *N satisfies HR if and only if there exists $\varepsilon \in \{\pm 1\}$ such that for all $i \geq 0$ with $d = \min + 2i \leq 0$ the form $\langle z^{-d}x, y \rangle$ on N^d has signature $\varepsilon(\tau_{\leq i} f)(-1)$ (by definition $\tau_{\leq d}(\sum \alpha_j v^j) = \sum_{j \leq d} \alpha_j v^j$).*

Proof. Let $d = \min + 2i \leq 0$. The decomposition in Lemma 4.3 gives a decomposition

$$N^d = z^i P^{\min} \oplus \dots \oplus z P^{d-2} \oplus P^d.$$

This decomposition is orthogonal for Lefschetz forms and $z: N^{d-2} \rightarrow N^d$ is an isometry. The lemma now follows: fixing the signature of the Lefschetz forms on N^d for all $d \leq 0$ is equivalent to fixing the signature on P^d for all $d \leq 0$. \square

4.3. Weak Lefschetz

Let N_K and N'_K be two finitely generated free graded K -modules equipped with non-degenerate symmetric forms $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$. Let $N \subset N_K$ and $N' \subset N'_K$ be lattices generated in degrees ≤ 0 .

The following proposition provides a useful tool for establishing hard Lefschetz inductively (it is essentially a restatement of [11, Lemma 2.3]):

PROPOSITION 4.7. (Weak Lefschetz substitute) *Let $d: N \rightarrow N'[1]$ and $d': N' \rightarrow N[1]$ be maps such that*

- (1) *d and d' are adjoint (i.e. $\langle d(n), n' \rangle' = \langle n, d'(n') \rangle$ for all $n \in N, n' \in N'$);*
- (2) *$d' \circ d$ is equal to multiplication by $0 \neq \beta \in A$.*

Then if N' satisfies HR then N satisfies hard Lefschetz.

Proof. First note that d is injective by (2). Now assume for contradiction that $\langle \cdot, \cdot \rangle$ does not satisfy hard Lefschetz. In other words, there exists $0 \neq m \in N$ of degree $i \leq 0$ such that $\langle m, \text{deg}_{\leq i} N \rangle = 0$. By assumption, $\text{deg}_{\leq 0} N = N$ and $\langle \cdot, \cdot \rangle$ is non-degenerate, so we may assume that $i < 0$. Then $0 \neq d(m) \in (N')^{i+1}$ and for all $m' \in \text{deg}_{\leq i-1} N'$ we have

$$\langle d(m), m' \rangle' = \langle m, d'(m') \rangle = 0$$

because $d'(m') \in \text{deg}_{\leq i} N$. In particular, $d(m)$ is orthogonal to $\text{deg}_{\leq i-1} N'$ (and even to $\text{deg}_{\leq i} N'$ because $(N')^i = 0$, as N' satisfies HR and hence is parity). In particular $d(m) \in P^{i+1} \subset (N')^{i+1}$. Hence, as N' satisfies HR, we have

$$0 \neq \langle d(m), d(m) \rangle' = \langle m, (d' \circ d)(m) \rangle = \beta \langle m, m \rangle$$

which contradicts $\langle m, \text{deg}_{\leq i} N \rangle = 0$. □

5. Moment graph sheaves on the projective line

In this section we study certain sheaves on the moment graph of \mathbb{P}^1 , which we dub \mathbb{P}^1 -sheaves. This provides a useful language for discussing certain local calculations with Soergel bimodules.

Remark 5.1. Although we do not discuss the general theory below, our discussion has been strongly influenced by the Braden–MacPherson and Fiebig theory of sheaves on moment graphs [6], [13], [14].

5.1. \mathbb{P}^1 -sheaves

Let $A = \mathbb{R}[z]$ and $K = \mathbb{R}[z^{\pm 1}]$ as above.

Definition 5.2. A sheaf on the moment graph of \mathbb{P}^1 is a collection M of

- (1) finitely generated graded A -modules M_0, M_∞ and $M_{\mathbb{C}^*}$;
- (2) graded A -module morphisms $\varrho_0: M_0 \rightarrow M_{\mathbb{C}^*}$ and $\varrho_\infty: M_\infty \rightarrow M_{\mathbb{C}^*}$

such that $M_{\mathbb{C}^*}$ is annihilated by $z \in A$.

The category of sheaves on the moment graph of \mathbb{P}^1 is a graded (with shift functor [1]), additive category in an obvious way.

Definition 5.3. Let M be a sheaf on the moment graph of \mathbb{P}^1 . We say that M is a \mathbb{P}^1 -sheaf if M_0 and M_∞ are free A -modules, ϱ_0 is surjective and ϱ_∞ is isomorphic to the quotient map $M_\infty \rightarrow M_\infty/(z)$.

Remark 5.4. Let \mathbb{C}^* act non-trivially and linearly on \mathbb{P}^1 . Any object in the constructible \mathbb{C}^* -equivariant derived category of \mathbb{P}^1 yields modules M_0, M_∞ and $M_{\mathbb{C}^*}$ over $H_{\mathbb{C}^*}^*(\text{pt})=A$ by taking equivariant hypercohomology of the stalks at $0, \infty$ and \mathbb{C}^* [6], [16]. This explains the name.

Remark 5.5. In Fiebig’s language, \mathbb{P}^1 -sheaves are the Braden–MacPherson sheaves on the moment graph of \mathbb{P}^1 . However we prefer the term \mathbb{P}^1 -sheaf in this context because \mathbb{P}^1 -sheaves are quite simple objects (in contrast to Braden–MacPherson sheaves on general moment graphs).

The two most important examples of sheaves on the moment graph of \mathbb{P}^1 are the skyscraper at 0 ($M_0=A, M_{\mathbb{C}^*}=M_\infty=0$) which we will call simply the *skyscraper*, and the *constant sheaf* ($M_0=M_\infty=A, M_{\mathbb{C}^*}=A/(z), \varrho_0, \varrho_\infty$ the canonical quotient maps). Both are \mathbb{P}^1 -sheaves. In fact, we have the following.

LEMMA 5.6. Any \mathbb{P}^1 -sheaf is (non-canonically) isomorphic to a direct sum of shifts of skyscraper and constant sheaves. Hence any indecomposable \mathbb{P}^1 -sheaf is isomorphic (up to shift) to a skyscraper or constant sheaf.

Proof. Exercise. □

Let M denote a \mathbb{P}^1 -sheaf. The *global sections* of M are

$$M_{0,\infty} := \{(m_0, m_\infty) \in M_0 \oplus M_\infty : \varrho_0(m_0) = \varrho_\infty(m_\infty)\} \subset (M_0 \oplus M_\infty),$$

which we regard as a left A -module via $r \cdot (m_0, m_\infty) = (rm_0, rm_\infty)$. We have

$$K \otimes_A M_{0,\infty} = K \otimes_A M_0 \oplus K \otimes_A M_\infty. \tag{5.1}$$

Remark 5.7. By Lemma 5.6, if the graded ranks of M_0 and M_∞ are $p_0, p_\infty \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ respectively, then the graded rank of $M_{0,\infty}$ is

$$(p_0 - p_\infty) + (1 + v^2)p_\infty = p_0 + v^2 p_\infty.$$

More generally we consider the *structure algebra*

$$Z := \{(r_0, r_\infty) \in A \oplus A : r_0 = r_\infty \bmod (z)\}.$$

Of course this is nothing other than the global sections of the constant sheaf. It is a ring via pointwise multiplication. Moreover, one may check that Z acts on the global sections of any \mathbb{P}^1 -sheaf via $(r_0, r_\infty) \cdot (m_0, m_\infty) = (r_0 m_0, r_\infty m_\infty)$ for $(m_0, m_\infty) \in M_{0,\infty}$.

Below a special role will be played by the action of degree-2 elements of Z on the global sections of \mathbb{P}^1 -sheaves (“Lefschetz operators”). Of course

$$Z^2 = \mathbb{R}z \oplus \mathbb{R}z.$$

We define the *ample cone* in Z^2 to be

$$Z^2_{\text{ample}} := \{(\lambda_0, \lambda_\infty) = (az, bz) \in Z^2 : 0 < b < a\}.$$

5.2. Polarised \mathbb{P}^1 -sheaves

Let M be a \mathbb{P}^1 -sheaf. A *polarisation* of M is a pair of symmetric graded K -valued A -bilinear forms:

$$\langle \cdot, \cdot \rangle^0 : M_0 \times M_0 \longrightarrow K \quad \text{and} \quad \langle \cdot, \cdot \rangle^\infty : M_\infty \times M_\infty \longrightarrow K.$$

A polarisation is *non-degenerate* if both $\langle \cdot, \cdot \rangle^0$ and $\langle \cdot, \cdot \rangle^\infty$ are non-degenerate over K . A *polarised \mathbb{P}^1 -sheaf* is a \mathbb{P}^1 -sheaf together with a non-degenerate polarisation.

A polarisation of M induces an A -bilinear form

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle^0 + \langle \cdot, \cdot \rangle^\infty : M_{0,\infty} \times M_{0,\infty} \longrightarrow K$$

on the global sections of M . We have

$$\langle \gamma m, m' \rangle = \langle m, \gamma m' \rangle$$

for all $m, m' \in M_{0,\infty}$ and $\gamma \in Z$. By (5.1) we see that over K the form $\langle \cdot, \cdot \rangle$ is just the direct sum of $\langle \cdot, \cdot \rangle^0$ and $\langle \cdot, \cdot \rangle^\infty$. In particular, $\langle \cdot, \cdot \rangle$ is non-degenerate if the polarisation is.

5.3. Hard Lefschetz

Let M be a polarised \mathbb{P}^1 -sheaf. We assume that the global sections of M are generated in degrees ≤ 0 .

We say that $\gamma \in Z^2$ *satisfies hard Lefschetz* on M if and only if for all $d \leq 0$ the form $\langle \gamma^{-d} x, y \rangle$ on $\text{deg}_{\leq d} M_{0,\infty}$ is non-degenerate (i.e. the determinant of its Gram matrix is invertible \Leftrightarrow non-zero in K). We say that M *satisfies hard Lefschetz* if γ satisfies hard Lefschetz on M for all $\gamma \in Z^2_{\text{ample}}$.

Remark 5.8. See Remark 4.1 for some motivation for this definition.

Recall that $M_{0,\infty}$ is equipped with a non-degenerate form given by the sum of the forms $\langle \cdot, \cdot \rangle^0$ and $\langle \cdot, \cdot \rangle^\infty$. Let $M_{0,\infty}^! \subset K \otimes_A M_0 \oplus K \otimes_A M_\infty$ denote the dual lattice. Because $M_{0,\infty}$ is generated in degrees ≤ 0 , $M_{0,\infty}^!$ is generated in degrees ≥ 0 and so $M_{0,\infty}^! \subset M_{0,\infty}$. We set

$$H_{0,\infty} := M_{0,\infty} / (zM_{0,\infty}^!).$$

Any $\gamma \in Z^2$ preserves $M_{0,\infty}$ and $M_{0,\infty}^!$ and hence induces a degree-2 operator on $H_{0,\infty}$. The above definition is equivalent to γ satisfying hard Lefschetz in the usual sense (i.e. $\gamma^i: H_{0,\infty}^{-i} \rightarrow H_{0,\infty}^i$ is an isomorphism for all $i \geq 0$).

Remark 5.9. The condition for the \mathbb{P}^1 -sheaf M to satisfy hard Lefschetz is not the same as requiring that its global sections $M_{0,\infty}$ satisfy hard Lefschetz (in the sense of §4.1). Indeed, $M_{0,\infty}$ satisfies hard Lefschetz if and only if $\gamma = (z, z)$ satisfies hard Lefschetz on $H_{0,\infty}$, whereas M satisfies hard Lefschetz if and only if (az, bz) satisfies hard Lefschetz on $H_{0,\infty}$, for all $0 < b < a$. Hence the condition for the global sections $M_{0,\infty}$ to satisfy hard Lefschetz is a “degeneration to a wall” of M satisfying hard Lefschetz.

Example 5.10. We consider the simplest non-trivial example. Let M be a constant \mathbb{P}^1 -sheaf generated in degree m for some $m \leq -2$: $M_0 = M_\infty = A[-m]$, $M_{\mathbb{C}^*} = A/(z)[-m]$, $\varrho_0 = \varrho_\infty$ the quotient maps. (The condition $m \leq -2$ is to ensure that the global sections are generated in degrees ≤ 0 .) Equip M with the polarisation

$$\langle 1, 1 \rangle^0 = \lambda_0 z^m \text{ and } \langle 1, 1 \rangle^\infty = \lambda_\infty z^m \text{ for some } \lambda_0, \lambda_\infty \in \mathbb{R}.$$

We assume the polarisation is non-degenerate (i.e. $\lambda_0 \neq 0 \neq \lambda_\infty$). The global sections of M are

$$M_{0,\infty} = A \cdot (1, 1) \oplus A \cdot (z, 0)$$

with the generators in degrees m and $m+2$, respectively. Hence,

$$\text{deg}_{\leq d} M_{0,\infty} = \begin{cases} 0, & \text{if } d < m, \\ A \cdot (1, 1), & \text{if } d = m, m+1, \\ M_{0,\infty}, & \text{if } d \geq m+2. \end{cases}$$

Let $\gamma = (az, bz) \in Z^2$. We calculate the forms $\langle \gamma^{-d} x, y \rangle$ on $\text{deg}_{\leq d} M_{0,\infty}$ in the above basis:

$$\begin{aligned} d = m, m+1: & \quad ((\lambda_0 a^{-d} + \lambda_\infty b^{-d}) z^{-d+m}), \\ m+2 \leq d \leq 0: & \quad \begin{pmatrix} (\lambda_0 a^{-d} + \lambda_\infty b^{-d}) z^{-d+m} & \lambda_0 a^{-d} z^{-d+1+m} \\ \lambda_0 a^{-d} z^{-d+1+m} & \lambda_0 a^{-d} z^{-d+2+m} \end{pmatrix}. \end{aligned}$$

Calculating determinants we conclude that γ satisfies hard Lefschetz on M if and only if

$$\begin{aligned} \lambda_0 a^{-d} + \lambda_\infty b^{-d} &\neq 0 && \text{for } d = m, m+1, \\ \lambda_0 \lambda_\infty a^{-d} b^{-d} &\neq 0 && \text{for } m+2 \leq d \leq 0. \end{aligned}$$

For $\gamma \in Z^2_{\text{ample}}$ the second condition is automatic. The first condition holds for all $\gamma \in Z^2_{\text{ample}}$ (i.e. for all $0 < b < a$) if and only if either:

- (1) λ_0 and λ_∞ have the same sign, or
- (2) λ_0 and λ_∞ have opposite signs and $|\lambda_0| \geq |\lambda_\infty|$.

Below it will be the second case that is relevant. In case (2) the global sections satisfy hard Lefschetz if and only if we have strict inequality $|\lambda_0| > |\lambda_\infty|$. This is an illustration of Remark 5.9.

5.4. Hodge–Riemann

Let M be a polarised \mathbb{P}^1 -sheaf as in the previous section (i.e. the global sections of M are generated in degrees ≤ 0).

Let $\gamma \in Z^2$ and assume that γ satisfies hard Lefschetz on M . For $d \leq 0$ define the γ -primitive subspaces:

$$P_\gamma^d := (\gamma^{-d+1} \text{deg}_{\leq d-1} M_{0,\infty})^\perp \cap M_{0,\infty}^d = (\gamma^{-d+1} M_{0,\infty}^{d-2})^\perp \cap M_{0,\infty}^d \subset M_{0,\infty}^d.$$

The following is an easy application of Gram–Schmidt orthogonalisation:

LEMMA 5.11. *We have a decomposition $M_{0,\infty} = \bigoplus_{d \leq 0} \mathbb{R}[\gamma] \cdot P_\gamma^d$.*

Warning 5.12. Unless $\gamma = (az, az)$, the subspaces $\mathbb{R}[\gamma] \cdot P_\gamma^d \subset M_{0,\infty}$ are not A -submodules in general and the above decomposition need not be orthogonal (although it is orthogonal between degrees d and $-d$).

We say that $\gamma \in Z^2$ satisfies HR on M if

- (1) M_0 and M_∞ are either both even or both odd (hence the global sections $M_{0,\infty}$ are either even or odd);
- (2) if \min denotes the minimal non-zero degree of $M_{0,\infty}$ then there exists $\varepsilon \in \{\pm 1\}$ such that, for all $d = \min + 2i \leq 0$, the form $\langle \gamma^{-d} p, p' \rangle$ on P_γ^d is $\varepsilon(-1)^i$ -definite.

We say that M satisfies HR if all $\gamma \in Z^2_{\text{ample}}$ satisfy HR on M .

Remark 5.13. Because $M_{0,\infty}$ is parity we can write the graded rank of $M_{0,\infty}$ as $v^{\min} f(v^2)$ for some $f \in \mathbb{Z}_{\geq 0}[v]$, where $\min \in \mathbb{Z}$ denotes the minimal non-zero degree of $M_{0,\infty}$. Fix $d \leq 0$ with $d = \min + 2i \leq 0$. We have a decomposition

$$M_{0,\infty}^d = P_\gamma^d \oplus \gamma P_\gamma^{d-2} \oplus \dots \oplus \gamma^i P_\gamma^{\min}.$$

From the definitions it follows that this decomposition is orthogonal with respect to the form $\langle x, \gamma^{-d}y \rangle$. Moreover, the induced form on the subspace

$$\gamma P_\gamma^{d-2} \oplus \dots \oplus \gamma^i P_\gamma^{\min} \subset M_{0,\infty}^d$$

agrees with the form $\langle x, \gamma^{-d+2}y \rangle$ on $M_{0,\infty}^{d-2}$ (i.e. $\gamma: M_{0,\infty}^{d-2} \rightarrow M_{0,\infty}^d$ is an isometry). In particular, we see that γ satisfies HR on M if and only if there exists $\varepsilon \in \{\pm 1\}$ such that the signature of $\langle x, \gamma^{-d}y \rangle$ on $M_{0,\infty}^d$ is $\varepsilon(\tau_{\leq i} f)(-1)$ for all $\min \leq d = \min + 2i \leq 0$. (See Lemma 4.6.)

Remark 5.14. The form $\langle \cdot, \cdot \rangle$ on $M_{0,\infty}$ induces in a natural way an \mathbb{R} -valued form on $H_{0,\infty} = M_{0,\infty}/(zM_{0,\infty}^1)$. Then γ satisfies HR if and only if γ induces a Lefschetz operator satisfying HR on $H_{0,\infty}$ (in the usual sense).

Example 5.15. We continue the example of the polarised constant sheaf begun in Example 5.10. The form $\langle x, \gamma^{-m}y \rangle$ on $M_{0,\infty}^m$ is $(\lambda_0 a^{-m} + \lambda_\infty b^{-m})$. The form $\langle x, \gamma^{-m-2}y \rangle$ on $M_{0,\infty}^{m+2}$ in the basis $\{(z, 0), (0, z)\}$ is

$$\begin{pmatrix} \lambda_0 a^{-m-2} & 0 \\ 0 & \lambda_\infty b^{-m-2} \end{pmatrix}.$$

For HR to be satisfied in degree $m+2$ this matrix must have signature zero. Hence if $\gamma \in Z_{\text{ample}}^2$ then λ_0 and λ_∞ must have opposite signs. We conclude that M satisfies HR if and only if λ_0 and λ_∞ have opposite signs, and $|\lambda_0| \geq |\lambda_\infty|$. The global sections $M_{0,\infty}$ satisfy HR if and only if λ_0 and λ_∞ have opposite signs and $|\lambda_0| > |\lambda_\infty|$.

It is clear that if γ satisfies hard Lefschetz or HR on M then so does any positive scalar multiple of γ . Hence the following lemma is easy.

LEMMA 5.16. *Let M denote a polarised \mathbb{P}^1 -sheaf whose global sections are generated in degrees ≤ 0 . Suppose that for all $1 < c$ there exists $0 < b < a$ such that $c = a/b$ and $\gamma = (az, bz)$ satisfies hard Lefschetz (resp. HR) on M . Then M satisfies hard Lefschetz (resp. HR).*

5.5. Weak Lefschetz

The following is the analogue for \mathbb{P}^1 -sheaves of Proposition 4.7.

PROPOSITION 5.17. (Weak Lefschetz substitute for \mathbb{P}^1 -sheaves) *Let M and M' be two polarised \mathbb{P}^1 -sheaves and fix $\gamma = (\lambda_0, \lambda_\infty) \in Z^2$ such that $\lambda_0, \lambda_\infty$ are both non-zero. Assume that we are given morphisms $d: M \rightarrow M'[1]$ and $d': M' \rightarrow M[1]$ such that*

- (1) *d and d' are adjoint (i.e. $\langle dm, m' \rangle^? = \langle m, d'm' \rangle^?$ for $? \in \{0, \infty\}$);*
- (2) *$d' \circ d$ is equal to multiplication by γ .*

Suppose that γ satisfies HR on M' . Then γ satisfies hard Lefschetz on M .

Proof. As in the proof of Proposition 4.7, condition (2) implies that $d_0: M_0 \rightarrow M'_0[1]$ and $d_\infty: M_\infty \rightarrow M'_\infty[1]$ are injective.

Assume by contradiction that γ does not satisfy hard Lefschetz on M . Then there exists $0 \neq m \in M_{0,\infty}$ of degree $-i$ for $i \geq 0$ such that

$$\langle \gamma^i m, \text{deg}_{\leq -i} M_{0,\infty} \rangle = 0. \tag{5.2}$$

Because $\langle \cdot, \cdot \rangle$ on M is non-degenerate and $\text{deg}_{\leq 0} M_{0,\infty} = M_{0,\infty}$ we must have $i > 0$. Then $0 \neq dm \in (M')_{0,\infty}^{-i+1}$ and for all $m' \in \text{deg}_{\leq -i-1}(M'_{0,\infty})$ we have

$$\langle d(m), \gamma^i m' \rangle = \langle m, \gamma^i d'(m') \rangle = \langle \gamma^i m, d'(m') \rangle = 0.$$

Hence $d(m)$ is orthogonal to $\gamma^i((M'_{0,\infty})^{-i-1})$. Also, $(M'_{0,\infty})^{-i} = 0$ as M'_0 and M'_∞ are either both even or both odd. Thus $d(m) \in P_\gamma^{-i+1} \subset (M'_{0,\infty})^{-i+1}$. Because γ satisfies HR on M' we have

$$0 \neq \langle \gamma^{i-1} d(m), d(m) \rangle = \langle \gamma^{i-1} m, (d' \circ d)(m) \rangle = \langle m, \gamma^i m \rangle.$$

This contradicts (5.2). □

Remark 5.18. The above proposition reduces to Proposition 4.7 if M and M' are skyscraper sheaves.

5.6. Opposite signs and the limit lemma

For the Hodge–Riemann relations to have a hope of holding one needs to place some assumptions on the signs at 0 and ∞ . (We have already seen a hint of this in Example 5.15. This will become clearer in the next section, where we discuss the structure theory of polarised \mathbb{P}^1 -sheaves.)

We say that a polarised \mathbb{P}^1 -sheaf M is polarised with *opposite signs* if

- (1) M_0 and M_∞ are either both even or both odd;
- (2) the global sections of M are generated in degrees ≤ 0 ;
- (3) both $\langle \cdot, \cdot \rangle^0$ and $\langle \cdot, \cdot \rangle^\infty$ satisfy HR;
- (4) if we denote by $P_0^d \subset M_0^d$ and $P_\infty^d \subset M_\infty^d$ the primitive subspaces, then, for all $d \leq 0$, the restriction of $\langle \cdot, \cdot \rangle^0$ to P_0^d and $\langle \cdot, \cdot \rangle^\infty$ to P_∞^d are definite of opposite signs.

Let N be a free A -module generated in degrees ≤ -2 and equipped with a K -valued non-degenerate form $\langle \cdot, \cdot \rangle_N: N \times N \rightarrow K$ satisfying HR. We can build a constant \mathbb{P}^1 -sheaf out of N by setting $M_0 = M_\infty = N$ and $M_{\mathbb{C}^*} = N/(z)$ with $\varrho_0, \varrho_\infty$ being the quotient maps. We can equip M with a polarisation by setting $\langle \cdot, \cdot \rangle^0 = \langle \cdot, \cdot \rangle_N = -\langle \cdot, \cdot \rangle^\infty$. Because N satisfies HR this polarisation has opposite signs. A \mathbb{P}^1 -sheaf which is isomorphic (isometrically for polarisations) to such an M we will call *polarised constant*.

Remark 5.19. In the following lemma the “opposite signs” assumption is crucial. It occurs in a large class of examples coming from Soergel bimodules (as we will explain). We do not properly understand its geometric meaning.

LEMMA 5.20. (Limit lemma) *Let M be a polarised \mathbb{P}^1 -sheaf with opposite signs. Consider $\gamma=(az,bz)\in Z^2$ with $0<b<a$. Then γ satisfies HR on M for $a/b\gg 0$. Moreover the signs agree with the signs on M_0 : if $m=(m_0,m_\infty)$ denotes a non-zero element of minimal degree $-d$ in $M_{0,\infty}$ then $\langle \gamma^d m, m \rangle$ and $\langle m_0, m_0 \rangle^0$ have the same sign for $a/b\gg 0$. (The map $m\mapsto m_0$ is an isomorphism in degree $-d$, as follows from Lemma 5.6.)*

This lemma will be obvious later (see Lemma 5.27) once we have developed the structure theory of polarised \mathbb{P}^1 -sheaves.

In the following the assumptions on M are as in Lemma 5.20.

COROLLARY 5.21. *Suppose that $\gamma=(az,z)$ satisfies hard Lefschetz on M for all $a\in I$, where $I\subset\mathbb{R}$ is a connected subset which is not bounded above. Then γ satisfies HR on M for all $a\in I$. In particular, if $I=[1,\infty)$ then the global sections $M_{0,\infty}$ satisfy HR.*

Proof. For any fixed $d\leq 0$ the form $(x,y)\mapsto\langle x,\gamma^{-d}y \rangle$ on $M_{0,\infty}^d$ varies continuously in γ . If γ satisfies hard Lefschetz for all $a\in I$ then these forms are non-degenerate, and the previous lemma says that for $a\gg 0$ these forms have signatures given by the Hodge–Riemann relations (see Remark 5.13). The lemma now follows, as the signature of a continuous family of real non-degenerate forms is constant. \square

5.7. Structure theory of polarised \mathbb{P}^1 -sheaves

Throughout this section M denotes a polarised \mathbb{P}^1 -sheaf. We assume in addition that M_0 and M_∞ satisfy HR (with respect to the forms $\langle \cdot, \cdot \rangle^0$ and $\langle \cdot, \cdot \rangle^\infty$).

The goal of this section is to show that M admits a canonical decomposition into simpler pieces. That is, we will see that the decomposition in Lemma 5.6 becomes canonical in the presence of a polarisation satisfying HR at 0 and ∞ .

LEMMA 5.22. *We have a canonical decomposition*

$$M = M' \oplus N$$

such that

- (1) N is a skyscraper, i.e. $N_\infty = N_{\mathbb{C}^*} = 0$;
- (2) the induced decomposition of M_0 is orthogonal for $\langle \cdot, \cdot \rangle^0$;
- (3) the induced map $M'_0 \rightarrow M'_{\mathbb{C}^*} = M_{\mathbb{C}^*}$ is a projective cover.

Let $\gamma \in \{0, \infty\}$. By assumption M_γ satisfies hard Lefschetz. In particular it is generated in degrees ≤ 0 . For $d \leq 0$ let $P_\gamma^d \subset M_\gamma^d$ denote the primitive subspace (see §4.2) and set $P_\gamma = \bigoplus P_\gamma^d \subset M_\gamma$.

Proof. Let L denote the kernel of the composition $P_0 \hookrightarrow M_0 \xrightarrow{\varrho_0} M_{\mathbb{C}^*}$. Let $L^\perp \subset P_0$ denote the orthogonal to L under the Lefschetz form (see §4.2) degree by degree. By our HR assumption, each Lefschetz form on P_0 is definite in any fixed degree. Hence $P_0 = L \oplus L^\perp$. This leads to a canonical decomposition (see Lemma 4.3)

$$M_0 = A \otimes_{\mathbb{R}} L \oplus A \otimes_{\mathbb{R}} L^\perp.$$

Hence we can write our sheaf as a direct sum $M = N \oplus M'$ where N is the skyscraper sheaf at zero associated with $A \otimes_{\mathbb{R}} L$ (i.e. $N_0 = A \otimes_{\mathbb{R}} L$, $N_{\mathbb{C}^*} = N_\infty = 0$). (1) and (2) are now clear. (3) follows because the composition $L^\perp \rightarrow M_0 \rightarrow M_{\mathbb{C}^*} = M'_{\mathbb{C}^*}$ is an isomorphism by construction. □

LEMMA 5.23. *Let M be as above and assume additionally that $\varrho_0: M_0 \rightarrow M_{\mathbb{C}^*}$ is a projective cover. We have a canonical decomposition*

$$M = \bigoplus_{i \leq 0} M_i$$

where each M_i is isomorphic to a direct sum of constant sheaves generated in degree i (ignoring forms). Moreover the induced decomposition of M_0 (resp. M_∞) is orthogonal with respect to $\langle \cdot, \cdot \rangle^0$ (resp. $\langle \cdot, \cdot \rangle^\infty$).

Proof. Under the assumptions of the lemma the induced maps

$$P_0 \longrightarrow M_{\mathbb{C}^*} \longleftarrow P_\infty$$

are isomorphisms. Now the canonical decompositions

$$M_0 = \bigoplus_{i \leq 0} A \otimes_{\mathbb{R}} P_0^i \quad \text{and} \quad M_\infty = \bigoplus_{i \leq 0} A \otimes_{\mathbb{R}} P_\infty^i$$

lead to the desired decomposition. □

Remark 5.24. Two forms on a real vector space may be simultaneously diagonalised if one form is definite. (I thank Pavel Etingof for this remark.) Hence we could further decompose our polarised \mathbb{P}^1 -sheaf into a direct sum of polarised sheaves of rank 1. The decomposition of Lemma 5.23 is enough for our needs.

5.8. Hodge–Riemann revisited

One can use the above structure theory to give a simple criterion for HR to be satisfied.

Let M be a \mathbb{P}^1 -sheaf which is polarised with opposite signs, and whose global sections are generated in degrees ≤ 0 . We would like to know when the global sections of M satisfy HR with the same signs as those on M_0 .

Let $M = N \oplus M'$ be the decomposition of Lemma 5.22 (so N is a skyscraper). It is easy to see that the Hodge–Riemann bilinear relations are always satisfied (with the correct sign) for the summand of the global sections coming from N . Hence we may assume that $\varrho_0: M_0 \rightarrow M_{\mathbb{C}^*}$ is a projective cover. By Lemma 5.23 we may even assume that M is of the following form:

- (1) $M_{\mathbb{C}^*} = V$ for some finite dimensional graded real vector space concentrated in fixed degree $d \leq -2$;
- (2) $M_0 = M_\infty = A \otimes_{\mathbb{R}} V$;
- (3) there exist symmetric forms $(\cdot, \cdot)^0$ and $(\cdot, \cdot)^\infty$ on V which are definite of opposite signs and such that the local forms are given by

$$\langle 1 \otimes v, 1 \otimes v' \rangle^0 = (v, v')^0 z^d \quad \text{and} \quad \langle 1 \otimes v, 1 \otimes v' \rangle^\infty = (v, v')^\infty z^d$$

for all $v, v' \in V$.

LEMMA 5.25. *The global sections of M satisfy HR (with the same signs as M_0) if and only if the form $(\cdot, \cdot)^0 + (\cdot, \cdot)^\infty$ on V is definite (of the same sign as $(\cdot, \cdot)^0$).*

Remark 5.26. Informally, the global sections of a \mathbb{P}^1 -sheaf which is polarised with opposite signs satisfies HR if the “form at 0 dominates the form at ∞ ”. This will be a subtle question in general!

Proof. The graded rank of $M_{0,\infty}$ is $v^d + v^{d+2}$. To verify hard Lefschetz and HR it is enough to show that the form $z^{-m} \langle x, y \rangle$ on $M_{0,\infty}^m$ is non-degenerate for $m = d, d+2$, and that its signature is the same as that of $(\cdot, \cdot)^0$ on $M_{0,\infty}^d$, and is 0 on $M_{0,\infty}^{d+2}$ (see Lemma 4.6).

The global sections of degree d are given by the diagonal $v \mapsto (1 \otimes v, 1 \otimes v)$. The restriction of $\langle \cdot, \cdot \rangle$ to $M_{0,\infty}^d$ is given by

$$\langle (1 \otimes v, 1 \otimes v), (1 \otimes v', 1 \otimes v') \rangle = ((v, v')^0 + (v, v')^\infty) z^d.$$

Hence the form $z^{-d} \langle x, y \rangle$ is non-degenerate and the Hodge–Riemann relations are satisfied in this degree (with the correct signs) if and only if $(v, v)^0 + (v, v)^\infty$ is non-zero and of the same sign as $(v, v)^0$, for all $0 \neq v \in V$.

The map $(v, v') \mapsto (z \otimes v, z \otimes v')$ gives an isomorphism between $V \oplus V$ and the global sections in degree $d+2$. This isomorphism identifies the form $z^{-d-2} \langle x, y \rangle$ on $M_{0,\infty}^{d+2}$ with the direct sum of the forms $(\cdot, \cdot)^0$ and $(\cdot, \cdot)^\infty$ on $V \oplus V$. So the non-degeneracy and HR relations in this degree follow automatically from our assumption that $(\cdot, \cdot)^0$ and $(\cdot, \cdot)^\infty$ are definite of opposite signs. \square

The following lemma is an equivalent formulation of Lemma 5.20.

LEMMA 5.27. *Suppose that M is a polarised \mathbb{P}^1 -sheaf with opposite signs. For non-zero $a > b > 0$ consider the rescaled polarisations $a \langle \cdot, \cdot \rangle^0$ on M_0 and $b \langle \cdot, \cdot \rangle^\infty$ on M_∞ . Then if $a/b \gg 0$, the global sections $M_{0,\infty}$ satisfy HR with signs agreeing with those of M_0 (see Lemma 5.20).*

Proof. By the above structure theory we have an orthogonal decomposition of M into a direct sum of skyscraper sheaves (satisfying HR) and constant sheaves of the form of Lemma 5.25. In the notation of Lemma 5.25 $a \langle \cdot, \cdot \rangle^0 + b \langle \cdot, \cdot \rangle^\infty$ is definite of the same sign as $(\cdot, \cdot)^0$ if $a/b \gg 0$. Now the result follows from Lemma 5.25. \square

Recall the notion of a polarised constant sheaf M (see §5.6).

LEMMA 5.28. (HR in constant case) *If M is polarised constant then M satisfies HR.*

Proof. By Lemma 4.3 we may assume that M_0 and M_∞ are generated in one degree. Now let $\gamma = (az, bz) \in Z_{\text{ample}}^2$. We have to verify that the form $\langle \gamma^{-m} x, y \rangle$ on $M_{0,\infty}^m$ is non-degenerate with the correct signature for $m \leq 0$. As in Lemma 5.25 we can reduce to the case of the minimal non-zero degree d , in which case we are asking whether $a^{-d} \langle \cdot, \cdot \rangle^0 + b^{-d} \langle \cdot, \cdot \rangle^\infty$ is definite of the same sign as $(\cdot, \cdot)^0$. However this is the case because $\langle \cdot, \cdot \rangle^\infty = -\langle \cdot, \cdot \rangle^0$ (M is assumed polarised constant) and $a > b > 0$. \square

6. Soergel bimodule background

6.1. Bimodules

Let R be the regular functions on \mathfrak{h} , as above. We will work mostly inside the category $R\text{-bim}$ of graded R -bimodules (with degree zero morphisms) which are finitely generated as both left and right R -modules. Given $M, N \in R\text{-bim}$ we write

$$\text{Hom}^\bullet(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(M, N[i])$$

for the graded vector space of morphisms of all degree (and similarly for other graded objects, for example graded left R -modules).

The category $R\text{-bim}$ is a monoidal category via tensor product of bimodules. We denote the monoidal structure simply by juxtaposition: given $M, N \in R\text{-bim}$ their tensor product is

$$MN := M \otimes_R N.$$

Given elements $m \in M$ and $n \in N$, we abbreviate $mn := m \otimes n \in MN$. We also employ this notation for morphisms: given $f: M \rightarrow M'$ and $g: N \rightarrow N'$ the (horizontal) tensor product of these two morphisms is written $fg: MN \rightarrow M'N'$. Following standard practice we will often use the symbol denoting an object to also denote its identity morphism. For example fN denotes the morphism $f \text{id}_N: MN \rightarrow M'N$. Given $r \in M$, the morphism obtained by left (resp. right) multiplication by r is denoted rM (resp. Mr).

Given an R -bimodule M , its dual is

$$\mathbb{D}M := \text{Hom}_{R_-}^\bullet(M, R)$$

(homomorphisms of all degrees of left R -modules). Then $\mathbb{D}M$ is a graded R -bimodule via $(r \cdot f)(m) = f(rm)$ and $(f \cdot r)(m) = f(mr)$. This definition is only sensible for bimodules which are free and finitely generated as graded left R -modules. This will always be the case below. For such bimodules the natural morphism $M \rightarrow \mathbb{D}(\mathbb{D}(M))$ is an isomorphism.

6.2. Soergel bimodules

For background on Soergel bimodules see [28], [10], [11], [12] and the references therein.

We write \mathcal{B} for the category of Soergel bimodules. By definition \mathcal{B} is the full graded additive monoidal Karoubian subcategory of $R\text{-bim}$ generated by the bimodules

$$B(s) := R \otimes_{R^s} R[1].$$

Given an expression $\underline{w} := s_1 s_2 \dots s_m$ we denote the corresponding Bott–Samelson bimodule by

$$B(\underline{w}) := B(s_1)B(s_2) \dots B(s_m).$$

If \underline{w} is reduced then $B(\underline{w})$ contains a unique summand which is not isomorphic to a shift of a summand of any Bott–Samelson bimodule $B(\underline{w}')$ for a shorter expression \underline{w}' . We denote (the isomorphism class of) this bimodule by $B(w)$. Then the set

$$\{B(w) : w \in W\}$$

give representatives for the isomorphism classes of indecomposable self-dual Soergel bimodules, and any indecomposable bimodule is isomorphic to $B(w)[m]$ for some $w \in W$ and $m \in \mathbb{Z}$.

In this paper we arbitrarily choose to consider Soergel bimodules predominantly as left modules.

Warning 6.1. This emphasis on left over right is the opposite to the choice made in [11]. It simplifies the notation a little in what follows. We have tried to include warnings like this one when the conventions of the current paper differ from those of [11].

We define some elements and simple morphisms between Soergel bimodules that will play an important role in this paper. Consider the elements

$$c_{\text{id}} := 1 \otimes 1 \in B(s)^{-1} \quad \text{and} \quad c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in B(s)^1.$$

These are easily seen to give a basis for $B(s)$ as a left or right R -module. One checks easily that $c_s r = r c_s$ for $r \in R$. Define the maps:

$$\begin{aligned} m: B(s) &\longrightarrow R[1], & \text{and} & & \mu: R[-1] &\longrightarrow B(s), \\ f \otimes g &\longmapsto fg, & & & f &\longmapsto f c_s. \end{aligned}$$

(These are the units and counits (“dot” maps) of a Frobenius algebra structure on $B(s)$, see [10] and [12].) We have the “polynomial sliding relation” which for $\lambda \in \mathfrak{h}^*$ takes the form

$$B(s)\lambda = s(\lambda)B(s) + \langle \lambda, \alpha_s^\vee \rangle (\mu \circ m). \tag{6.1}$$

6.3. Support, stalk, costalk

Any $M \in R$ -bim can be regarded as a coherent sheaf on $\mathfrak{h} \times \mathfrak{h}$ (remember that R is commutative, so R -bimodules are the same thing as $R \otimes R$ -modules). For $x \in W$ consider its “twisted graph”:

$$\text{Gr}_x := \{(x\lambda, \lambda) : \lambda \in \mathfrak{h}\} \subset \mathfrak{h} \times \mathfrak{h}.$$

One may identify the regular functions on Gr_x with the bimodule $R(x)$ which is free of rank 1 as a left R -module, and has right action given by

$$b \cdot r = x(r)b$$

for $b \in R(x)$ and $r \in R$.

Given any subset $X \subset W$ we set $\text{Gr}_X := \cup_{x \in X} \text{Gr}_x$. Given a subset $X \subset W$ we write B_X (resp. $B_X^!$) for the stalk (resp. costalk, i.e. sections with support) of B along Gr_X . We write B_x instead of $B_{\{x\}}$ and $B_x^!$ instead of $B_{\{x\}}^!$. We have

$$B_x^! = \text{Hom}^\bullet(R(x), B) \quad \text{and} \quad B_x = R(x) \otimes_{R \otimes R} B$$

(where in the second equality we regard $R(x)$ and B as graded $R \otimes R$ -modules). Given $x, y \in W$ we write $R(x, y)$ for the regular functions on $\text{Gr}_x \cup \text{Gr}_y$ and $B_{x,y}$ for $B_{\{x,y\}}$. We have $B_{x,y} = B \otimes_{R \otimes R} R(x, y)$.

Remark 6.2. The modules B_X and $B_X^!$ are denoted $\Gamma^X B$ and $\Gamma_X B$ in [28].

Warning 6.3. Stalks and costalks appear so frequently in the present work that we decided to denote them B_x and $B_x^!$. Let us emphasise that the indecomposable self-dual bimodule parameterised by $y \in W$ will be denoted $B(y)$ in this paper (and not B_y as in [28] and [11]). We hope that this does not cause confusion for the reader.

For any Soergel bimodule B the stalks and costalks B_x and $B_x^!$ are free as left R -modules [28, Theorem 5.15] and we have canonical inclusions and projections

$$B_x^! \hookrightarrow B \quad \text{and} \quad B \twoheadrightarrow B_x,$$

which split when regarded as morphisms of left R -modules (see the proof of [28, Proposition 6.4]). Recall that we write $Q = R[1/\Phi]$ for the localisation of R at all roots.

Taking the direct sum over the canonical maps we obtain injections

$$\bigoplus_{w \in W} B_w^! \hookrightarrow B \hookrightarrow \bigoplus_{w \in W} B_w,$$

and both maps become isomorphisms after applying $Q \otimes_R (\cdot)$. (This is not difficult to check for Bott–Samelson bimodules, from which the general case follows.) In particular the composition

$$i_x : B_x^! \longrightarrow B_x$$

is an injection, which becomes an isomorphism after tensoring with Q .

In what follows it will be convenient to consider the injection (an isomorphism over Q)

$$i : B \hookrightarrow \bigoplus_{w \in W} B_w, \quad b \mapsto (b_w). \tag{6.2}$$

(Finitely many B_w are non-zero.)

6.4. Polarisation

An *invariant form* on a Soergel bimodule B means a symmetric graded bilinear form

$$\langle \cdot, \cdot \rangle : B \times B \longrightarrow R$$

such that $\langle rb, b' \rangle = \langle b, rb' \rangle = r \langle b, b' \rangle$ and $\langle br, b' \rangle = \langle b, b'r \rangle$ for all $b, b' \in B$ and $r \in R$ (note the left/right asymmetry).

Warning 6.4. This does not agree with the terminology “invariant form” in [11], where the roles of the left and right actions are interchanged.

An invariant form on a Soergel bimodule B is *non-degenerate* if it induces an isomorphism $B \xrightarrow{\sim} \mathbb{D}B$. A *polarisation* of a Soergel bimodule B is a non-degenerate invariant form $\langle \cdot, \cdot \rangle_B$ on B . Throughout a *polarised Soergel bimodule* will mean a Soergel bimodule B together with a fixed non-degenerate invariant form $\langle \cdot, \cdot \rangle_B$. We will denote a polarised Soergel bimodule by $(B, \langle \cdot, \cdot \rangle_B)$ or simply B (in which case the form is implicit).

Let \underline{w} be an expression. The set

$$\{c_\pi := c_{u_1}c_{u_2} \dots c_{u_m} : \underline{\pi} = u_1 \dots u_m \text{ a subexpression of } \underline{w}\}$$

gives a basis of $B(\underline{w})$ as a free left R -module. We define the *intersection form* on $B(\underline{w})$ to be

$$\langle f, g \rangle_{B(\underline{w})} = \text{Tr}(fg),$$

where $\text{Tr}(fg)$ denotes the coefficient of $c_{\underline{w}}$ in the above basis, and fg denotes the product of f and g in $B(\underline{w})$ (a ring). Then $\langle \cdot, \cdot \rangle_{B(\underline{w})}$ is a non-degenerate invariant form on $B(\underline{w})$ (see [11, §3.4] and [11, Lemma 3.8], remembering to switch left and right actions). Unless we state explicitly otherwise, we will always regard Bott–Samelson bimodules as polarised with respect to their intersection forms.

An important case below will be given by the intersection form on $B(s)$. In the left basis $\{c_{\text{id}}, c_s\}$ of §6.2 we have

$$\langle c_{\text{id}}, c_{\text{id}} \rangle = 0, \quad \langle c_{\text{id}}, c_s \rangle = \langle c_s, c_{\text{id}} \rangle = 1 \quad \text{and} \quad \langle c_s, c_s \rangle = \alpha_s. \tag{6.3}$$

Given two Soergel bimodules B_1 and B_2 equipped with invariant forms $\langle \cdot, \cdot \rangle_{B_1}$ and $\langle \cdot, \cdot \rangle_{B_2}$ it is easy to check that we get an invariant form on B_1B_2 via

$$\langle b_1b_2, b'_1b'_2 \rangle_{B_1B_2} = \langle b_1 \langle b_2, b'_2 \rangle_{B_2}, b'_1 \rangle_{B_1} = \langle b_1, b'_1 \langle b_2, b'_2 \rangle_{B_2} \rangle_{B_1}.$$

One may also check that if $\langle \cdot, \cdot \rangle_{B_1}$ and $\langle \cdot, \cdot \rangle_{B_2}$ are non-degenerate, then so is $\langle \cdot, \cdot \rangle_{B_1B_2}$. (This is clear after choosing bases and dual bases for B_1 and B_2 .) In particular, if B_1 and B_2 are polarised, then so is B_1B_2 .

Remark 6.5. This construction is associative in an obvious sense. One may check that it returns the intersection form on a Bott–Samelson bimodule, starting from the intersection form on each of the $B(s)$ factors.

6.5. Positive polarisations

Recall that a Soergel bimodule B is *perverse* if

$$B = \bigoplus B(y)^{\oplus m_y}$$

for some $m_y \in \mathbb{Z}_{\geq 0}$. (That is, B is isomorphic to a direct sum of indecomposable self-dual Soergel bimodules without shifts.)

Recall that Soergel’s conjecture combined with Soergel’s hom formula [28, Theorem 5.15] implies that

$$\text{Hom}(B(x), B(y)) = \begin{cases} \mathbb{R}, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases} \tag{6.4}$$

Hence if B is any perverse Soergel bimodule we have a canonical “isotypic” decomposition

$$B = \bigoplus V(z) \otimes_{\mathbb{R}} B(z) \tag{6.5}$$

for real (degree-zero) vector spaces $V(z)$. The following important fact will be used repeatedly in what follows (it also played a key role in [11]).

LEMMA 6.6. *The decomposition (6.5) is orthogonal for any invariant form on B .*

Proof. An invariant form yields a morphism $B \rightarrow \mathbb{D}B \cong B$ which must respect (6.5) by (6.4). □

Remark 6.7. Similar arguments show that giving an invariant form on B is the same thing as giving a symmetric form on each $V(z)$, once one has fixed the intersection form on each $B(z)$.

In particular, a polarisation of $B(z)$ induces an isomorphism $B(z) \xrightarrow{\sim} \mathbb{D}B(z) = B(z)$, and hence is unique up to a scalar. As in [11] we choose for every $y \in W$ an embedding of $B(y)$ as a summand in $B(\underline{y})$ for some reduced expression \underline{y} of y . Restricting the intersection form on $B(\underline{y})$ yields a polarisation of $B(y)$. (The non-degenerate intersection form gives an isomorphism $\phi: B(y) \xrightarrow{\sim} \mathbb{D}B(y)$. As the summand $B(y) \subset B(\underline{y})$ is unique up to isomorphism, ϕ restricts to an isomorphism $\phi: B(y) \xrightarrow{\sim} \mathbb{D}B(y)$. Hence the restriction to $B(y)$ is non-degenerate.) We call this polarisation the intersection form (it is well-defined up to a positive scalar).

Let $(B, \langle \cdot, \cdot \rangle)$ be a polarised Soergel bimodule. We say that B is *positively polarised* if it satisfies the following conditions:

- (1) B is perverse and vanishes in even or odd degree;
- (2) if we fix a decomposition as in (6.5) and let $z \in W$ be maximal such that $m_z \neq 0$ then the induced form on each $V(y)$ is $(-1)^{(\ell(z) - \ell(y))/2}$ times a positive definite form.

(Our assumption that B vanishes in even or odd degree forces all elements of $\{\ell(y) : m_y \neq 0\}$ to have the same parity, and hence $\frac{1}{2}(\ell(z) - \ell(y))$ in (2) makes sense.)

Remark 6.8. Suppose $y \in W$ and $s \in S$ with $ys > y$. Then $B(y)B(s)$ is perverse (as follows from Soergel’s conjecture), and has a natural form induced from the intersection forms on $B(y)$ and $B(s)$ (see §6.4). This yields a positive polarisation [11, Proposition 6.12].

6.6. Adjoint

Let B and B' be polarised Soergel bimodules. Given a map $f: B \rightarrow B'[m]$ (i.e. f is a degree m map from B to B') we denote by $f^*: B' \rightarrow B[m]$ the adjoint map. It is uniquely determined by the property

$$\langle b, f^*(b') \rangle_B = \langle f(b), b' \rangle_{B'}$$

for all $b \in B$ and $b' \in B'$. In particular $f = (f^*)^*$.

Recall the “dot” maps $m: B(s) \rightarrow R[1]$ and $\mu: R \rightarrow B(s)[1]$ from §6.2. An easy calculation shows that (with respect to the intersection forms on $B(s)$ and R)

$$m = \mu^*. \tag{6.6}$$

Let B_1 and B_2 be two polarised Soergel bimodules. Then, if

$$f_1: B_1 \rightarrow B'_1[i] \quad \text{and} \quad f_2: B_2 \rightarrow B'_2[i']$$

are morphisms, then

$$(f_1 f_2)^* = f_1^* f_2^*: B'_1 B'_2 \rightarrow B_1 B_2 [i + i']. \tag{6.7}$$

6.7. Local forms

Now suppose that B is polarised via

$$\langle \cdot, \cdot \rangle: B \times B \rightarrow R.$$

By extension of scalars, we obtain a form

$$\langle \cdot, \cdot \rangle_Q: Q \otimes_R B \times Q \otimes_R B \rightarrow Q.$$

LEMMA 6.9. *The form $\langle \cdot, \cdot \rangle_Q$ is orthogonal with respect to the decomposition in (6.2).*

Proof. Suppose that $b \in B_x$ and $b' \in B_y$. Then, for all $r \in R$, we have

$$r \langle b, b' \rangle = \langle b, rb' \rangle = \langle b, b' y^{-1}(r) \rangle = \langle by^{-1}(r), b' \rangle = \langle xy^{-1}(r)b, b' \rangle = xy^{-1}(r) \langle b, b' \rangle.$$

Hence if $\langle b, b' \rangle \neq 0$ then $x = y$ (remember that R is an integral domain and $W \rightarrow \text{Aut}(R)$ is faithful). □

Definition 6.10. We write $\langle \cdot, \cdot \rangle_B^w$ (or $\langle \cdot, \cdot \rangle^w$ if the context is clear) for the induced Q -valued form on B_w and call it the *local intersection form*.

Remark 6.11. This local intersection form is not the same as the local intersection form considered in [11]. In fact, the local intersection forms considered in [11] may be “embedded” into those above. We will not discuss this here, but see §7.5.

The following proposition summarises the key properties of the local intersection form.

PROPOSITION 6.12. (1) For all $b, b' \in B$ we have

$$\langle b, b' \rangle = \sum_{w \in W} \langle b_w, b'_w \rangle^w.$$

(2) $\langle \cdot, \cdot \rangle^w$ induces a non-degenerate graded form on $Q \otimes_R B_w$.

(3) $B_w, B_w^! \subset Q \otimes_R B_w$ are dual lattices with respect to $\langle \cdot, \cdot \rangle^w$.

Proof. Statements (1) and (2) follow from the definitions. For (3) note that, by (1) and Lemma 6.9, $\langle B_w^!, B_w \rangle^w = \langle B_w^!, B \rangle \subset R$. Hence our non-degenerate form gives an injection

$$B_w^! \longrightarrow (B_w)^*.$$

Now, if we compare graded ranks (given by Soergel’s hom formula), we see that our map is an isomorphism. □

6.8. Local induced forms

Throughout this section we fix a Soergel bimodule B . The goal is to relate two forms on the Soergel bimodule $BB(s)$.

PROPOSITION 6.13. For any Soergel bimodule B , $x \in W$ and $s \in S$ we have a canonical identification $(BB(s))_x = B_{x, xs}[1]$ (as left R -modules).

Proof. For the proof let us work in the category of $R \otimes R$ -modules, viewing all R -bimodules as $R \otimes R$ -modules. We have (all unspecified tensor products are over \mathbb{R})

$$\begin{aligned} (BB(s)) \otimes_{R \otimes R} R(x)[-1] &= B \otimes_{R \otimes R} (R \otimes R \otimes_{R \otimes R^s} R \otimes R) \otimes_{R \otimes R} R(x) \\ &= B \otimes_{R \otimes R} (R \otimes R \otimes_{R \otimes R^s} R(x)) \\ &= B \otimes_{R \otimes R} (R(x)B(s))[-1] \\ &= B \otimes_{R \otimes R} R(x, xs) \\ &= B_{x, xs}. \end{aligned}$$

(We have used the isomorphism $R(x)B(s)[-1] = R(x, xs)$. This follows from the identity $B(s)[-1] = R(\text{id}, s)$, which can be checked by hand.) The proposition now follows. □

Remark 6.14. Recall that the invariant ring R^s is the ring of regular functions on the quotient $\mathfrak{h}/\langle s \rangle$. In the language of coherent sheaves the functor of tensoring on the right with $B(s)$ is isomorphic to $\pi^* \pi_*[1]$, where $\pi: \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h} \times \mathfrak{h}/\langle s \rangle$ is the quotient map. The above proof is an algebraic translation of simple facts about the effect of push-forward and pull-back on stalks.

LEMMA 6.15. *The natural map $B_{x,xs} \rightarrow B_x \oplus B_{xs}$ is injective.*

Proof. The map in question becomes an isomorphism after applying $Q \otimes_R(\cdot)$. Hence it is enough to show that $B_{x,xs}$ is torsion free as a left R -module. However this follows from the previous proposition, as the stalks and costalks of Soergel bimodules are free as left R -modules [28, Theorem 5.15]. \square

LEMMA 6.16. *Let $f, g, h \in R$. For any $x \in W$ we have a commutative diagram*

$$\begin{array}{ccccccc}
 BB(s) & \twoheadrightarrow & (BB(s))_x & \xrightarrow{\sim} & B_{x,xs} & \hookrightarrow & B_x \oplus B_{xs} \\
 \downarrow fBgB(s)h & & \downarrow & & \downarrow & & \downarrow \gamma \\
 BB(s) & \twoheadrightarrow & (BB(s))_x & \xrightarrow{\sim} & B_{x,xs} & \hookrightarrow & B_x \oplus B_{xs}
 \end{array}$$

where $\gamma = (fx(g)x(h), fx(g)xs(h)) \in R \oplus R$. (The two isomorphisms are those of Proposition 6.13. All other horizontal maps are canonical.)

Proof. This follows easily by chasing $fbgb'h \otimes 1$ through the identifications in the proof of Proposition 6.13. \square

Now let us assume that B is polarised by $\langle \cdot, \cdot \rangle_B$. Then $B_{x,xs}$ carries a form induced by the sum of the two local intersection forms on B_x and B_{xs} under the inclusion $B_{x,xs} \hookrightarrow B_x \oplus B_{xs}$. On the other hand $(BB(s))_x$ carries a local intersection form (coming from the induced form on $BB(s)$). The following proposition relates these forms.

PROPOSITION 6.17. *Let B be a polarised Soergel bimodule. Then, under the identification*

$$(BB(s))_x = B_{x,xs}[1] \subset (B_x \oplus B_{xs})[1]$$

of Proposition 6.13, we have

$$\langle \cdot, \cdot \rangle_{BB(s)}^x = \frac{1}{x\alpha_s} (\langle \cdot, \cdot \rangle_B^x + \langle \cdot, \cdot \rangle_B^{xs}).$$

Proof. Recall that for a general Soergel bimodule B' we denote the map $B' \rightarrow \bigoplus B'_x$ by $b \mapsto (b_x)$. For the course of the proof let j_x denote the composition

$$j_x: (BB(s))_x \xrightarrow{\sim} B_{x,xs}[1] \hookrightarrow (B_x \oplus B_{xs})[1],$$

where the first map is the identification of Proposition 6.13 and the second map is the inclusion. We have (as follows from a simple calculation)

$$j_x(bc_{\text{id}}) = (b_x, b_{x_s}), \quad (6.8)$$

$$j_x(bc_s) = (b_x\alpha_s, 0) = x\alpha_s(b_x, 0). \quad (6.9)$$

For the course of the proof let us write $\langle \cdot, \cdot \rangle_{\text{ind}}^x$ for the form displayed on the right-hand side in the proposition. We want to show $\langle \cdot, \cdot \rangle_{BB(s)}^x = \langle \cdot, \cdot \rangle_{\text{ind}}^x$. In order to check this it is enough to show that if we define a form on $BB(s)$ via

$$\langle b, b' \rangle_{\text{ind}} := \sum_{x \in W} \langle j_x(b), j_x(b') \rangle_{\text{ind}}^x$$

then we have

$$\langle \cdot, \cdot \rangle_{\text{ind}} = \langle \cdot, \cdot \rangle_{BB(s)}.$$

We do this by checking the defining properties of the induced form:

$$\langle bc_{\text{id}}, b'c_{\text{id}} \rangle_{BB(s)} = 0 \quad (6.10)$$

$$\langle bc_s, b'c_{\text{id}} \rangle_{BB(s)} = \langle bc_{\text{id}}, b'c_s \rangle_{BB(s)} = \langle b, b' \rangle_B \quad (6.11)$$

$$\langle bc_s, b'c_s \rangle_{BB(s)} = \langle b, b'\alpha_s \rangle_B = \langle b\alpha_s, b' \rangle_B \quad (6.12)$$

(these formulas follow from (6.3) and the definition of the induced form).

Firstly, by (6.9) we have (for any $b, b' \in B$)

$$\begin{aligned} \langle bc_{\text{id}}, b'c_{\text{id}} \rangle &= \sum_{x \in X} \langle j_x(bc_{\text{id}}), j_x(b'c_{\text{id}}) \rangle_{\text{ind}}^x = \sum_{x \in W} \langle (b_x, b_{x_s}), (b'_x, b'_{x_s}) \rangle_{\text{ind}}^x \\ &= \sum_{x \in W} \frac{1}{x\alpha_s} (\langle b_x, b'_x \rangle_B^x + \langle b_{x_s}, b'_{x_s} \rangle_B^{x_s}) = 0 \end{aligned}$$

(The last line follows by breaking the sum into two pieces corresponding to $x_s > x$ and $x_s < x$ and using that $s(\alpha_s) = -\alpha_s$.) This gives (6.10).

For (6.11) we have

$$\begin{aligned} \langle bc_s, b'c_{\text{id}} \rangle_{\text{ind}} &= \sum_{x \in W} \langle j_x(bc_s), j_x(b'c_{\text{id}}) \rangle_{\text{ind}}^x = \sum_{x \in W} \frac{1}{x\alpha_s} \langle b_x\alpha_s, b'_x \rangle_B^x \\ &= \sum_{x \in W} \langle b_x, b'_x \rangle_B^x = \langle b, b' \rangle_B. \end{aligned}$$

(We use that $b''\alpha_s = x(\alpha_s)b''$ for all $b'' \in B_x$.) An almost identical calculation shows that $\langle bc_{\text{id}}, b'c_s \rangle_{\text{ind}} = \langle b, b' \rangle_B$.

For (6.12) we have

$$\langle bc_s, b'c_s \rangle_{\text{ind}} = \sum_{x \in W} \frac{1}{x\alpha_s} \langle b_x\alpha_s, b'_x\alpha_s \rangle_{\text{ind}}^x = \sum_{x \in W} \langle b_x, b'_x\alpha_s \rangle_B^x = \langle b, b'\alpha_s \rangle_B.$$

Hence $\langle \cdot, \cdot \rangle_{\text{ind}} = \langle \cdot, \cdot \rangle_{BB(s)}$ and the proposition follows. \square

6.9. Local intersection forms and the equivariant multiplicity

Recall the nil Hecke ring from §3.4. For any expression $\underline{y}=u \dots ts$ define $e_{x,\underline{y}} \in Q$ via

$$D_{\underline{y}} := D_u \dots D_t D_s = \sum e_{x,\underline{y}} \delta_x.$$

Then $e_{x,\underline{y}} = e_{x,y}$ if \underline{y} is reduced and is zero otherwise (see §3.4). Let $c_{x,\underline{y}}$ denote the image of $1 \otimes 1 \otimes \dots \otimes 1$ in $B(\underline{y})_x$ and let $\langle \cdot, \cdot \rangle_{B(\underline{y})}^x$ denote the local intersection form on $B(\underline{y})_x$.

LEMMA 6.18. *We have $\langle c_{x,\underline{y}}, c_{x,\underline{y}} \rangle_{B(\underline{y})}^x = e_{x,\underline{y}}$.*

Proof. We prove the lemma by induction on the length of \underline{y} , with the case of the empty sequence being straightforward. Let $\underline{y}' := u \dots t$ denote the expression obtained from \underline{y} by deleting the final s . Under the identifications and injection

$$B(\underline{y})_x = (B(\underline{y}')B(s))_x = B(\underline{y}')_{x,xs} \hookrightarrow B(\underline{y}')_x \oplus B(\underline{y}')_{xs},$$

one checks that $c_{x,\underline{y}}$ maps to $(c_{x,\underline{y}'}, c_{xs,\underline{y}'})$. By Proposition 6.17 and induction we have

$$\langle c_{x,\underline{y}}, c_{x,\underline{y}} \rangle_{B(\underline{y})}^x = \frac{1}{x\alpha_s} (\langle c_{x,\underline{y}'}, c_{x,\underline{y}'} \rangle_{B(\underline{y}')}^x + \langle c_{xs,\underline{y}'}, c_{xs,\underline{y}'} \rangle_{B(\underline{y}')}^{xs}) = \frac{1}{x\alpha_s} (e_{x,\underline{y}'} + e_{xs,\underline{y}'}) = e_{x,\underline{y}},$$

where the last equality follows by expanding $D_{\underline{y}'} D_s$. □

Recall that for all y we have fixed a realisation of $B(y)$ as a summand of $B(\underline{y})$, for some reduced expression \underline{y} for y . Let us denote by c_{bot} (resp. $c_{x,y}$) the image of $1 \otimes \dots \otimes 1$ in $B(y)$ (resp. $B(\underline{y})_x$). Because $B(y)^{-\ell(y)}$ is 1-dimensional, c_{bot} and $c_{x,y}$ are well defined up to a non-zero scalar.

THEOREM 6.19. *We have $\langle c_{x,y}, c_{x,y} \rangle_{B(\underline{y})}^x = \gamma e_{x,y}$ for some $\gamma \in \mathbb{R}_{>0}$.*

Proof. Let us denote by $i: B(y) \hookrightarrow B(\underline{y})$ our fixed realisation of $B(y)$ as a summand of $B(\underline{y})$. Recall that the intersection form on $B(y)$ is defined as the restriction of the intersection form on $B(\underline{y})$. Hence we need to calculate $\langle i(c_{x,y}), i(c_{x,y}) \rangle_{B(\underline{y})}^x$. However $B(y)_x$ and $B(\underline{y})_x$ are both generated in degrees $\geq -\ell(y)$ and are of dimension 1 in degree $-\ell(y)$. It follows that $i(c_{x,y}) = c_{x,\underline{y}}$ and the theorem follows from the previous lemma. □

Remark 6.20. Actually one may prove the existence of elements $c_{\text{bot}} \in B(y)^{-\ell(y)}$ and $c_{x,y} \in B(\underline{y})_x^{-\ell(y)}$ which are canonical up to sign, once one has fixed a positive polarisation on $B(y)$. With this choice the scalar factor γ in the above theorem disappears. One proceeds as follows: for any reduced expression \underline{y} the positive integer N appearing in the the proof of [11, Lemma 3.10] is easily seen to depend only on y . Now c_{bot} (and hence $c_{x,y}$) is fixed up to sign by requiring that $\langle c_{\text{bot}}, \varrho^{\ell(y)} c_{\text{bot}} \rangle_{B(\underline{y})} = N$.

6.10. Soergel bimodules and \mathbb{P}^1 -sheaves

Let B be a Soergel bimodule and fix $x \in W$ and $s \in S$ with $x < xs$.

With this data we may associate a \mathbb{P}^1 -sheaf $M(B, x, xs)$ as follows (we set $M := M(B, x, xs)$ to simplify notation, and why we obtain a \mathbb{P}^1 -sheaf will be explained later):

- (1) $M_0 = A \otimes_R B_x[1]$ and $M_\infty = A \otimes_R B_{xs}[1]$;
- (2) $M_{\mathbb{C}^*}$ is defined as the push-out of (left, graded) A -modules:

$$\begin{array}{ccc} A \otimes_R B_{x,xs}[1] & \longrightarrow & A \otimes_R B_{xs}[1] \\ \downarrow & & \downarrow \\ A \otimes_R B_x[1] & \longrightarrow & M_{\mathbb{C}^*}; \end{array}$$

(3) $\varrho_0: M_0 \rightarrow M_{\mathbb{C}^*}$ and $\varrho_\infty: M_\infty \rightarrow M_{\mathbb{C}^*}$ are the maps occurring in the above push-out diagram.

Now $B_{x,xs} \rightarrow B_x \oplus B_{xs}$ is an injective map of free R -modules which is an isomorphism over Q and hence $A \otimes_R B_{x,xs} \rightarrow A \otimes_R B_x \oplus A \otimes_R B_{xs}$ is injective. Hence we have a canonical isomorphism

$$A \otimes_R B_{x,xs}[1] = M_{0,\infty}. \tag{6.13}$$

Also, as the inclusion $B_{x,xs} \rightarrow B_x \oplus B_{xs}$ becomes an isomorphism after inverting $x(\alpha_s)$, $M_{\mathbb{C}^*}$ is annihilated by $0 \neq \sigma(x(\alpha_s))$. Hence we indeed have a sheaf on the moment graph of \mathbb{P}^1 .

PROPOSITION 6.21. $M = M(B, x, xs)$ is a \mathbb{P}^1 -sheaf.

Proof. Deferred until the next section. □

Remark 6.22. Alternatively, one can deduce Proposition 6.21 from results of Fiebig (indeed it was Fiebig’s work that led the author to consider \mathbb{P}^1 -sheaves). In [14, Proposition 7.1] Fiebig shows that one may obtain B as the global sections of a sheaf \mathcal{B} on the moment graph of W (we refer the reader to [14] for unexplained terminology). The \mathbb{P}^1 -sheaf defined above is obtained by restricting \mathcal{B} to the directed subgraph $x \rightarrow xs$ and applying $A \otimes_R (\cdot)$. It now follows from [14, Proposition 7.4] that we obtain a \mathbb{P}^1 -sheaf.

If B carries a polarisation, then we can equip M with a polarisation via

- (1) $\sigma(\langle \cdot, \cdot \rangle^x) / \sigma(x(\alpha_s))$ on M_0 ;
- (2) $\sigma(\langle \cdot, \cdot \rangle^{xs}) / \sigma(x(\alpha_s)) = -\sigma(\langle \cdot, \cdot \rangle^{xs}) / \sigma(xs(\alpha_s))$ on M_∞ .

Combining Proposition 6.13 and (6.13), we have an identification

$$M_{0,\infty} = A \otimes_R (BB(s))_x, \tag{6.14}$$

and by Proposition 6.17 we conclude the following result.

LEMMA 6.23. (6.14) is an isometry.

6.11. Proof of Proposition 6.21

We keep the notation of the previous section. Our goal is to show that $M := M(B, x, xs)$ is a \mathbb{P}^1 -sheaf. This is immediate from the following proposition.

PROPOSITION 6.24. *Given a Soergel bimodule B , $x \in W$ and $s \in S$ with $x < xs$, $B_{x,xs}$ is isomorphic to a direct sum of shifts of $R(x, xs)$ and $R(x)$.*

Before giving the proof we need some terminology. We say that a graded R -bimodule E has a *biflag* (with respect to x and xs) if it admits filtrations

$$C \hookrightarrow E \twoheadrightarrow D \quad \text{and} \quad D' \hookrightarrow E \twoheadrightarrow C' \tag{6.15}$$

such that C and C' (resp. D and D') are isomorphic to direct sums of shifts of $R(x)$ (resp. $R(xs)$). If E has a biflag then $C = E_x^!$, $C' = E_x$, $D = E_{xs}$ and $D' = E_{xs}^!$. Hence, the filtrations (6.15) are canonical, if they exist.

Proof. For any Soergel bimodule B , $B_{x,xs}$ has a biflag (see the last three lines of the proof of [28, Proposition 6.4]). It follows from the proposition below that $B_{x,xs}$ is isomorphic to a direct sum of shifts of $R(x)$, $R(xs)$ and $R(x, xs)$. Finally, one can use [28, Lemma 6.10] to rule out any occurrences of $R(xs)$. □

PROPOSITION 6.25. *Suppose that E is a graded R -bimodule, and that E has a biflag. Then E is isomorphic to a direct sum of shifts of $R(x)$, $R(xs)$ and $R(x, xs)$.*

We are grateful to Wolfgang Soergel for providing the following proof.

Proof. To simplify notation, set $v = x$ and $w = xs$. It will be clear in the proof that the only assumptions we need on v and w is that $\dim(\text{Gr}_v \cap \text{Gr}_w) + 1 = \dim \text{Gr}_w = \dim \text{Gr}_v$. In the proof, Ext^1 refers to degree-zero extensions of graded $R \otimes R$ -modules.

Each choice of linear form $\theta \in (\mathfrak{h} \oplus \mathfrak{h})^*$ with $\theta|_{\text{Gr}_v} \neq 0$ and $\theta|_{\text{Gr}_w} = 0$ gives an extension

$$R(v)[-2] \xrightarrow{\theta} R(v, w) \twoheadrightarrow R(w). \tag{6.16}$$

Moreover, if we let I denote the regular functions on $\text{Gr}_v \cap \text{Gr}_w$ then, by [28, Lemma 5.8], we have an identification of graded I -modules

$$\bigoplus_{m \in \mathbb{Z}} \text{Ext}^1(R(w), R(v)[-2+m]) = I \tag{6.17}$$

mapping the class of (6.16) to $1 \in I$. We fix such a θ and hence an identification (6.17).

Let us fix sequences $m_1 \geq \dots \geq m_f$ and $n_1 \geq \dots \geq n_g$ of integers. By additivity and (6.17), we have an identification

$$\begin{aligned} \text{Ext}^1\left(\bigoplus_{j=1}^g R(w)[n_j], \bigoplus_{i=1}^f R(v)[m_i-2]\right) &= \bigoplus_{i,j} \text{Ext}^1(R(w)[n_j], R(v)[m_i-2]) \\ &= \bigoplus_{i,j} I^{m_i-n_j}. \end{aligned}$$

So now assume that E has a biflag. In particular, there exists a (homogenous degree-zero) extension

$$\bigoplus_{i=1}^f R(v)[m_i-2] \hookrightarrow E \twoheadrightarrow \bigoplus_{j=1}^g R(w)[n_j] \tag{6.18}$$

for certain m_i and n_j as above. Via the above identification, such an extension is determined by a matrix with entries

$$C_{ji} \in \text{Ext}^1(R(w)[n_j], R(v)[m_i-2]) = I^{m_i-n_j}.$$

As $I^{<0} = 0$ and $I^0 = \mathbb{R}$, it follows that our matrix is block upper-triangular (i.e. $C_{ji} = 0$ if $m_i < n_j$) with scalar matrices on the diagonal (i.e. $C_{ji} \in I^0 = \mathbb{R}$ if $m_i = n_j$).

Now I is even and so $I^{m_i-n_j} = 0$ if m_i and n_j are not of the same parity. In particular we may assume without loss of generality that all m_i and n_j are of the same parity. Moreover, if there exists i and j with $n_j = m_i$ and $0 \neq C_{ji} \in \mathbb{R}$, then we change bases on the left and right of (6.18) above to ensure that $C_{ji} = 1$ and $C_{j'i'} = 0 = C_{j'i}$ for all $i' \neq i$ and $j' \neq j$. In this case our extension decomposes as $E = R(v, w) \oplus E'$ and we can continue with E' in place of E .

Hence we may assume without loss of generality that our matrix is block upper-triangular (i.e. $C_{ji} = 0$ if $m_i < n_j$) with zeroes on the diagonal (i.e. $C_{ji} = 0$ if $m_i = n_j$). Under these new assumptions we see that if $m_1 \leq n_1$ then $C_{1i} = 0$ for all i and so our extension splits as $E = R(w)[n_1] \oplus E'$; again we are done by induction. So we may assume that $m_1 > n_1$. By assumption E has the biflag property, and hence if we consider the filtration

$$E_w^! \hookrightarrow E \twoheadrightarrow E/E_w^!,$$

we can find isomorphisms

$$E_w^! \cong \bigoplus_{j=1}^g R(w)[n'_j-2] \quad \text{with } n'_1 \geq \dots \geq n'_g$$

and

$$E/E_w^! \cong \bigoplus_{i=1}^f R(v)[m_i'] \quad \text{with } m_1' \geq \dots \geq m_f'.$$

Multiplication by θ and the canonical quotient map give injections

$$E/E_w^![-2] \xrightarrow{\theta} E_v^! \twoheadrightarrow E/E_w^!.$$

(The first map is injective because $E/E_w^!$ is isomorphic to a direct sum of shifts of $R(v)$, upon which multiplication by θ is injective. For the second map, note that every non-zero element of E is either non-zero in E_w or is contained in $E_v^!$, and thus has support containing either Gr_v or Gr_w . Hence $E_v^! \cap E_w^! = 0$, which implies that the second map is injective.) Similarly, after choosing $\varkappa \in (\mathfrak{h} \oplus \mathfrak{h})^*$ with $\varkappa|_{\text{Gr}_w} \neq 0$ and $\varkappa|_{\text{Gr}_v} = 0$, we have injections

$$E/E_v^![-2] \xrightarrow{\varkappa} E_w^! \twoheadrightarrow E/E_v^!.$$

By Lemma 6.26 below we have $m_i' \in \{m_i, m_i - 2\}$ and $n_j' \in \{n_j, n_j + 2\}$. If we consider the graded rank of E as an R -module, we deduce that

$$\sum_{i=1}^f v^{-m_i+2} + \sum_{j=1}^g v^{-n_j} = \sum_{i=1}^f v^{-m_i'} + \sum_{j=1}^g v^{-n_j'+2}.$$

Under our assumption $m_1 > n_1$, we see (by considering terms of minimal degrees on both sides) that $m_1' = m_1$ is impossible, and hence $m_1' = m_1 - 2$. Therefore the smallest non-zero degree of $E/E_w^!$ is $2 - m_1$, and the injection

$$R(v)[m_1 - 2] \hookrightarrow \bigoplus_{i=1}^f R(v)[m_i - 2] = E_v^! \hookrightarrow E/E_w^!$$

splits. The result now follows by induction on the graded rank of E . □

LEMMA 6.26. *Let $m_1 \geq \dots \geq m_f$ and $m_1' \geq \dots \geq m_g'$ and suppose that we have an injection*

$$\bigoplus_{i=1}^f R[m_i] \hookrightarrow \bigoplus_{i=1}^g R[m_i'].$$

Then $f \leq g$ and $m_i \leq m_i'$ for all $1 \leq i \leq f$.

Proof. Left to the reader. □

6.12. Statements of local Hodge theory

The proof of local hard Lefschetz is an induction relying on some auxiliary statements which are interesting in their own right. In this section we state these properties.

Let $(B, \langle \cdot, \cdot \rangle)$ denote a polarised Soergel bimodule. We say that B *satisfies local hard Lefschetz* (resp. *satisfies local HR*) if for all $x \in W$ the pair $(A \otimes_R B_x, A \otimes_R \langle \cdot, \cdot \rangle_B^x)$ satisfies hard Lefschetz (resp. satisfies hard Lefschetz and HR). We say that B *satisfies local HR with standard signs* if B satisfies local HR and for all $x \in W$ we have

$$(-1)^{\ell(x)} \sigma(\langle c, c \rangle_B^x) > 0,$$

where $0 \neq c \in B_x$ denotes an element of minimal degree. (The term on the left-hand side is a scalar times a power of z ; our notation means that this scalar is positive.)

To simplify notation in inductive steps we employ the following notation:

- hL(y): $B(y)$ satisfies local hard Lefschetz.
- HR(y): $B(y)$ satisfies local HR with standard signs.

(As always we regard $B(y)$ as polarised with respect to its intersection form.) Given a subset $X \subset W$ we write hL(X) (resp. hL($\leq x$)) to mean that hL(y) for all $y \in X$ (resp. for all $y \leq x$). In a similar way we define HR(X) and HR($\leq x$).

Fix $s \in S$. In §6.10 we explained how to associate with a polarised Soergel bimodule $(B, \langle \cdot, \cdot \rangle)$ and $x \in W$ with $x < xs$ a polarised \mathbb{P}^1 -sheaf $M(B, x, xs)$. We say that B *satisfies local hard Lefschetz* (resp. *satisfies local HR*) *in the s direction* if

- (1) B satisfies local hard Lefschetz (resp. local HR);
- (2) for all $x \in W$ with $x < xs$ the polarised \mathbb{P}^1 -sheaf $M(B, x, xs)$ satisfies hard Lefschetz (resp. satisfies HR).

We abbreviate:

- hL(y) $_s$: $B(y)$ satisfies local hard Lefschetz in the s direction.
- HR(y) $_s$: HR(y) holds and $B(y)$ satisfies local HR in the s direction.

7. Proof

7.1. Outline of the proof

With the terminology of the previous section, the main result of this paper is the following.

THEOREM 7.1. *For all $y \in W$, HR(y) holds.*

We now outline the structure of the argument. Throughout, $y \in W$ and $s \in S$ is a simple reflection.

The following are the key statements, which rely on weak Lefschetz style induction.

CLAIM 7.2. (Proposition 7.15) $\text{HR}(<y)$ implies $\text{hL}(y)$.

CLAIM 7.3. (Proposition 7.17) If $ys > y$, then $\text{HR}(<y)_s + \text{HR}(y)$ implies $\text{hL}(y)_s$.

The following are “limit lemma” style arguments, which are easier.

CLAIM 7.4. If $ys > y$, then $\text{HR}(y) + \text{hL}(y)_s + \text{hL}(\leqslant ys)$ implies $\text{HR}(ys)$.

Proof. Firstly, if $B(ys)_x$ satisfies HR, then it satisfies HR with standard signs, by Theorem 6.19 and Corollary 3.7. Hence it is enough to check that $B(ys)_x$ and $B(ys)_{xs}$ satisfy HR, for all $x < xs$. Because $ys > y$, $B(y)B(s)$ is perverse (see Remark 6.8), $B(ys)$ is a summand of $B(y)B(s)$ and we have an isometry (see Lemma 6.23)

$$A \otimes_R (B(y)B(s))_x = M_{0,\infty}, \tag{7.1}$$

where $M = M(B(y), x, xs)$ is the \mathbb{P}^1 -sheaf associated with $B(y)$, and $x < xs$. Moreover, by Proposition 6.17 we have a canonical identification $B(y)B(s)_x = B(y)B(s)_{xs}$ which is -1 times an isometry (i.e. $\langle \cdot, \cdot \rangle^x = -\langle \cdot, \cdot \rangle^{xs}$ under this identification). By Lemma 6.6, $B(ys)$ is an orthogonal summand of $B(y)B(s)$ and we can apply Lemma 4.5 to conclude that it is enough to prove that $M_{0,\infty}$ satisfies HR for all x as above.

In other words, if multiplication by (z, z) on $M_{0,\infty}$ satisfies HR, then $\text{HR}(ys)$ holds. By our assumptions $\text{hL}(y)_s$ and $\text{hL}(\leqslant ys)$, multiplication by (az, z) on M satisfies hard Lefschetz for all $a \geqslant 1$. By $\text{HR}(y)$, the polarised \mathbb{P}^1 -sheaf M is easily seen to have opposite signs ($M_{0,\infty}$ is generated in degrees $\leqslant 0$ by (7.1) and the fact that $B(y)B(s)$ is perverse), and now the result follows by Corollary 5.21. □

CLAIM 7.5. If $ys > y$, then $\text{hL}(y)_s + \text{HR}(y)$ implies $\text{HR}(y)_s$.

Proof. Let $x < xs$ and $M = M(B(y), x, xs)$ be as in the previous proof. We need to check that M satisfies HR. We saw in the previous proof that $M_{0,\infty}$ is generated in degrees $\leqslant 0$. Now,

- (1) $\text{HR}(y)$ implies that M is polarised with opposite signs;
- (2) $\text{hL}(y)_s$ implies that multiplication by (az, z) on $M_{0,\infty}$ satisfies hard Lefschetz for $a > 1$.

The result now follows from Corollary 5.21 (with $I = (1, \infty)$). □

The following is straightforward (“constant case”).

CLAIM 7.6. (Proposition 7.11) If $ys < y$, $\text{HR}(\leqslant y)$ implies $\text{HR}(y)_s$.

From these claims we deduce the following.

CLAIM 7.7. *If $ys > y$, then $\text{HR}(<ys) + \text{hL}(y)_s$ implies $\text{HR}(ys)$.*

Proof. We have

$$\begin{aligned} \text{HR}(<ys) + \text{hL}(y)_s &\implies \text{HR}(<ys) + \text{hL}(\leq ys) + \text{hL}(y)_s && \text{(by Claim 7.2)} \\ &\implies \text{HR}(y) + \text{hL}(\leq ys) + \text{hL}(y)_s \\ &\implies \text{HR}(ys) && \text{(by Claim 7.4). } \quad \square \end{aligned}$$

We also have the following result.

CLAIM 7.8. *$\text{HR}(y) + \text{HR}(<y)_s$ implies $\text{HR}(y)_s$.*

Proof. If $ys < y$ then this follows from Claim 7.6 (remember that $\text{HR}(<y)_s$ includes $\text{HR}(<y)$ by definition). So we may assume that $ys > y$. Now $\text{hL}(y)_s$ holds by Claim 7.3, and then we are done by Claim 7.5. \square

Now we can give the proof of Theorem 7.1 (assuming the above statements).

Proof of Theorem 7.1. Let X denote an ideal in the Bruhat order, and for all $x \in X$ assume that $\text{HR}(x)$ and $\text{HR}(x)_s$ hold for all $s \in S$. If $X \neq W$, then we can choose $y' \in W \setminus X$ of minimal length and $s \in S$ with $y := y's < y'$. Now $y \in X$, so $\text{HR}(ys)$ holds by Claim 7.7, and then Claim 7.8 tells us that $\text{HR}(ys)_t$ holds for all $t \in S$. Hence we can add ys to our set X .

One may check directly that $\text{HR}(\text{id})$ and $\text{HR}(\text{id})_s$ hold for all $s \in S$. The above induction tells us that $\text{HR}(x)$ and $\text{HR}(x)_t$ hold for all $x \in W$ and $t \in S$. The theorem now follows. \square

7.2. Easy cases

In this section we make some easy observations which are used in the proof.

LEMMA 7.9. *Suppose that B is positively polarised and that $\text{HR}(y)$ holds for all indecomposable summands $B(y)$ of B . Then B satisfies local HR.*

Proof. Consider the canonical decomposition $B = \bigoplus V(y) \otimes_{\mathbb{R}} B(y)$ of § 6.5. Let z be maximal such that $V(z) \neq 0$. Fix $x \in W$. We want to show that $(A \otimes_{\mathbb{R}} B_x, A \otimes_{\mathbb{R}} \langle \cdot, \cdot \rangle_B^x)$ satisfies HR. Our decomposition induces a decomposition

$$A \otimes_{\mathbb{R}} B_x = \bigoplus A \otimes_{\mathbb{R}} (V(y) \otimes_{\mathbb{R}} B(y))_x.$$

Now let $p \in A \otimes_R (V(y) \otimes_{\mathbb{R}} B(y))_x$ be a primitive element in degree $d = -\ell(y) + 2d'$. Let $c = z^{-d} \langle p, p \rangle^x \in \mathbb{R}$. Then, by $HR(y)$ and the definition of positively polarised (see § 6.5) we see that

$$0 < (-1)^{(\ell(z) - \ell(y))/2} (-1)^{d' + \ell(x)} c = (-1)^{(\ell(z) + d)/2 + \ell(x)} c.$$

Hence the sign of c depends only on $\ell(z)$, $\ell(x)$ and d , and not on y , and hence B satisfies local HR. □

The proof of the following analogue of the previous lemma for \mathbb{P}^1 -sheaves is similar, and is left to the reader.

LEMMA 7.10. *Suppose that B is positively polarised and that $HR(y)_s$ holds for all indecomposable summands $B(y)$ of B . Then B satisfies local HR in the s direction.*

PROPOSITION 7.11. *Suppose that $ys < y$ with $y \in W$ and $s \in S$. If $HR(\leq y)$ holds then for all $x < xs$ the polarised \mathbb{P}^1 -sheaf $M(B(y), x, xs)$ is polarised constant, and hence satisfies HR.*

Proof. Our first step is to prove that the \mathbb{P}^1 -sheaf $M := M(B(ys)B(s), x, xs)$ is polarised constant. Let us check that the stalks are generated in degrees ≤ -2 . Because of the shift involved in the definition of M , this is equivalent to checking that $B(ys)B(s)_x$ and $B(ys)B(s)_{xs}$ are generated in degrees ≤ -1 . However $B(ys)B(s)$ is perverse (see Remark 6.8), and it follows from [28, Theorem 5.3] and the solution of Soergel’s conjecture that the stalks of any $B(z)$ with $z \neq \text{id}$ are generated in degrees ≤ -1 . Thus the claim follows from the fact that $B(\text{id})$ is not a summand of $B(ys)B(s)_x$ (all summands $B(z)$ satisfy $zs < z$).

Note that $B(ys)B(s)$ is positively polarized by Remark 6.8. By Lemma 7.9 and our assumption $HR(\leq y)$, we conclude that $B(ys)B(s)$ satisfies local HR. By Proposition 6.13 we have

$$(B(ys)B(s))_x = B(ys)_{x,xs}[1] = (B(ys)B(s))_{xs}.$$

By Proposition 6.17 we see that under the above identifications we have

$$\langle \cdot, \cdot \rangle_{B(ys)B(s)}^x = -\langle \cdot, \cdot \rangle_{B(ys)B(s)}^{xs}.$$

It follows that the \mathbb{P}^1 -sheaf M is polarised constant.

Now we have a canonical and orthogonal decomposition (see Lemma 6.6)

$$B(ys)B(s) = B(y) \oplus E$$

where E is a polarised Soergel bimodule with all indecomposable summands isomorphic to $B(z)$ for $z < ys < y$. Now it is not difficult to see that any orthogonal summand of a polarized constant sheaf is polarized constant. In particular the summand $M(B(y), x, xs)$ of M is polarised constant. Hence $M(B(y), x, xs)$ satisfies HR by Lemma 5.28. □

PROPOSITION 7.12. For $y \in W$, $(A \otimes_R B(y)_y, A \otimes_R \langle \cdot, \cdot \rangle_{B(y)}^y)$ satisfies HR with standard signs.

Proof. By Soergel’s character formula [28, Theorem 5.3], $B(y)_y$ is free of graded rank $v^{-\ell(y)}$. Thus

$$(A \otimes_R B(y)_y, A \otimes_R \langle \cdot, \cdot \rangle_{B(y)}^y)$$

satisfies hard Lefschetz, as this is automatic for A -modules of rank 1. Moreover, in the notation of §6.9, $c_{y,y} \in B(y)_y$ is a generator and

$$\langle c_{y,y}, c_{y,y} \rangle_{B(y)}^y = \gamma e_{y,y} = \gamma (-1)^{\ell(y)} \prod_{t \in L_T(y)} \frac{1}{\alpha_t} \quad \text{for some } \gamma \in \mathbb{R}_{>0},$$

by Theorem 6.19 and Proposition 3.6 (2). Applying σ (and using that $\sigma(\alpha) > 0$ for $\alpha \in \Phi^+$) yields the result. □

Remark 7.13. More generally, the above proof works whenever $B(y)_x$ is free of rank 1 (the “rationally smooth case”).

7.3. Non-deformed case

PROPOSITION 7.14. Suppose that $\underline{y} = s_1 \dots s_m$ is a reduced expression and $\lambda \in \mathfrak{h}^*$ is such that

$$\langle s_{i+1} \dots s_m(\lambda), \alpha_{s_i}^\vee \rangle > 0$$

for all $1 \leq i \leq m$. Then there exists a positively polarised Soergel bimodule $(B', \langle \cdot, \cdot \rangle_{B'})$ all of whose summands are isomorphic to $B(x)$ with $x < y$ and a map

$$d: B(y) \longrightarrow B'[1]$$

such that

$$d^* \circ d = B(y)\lambda - (y\lambda)B(y) \tag{7.2}$$

(as always, $B(y)$ is polarised with its intersection form).

The proof follows the same lines as the proof of [11, Theorem 6.21]. During the proof we need the perverse filtration and the functors $\tau_{\leq i}$ of [11, §6.3].

Proof. We prove the proposition by induction on m . The statement makes sense for $m=0$ (so $y=\text{id}$). In this case we can take $B'=0$.

Now assume $m \geq 1$. Let $z = ys_m$ and $s = s_m$ so $y = zs$. Then we can apply induction with $\underline{z} = s_1 \dots s_{m-1}$ and $s\lambda \in \mathfrak{h}^*$ to find a positively polarised bimodule $(D, \langle \cdot, \cdot \rangle_D)$ and a map

$$d': B(z) \longrightarrow D[1]$$

such that

$$(d')^* \circ d' = B(z)s(\lambda) - z(s(\lambda))B(z) = B(z)s(\lambda) - y(\lambda)B(z).$$

Now consider the map

$$d = \begin{pmatrix} \sqrt{\langle \lambda, \alpha_s^\vee \rangle} B(z)m \\ d'B(s) \end{pmatrix} : B(z)B(s) \longrightarrow (B(z) \oplus DB(s))[1].$$

(The target is polarised with respect to the intersection form on $B(z)$ and the induced form on $DB(s)$.) We have $m^* = \mu$ (see (6.6)) and hence the adjoint of d is (see (6.7))

$$d^* = \left(\sqrt{\langle \lambda, \alpha_s^\vee \rangle} B(z)\mu \quad (d')^* B(s) \right),$$

and hence

$$\begin{aligned} d^* \circ d &= \langle \lambda, \alpha_s^\vee \rangle B(z)(\mu \circ m) + ((d')^* \circ d')B(s) \\ &= \langle \lambda, \alpha_s^\vee \rangle B(z)(\mu \circ m) + B(z)s(\lambda)B(s) - y(\lambda)B(z)B(s) \\ &= B(z)B(s)\lambda - y(\lambda)B(z)B(s), \end{aligned}$$

by the “polynomial sliding” relation (6.1).

In particular d satisfies the relation (7.2). We now need to show that we can replace $B(z) \oplus DB(s)$ by a perverse summand whilst keeping the relation (7.2).

Let us choose a decomposition $D = \bigoplus B(u)^{\oplus m_u}$ with $m_u \in \mathbb{Z}_{\geq 0}$ and define

$$D^\uparrow = \bigoplus_{us > u} B(u)^{\oplus m_u} \quad \text{and} \quad D^\downarrow = \bigoplus_{us < u} B(u)^{\oplus m_u}.$$

Then we have a canonical, orthogonal decomposition (see Lemma 6.6)

$$D = D^\uparrow \oplus D^\downarrow.$$

The bimodule $D^\uparrow B(s)$ is perverse. We have (canonically and orthogonally)

$$B(z)B(s) = B(y) \oplus E \tag{7.3}$$

for some perverse Soergel bimodule E (see Remark 6.8). Moreover the restriction of the intersection form on $B(z)B(s)$ yields the intersection form on $B(y)$, up to a positive scalar multiple. By rescaling the inclusion $B(y) \hookrightarrow B(z)B(s)$ if necessary we may assume that this scalar multiple is 1.

We have a (non-canonical and non-orthogonal) decomposition

$$D^\downarrow B(s) = D^\downarrow[1] \oplus D^\downarrow[-1].$$

Consider the maps induced by d and d^* on the (canonical) summands $B(y) \subset B(z)B(s)$ and $D^\downarrow B(s) \subset B(z) \oplus DB(s)$:

$$B(y) \xrightarrow{f} D^\downarrow B(s)[1] \xrightarrow{f^*} B(y)[2].$$

All summands of D^\downarrow are isomorphic to B_x with $x < y$. Hence f lands in

$$\tau_{\leq -1}(D^\downarrow B(s)[1]) = D^\downarrow[2].$$

Similarly, f^* is zero on $\tau_{\leq -1}(D^\downarrow B(s)[1])$. In particular,

$$f^* \circ f = 0. \tag{7.4}$$

Let us write the matrix of d with respect to these decompositions as

$$d = \begin{pmatrix} a & b \\ c & d \\ f & g \end{pmatrix} : B(y) \oplus E \longrightarrow (B(z) \oplus D^\uparrow B(s) \oplus D^\downarrow B(s))[1].$$

Then

$$d^* = \begin{pmatrix} a^* & c^* & f^* \\ b^* & d^* & g^* \end{pmatrix}.$$

The computation of $d^* \circ d$ above and (7.4) imply that

$$a^* \circ a + c^* \circ c = a^* \circ a + c^* \circ c + f^* \circ f = B(y)\lambda - y(\lambda)B(y). \tag{7.5}$$

Now define d_{sub} to be the composition

$$B(y) \longrightarrow B(z)B(s) = B(y) \oplus E \xrightarrow{d} (B(z) \oplus D^\uparrow B(s) \oplus D^\downarrow B(s))[1] \longrightarrow (B(z) \oplus D^\uparrow B(s))[1],$$

where the first (resp. last map) is the inclusion (resp. projection) with respect to the above decompositions. By the orthogonality of these decompositions the adjoints of the first (resp. last) map is the projection (resp. inclusion). By [11, Proposition 6.12] the bimodule $D^\uparrow B(s)$ is positively polarised, and $B(z)$ is clearly positively polarised.

Finally, by (7.5),

$$d_{\text{sub}}^* \circ d_{\text{sub}} = a^* \circ a + c^* \circ c = B(y)\lambda - y(\lambda)B(y).$$

Now we are done: we can take $d = d_{\text{sub}}$ and $B' = B(z) \oplus D^\uparrow B(s)$, which is perverse and positively polarised. (It is easy to check that the signs on the two summands match up.) □

PROPOSITION 7.15. $\text{HR}(\langle y \rangle)$ implies $\text{hL}(y)$.

Proof. The fact that hard Lefschetz is true for $A \otimes_R B(y)_y$ follows from Proposition 7.12.

It remains to show hard Lefschetz for $x < y$. Let us apply the above proposition with $\lambda = \varrho$. Taking the stalk at x and applying $A \otimes_R (\cdot)$ we see that we have a map

$$A \otimes_R B(y)_x \xrightarrow{d} A \otimes_R B'_x[1]$$

of free A -modules equipped with forms $A \otimes \langle \cdot, \cdot \rangle_{B(y)_x}^x$ and $A \otimes \langle \cdot, \cdot \rangle_{B'_x}^x$, which are symmetric and non-degenerate over K and such that $d^* \circ d$ is equal to left multiplication by

$$\sigma(x(\lambda) - y(\lambda)).$$

By Lemma 3.2, $\sigma(x(\lambda) - y(\lambda)) > 0$, and in particular is non-zero. Moreover, as B' is positively polarised, Lemma 7.9 ensures that $A \otimes_R B'_x$ satisfies HR. Now we can apply Proposition 4.7 to conclude that $A \otimes_R B(y)_x$ satisfies hard Lefschetz. The proposition follows. \square

7.4. Deformed case

PROPOSITION 7.16. Let $\underline{y} = s_1 \dots s_m$ be a reduced expression and $s \in S$ be such that $ys > y$. Let $\lambda \in \mathfrak{h}^*$ be such that

$$\langle \lambda, \alpha_s^\vee \rangle > 0$$

and

$$\langle s_{i+1} \dots s_m s(\lambda), \alpha_{s_i}^\vee \rangle > 0$$

for all $1 \leq i \leq m$.

Then, for any $0 \leq a < 1$, there exist a positively polarised bimodule $(B', \langle \cdot, \cdot \rangle_{B'})$, all of whose summands are isomorphic to $B(x)$ with $x < y$, and a map

$$d: B(y)B(s) \longrightarrow (B(y) \oplus B'B(s))[1]$$

such that

$$d^* \circ d = B(y)B(s)\lambda - B(y)(as(\lambda))B(s) - (1-a)ys(\lambda)B(y)B(s) \tag{7.6}$$

($B(y)$ is polarised with its intersection form).

Proof. We want to find d such that

$$\begin{aligned} d^* \circ d &= B(y)B(s)\lambda - B(y)(as(\lambda))B(s) - (1-a)ys(\lambda)B(y)B(s) \\ &= B(y)(\mu \circ m)\langle \lambda, \alpha_s^\vee \rangle + B(y)(1-a)s(\lambda)B(s) - (1-a)ys(\lambda)B(y)B(s) \end{aligned} \tag{7.7}$$

(we have used (6.1)). Applying Proposition 7.14 with $y=s_1 \dots s_m$ and $(1-a)s(\lambda) \in \mathfrak{h}^*$ gives us a positively polarised bimodule B' , all of whose summands are isomorphic to $B(x)$ with $x < y$, and a map

$$d': B(y) \longrightarrow B'[1]$$

such that

$$(d')^* \circ d' = B(y)(1-a)s(\lambda) - (1-a)ys(\lambda)B(y).$$

Now, if we set

$$d := \begin{pmatrix} \sqrt{\langle \lambda, \alpha_s^\vee \rangle} B(y)m \\ d'B(s) \end{pmatrix} : B(y)B(s) \longrightarrow (B(y) \oplus B'B(s))[1],$$

then

$$d^* = \left(\sqrt{\langle \lambda, \alpha_s^\vee \rangle} B(y)\mu \quad (d')^* B(s) \right)$$

and

$$\begin{aligned} d^* \circ d &= \langle \lambda, \alpha_s^\vee \rangle B(y)(\mu \circ m) + ((d')^* \circ d')B(s) \\ &= \langle \lambda, \alpha_s^\vee \rangle B(y)(\mu \circ m) + B(y)(1-a)s(\lambda)B(s) - (1-a)ys(\lambda)B(y)B(s) \end{aligned}$$

as required. □

PROPOSITION 7.17. *If $ys > y$, $\text{HR}(\langle y \rangle_s) + \text{HR}(y)$ implies $\text{hL}(y)_s$.*

Proof. Let $\lambda \in \mathfrak{h}^*$ and

$$d: B(y)B(s) \longrightarrow (B(y) \oplus B'B(s))[1]$$

be as in the statement of the previous proposition (for some fixed $0 \leq a < 1$). Fix $x < xs$ and let us take the stalk of x (we abuse notation and continue to denote these maps by the same symbols):

$$\begin{aligned} d: (B(y)B(s))_x &\longrightarrow (B(y)_x \oplus B'B(s)_x)[1] \\ d^*: (B(y)_x \oplus B'B(s)_x) &\longrightarrow (B(y)B(s))_x[1]. \end{aligned}$$

We claim that we can obtain $A \otimes_R d$ and $A \otimes_R d^*$ as the global sections of a pair of adjoint maps

$$\tilde{d}: M \rightarrow N[1] \quad \text{and} \quad \tilde{d}^*: N \rightarrow M[1] \tag{7.8}$$

of polarised \mathbb{P}^1 -sheaves. Let B' and d' be as in the proof of the previous proposition. Consider the following polarised \mathbb{P}^1 -sheaves:

- (1) $M := M(B(y), x, xs)$, polarised as in §6.10.
- (2) $N' :=$ the skyscraper at 0 with stalk $A \otimes_R B(y)_x$ (i.e. $N'_0 = A \otimes_R B(y)_x$, $N_{\mathbb{C}^*} = 0$, $N'_\infty = 0$ and polarisation $\sigma(\langle \cdot, \cdot \rangle_{B(y)_x}^x)$ on N'_0).
- (3) $N'' := M(B', x, xs)$, polarised as in §6.10.

We have a natural map $\tilde{d}_1: M \rightarrow N'[1]$ given by $(b_0, b_\infty) \mapsto \sqrt{\langle \lambda, \alpha_s^\vee \rangle}(b_0, 0)$. The adjoint \tilde{d}_1^* is given by $(b_0, 0) \mapsto \sqrt{\langle \lambda, \alpha_s^\vee \rangle}(\sigma(x(\alpha_s))b_0, 0)$. Under the identification (6.14), one checks that on global sections the maps \tilde{d}_1 and \tilde{d}_1^* agree with the maps

$$A \otimes_R (B(y)B(s))_x \longrightarrow A \otimes_R B(y)_x[1] \quad \text{and} \quad A \otimes_R B(y)_x \longrightarrow A \otimes_R (B(y)B(s))_x[1]$$

induced by

$$\sqrt{\langle \lambda, \alpha_s^\vee \rangle} B(y)m \quad \text{and} \quad \sqrt{\langle \lambda, \alpha_s^\vee \rangle} B(y)\mu.$$

The map $d': B(y) \rightarrow B'[1]$ induces a map $\tilde{d}_2: M \rightarrow N''[1]$ of polarised \mathbb{P}^1 -sheaves. We denote its adjoint by \tilde{d}_2^* . Under the identification (6.14), the map \tilde{d}_2 agrees on global sections with the map $A \otimes_R (B(y)B(s))_x \rightarrow A \otimes_R (B'B(s))_x[1]$ induced by $d'B(s)$. By Lemma 6.23, \tilde{d}_2^* agrees on global sections with the map

$$A \otimes_R (B'B(s))_x \longrightarrow A \otimes_R (B(y)B(s))_x[1]$$

induced by $(d')^*B(s)$. Hence, if we set $N := N' \oplus N''$ and $\tilde{d} := \tilde{d}_1 + \tilde{d}_2$, we have constructed our desired maps in (7.8).

By the previous proposition

$$d^* \circ d = B(y)B(s)\lambda - B(y)(as(\lambda))B(s) - (1-a)ys(\lambda)B(y)B(s),$$

and hence, under the injection $B(y)B(s)_x \hookrightarrow B(y)_x \oplus B(y)_{xs}$, Lemma 6.16 implies that $d^* \circ d$ agrees with multiplication by

$$(x(\lambda) - a(xs(\lambda)) - (1-a)(ys(\lambda)), xs(\lambda) - a(xs(\lambda)) - (1-a)(ys(\lambda))).$$

Thus, for all $b \in M_{0,\infty}$, we have the relation

$$(d^* \circ d)(b) = \gamma \cdot b,$$

where $\gamma = (\lambda_0, \lambda_\infty) \in Z^2$ is given by

$$\begin{aligned} \lambda_0 &= \sigma((x-ys)(\lambda) - a(xs-ys)(\lambda)), \\ \lambda_\infty &= \sigma((xs-ys)(\lambda) - a(xs-ys)(\lambda)) = \sigma((1-a)(xs-ys)(\lambda)). \end{aligned} \tag{7.9}$$

Using Lemma 3.2, one may check that $\gamma \in Z_{\text{ample}}^2$ for all dominant regular λ and $0 \leq a < 1$.

Now B' is positively polarised and by our assumption $\text{HR}(\langle y \rangle_s)$, B' satisfies HR in the s direction, by Lemma 7.10. Hence N'' satisfies HR. Also $B(y)$ satisfies local HR by assumption, and so N' satisfies HR. Hence N satisfies HR (one checks easily that the signs on N' and N'' match). We deduce from Proposition 5.17 that $M(B(y), x, xs)$ satisfies hard Lefschetz for all pairs $(\lambda_0, \lambda_\infty)$ above.

We will see in the lemma below that we can vary λ and a so that λ_0/λ_∞ takes on all values in $(1, \infty)$. Hence $hL(y)_s$ holds, by Lemma 5.16. \square

LEMMA 7.18. *For varying dominant regular $\lambda \in \mathfrak{h}^*$ and $0 \leq a < 1$, λ_0/λ_∞ (see (7.9)) takes on all values in $(1, \infty)$.*

Proof. By continuity and the intermediate value theorem it is enough to show that by varying λ and a we can get values which are both arbitrarily large and arbitrarily close to 1. We have

$$\lambda_0 - \lambda_\infty = \sigma(x(\lambda) - xs(\lambda)) = \langle \lambda, \alpha_s^\vee \rangle \sigma(x(\alpha_s)),$$

and hence

$$\frac{\lambda_0}{\lambda_\infty} = \frac{\lambda_\infty + (\lambda_0 - \lambda_\infty)}{\lambda_\infty} = \frac{(1-a)C + \langle \lambda, \alpha_s^\vee \rangle \sigma(x(\alpha_s))}{(1-a)C},$$

where $C = \sigma((xs - ys)(\lambda))$. Hence if we choose $\lambda = \varrho$ then $\lambda_0/\lambda_\infty \rightarrow \infty$ as $a \rightarrow 1$.

On the other hand, if we take $a=0$ and let λ approach the s -wall (so that $\langle \lambda, \alpha_s^\vee \rangle \rightarrow 0$) then we see that $\lambda_0/\lambda_\infty \rightarrow 1$. The result now follows. \square

7.5. Soergel's conjecture

In this section we discuss how these arguments can be adapted to deduce Soergel's conjecture. Unfortunately the most difficult parts of the proof take the same road as [11], so this cannot be considered a new proof. In fact, in the author's opinion the current paper is strictly more complicated than [11]. For this reason we only give a sketch.

In the following (as in [11]) we fix $x \in W$ and $s \in S$ with $xs > x$ and assume Soergel's conjecture for all $y < xs$.

PROPOSITION 7.19. *If $B := B(x)B(s)$ satisfies local hard Lefschetz then Soergel's conjecture holds for $B(xs)$.*

Remark 7.20. This proposition seems to have first been observed by Soergel and Fiebig a number of years ago. We will see in the proof that the proposition is not if and only if. It is not clear to the author how much stronger local hard Lefschetz is. (There is also the question of the choice of specialisation parameter ϱ^\vee .)

Proof. Fix $y < xs$ and consider the inclusion

$$i_y: B_y^! \hookrightarrow B_y.$$

Then $B_y^!$ is generated in degrees $\geq \ell(y)$. Soergel shows that the image is contained in $B_y \cdot p_y$ for some (explicit) product of $\ell(y)$ roots p_y . Moreover, by [28, Lemma 7.1 (3)], Soergel’s conjecture for $B(xs)$ (under the assumption of Soergel’s conjecture for all $B(y)$ with $y < xs$) is equivalent to the above inclusion inducing an inclusion (then necessarily an isomorphism in degree $\ell(y)$)

$$(B_y^!)^{\ell(y)} \hookrightarrow \mathbb{R} \otimes_R (B_y \cdot p_y)$$

for all $y < xs$.

If we specialise via $\sigma: R \rightarrow A = \mathbb{R}[z]$, we see that Soergel’s conjecture is equivalent to the natural map inducing an inclusion

$$(A \otimes_R B_y^!)^{\ell(y)} \hookrightarrow \mathbb{R} \otimes_R (z^{\ell(y)} \otimes_A B_y).$$

This is the case if and only if

$$(1 \otimes B_y^!) \cap (z^{\ell(y)+1} \otimes B_y^{-\ell(y)-2}) = 0.$$

In other words, if we set $H := (A \otimes_R B_y) / (z \otimes_R B_y^!)$, we want

$$(\ker(z \cdot) \cap H^{\ell(y)}) \cap (z^{\ell(y)+1} H^{-\ell(y)-2}) = 0,$$

or in other words that multiplication by $z^{\ell(y)+2}$ should give an isomorphism

$$H^{-\ell(y)-2} \xrightarrow{\sim} H^{\ell(y)+2}.$$

This is clearly the case if z satisfies hard Lefschetz on H , which is the case if B satisfies local hard Lefschetz (essentially by definition, see Lemma 4.2). □

It seems likely that one could adapt the proof over the last few pages to prove local hard Lefschetz for $B(x)B(s)$ assuming only statements (Soergel’s conjecture, local hard Lefschetz and HR, etc.) for elements $y \leq x$. One could then use the above proposition to deduce Soergel’s conjecture and continue the induction. However the key ideas would still be those of [11] and this paper is already complicated enough!

8. Some calculations in \mathfrak{sl}_4

The goal of this section is to give a few examples of local intersection forms and see the connection to the Jantzen filtration.

Let $\mathfrak{g}=\mathfrak{sl}_4(\mathbb{R})$, $\mathfrak{b}\subset\mathfrak{g}$ be the Borel subalgebra of upper-triangular matrices, and $\mathfrak{h}\subset\mathfrak{b}$ be the Cartan subalgebra of diagonal matrices. Denote by α_s, α_t and α_u the simple roots in \mathfrak{h}^* and by s, t and u the corresponding simple reflections in the Weyl group W . (Our normalisation is such that $su=us$.) Let $\alpha_s^\vee, \alpha_t^\vee$ and $\alpha_u^\vee\in\mathfrak{h}$ denote the simple coroots.

Using the realisation \mathfrak{h} , we can define the category of Soergel bimodules for W and the theory of this paper applies. We work over \mathbb{R} so that we can discuss signatures.

8.1. The strategy

Recall the definition of the local intersection form. We start with a polarised Soergel bimodule $(B, \langle \cdot, \cdot \rangle_B)$. Then $\langle \cdot, \cdot \rangle_B$ induces an R -valued symmetric form on the costalk $B_y^!$ by restriction. The inclusion

$$B_y^! \hookrightarrow B_y$$

is an isomorphism over Q and realises $B_y^!$ and B_y as dual lattices in $Q\otimes_R B_y$. The R -valued form on $B_y^!$ then induces a Q -valued form on B_y , which is the local intersection form.

We use the following lemma to calculate the local intersection form.

LEMMA 8.1. *Let e_1, \dots, e_m denote a graded R -basis for $B_y^!$, and let e_1^*, \dots, e_m^* denote the dual basis of B_y . If $M:=\langle e_i, e_j \rangle_B$ denotes the Gram matrix of $\langle \cdot, \cdot \rangle_B$ on $B_y^!$ in the basis e_1, \dots, e_m , then the Gram matrix of $\langle \cdot, \cdot \rangle_B$ on B_y in the basis e_1^*, \dots, e_m^* is given by M^{-1} .*

Hence one needs to calculate a basis of $B_y^!$ and then compute the restriction of $\langle \cdot, \cdot \rangle_B$ to it. Finding a basis for $B_y^!$ is a linear algebra problem (one knows the graded rank from a calculation in the Hecke algebra). However this can be tricky in practice.

Below we will only consider the case $y=\text{id}$, in which case $B_{\text{id}}^!$ can be calculated easily using Soergel calculus [12], as we will see. (Actually the restriction $y=\text{id}$ is not necessary, but we don't go into that here.) In the following we will use the notation of [12, §2 and §6] concerning expressions and light leaves morphisms. We will denote subexpressions by the corresponding *01-sequence* (see [12, §2.4]). See [18, §2.10] for a sample calculation of local intersection forms using light leaves morphisms and Soergel calculus.

8.2. The first singular Schubert variety

We calculate the local intersection form for $B=B(tsut)$ at $y=id$.

In this case $B(t)B(s)B(u)B(t)$ is indecomposable (as follows from a calculation in the Hecke algebra) and hence $B=B(t)B(s)B(u)B(t)$. There are two subexpressions of $\underline{x}=tsut$ for $y=id$: $\underline{e}=0000$ and $\underline{f}=1001$ of defects 4 and 2 respectively. The corresponding light leaf maps (with colour coding s, t and u) are as follows:



Hence $\{l_{\underline{e}}, l_{\underline{f}}\}$ give a left (or right) R -basis for $\text{Hom}^*(R, B)=B_{id}^!$. Pairing the light leaf maps gives the matrix of the restriction of the intersection form on $B_{id}^!$:

$$\begin{pmatrix} \alpha_t^2 \alpha_s \alpha_u & \alpha_s \alpha_u \alpha_t \\ \alpha_s \alpha_u \alpha_t & -\alpha_t \alpha_0 \end{pmatrix}$$

(where $\alpha_0=\alpha_s+\alpha_t+\alpha_u$). The determinant of this matrix is

$$\det = -\alpha_t^2 \alpha_s \alpha_u (\alpha_s + \alpha_t) (\alpha_t + \alpha_u).$$

Inverting this matrix gives the matrix of the (Q -valued) form on B_{id} :

$$E = \frac{1}{\det} \begin{pmatrix} -\alpha_t \alpha_0 & -\alpha_s \alpha_t \alpha_u \\ -\alpha_s \alpha_t \alpha_u & \alpha_t^2 \alpha_s \alpha_u \end{pmatrix} = \begin{pmatrix} \frac{\alpha_0}{\alpha_s \alpha_t \alpha_u (\alpha_s + \alpha_t) (\alpha_t + \alpha_u)} & \frac{1}{\alpha_t (\alpha_s + \alpha_t) (\alpha_t + \alpha_u)} \\ \frac{1}{\alpha_t (\alpha_s + \alpha_t) (\alpha_t + \alpha_u)} & \frac{-1}{(\alpha_s + \alpha_t) (\alpha_t + \alpha_u)} \end{pmatrix}.$$

The determinants of the leading principal minors are

$$E_{1,1} = \frac{\alpha_0}{\alpha_s \alpha_t \alpha_u (\alpha_s + \alpha_t) (\alpha_t + \alpha_u)} \quad \text{and} \quad \det E = \frac{1}{\det}.$$

We conclude that for any regular dominant coweight $\varrho^\vee \in \mathfrak{h}$ the leading principal minors are >0 and <0 respectively. Hence the Hodge–Riemann relations are satisfied. Also $E_{1,1}$ agrees with the equivariant multiplicity at $y=id$ in the Schubert variety indexed by $x=tsut$ (see Theorem 6.19).

8.3. The second singular Schubert variety

Let $\underline{x}=sutsu$. Then

$$BS := B(s)B(u)B(t)B(s)B(u) \cong B(sutsu) \oplus B(su)[1] \oplus B(su)[-1].$$

One possible choice of idempotent projector to $B=B(sutsu)$ in $\text{End}(BS)$ is the following morphism (with colour coding s, t and u as above):

$$e = \left(\begin{array}{c} | \\ | \\ | \\ | \\ | \\ | \\ | \\ | \end{array} \right) + \left(\begin{array}{c} \text{---} \bullet \text{---} \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \\ \diagup \quad \diagdown \\ \text{---} \bullet \text{---} \\ | \\ | \end{array} \right).$$

There are four subexpressions of \underline{x} for $x=\text{id}$: 00000, 10010, 01001 and 11011 of defects 5, 3, 3 and 1, respectively. The light leaf morphisms corresponding to the subexpressions 10010 and 11011 give zero when composed with the idempotent e , and the light leaf morphisms l_5 and l_3 corresponding to 00000 and 01001 give a basis for $B_{\text{id}}^!$ after composition with e . The matrix of the intersection form on $B_{\text{id}}^!$ is given by

$$\begin{pmatrix} \alpha_s \alpha_t \alpha_u^2 (\alpha_s + \alpha_t) & \alpha_s \alpha_t \alpha_u (\alpha_s + \alpha_t) \\ \alpha_s \alpha_t \alpha_u (\alpha_s + \alpha_t) & -\alpha_s \alpha_u (\alpha_s + 2\alpha_t + \alpha_u) \end{pmatrix},$$

with determinant $-\alpha_s^2 \alpha_t \alpha_u^2 (\alpha_s + \alpha_t)(\alpha_t + \alpha_u)\alpha_0$. Inverting this matrix gives the intersection form on B_{id} :

$$E = \begin{pmatrix} \frac{\alpha_s + 2\alpha_t + \alpha_u}{\alpha_s \alpha_t \alpha_u (\alpha_s + \alpha_t)(\alpha_t + \alpha_u)\alpha_0} & \frac{1}{\alpha_s \alpha_u (\alpha_t + \alpha_u)\alpha_0} \\ \frac{1}{\alpha_s \alpha_u (\alpha_t + \alpha_u)\alpha_0} & \frac{-1}{\alpha_s (\alpha_t + \alpha_u)\alpha_0} \end{pmatrix}.$$

Again we see that $E_{1,1}$ agrees with the equivariant multiplicity and that the two leading principal minors have signatures 1 and 0 under any specialisation determined by a regular dominant coweight. Hence the Hodge–Riemann bilinear relations are satisfied.

This example has an interesting feature not seen in the previous case. Since the numerator of $E_{1,1}$ is not a product of roots, there exist regular $\gamma^\vee \in \mathfrak{h}$ such that the evaluation of $E_{1,1}$ at γ^\vee gives zero (i.e. $\langle \alpha_s + 2\alpha_t + \alpha_u, \gamma^\vee \rangle = 0$). For such γ^\vee local hard Lefschetz fails. (All that matters in this example is $E_{1,1}$. Even if the reader did not follow the above calculation, one can calculate $E_{1,1}$ easily using the nil Hecke ring and Theorem 6.19.)

As explained in the introduction, via the work of Soergel and Kübel this example implies that the Jantzen filtration behaves differently for a choice of deformation direction corresponding to γ^\vee . We analyse this directly. (Actually, the example we will consider corresponds to the local intersection form of B at $y=su$, not $y=\text{id}$ (this gives a weight space of more manageable dimension). However the behaviour is very similar to the above.)

8.4. Examples of the Jantzen filtration

We keep the notation above, except that now we work over \mathbb{C} . Let

$$\varrho = \frac{1}{2}(3\alpha_s + 4\alpha_t + 3\alpha_u)$$

denote the half sum of the positive roots and let

$$x \cdot \lambda := x(\lambda + \varrho) - \varrho$$

denote the dot action of W on \mathfrak{h}^* . We work in \mathcal{O}_0 , the principal block of category \mathcal{O} (i.e. all modules are \mathfrak{g} -finitely generated, \mathfrak{b} -integrable, \mathfrak{h} -semisimple and have the same central character as the trivial representation). We denote the Verma and simple module of highest weight $x \cdot 0$ by $\Delta(x)$ and $L(x)$.

Motivated by the previous section, we consider the Verma module $\Delta(su)$ of highest weight $su \cdot 0 = -\alpha_s - \alpha_u$ and its weight space at

$$\lambda = sutsu \cdot 0 = -3\alpha_s - 3\alpha_t - 3\alpha_u.$$

The dimension of the λ -weight space is the number of Kostant partitions of

$$-(\alpha_s + \alpha_u) - \lambda = 2\alpha_s + 3\alpha_t + 2\alpha_u,$$

which is 13.

Let $\Delta := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} S(\mathfrak{h})$ denote the universal Verma module (of highest weight univ). Computing the Shapovalov form on the weight space univ $-\nu$, with $\nu = su \cdot 0 - \lambda$, gives a 13×13 matrix of polynomials in $S(\mathfrak{h})$. One can compute this matrix of polynomials via computer (I used `magma`). Specialising via a highest weight $\mu: S(\mathfrak{h}) \rightarrow \mathbb{C}$ gives the Shapovalov form on the weight space $\mu - \nu$.

If we choose to deform via ϱ we get Jantzen filtration layers of dimensions

$$7, \quad 3, \quad 2 \quad \text{and} \quad 1.$$

Now choose $\gamma \in \mathfrak{h}^*$ such that γ does not vanish on any coroot but

$$\gamma(\alpha_s^\vee + 2\alpha_t^\vee + \alpha_u^\vee) = 0.$$

With this choice of deformation direction the Jantzen filtration layers have dimensions

$$7, \quad 2, \quad 4 \quad \text{and} \quad 0.$$

By Kazhdan–Lusztig theory, we have (in the Grothendieck group of \mathcal{O}_0)

$$\begin{aligned} \Delta(su) = & L(su) + L(stu) + L(tsu) + L(sut) + L(uts) \\ & + L(suts) + L(stut) + L(stsu) + L(tuts) + 2L(stuts) + \dots \end{aligned}$$

(where ... consists of terms $L(x)$ with $x > sutus$). Taking the dimension of the λ -weight space we get (again by Kazhdan–Lusztig theory)

$$13 = 7 + 0 + 0 + 2 + 0 + 1 + 1 + 0 + 0 + 2.$$

By Kazhdan–Lusztig theory, one expects the following Jantzen filtration layers

$$\begin{array}{rcl} & L(su) & 0 \\ & L(stu) \oplus L(tsu) \oplus L(sut) \oplus L(uts) \oplus L(stuts) & 1 \\ \Delta(su) = & L(suts) \oplus L(stut) \oplus L(stsu) \oplus L(tuts) \oplus \dots & 2 \\ & L(stuts) \oplus \dots & 3 \\ & \vdots & \vdots \end{array}$$

where we have omitted any terms $L(x)$ with $x > sutus$.

Taking dimensions of weight spaces gives

$$\begin{array}{r} 7 \\ 13 = \begin{array}{r} 0+0+2+0+1 \\ 1+1+0+0 \end{array} \\ 1. \end{array}$$

This matches the above calculation of filtration layers 7, 3, 2 and 1.

One sees what happens when the Jantzen filtration degenerates. The two subquotients isomorphic to $L(stuts)$ which generically occur in degrees 1 and 3 “slide together” into degree 2 so that gr_2 is no longer semi-simple.

9. List of notation

The most important cast members, in order of appearance:

$[m]$	the shift of grading functor, §2.1
$\text{deg}_{\leq i}$	a term in the degree filtration, §2.3
(W, S)	the fixed Coxeter system, §3.1
ℓ, \leq	the length function and Bruhat order, §3.1
x, u	an expression, a subexpression, §3.1
\mathfrak{h}	the reflection faithful representation of \mathfrak{h} , §3.2
ϱ, ϱ^\vee	fixed dominant regular elements of \mathfrak{h}^* and \mathfrak{h} , §3.2
R, Q	the regular functions on \mathfrak{h} and its localisation at Φ , §3.3
∂_s	a divided difference operator, §3.3
A, K	the rings $\mathbb{R}[z]$ and $\mathbb{R}[z^{\pm 1}]$, §3.3
σ	the homomorphism $R \rightarrow A$ determined by ϱ^\vee , §3.3
P^{-d}	primitive subspaces, §4.2
$e_{x,y}$	an equivariant multiplicity, §3.4
M, M_0, M_∞	a \mathbb{P}^1 -sheaf and its stalks, §5.2
Z, Z^2	the structure algebra and its degree two elements, §5.2
Z_{ample}^2	the ample cone in Z^2 , §5.2.
$B(w), B(y)$	a Bott–Samelson (resp. indecomposable) Soergel bimodule, §6.2
c_{id}, c_s	elements in Soergel bimodules $B(s)$, §6.2
m, μ	“dot” maps $B(s) \rightarrow R[1]$ and $R \rightarrow B(s)[1]$, §6.2
B_x, B_x^1	the stalk and costalk of a Soergel bimodule, §6.3
i_x	the inclusion $B_x^1 \rightarrow B_x$, §6.3
Gr_x	the (twisted) graph of x inside $\mathfrak{h} \times \mathfrak{h}$, §6.3
$M(B, x, xs)$	the \mathbb{P}^1 -sheaf associated with B and $x < xs$, §6.10

References

- [1] BARBASCH, D., Filtrations on Verma modules. *Ann. Sci. École Norm. Sup.*, 16 (1983), 489–494.
- [2] BEILINSON, A. & BERNSTEIN, J., A proof of Jantzen conjectures, in *I. M. Gel'fand Seminar*, Adv. Soviet Math., 16, pp. 1–50. Amer. Math. Soc., Providence, RI, 1993.
- [3] BEILINSON, A., GINZBURG, V. & SOERGEL, W., Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9 (1996), 473–527.
- [4] BERNSTEIN, J. & LUNTS, V., *Equivariant Sheaves and Functors*. Lecture Notes in Mathematics, 1578. Springer, Berlin–Heidelberg, 1994.
- [5] BOSMA, W., CANNON, J. & PLAYOUST, C., The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24 (1997), 235–265.

- [6] BRADEN, T. & MACPHERSON, R., From moment graphs to intersection cohomology. *Math. Ann.*, 321 (2001), 533–551.
- [7] BRION, M., Equivariant cohomology and equivariant intersection theory, in *Representation Theories and Algebraic Geometry* (Montreal, QC, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, pp. 1–37. Kluwer, Dordrecht, 1998.
- [8] DE CATALDO, M. A. A. & MIGLIORINI, L., The hard Lefschetz theorem and the topology of semismall maps. *Ann. Sci. École Norm. Sup.*, 35 (2002), 759–772.
- [9] — The Hodge theory of algebraic maps. *Ann. Sci. École Norm. Sup.*, 38 (2005), 693–750.
- [10] ELIAS, B. & KHOVANOV, M., Diagrammatics for Soergel categories. *Int. J. Math. Math. Sci.*, 2010 (2010), Art. ID 978635, 58 pp.
- [11] ELIAS, B. & WILLIAMSON, G., The Hodge theory of Soergel bimodules. *Ann. of Math.*, 180 (2014), 1089–1136.
- [12] — Soergel calculus. *Represent. Theory*, 20 (2016), 295–374.
- [13] FIEBIG, P., Kazhdan–Lusztig combinatorics via sheaves on Bruhat graphs, in *Algebraic and Geometric Combinatorics*, Contemp. Math., 423, pp. 195–204. Amer. Math. Soc., Providence, RI, 2006.
- [14] — The combinatorics of Coxeter categories. *Trans. Amer. Math. Soc.*, 360 (2008), 4211–4233.
- [15] — An upper bound on the exceptional characteristics for Lusztig’s character formula. *J. Reine Angew. Math.*, 673 (2012), 1–31.
- [16] FIEBIG, P. & WILLIAMSON, G., Parity sheaves, moment graphs and the p -smooth locus of Schubert varieties. *Ann. Inst. Fourier (Grenoble)*, 64 (2014), 489–536.
- [17] GABBER, O. & JOSEPH, A., Towards the Kazhdan–Lusztig conjecture. *Ann. Sci. École Norm. Sup.*, 14 (1981), 261–302.
- [18] HE, X. & WILLIAMSON, G., Soergel calculus and Schubert calculus. To appear in *Bull. Inst. Math. Acad. Sinica*.
- [19] JANTZEN, J. C., *Moduln mit einem höchstem Gewicht*. Lecture Notes in Mathematics, 750. Springer, Berlin–Heidelberg, 1979.
- [20] JUTEAU, D. & WILLIAMSON, G., Kumar’s criterion modulo p . *Duke Math. J.*, 163 (2014), 2617–2638.
- [21] KOSTANT, B. & KUMAR, S., The nil Hecke ring and cohomology of G/P for a Kac–Moody group G . *Adv. in Math.*, 62 (1986), 187–237.
- [22] KÜBEL, J., From Jantzen to Andersen filtration via tilting equivalence. *Math. Scand.*, 110 (2012), 161–180.
- [23] — Tilting modules in category \mathcal{O} and sheaves on moment graphs. *J. Algebra*, 371 (2012), 559–576.
- [24] KUMAR, S., The nil Hecke ring and singularity of Schubert varieties. *Invent. Math.*, 123 (1996), 471–506.
- [25] SOERGEL, W., Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3 (1990), 421–445.
- [26] — Character formulas for tilting modules over Kac–Moody algebras. *Represent. Theory*, 2 (1998), 432–448.
- [27] — Langlands’ philosophy and Koszul duality, in *Algebra—Representation Theory* (Constanta, 2000), NATO Sci. Ser. II Math. Phys. Chem., 28, pp. 379–414. Kluwer, Dordrecht, 2001.
- [28] — Kazhdan–Lusztig–Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu*, 6 (2007), 501–525.
- [29] — Andersen filtration and hard Lefschetz. *Geom. Funct. Anal.*, 17 (2008), 2066–2089.

GEORDIE WILLIAMSON
University of Sydney
School of Mathematics and Statistics
Carslaw Building F07
Sydney, NSW 2006
Australia
g.williamson@sydney.edu.au

Received October 29, 2014

Received in revised form May 5, 2016