ON THE RELATION OF THE ABELIAN TO THE JACOBIAN ELLIPTIC FUNCTIONS

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1. The three Jacobian elliptic functions $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$, their reciprocals and their quotients form a system of twelve functions which I have found it convenient to represent by a uniform notation viz. putting $\frac{\operatorname{sn} x}{\operatorname{cn} x} = \operatorname{sc} x$, $\frac{\operatorname{cn} x}{\operatorname{dn} x} = \operatorname{cd} x$, $\frac{1}{\operatorname{sn} x} = \operatorname{ns} x$, &c, we have twelve functions, each denoted by a functional sign consisting of two of the letters s, c, d, n, and forming the following four groups, the members of each group having the same final letter:

$$\operatorname{sn} x, \operatorname{cn} x, \operatorname{dn} x\,;\ \operatorname{cd} x, \operatorname{sd} x, \operatorname{nd} x\,;\ \operatorname{dc} x, \operatorname{nc} x, \operatorname{sc} x\,;\ \operatorname{ns} x, \operatorname{ds} x, \operatorname{cs} x.$$

Each of these four groups might have been selected as the standard group, the members of the other groups being derived from it merely as reciprocals and quotients. The actual selection of the first group by JACOBI was due to the fact that $(1 - x^2)(1 - c^2x^2)$ was LEGENDRE's standard form of the general quartic function, this form having been chosen by him in order that the denominator of the integral might be reducible to the form $\sqrt{1 - c^2 \sin^2 \varphi}$.

2. In treating the Jacobian theory in my lectures on Elliptic Functions, I have been accustomed for many years to employ the twelve elliptic functions sn, cn, cd, dc, &c. and the four corresponding Zeta functions denoted, by a somewhat analogous notation, by zn, zd, zc, zs. These

sixteen functions form a complete system, of which, however, only twelve are doubly periodic.[1]

The s-group has some special properties, but with this exception all the four groups of the elliptic functions are similar to one another in all essential respects, and any one of the four might have been selected as the standard group.

We can pass from any one group to the other three either by forming the reciprocals and quotients of the members of the original group as indicated by the notation (i. e. $\frac{\operatorname{sd} x}{\operatorname{nd} x} = \frac{\operatorname{sc} x}{\operatorname{nc} x} = \frac{1}{\operatorname{ns} x} = \operatorname{sn} x$), or by increasing the arguments by K, iK', $K + iK'$. Thus, taking for example the n-group, we have

$$\operatorname{sn}(x+K) = \operatorname{cd} x, \qquad \operatorname{sn}(x+iK') = \frac{1}{k}\operatorname{ns} x,$$

$$\operatorname{cn}(x+K) = -k'\operatorname{sd} x, \qquad \operatorname{cn}(x+iK') = -\frac{i}{k}\operatorname{ds} x,$$

$$\operatorname{dn}(x+K) = k'\operatorname{nd} x, \qquad \operatorname{dn}(x+iK') = -i\operatorname{cs} x,$$

$$\operatorname{sn}(x+K+iK') = \frac{1}{k}\operatorname{dc} x,$$

$$\operatorname{cn}(x+K+iK') = -\frac{ik'}{k}\operatorname{nc} x,$$

$$\operatorname{dn}(x+K+iK') = ik\operatorname{sc} x.$$

3. The algebraical relations connecting the three members of each group are:

$$\operatorname{sn}^2 x + \operatorname{cn}^2 x = 1, \qquad k'^2\operatorname{sd}^2 x + \operatorname{cd}^2 x = 1,$$

$$k^2\operatorname{sn}^2 x + \operatorname{dn}^2 x = 1, \qquad \operatorname{nd}^2 x - k^2\operatorname{sd}^2 x = 1,$$

$$\operatorname{dn}^2 x - \operatorname{cn}^2 x = k'^2, \qquad k^2\operatorname{cd}^2 x + k'^2\operatorname{nd}^2 x = 1,$$

$$\operatorname{nc}^2 x - \operatorname{sc}^2 x = 1, \qquad \operatorname{ns}^2 x - \operatorname{cs}^2 x = 1,$$

$$\operatorname{dc}^2 x - k'^2\operatorname{sc}^2 x = 1, \qquad \operatorname{ns}^2 x - \operatorname{ds}^2 x = k^2,$$

$$\operatorname{dc}^2 x - k'^2\operatorname{nc}^2 x = k^2, \qquad \operatorname{ds}^2 x - \operatorname{cs}^2 x = k'^2,$$

[1] This system of twelve or sixteen functions is considered in the Messenger of Mathematics vol. XI, 81—95, 120—138 (1881—2), XV, 92—148 (1885), XVI, 67—86 (1886), XVII, 1—18 (1887), XVIII, 1—84 (1888).

and the formulæ giving the derivatives of the functions with respect to x are:

$$\begin{array}{ll} \mathrm{sn}' x = \mathrm{cn}\, x\, \mathrm{dn}\, x, & \mathrm{cd}' x = -k'^2\, \mathrm{sd}\, x\, \mathrm{nd}\, x, \\ \mathrm{cn}' x = -\mathrm{dn}\, x\, \mathrm{sn}\, x, & \mathrm{sd}' x = \mathrm{nd}\, x\, \mathrm{cd}\, x, \\ \mathrm{dn}' x = -k^2\, \mathrm{sn}\, x\, \mathrm{cn}\, x, & \mathrm{nd}' x = k^2\, \mathrm{cd}\, x\, \mathrm{sd}\, x, \\ \mathrm{dc}' x = k'^2\, \mathrm{nc}\, x\, \mathrm{sc}\, x, & \mathrm{ns}' x = -\mathrm{ds}\, x\, \mathrm{cs}\, x, \\ \mathrm{nc}' x = \mathrm{sc}\, x\, \mathrm{dc}\, x, & \mathrm{ds}' x = -\mathrm{cs}\, x\, \mathrm{ns}\, x, \\ \mathrm{sc}' x = \mathrm{dc}\, x\, \mathrm{nc}\, x, & \mathrm{cs}' x = -\mathrm{ns}\, x\, \mathrm{ds}\, x. \end{array}$$

Certain multipliers such as k, k', &c. are connected with some of the functions, and attaching these quantities to the functions to which they belong, the four groups become

$$k\, \mathrm{sn}\, x\,,\ k\, \mathrm{cn}\, x\,,\ \mathrm{dn}\, x\,;\ k\, \mathrm{cd}\, x\,,\ kk'\, \mathrm{sd}\, x\,,\ k'\, \mathrm{nd}\, x\,;\ \mathrm{dc}\, x\,,\ k'\, \mathrm{nc}\, x\,,\ k'\, \mathrm{sc}\, x;$$

$$\mathrm{ns}\, x\,,\ \mathrm{ds}\, x\,,\ \mathrm{cs}\, x,$$

which, except for sign and the multiplier i, transform into one another by the addition of K, iK', $K + iK'$ to the argument.

4. It will be apparent by inspection of the above formulæ, and it becomes still more evident in working systematically with the twelve functions that, although each group might properly have been selected as the standard, still the groups differ from each other in points of detail which affect their convenience in use.

The s-group is the most regular, and is free from k-coefficients, but, like the cotangent and cosecant in trigonometry, the functions become infinite when x is zero. The c- and d-groups are very much alike. Although the d-group is the most encumbered by external factors, k and k' are involved in a quasi-symmetrical manner, and on this account this group appears to be preferable to the n- or c-group. In the s-group all three functions are uneven; in the other three groups one is uneven and two are even.

5. It is evident that the Abelian elliptic functions φ, f, F bear a close general resemblance to sn, cn, dn respectively, but the actual relations

between φ and sn, f and cn, &c. are complicated and inconvenient, the modulus of the Jacobian functions being imaginary. These relations are

$$\varphi(x) = \frac{\operatorname{sn} cx}{c}$$

$$f(x) = \operatorname{cn} cx,$$

$$F(x) = \operatorname{dn} cx,$$

where k, the modulus of the Jacobian functions, $= \frac{ie}{c}$. The corresponding relations between the periods are also complicated viz.

$$\omega = \frac{2K}{c}, \qquad \varpi = \frac{2(iK - K')}{c}$$

the modulus k being $\frac{ie}{c}$ as before.

Thus, in spite of the general similarity between the functions, the direct transition from φ, f, F to sn, cn, dn is by no means convenient.

6. The special object of this note is to point out that it is quite otherwise if we identify φ, f, F not with sn, cn, dn but with sd, cd, nd; in fact these two sets of functions are practically the same and the transition is extremely simple.

Thus, if we put $a^2 = c^2 + e^2$, c and e being Abel's c and e, and take

$$k = \frac{e}{a}, \quad \text{so that} \quad k' = \frac{c}{a},$$

then

$$\varphi(x) = \frac{\operatorname{sd} ax}{a},$$

$$f(x) = \operatorname{cd} ax,$$

$$F(x) = \operatorname{nd} ax,$$

and the relations between the periods are

$$\omega = \frac{2K}{a}, \qquad \varpi = \frac{2K'}{a}.$$

If we suppose c and e connected by the relation $c^2 + e^2 = 1$, so that

$a = 1$, then $\varphi(x)$, $f(x)$, $F(x)$ are the same as $\operatorname{sd} x$, $\operatorname{cd} x$, $\operatorname{nd} x$ and $\frac{1}{2}\omega$ and $\frac{1}{2}\varpi$ are the same as K and K', c and e being k' and k.

Thus, just as Jacobi selected the n-group, so did Abel select the d-group. The only other difference is that Abel's functions depend upon two disposable constants c and e, and Jacobi's upon only one k.

7. As already mentioned the d-group seems preferable to the n-group on account of its greater regularity. It has also certain other advantages; for example, considering the change of the argument x into ix, the formulæ for the twelve functions are:

$$\begin{aligned}
\operatorname{sn} ix &= i \operatorname{sc}(x, k'), & \operatorname{cd} ix &= \operatorname{nd}(x, k'),\\
\operatorname{cn} ix &= \operatorname{nc}(x, k'), & \operatorname{sd} ix &= i \operatorname{sd}(x, k'),\\
\operatorname{dn} ix &= \operatorname{dc}(x, k'), & \operatorname{nd} ix &= \operatorname{cd}(x, k'),\\
\operatorname{dc} ix &= \operatorname{dn}(x, k'), & \operatorname{ns} ix &= -i \operatorname{cs}(x, k'),\\
\operatorname{nc} ix &= \operatorname{cn}(x, k'), & \operatorname{ds} ix &= -i \operatorname{ds}(x, k'),\\
\operatorname{sc} ix &= i \operatorname{sn}(x, k'), & \operatorname{cs} ix &= -i \operatorname{ns}(x, k').
\end{aligned}$$

Thus when the argument is multiplied by i the n-group transforms into the c-group and vice versa, but the d-group and the s-group transform each into itself. On account of the property $\operatorname{sd} ix = i \operatorname{sd}(x, k')$ the function $\operatorname{sd} x$ is perhaps the most convenient of the twelve to select when the choice is quite free as, e. g., when it is required to select an elliptic function as a subsidiary function.

8. The Jacobian function $\operatorname{sn} x$ was obtained by the inversion of the integral

$$\int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

(this being Legendre's form with k instead of c), and the Abelian function $\varphi(x)$ by the inversion of the integral

$$\int_0^x \frac{dx}{\sqrt{(1 - c^2x^2)(1 + e^2x^2)}}$$

and, except as regards sign, $\mathrm{cn}\,x$ and $\mathrm{dn}\,x$ may be defined by

$$\mathrm{cn}^2 x = 1 - \mathrm{sn}^2 x, \qquad \mathrm{dn}^2 x = 1 - k^2 \mathrm{sn}^2 x,$$

and $f(x)$ and $F(x)$ by

$$f^2(x) = 1 - c^2\varphi^2(x), \qquad F^2(x) = 1 + e^2\varphi^2(x).$$

JACOBI's K was defined by

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}},$$

K' being the same function of k', and ABEL's quarter-periods were defined by

$$\frac{1}{2}\omega = \int_0^{\frac{1}{c}} \frac{dx}{\sqrt{(1 - c^2x^2)(1 + e^2x^2)}}, \qquad \frac{1}{2}\varpi = \int_0^{\frac{1}{e}} \frac{dx}{\sqrt{(1 - e^2x^2)(1 + c^2x^2)}}.$$

9. ABEL's selection of the d-group was not accidental. After stating in his 'Recherches' that his object is to consider the inverse of LEGENDRE's integral, he mentions that he had remarked that the formulæ become simpler by supposing c^2 negative in LEGENDRE's form $(1 - x^2)(1 - c^2x^2)$. He therefore puts $c^2 = -e^2$ and, for greater symmetry replaces $1 - x^2$ by $1 - c^2x^2$.

The property of $\varphi(x)$ which is equivalent to $\mathrm{sd}\,x = i\,\mathrm{sd}(x, k')$ was noticed at the very outset by ABEL who shows that, by the interchange of c and e, $\frac{\varphi(xi)}{i}$ is changed into $\varphi(x)$; whence also by the same interchange, $f(xi)$ and $F(xi)$ are changed into $F(x)$ and $f(x)$.[1]

10. Every elliptic function is of course the inverse of an integral of the form $\int \frac{dx}{\sqrt{X}}$, where X is a quartic (or cubic) function of x. The

[1] ABEL's form of elliptic function is peculiarly well adopted to the geometry of the lemniscate, and this may have helped to influence his choice of the form of integral, which reduces to that giving the length of the arc of a lemniscate when $c = e = 1$. The relation between r the radius vector, and s the arc of the lemniscate $r^2 = a^2 \cos 2\theta$, both measured from the pole, is $r = \frac{a}{\sqrt{2}}\,\mathrm{sd}\left(s\sqrt{2}, \frac{1}{\sqrt{2}}\right)$.

values of X which correspond to the twelve functions (as is evident from the formulæ in § 3) are:

$$
\begin{array}{llll}
\operatorname{sn} x\,,\ X = & \sqrt{(1-x^2)(1-k^2x^2)}\,, & \operatorname{cd} x\,,\ X = & -\sqrt{(1-x^2)(1-k^2x^2)}\,,\\
\operatorname{cn} x\,,\ X = & -\sqrt{(1-x^2)(k'^2+k^2x^2)}\,, & \operatorname{sd} x\,,\ X = & \sqrt{(1+k^2x^2)(1-k'^2x^2)}\,,\\
\operatorname{dn} x\,,\ X = & -\sqrt{(1-x^2)(x^2-k'^2)}\,, & \operatorname{nd} x\,,\ X = & \sqrt{(1-k'^2x^2)(x^2-1)}\,,\\
\operatorname{dc} x\,,\ X = & \sqrt{(x^2-1)(x^2-k^2)}\,, & \operatorname{ns} x\,,\ X = & -\sqrt{(x^2-1)(x^2-k^2)}\,,\\
\operatorname{nc} x\,,\ X = & \sqrt{(x^2-1)(k^2+k'^2x^2)}\,, & \operatorname{ds} x\,,\ X = & -\sqrt{(x^2+k^2)(x^2-k'^2)}\,,\\
\operatorname{sc} x\,,\ X = & \sqrt{(1+x^2)(1+k'^2x^2)}\,, & \operatorname{cs} x\,,\ X = & -\sqrt{(x^2+1)(x^2+k'^2)}\,.
\end{array}
$$

Among the groups the *s*-group is the most regular, but among the twelve functions the value of X is most symmetrical in the case of $\operatorname{sd} x$ and $\operatorname{ds} x$.

11. The addition formulæ for the Jacobian *n*-group are so well known that they need not be written down; those for the *c*-group which closely resemble them (the principal difference being the substitution of k'^2 for k^2) are:

$$\operatorname{sc}(u+v) = \frac{\operatorname{sc}u\operatorname{dc}v\operatorname{nc}v + \operatorname{sc}v\operatorname{dc}u\operatorname{nc}u}{1-k'^2\operatorname{sc}^2u\operatorname{sc}^2v}\,,$$

$$\operatorname{nc}(u+v) = \frac{\operatorname{nc}u\operatorname{nc}v + \operatorname{sc}u\operatorname{dc}u\operatorname{sc}v\operatorname{dc}v}{1-k'^2\operatorname{sc}^2u\operatorname{sc}^2v}\,,$$

$$\operatorname{dc}(u+v) = \frac{\operatorname{dc}u\operatorname{dc}v + k'^2\operatorname{sc}u\operatorname{nc}u\operatorname{sc}v\operatorname{nc}v}{1-k'^2\operatorname{sc}^2u\operatorname{sc}^2v}\,,$$

For the *d*-group the formulæ are

$$\operatorname{sd}(u+v) = \frac{\operatorname{sd}u\operatorname{cd}v\operatorname{nd}v + \operatorname{sd}v\operatorname{cd}u\operatorname{nd}u}{1+k^2k'^2\operatorname{sd}^2u\operatorname{sd}^2v}\,,$$

$$\operatorname{cd}(u+v) = \frac{\operatorname{cd}u\operatorname{cd}v - k^2\operatorname{sd}u\operatorname{sd}v\operatorname{nd}u\operatorname{nd}v}{1+k^2k'^2\operatorname{sd}^2u\operatorname{sd}^2v}\,,$$

$$\operatorname{nd}(u+v) = \frac{\operatorname{nd}u\operatorname{nd}v + k'^2\operatorname{sd}u\operatorname{sd}v\operatorname{cd}u\operatorname{cd}v}{1+k^2k'^2\operatorname{sd}^2u\operatorname{sd}^2v}\,,$$

being practically the same as Abel's formulæ, with k' and k in place of c and e.

For the s-group the formulæ are

$$\mathrm{ns}(u+v)=\frac{\mathrm{ns}\,u\,\mathrm{ds}\,v\,\mathrm{cs}\,v-\mathrm{ns}\,v\,\mathrm{ds}\,u\,\mathrm{cs}\,u}{\mathrm{ns}^2v-\mathrm{ns}^2u}, \qquad \mathrm{ds}(u+v)=\frac{\mathrm{ds}\,u\,\mathrm{cs}\,v\,\mathrm{ns}\,v-\mathrm{ds}\,v\,\mathrm{cs}\,u\,\mathrm{ns}\,u}{\mathrm{ns}^2v-\mathrm{ns}^2u},$$

$$\mathrm{cs}(u+v)=\frac{\mathrm{cs}\,u\,\mathrm{ns}\,v\,\mathrm{ds}\,v-\mathrm{cs}\,v\,\mathrm{ns}\,u\,\mathrm{ds}\,u}{\mathrm{ns}^2v-\mathrm{ns}^2u}.$$

Thus the formulæ for the d-group are more symmetrical than for the n- or c-group, but not so symmetrical as those for the s-group.

12. Although in this note I have spoken of selecting one of the groups as the standard group, because such a selection was made by JACOBI and by ABEL, and because otherwise the number of formulæ required is considerably increased, it should be noted that for the complete development of the theory, it is necessary to consider the twelve functions as a whole, without giving special prominence by notation or otherwise to any one group. For this purpose the two-letter notation is very convenient as all twelve functions are placed on exactly the same footing: it also adapts itself naturally to the fact that the elliptic functions are the quotients of theta functions siz. we may put $\mathrm{sn}\,x=\frac{\mathrm{s}(x)}{\mathrm{n}(x)}$, $\mathrm{cn}\,x=\frac{\mathrm{c}(x)}{\mathrm{n}(x)}$, $\mathrm{dn}\,x=\frac{\mathrm{d}(x)}{\mathrm{n}(x)}$ where $\mathrm{s}(x)$, $\mathrm{c}(x)$, $\mathrm{d}(x)$, $\mathrm{n}(x)$ differ from the Theta functions only by factors depending upon k.

13. With respect to the addition formulæ in § 11 the forms selected for the differents groups are those in which the denominators are rational (i. e. rational functions of any of the functions). There are however four forms[1] of the addition formulae for each group, and when all these forms are included, there is not much difference between the groups.

14. The Weierstrassian function $\wp(x)$ depends upon the s-group viz.

$$\wp(x)=e_1+\alpha^2\mathrm{cs}^2\alpha x=e_2+\alpha^2\mathrm{ds}^2\alpha x=e_3+\alpha^2\mathrm{ns}^2\alpha x$$

where

$$\alpha=\sqrt{e_1-e_3}, \qquad k=\sqrt{\frac{e_2-e_3}{e_1-e_3}}.$$

The corresponding functions derived from the c-, n-, and d-groups are $\wp(u+\omega)$, $\wp(u+\omega')$, and $\wp(u+\omega'')$ respectively.

[1] The four forms for the n-group are given in the Messenger, vol. IX, p. 106. The corresponding forms for the other groups are deducible from them at sight.