ON THE DISCONTINUITY OF ARBITRARY CONSTANTS THAT APPEAR AS MULTIPLIERS OF SEMI-CONVERGENT SERIES.

(A letter to the Editor.)

BY

G. G. STOKES
of CAMBRIDGE.

Cambridge, 23 April, 1902.

Dear Sir,

I regret that from circumstances which I need not detail the invitation with which you honoured me to write something for the collection of papers which are being put together in commemoration of ABEL has remained so long without reply.

At my age you will perhaps hardly expect me to produce something new and original. The subject ought to be one of pure mathematics, for it is in honour of ABEL, and most of my work refers to applications of mathematics. There is one thing I thought might perhaps do, but it has, I fear, been too long before the public to make it suitable. However, it is published in the Proceedings or Transactions of the Cambridge Philosophical Society, which are not, I believe, so widely known as many other serial works. I thought that just a short résumé of my results might not be wholly uninteresting.

The subject is the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series, the variable according to powers of which the series proceed being a mixed imaginary. My results are contained in three papers read before the Cambridge Philosophical Society in the years 1857, 1868 and 1889, which are published respectively in the Cambridge Philosophical Transactions Vol. X, p. 105, the Transactions Vol. XI, p. 412, and the Proceedings Vol. VI, p. 362. In these papers I have for the most part confined myself to the complete integral of the differential equation of the second order which is satisfied by Bessel's Functions; an equation which without loss of generality may be put under the form

$$(1) \qquad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{n^2}{x^2}y = y.$$

The variables x and y are taken to be mixed imaginaries, but I have confined myself to the case in which n is real. Although I have, as I said, limited myself to the integral of that particular differential equation, the method is, I believe, of much wider application. I have not taken n integral but (subject to the its being real) general. Putting $x = r(\cos\theta + i\sin\theta)$, I suppose the range of r and θ defined by the imparities

$$0 < r < \infty, \qquad -\infty < \theta < \infty.$$

It is well known that the integral of (I) may be put under the form

$$(2) \qquad y = Ax^n\left(1 + \frac{x^2}{2(2+2n)} + \ldots\right) + Bx^{-n}\left(1 + \frac{x^2}{2(2-2n)} + \ldots\right)$$

$$= AU + BV, \text{ say,}$$

or under the form

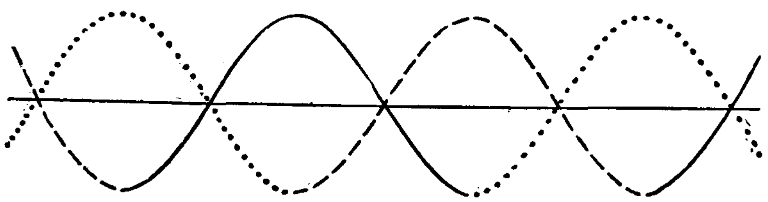
$$(3) \qquad y = Cx^{-\frac{1}{2}}e^x\left(1 + \frac{1^2-(2n)^2}{8x} + \ldots\right) + Dx^{-\frac{1}{2}}e^{-x}\left(1 - \frac{1^2-(2n)^2}{8x} + \ldots\right)$$

$$= Cu + Dv, \text{ say.}$$

The series (2) are always convergent, and completely define the function y over the whole range, but are not available for calculation when r is large, as they begin by diverging rapidly. The series (3) are always divergent (save when $2n$ is an odd integer, when they terminate), but begin by converging rapidly when r is large. But it is easy to see that (3) cannot be equivalent to (2) over the whole range of θ unless the constants C,

D have different values in different parts of the range. I have shown where and how they change discontinuously.

Of the functions u, v, let that be called the superior which has the real part of the index of the exponential positive, and that the inferior which has it negative. As θ increases in one direction, the functions u, v become alternately the superior and the inferior. I have shown that as θ changes, the constant C or D can only change when the index of the exponential in the function it multiplies is real and negative; and the change it then suffers is proportional, other circumstances being the same, to the coefficient of the other term, which of course is for that value of θ the superior term. The way in which the constants change with the value of θ may be illustrated by a pair of curves of sines drawn with inks of different colours (or else distinguished by a difference of marking) in the different parts. The ordinates may be taken to represent, for a



given value of r, the way in which the real part of the index changes with θ. A change of the coefficient is represented by a change in the colour of the ink (or of the marking of the line) with which the curves are drawn. The nature of the change is that in crossing, in the positive direction, one of the critical points represented by a change of colour or marking, the coefficient (C or D as the case may be) of the inferior term is increased by $2i \cos n\pi$, multiplied by the coefficient of the superior term.

The way in which the paradox of giving a discontinuous expression for a continuous function is explained is this. A semi-convergent series (considered numerically, and apart from its analytical form) defines a function only subject to a certain amount of vagueness, which is so much the smaller as the modulus of the variable according to inverse powers of which it proceeds is larger. I have shown that, in general (i. e. for general values of θ), the vagueness of the superior function ultimately, as r is increased, disappears in comparison with the whole value of the inferior term. But for the critical values of θ for which the index of the exponential is real the vagueness of the superior function becomes sufficient to swallow up the inferior function. As θ passes through the critical value, the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficient changed. The range during which the inferior term remains in a mist decreases indefinitely as the modulus r increases indefinitely.

We know the value of $Cu + Dv$, that is we know the values of C and D, for all values of θ provided we know them for a half period, say for $0 < \theta < \pi$; and it follows from what has been said that for the determination of C and D in terms of A and B we may take any value or values of θ within or at the edge of that range that we find convenient.

Suppose that in the neighbourhood of $\theta = 0$ we have only an inferior term Dv, C being 0, and that for $\theta = 0$ we can express D in terms of A and B. Then Dv will be a function continuous within the range $-\pi < \theta < \pi$. Again suppose that in the neighbourhood of $\theta = \pi$ we have only an inferior term Cu, D being 0, and that we express C in terms of A and B for $\theta = \pi$. Then Cu will express a continuous function within the range $0 < \theta < 2\pi$. Now superpose these results, and $Cu + Dv$, in which C and D have been expressed in terms of A and B, will express a function continuous within the range $0 < \theta < \pi$.

When there is only an inferior term Dv in the neighbourhood of $\theta = 0$, then D is constant for the range $-2\pi < \theta < 2\pi$, though it is only within the range $-\pi < \theta < \pi$ that Dv represents the continuous function y. Let the known integral of (I) in definite integrals P, Q be $EP + FQ$; then within the range last mentioned $Dv = EP + FQ$, and we may determine D in terms of E and F by putting $\theta = 0$. Similarly if in the neighbourhood of $\theta = \pi$ there is only an inferior term Cu, we may determine C in terms of E and F for the range $0 < \theta < 2\pi$ by putting $\theta = \pi$. And putting the two together we have $(0u + Dv) + (Cu + 0v)$, or $Cu + Dv$, equal to $EP + FQ$ within the range $0 < \theta < \pi$ provided C and D are two known linear functions of E and F. And as the linear relations between E, F and A, B are easily found, we have by eliminating E and F two linear relations between C, D and A, B which will make $Cu + Dv = AU + BV$ within the range $0 < \theta < \pi$, from whence can be found the relations between A, B in the ascending and C, D in the descending series for all values of θ.

But as I showed in my third paper these relations can be found *directly from the series* (2) and (3), by taking the *superior* terms for $\theta = 0$ and for $\theta = \pi$, the first giving C for the range $-\pi < \theta < \pi$, and the second giving D for the range $0 < \theta < 2\pi$, so that we need not know that it is possible to express the complete integral of (I) by means of definite integrals in the form $EP + FQ$. And this method I believe is of very general application.

I remain, Dear Sir, with the highest respect,

Yours very faithfully

G. G. STOKES.