FUNCTIONS OF A COMPLEX VARIABLE WITH ASSIGNED DERIVATIVES AT AN INFINITE NUMBER OF POINTS, AND AN ANALOGUE OF MITTAG-LEFFLER'S THEOREM.[1]

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1. Introduction.

The Problem of constructing a function of a real variable which is indefinitely differentiable and has all its derivatives assigned at one or more points has been studied by BOREL and BERNSTEIN.[2] In the complex plane we may no longer require the function to be differentiable in a deleted neighborhood of the point at which the derivatives are assigned which completely surround it, unless these derivatives are subject to the restrictions as to size which hold for the derivatives of an analytic function. We may, however, require it to be analytic in a sector having the given point as its vertex. The construction of the function in this case was discussed by RITT.[3] Later BESIKOWITSCH,[4] apparently ignorant of the work of Ritt, solved the problem by a slightly different method, and also obtained some approximation theorems, proving and generalizing a theorem stated by BIRKHOFF[5] in another connection. In the present paper

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[1] Presented to the American Mathematical Society, May 2, 1925.

[2] E. BOREL, Sur quelques points de la théorie des fonctions, Ann. de l'Ec. Norm., 1895, p. 38, or Fonctions de variables réelles (1905), p. 70. The problem here stated is not directly mentioned by Borel, but its solution is implicitly contained in his discussion of a related question. S. BERNSTEIN, Appendix to R. D'Adhémar, Principes de l'Analyse, vol. II, p. 272 (1913).

[3] J. F. RITT, On the Derivatives of a Function at a Point, Annals of Mathematics, 2nd series, vol. 18 (1916), p. 18.

[4] A. BESIKOWITSCH, Über analytische Funktionen mit vorgeschriebenen Werten ihrer Ableitungen. Mathematische Zeitschrift, vol. 21 (1924), p. 111.

[5] G. D. BIRKHOFF, The Generalized Riemann Problem, Proceedings of the American Academy of Arts and Sciences, vol. 49, (1913), p. 522.

we shall consider functions which are analytic in the entire complex plane, with the exception of certain branch cuts, and are infinitely differentiable at the branch points in the single cut sheet considered. At these branch points, the derivatives may be arbitrarily assigned, and we show the existence of the function, first when there is a single branch point, and later when they are infinite in number. Our chief theorem is: *Given an isolated infinite point set S, with derived set S'; a suitable set of branch cuts, one through each point of S (joining this to infinity or to some point of S'); and an enumerable infinity of numbers for each point; then there exists a function which is analytic in the cut plane, and at each of the points S has as the value of the function and its derivatives the numbers given at that point.*

It will be noticed that this theorem is somewhat analogous to that given by MITTAG-LEFFLER[1] for functions with assigned principal parts, the difference being that here we assign the derivatives, and in consequence must give up the requirement of analyticity in the entire plane, and insert the branch cuts. Our methods of proof are suggested by the proof of the earlier theorem.

By examining the magnitude of the absolute value of the function we construct in any given finite region, we obtain generalizations of the approximation theorems of Birkhoff and Besikowitsch.

2. The case of a single point.

We shall begin with the case in which the function and its derivatives are prescribed at only one point, and the region of analyticity of the function is the plane severed by a single branch cut, which we take as a straight line joining the given point with infinity. Our method is similar to that used by RITT[2], but we shall give the discussion in full, as we need a somewhat more general result than he obtains. Our object is the proof of:

Theorem I. *If in the complex plane a straight line be drawn from a given point out to infinity, there exists a function which is analytic at all points of the plane so cut except the given point, and at this point possesses derivatives of all*

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[1] G. MITTAG-LEFFLER, Sur la représentation analytique des fonctions monogènes uniformes, Acta Mathematica, vol. 4, (1884), p. 32.

[2] J. F. RITT, l. c., cf. the remark on p. 21.

orders, whose values, as well as that of the function at this point, may be arbitrarily assigned.

There is no loss of generality in taking the given point as the origin, and the branch cut as the negative real axis, since a transformation of the form $z = Ze^{i\theta} + A$ (which merely changes the derivatives of the function by constant factors) reduces the general case to this one. Let, then, $a_0, a_1, a_2 \ldots$ be the required values of the function and its derivatives at the origin. Select a set of real numbers $b_0, b_1, b_2, \ldots$ satisfying the conditions:

$$0 < b_n < 1 \text{ and } b_n < \frac{k}{|a_n|},$$

where k is a positive real number to be specified more precisely later. If $\sqrt[3]{z}$ means that branch of $z^{\frac{1}{3}}$ which is real for points on the positive real axis, and thence defined by continuity in the cut plane, the required function is:

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}\left(1 - e^{-\frac{b_n}{\sqrt[3]{z}}}\right).$$

To prove this we use the inequalities

$$|1 - e^C| = |C + C^2/2! + \cdots| < |C|(e-1) < 2|C| \text{ if } |C| \leq 1,$$

and

$$|1 - e^C| < 2 < 2|C|$$

if the real part of C is negative, and $|C| > 1$.

As the real part of $b_n/\sqrt[3]{z}$ is positive in the entire cut plane, we have:

$$\left|1 - e^{-\frac{b_n}{\sqrt[3]{z}}}\right| < 2\left|\frac{b_n}{\sqrt[3]{z}}\right|,$$

which shows that the nth term of the series for $F(z)$ is dominated by

$$\left|\frac{a_n z^n}{n!}\right| \cdot \left|\frac{2 b_n}{\sqrt[3]{z}}\right| < \frac{2k|z^n|}{n!|\sqrt[3]{z}|}.$$

Thus, inside a circle of radius R_2 and outside one of radius R_1, with the origin as center, the series is dominated by

$$\sum_{n=0}^{\infty} \frac{2kR_2^n}{n!\sqrt[3]{R_1}} = \frac{2k}{\sqrt[3]{R_1}} e^{R_2}.$$

Hence, by the Weierstrass test[1], it represents a function which is analytic inside this ring, or, since R_1 and R_2 are arbitrary, at all points of the severed plane except the origin.

Furthermore, if we omit the first term of the series, the remainder, inside the unit circle, is dominated by

$$\frac{2k(e^{|z|}-1)}{|\sqrt[3]{z}|} < 4k|z|^{\frac{2}{3}} < 4k,$$

using the inequality given above. This shows that, in the cut plane, the series is uniformly convergent inside the unit circle. In particular, we note that the function $F(z)$ is continuous at the origin, and $F(0)=a_0$.

The series obtained from $F(z)$ by termwise differentiation is:

$$F_1(z) = \sum_{n=1}^{\infty} \frac{a_n z^{n-1}}{(n-1)!}\left(1 - e^{-\frac{b_n}{\sqrt[3]{z}}}\right) - \sum_{n=0}^{\infty} \frac{a_n b_n z^n}{3\, n!\, z^{\frac{4}{3}}} e^{-\frac{b_n}{\sqrt[3]{z}}}.$$

The first of these series may be shown to be uniformly convergent by the methods used for the series $F(z)$, while the second, after the first term, is dominated by an exponential series. Hence this is also uniformly convergent in any circle of fixed radius with center at the origin. Hence, at any point of the severed plane distinct from the origin, $F_1(z)$ represents $F'(z)$, by a well known theorem on termwise differentiation of a series of analytic functions. At the origin we may write:

$$\int_0^z F_1(z)\,dz = F(z) - F(0),$$

from the uniform convergence of the series for $F_1(z)$, where for definiteness the integral is taken along the straight line joining z with the origin. Hence, from the continuity of $F_1(z)$, we have:

[1] See the note at the end of the paper, p. 385.

$$F_1(0) = \lim_{z \to 0} \frac{\int_0^z F_1(z)\, dz}{z} = \lim_{z \to 0} \frac{F(z) - F(0)}{z} = F'(0).$$

This shows that, in the severed plane, $F'(z)$ exists at the origin, and equals $F_1(0)$, or a_1.

Since later differentiations will merely give additional terms of essentially the same type as those appearing in $F_1(z)$, we may prove similarly that $F_n(z)$, the series obtained by differentiating termwise n times represents $F^n(z)$ at all points of the severed plane distinct from the origin, and at the origin $F^n(0)$ exists and equals a_n. Thus Theorem I is proved.

We have obtained above an upper bound for the function constructed holding in a ring with inner radius R_1 and outer radius R_2:

$$|F(z)| < \frac{2\,k}{\sqrt[3]{R_1}} e^{R_2}, \quad (R_1 < |z| < R_2).$$

By using our fundamental inequality, we also see that:

$$|F_1(z)| < \frac{2\,k}{R_1^{1/3}} e^{R_2} + \frac{k}{3\,R_1^{2/3}} e^{R_2}, \quad (R_1 < |z| < R_2).$$

In fact, each of the functions $F_n(z)$ is dominated, inside the ring in question, by an expression containing k as a factor. The remaining factor depend on R_1 and R_2 and, for fixed values of these, increases with n. If, however, we confine our attention to the first m derivatives, and fix R_1 and R_2, the values of the factors of k will be finite in number and hence bounded. Hence by choosing k, which was at our disposal, sufficiently small, we may make $F(z)$ and its first m derivatives arbitrarily small inside the ring in question. Since, moreover, any finite region having the origin as an exterior point may be included in a ring of this type, we have:

Theorem II. *The function of theorem I may be so chosen that, inside any finite region (not necessarily simply connected) having the origin as an exterior point, it, together with its first m derivatives, is in absolute value less than a preassigned quantity ε.*

In the above argument, we took the branch cut as a straight line. As the sole use of the branch cut was to restrict us to a branch of $z^{\frac{1}{3}}$ with positive real part, we might have taken any branch cut remaining entirely inside a sector of angle π. By using $z^{\frac{1}{2m+2}}$ instead of $z^{\frac{1}{3}}$, which would make no essential change in our argument, we could construct our function in the plane cut by any branch cut, such that no arc of the cut made more than m turns about the origin. This establishes the

Corollary. *The function of theorem I, or II, may be constructed when the branch cut, instead of being a straight line, is any curve joining the origin to infinity and such that the angle θ, defined along it continuously, is bounded in absolute value.*

3. Point sets without finite limit point.

In treating the case where the derivatives are assigned at an infinite number of points, we shall first confine our attention to point sets without a finite limit point. In consequence of this restriction, the points of our set may be enumerated according to their distance from the origin which we assume is not a point of the set. We number them P_1, P_2, . . . in such a way that:

$$|z_n| \leq |z_{n+1}|.$$

Through each of these points we draw a straight line to infinity, so selecting these halflines that they do not intersect. We might, for example, draw them all parallel in some direction not coinciding with one of the enumerable number of directions obtained by joining the given points in pairs. Or, we might select some point not on any of the joins of the points in pairs, and use the directions given by joining this point with the given point. In this case, if the point be chosen as the origin, the cut plane, in which our function is analytic, would be a Mittag-Leffler star. Denote the amplitude of the half-line through P_n by θ_n, and consider the function

$$A_n(z) = \sqrt[3]{\frac{e^{i\theta_n}}{z - z_n}}.$$

We take such a determination of the root that $A_n(z)$ is real on the prolongation of the branch cut through P_n. This insures that the real part of $A_n(z)$ be positive in the entire cut plane.

As $A_n(z)$ is analytic for $|z| < |z_n|$, we may write:

$$A_n(z) = a_{0n} + a_{1n}z + a_{2n}z^2 + \cdots = \sum_{i=0}^{\infty} a_{in}z^i, \quad (|z| < R_n < |z_n|).$$

Furthermore, the series converges uniformly for these values of z. Accordingly, if s_n is suitably chosen, and

$$B_n(z) = \sum_{i=0}^{s_n} a_{in} z^n - A_n(z),$$

we have:

$$|B_n(z)| < \varepsilon_n, \quad (|z| < R_n).$$

The ε_n are arbitrary, and we select them so that they form the terms of a converging series with sum η. The R_n are subject to the single condition that they be less than the $|z_n|$. As these last become infinite with n, we may, and shall, select the R_n so that they too become infinite with n. Now put:

$$C(z) = \sum_{n=1}^{\infty} B_n(z).$$

The function $C(z)$ is analytic at all points of the cut plane. For, on fixing a point Z, we will have $|Z| < R_n$ for some n, say $n=m$. The function $C(z)$ accordingly, in the neighborhood of Z, is the sum of a finite number of analytic functions and a uniformly convergent series of such functions, and hence is analytic. The argument still holds for a point on one bank of a branch cut, not a point P_n. At the branch points P_n, $C(z)$ is the sum of a convergent series, and a term which becomes infinite. Furthermore, from its construction, the real part of $C(z)$ becomes infinite negatively at the points P_n. It follows from this that

$$D(z) = e^{C(z)}$$

is analytic at all points of the severed plane except P_n, and at these points it vanishes, and possesses derivatives of all orders, which likewise vanish.

Next, form the function $C_n(z)$, similar to $C(z)$ but with the terms corresponding to P_n omitted. That is:

$$C_n(z) = C(z) - B_n(z).$$

We also form $D_n(z)$, similar to $D(z)$:

$$D_n(z) = e^{C_n(z)}.$$

Let the value of the function we are building up, $\Phi(z)$, and its derivatives, at the point P_n be denoted by:

$$b_{0n},\ b_{1n},\ b_{2n},\ \ldots$$

From these, we compute the value of the function $\Phi(z)/D_n(z)$ and its derivatives:

$$c_{0n},\ c_{1n},\ c_{2n},\ \ldots$$

Since the function we have multiplied in, $1/D_n(z)$ is analytic at the point P_n, all its derivatives are finite, and Leibnitz's theorem shows that all the c_{in} will be finite.

We now build up a function $E_n(z)$ which is analytic in the plane except for the cut through P_n, and has the numbers c_{in} as the values of it and its derivatives at P_n, which we may do by Theorem I. We shall also use Theorem II to keep the function and its first n derivatives bounded in the region outside a circle of radius ζ_n, and inside one of radius $2\,|z_n|$, $\zeta_n \leq |z-z_n| \leq 2\,|z_n|$. To select the bounds, we note that, as $|D_n(z)|$ is continuous in the region or regions bounded by the two circles just mentioned and the branch cuts which fall therein, and on the boundary as well if we regard the two banks of the branch cuts as separate, it has an upper bound there, say G_n^0. We bring it about that

$$|E_n(z)| < \varepsilon_n/G_n^0, \qquad \zeta_n \leq |z-z_n| \leq 2\,|z_n|.$$

This insures that in the region in question, in the cut plane,

$$|D_n(z)\cdot E_n(z)| < \varepsilon_n.$$

All the derivatives of $D_n(z)$ are likewise continuous in the region just used, and accordingly are bounded there. Let G_n^i be an upper bound for the ith derivative. If G is an upper bound for $G_n^0,\ G_n^1, \ldots\ G_n^n$, and we arrange that in the ring in question,

$$\zeta_n \leq |z-z_n| \leq 2\,|z_n|,$$

$E_n(z)$, as well as the absolute values of its first n derivatives, will be less than:

$$\frac{\varepsilon_n}{2^n G},$$

it will follow that $D_n(z) E_n(z)$ and its first n derivatives will be in absolute value less than ε_n, from Leibnitz's theorem. The ε_n are here, as before, the terms of a convergent series with sum η.

Now consider the function

$$F_n(z) = D_n(z) E_n(z).$$

At all points in the cut plane except the points P_i it is analytic. At all the points P_i except the point P_n it is zero, and all its derivatives exist and are zero. The differentiability of the product follows from that of the factors, and the zero values, entering from $D_n(z)$ are never cancelled out as $E_n(z)$ is analytic at all points except P_n. At P_n, the values of the function and its derivatives will be b_{0n}, b_{1n}, . . . from the way in which $E_n(z)$ was computed. No indeterminacy can occur, since $D_n(z)$ is analytic at P_n. Finally, outside the circle $|z - z_n| > \zeta_n$, and inside the circle $|z| < |z_n|$, the absolute value of $F_n(z)$ and of its first n derivatives, will be less than ε_n.

Finally we put

$$\Phi(z) = \sum_{n=1}^{\infty} F_n(z).$$

At any point distinct from the points P_n this represents an analytic function. For, if Z be such a point, we may select an m such that $|Z| < |z_n| - \zeta_n$ if $n > m$. Inside the circle about the origin with radius $|z_m| - \zeta_m$ the terms of the series for $\Phi(z)$ after the mth are dominated by the convergent series $\sum_{n=m+1}^{\infty} \varepsilon_n$, and accordingly represent an analytic function. In particular, the sum after m terms is analytic at Z. But the preceding terms, finite in number, are each analytic at Z, which proves our contention as to the analyticity of $\Phi(z)$ at Z.

At a point P_n, the function $\Phi(z)$ is continuous, and assumes the value b_{0n}. For, on taking an m such that $|z_n| < |z_p| - \zeta_p$ if $p > m$ we see that the series after the mth term is dominated by a convergent series, and accordingly is uniformly convergent and represents a continuous function. Its value is obtained by noting that all the terms are zero except $F_n(z_n)$ which equals b_{0n}.

We may prove the derived series uniformly convergent at P_n in a similar manner, and by the argument used in the proof of Theorem I, that termwise differentiation of the series is permissable at P_n. The only non-vanishing term is $F'_n(z_n)$, and we have $\Phi'(z)$ exists and equals b_{1n}.

The argument is capable of extension to any derivative. For the kth derivative, we must not only choose m so that $z_n < |z_p| - \zeta_p$ if $p > m$ but also so that $m > k$, since our bounds on the kth derivative only hold for terms after the kth.

At the beginning of our discussion we assumed that the origin was not a point P_i. This case is easily handled by considering, instead of $\Phi(z)$, the function $\Psi(z) = \Phi(z - \bar{z})$, where $\bar{z}$ is any point which is not in the set P_i, as our discussion enables us to construct a $\Psi(z)$ giving rise to the required $\Phi(z)$. Thus we have proved:

Theorem III. *Given an infinite set of points, without finite limit points, and a straight line joining each of these points to infinity, these straight lines having no common points, and an enumerable infinity of numbers for each point; then there exists a function which is analytic in the cut plane, and at each of the given points has as the value of the function and its derivatives, the numbers given for that point.*

We may include a condition of boundedness on the function and its derivatives, as was done in Theorem II. For, consider the region interior to a circle of radius R_2 about the origin, and exterior to a set of circles with centers at the points P_i interior to this circle, and radii ζ_i respectively. We take ζ_i above as those here given. Also, at each stage, instead of using the region

$$|z - z_n| < 2\,|z_n|$$

we use

$$|z - z_n| < 2\,|z_n| + R_2.$$

Further, instead of applying our bounds to the first n derivatives at each stage, we apply them to the first n, if $n > m$, and to the first m otherwise. If we do this, we shall find that the function $\Phi(z)$ finally arrived at has, in addition to its other properties, that of having its absolute value, and that of its first m derivatives, less than η. But this last was at our disposal. Finally, as any finite region having all the points P_i as exterior points may be included in a region bounded by circles of the kind just described, we obtain:

Theorem IV. *The function of Theorem III may be so chosen that, inside any finite region (not necessarily simply connected) having the given points as exterior points, it, together with its first m derivates is in absolute value less than a pre-assigned quantity.*

The extension of our branch cuts from straight lines to those winding around the origin only a finite number of times given in the Corollary to theorems I and II applies here as well. If we used the same exponent in all the terms $A_n(z)$, it would be necessary for the number of windings to be bounded for the branch cuts considered as a set. As, however, this exponent may be different for the different terms without changing the reasoning, we need merely require the branch cuts to be such that each only winds around the origin a finite number of times. The requirement that the branch cuts do not intersect, while necessary if we wish to keep our region simply connected, may be given up if we admit a function which is merely analytic in several regions. These remarks lead to the

Corollary. *The function of theorem III or IV may be constructed when the branch cuts, instead of being non-intersecting straight lines, are any curves joining the points to infinity in such a way that for any one such curve, the angle defined along it continuously, is bounded in absolute value. If the curves intersect, instead of arriving at a single analytic function, we may arrive at several, one for each region in which the cuts divide the plane, which collectively have the property of the single function previously obtained.*

4. General Isolated Point sets.

We next treat the problem we have just solved, where the given points may have finite limit points. We assume that the points are isolated, that is, that no one is itself a limit point. This restriction is obviously necessary. We also exclude the point infinity from the set, as no function exists which has all its derivatives and itself finite at infinity, unless the derivatives are all zero, and we are not concerned with such degenerate cases. Since the point set is isolated, it is necessarily enumerable. For, we may surround each point with a circle containing no other point. When we project on the sphere, the number of these circles of any one size is finite and accordingly we may enumerate the points according to the size of the projected circles. Let the enumerated point set be P_1, P_2, ... as before. The limit points of the collection, P' are of course not necessarily enumerable. We shall, however, associate one of the points P' with each of the points P_n as follows. Consider the distance from P_n to each of the points of P', and the number $1/|z_n|$ which we regard as measuring the »distance» to I, the point at infinity. Select h_n the minimum value of this distance, and one of the points for which it is reached (as it is, since the set P', being a derived set, is closed). We call this point P'_n. These points are of course not necessarily distinct, and the point at infinity, I, may occur as one of the P'_n even if it is not a limit point of the set P. As branch cuts we take lines joining P_n and P'_n. These might be taken as straight, in general necessitating intersections, or they might be taken curved lines satisfying the condition of the corollary. In the latter case, we may arrange that the plane, when cut, is no further subdivided than it was already by the set P'. Curved branch cuts will necessitate some slight changes in what follows as explained in connection with the corrolaries, since for simplicity we confine our discussion to the straight line case.

We are now ready to repeat the process of paragraph 3 for the case at hand. We define θ_n as the amplitude of the branch cut through P_n, and then obtain $A_n(z)$ as before. Instead of using a series in z to approximate to it, we use a series in $1/(z - z'_n)$ where z'_n is the number with image P'_n if $P'_n \neq I$. When P'_n is I, the point at infinity, we use the previous series. We write then:

$$A_n(z) = a_{0n} + \frac{a_{1n}}{z - z'_n} + \frac{a_{2n}}{(z - z'_n)^2} + \cdots.$$

The series may be obtained by putting $A_n(z) = A_n(Z)$, where $Z = 1/(z - z'_n)$, and finding the power series in Z. This shows that the series for $A_n(z)$ converges when $|z - z'_n| > |z_n - z'_n|$, that is, outside a circle with center P'_n and radius h_n. When $P'_n = I$, our previous series converged inside a circle of radius $1/h_n$.

We define

$$B_n(z) = \sum_{i=0}^{s_n} \frac{a_{in}}{(z - z'_n)^i} - A_n(z),$$

so choosing s_n that

$$|B_n(z)| < \varepsilon_n, \text{ if } |z - z'_n| > H_n > |z_n - z'_n| = h_n.$$

The numbers h_n approach zero as n becomes infinite. To see this, we observe that the number of points for which h_n is greater than any finite number h is finite. For, these points lie inside a circle of radius $1/h$ about the origin, and outside circles of radius h about the points P'_n. If they were infinite in number, they would have a limit point in this region, and accordingly for some of them h_n would be less than h. Since the H_n may be any numbers greater than the h_n, we may, and shall choose them so that they too approach zero as n becomes infinite. If $P'_n = I$, $H_n = 1/R_n$.

We may now define $C(z)$ in terms of $B_n(z)$ as before. It will have all the properties of the previous $C(z)$. For, as H_n is approaching zero, if we select any fixed point z, not a P_n or a P', we may find an m such that when $n > m$, H_n is less than $1/|z|$, and the minimum distance from z to P'. Accordingly we may break up the series into two parts, and prove the analyticity as before.

$D(z)$ is defined in terms of $C(z)$ as before, and retains its properties.

$C_n(z)$, $D_n(z)$ and $E_n(z)$ are formed as before. In constructing $E_n(z)$, we here arrange so that the bounds apply outside a circle of radius $\zeta_n < H_n$, and inside one of radius $R_n > 1/H_n$. This insures that the region of boundedness will eventually embrace any point not a P_n or a P', as n becomes infinite.

$F_n(z)$ and $\Phi(z)$ may be formed as before, and we obtain:

Theorem V. *Given an infinite set of points, no one being a limit point of the set, and a suitable set of branch cuts, one through each point of the set, and joining it with the point at infinity (assumed not to be in the original set) or the nearest point of the derived set; and an enumerable infinity of numbers for each point; then there exists a function which is analytic in the cut plane, and at each of the given points has as the value of the function and its derivatives the numbers given at that point.*

The reasoning which lead to theorem IV gives:

Theorem VI. *The function of theorem V may be so chosen that, inside any finite region (not necessarily simply connected) having the given points and those of the derived set as exterior points, it, together with its first m derivatives is in absolute value less than a pre-assigned quantity.*

5. The Generality of Our Function.

Theorem I, III and V above assert the existence of a function analytic in a certain region, and having assigned derivatives at one or more points. It is natural to inquire the relation of these functions to the most general function satisfying the given requirements. Owing to the character of the region of analyticity of our function, this question has no simple definitive answer. We may, however, state a partial answer to the question as follows.

If $\Phi(z)$ is the function we have constructed, and $\Psi(z)$ is any other function satisfying our requirements, we may write

$$\Phi(z) = \Psi(z) + X(z).$$

The function $X(z)$ will be analytic in the cut plane, and have all its derivatives existing, and equal to zero, at the points of the given set. We may go one step further, and put:

$$X(z) = D(z) \,.\, X(z).$$

$X(z)$ may now be any function which is analytic in the cut plane, and at the points of the given set has difference quotients which do not become infinite more rapidly than e^{z^2}. In particular, $X(z)$ may be any integral function, or a meromorphic function with all its poles at the given points. Our conditions on $X(z)$ make it fairly clear that the class of admissable functions is not any simple class.

6. Approximation Theorems.

Theorems II, IV and VI readily lead to approximation theorems of the Besikowitsch type. For, they establish the existence of functions with assigned derivatives, bounded in certain regions. By applying them to the difference between a function to be obtained, and a given function, making the necessary subtractions on the derivatives, we may construct functions approximating a given function in a region. As theorem VI includes II and IV, we merely state the approximation theorem obtained from it. It is:

Theorem VII. *Given a function analytic in a certain region, the function of theorem V may be formed so that, inside this region and exterior to a set of circles* *arbitrarily small drawn about such of the given points as fall in the region, it and its first m derivatives approximate the given analytic function.*

To bring out the force of this theorem, we shall state separately one interesting special case, namely that in which no points are inside of the region, but some are on the boundary. They must then be finite in number, in order to be isolated. The theorem is:

Theorem VIII. *Given a function analytic in a region, and a finite number of points on its boundary, a function can be found which is analytic inside the region, continuous and infinitely differentiable on the boundary, takes, with all its derivatives, assigned values at the given points, and in any region entirely inside the given one, approximates the given function.*

Note to p. 374, line 2.

The test here referred to consists in the application of the following two theorems:

Theorem A. (Weierstrass M-test for uniform convergence)

The infinite series

$$u_1(z) + u_2(z) + \cdots,$$

whose terms in the region R are functions of z, converges uniformly in this region, in case there exists a convergent series of positive terms, independent of z,

$$M_1 + M_2 + \cdots$$

such that, for each value of z in the region R, and for some value N, independent of z, the inequality

$$|u_n(z)| \leqq M_n$$

remains true if

$$n \geqq N.$$

Theorem B. (Weierstrass theorem on series)

Let

$$f(z) = u_1(z) + u_2(z) + \cdots$$

be an infinite series of functions, all of which are analytic in a region R. If the series converges uniformly in the region R, then it represents an analytic function in R.

For proofs of these theorems see, for example, Osgood, Funktionentheorie, vol. 1, Leipzig, 1912, p. 96 (for theorem A) and p. 303 (for theorem B). cf. also Weierstrass, Werke, vol. 1, p. 67 and vol. 2, p. 205.