FORMAL THEORY OF IRREGULAR LINEAR DIFFERENCE EQUATIONS.

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Introduction.

The general theory of ordinary linear differential equations with rational coefficients must be regarded now as known in its broad outlines: there exist in all cases whatsoever a full quota of formal series (normal and anormal) for each singular point; on the basis of these series, the behavior of the solutions in the neighbourhood of the singular points can be adequately characterized, their nature in the large can be determined by means of the monodromic group, and finally the inverse Riemann problem can be formulated and solved by direct use of the Fredholm theory of integral equations or otherwise.[1] The details have not been carried through except when the series which enter are of normal type, but the corresponding formulation and attack in the most general case are sufficiently evident.

On the other hand the situation is much less satisfactory for ordinary linear difference equations with rational coefficients. If these equations be written as a linear system,

$$(1) \qquad y_i(x+1) = \sum_{j=1}^{n} a_{ij}(x) y_j(x) \qquad (i = 1, 2, \ldots n),$$

in which the n^2 rational functions $a_{ij}(x)$ are analytic at $x = \infty$ or have a pole

[1] Cf. my paper, *The Generalized Riemann Problem for Linear Differential Equations and the Allied Problems for Linear Difference and q-Difference Equations*, Proc. Am. Acad. Arts and Sciences, vol. 49 (1913), pp. 521—568.

of maximum order μ there, then it is only the regular case in which the characteristic equation in ϱ,

$$(2) \qquad |\varrho e^{\mu}\delta_{ij} - a_{ij}^{(\mu)}| = 0$$

$$(\delta_{ij} = 1,\ i = j;\ \delta_{ij} = 0,\ i \neq j;\ a_{ij}^{(\mu)} = \lim_{x=\infty} a_{ij}(x)/x^{\mu}),$$

has n distinct roots $\varrho_1, \varrho_2, \ldots \varrho_n$, not zero, that has been treated adequately. To be sure, Nörlund, in his fundamental work on linear difference equations, has given a powerful general method by which solutions analytic in certain limited regions of the complex plane can be constructed always[1], but this method affords little indication as to the nature of those simplest analytic solutions, devoid of artificial singularities, which are of central theoretic importance.

The most general examination hitherto made of linear difference equations not of this special type is due to Adams.[2] He discusses the extent to which my own method of approach to the regular case[3] admits of extension to the irregular case, and finds that while similar results can be obtained in certain more general cases, yet the method appears to break down. In brief, he finds that so long as there are n types of formal series with elements of the form

$$x^{\mu^* x} \varrho^x x^r \left(a + \frac{b}{x} + \cdots\right) \qquad (\mu^* \leqq \mu),$$

the same method is applicable. Since these formal series are of the same kind as appear in the regular case, it is natural to group this particular irregular case closely with the regular case. In more general cases Adams only establishes highly restricted results. The difficulties which he meets with are of three types: (1) he does not arrive at a full quota of formal series in all cases[4]; (2) the sequences defining the 'determinant limits' converge, if at all, in much less extensive regions of the x plane than in the regular case; and (3) in consequence, the

[1] Cf. N. E. Nörlund, *Differenzenrechnung,* Berlin, 1924, chap. 10.

[2] C. R. Adams, *On the Irregular Cases of Linear Ordinary Difference Equations,* Trans. Am. Math. Soc., vol. 30 (1928), pp. 507—541. In this paper references to the work of Barnes, Horn, Batchelder, Perron, and Galbrun may be found.

[3] *General Theory of Linear Difference Equations,* Trans. Am. Math. Soc., vol. 12 (1911), pp. 243—284.

[4] Not even in the case $n = 2$, in which many but not all cases have been treated by Batchelder. Batchelder has not published these results.

determination of 'intermediate solutions' and 'principal solutions', as in my paper by means of series and contour integrals, fails.

It is the aim of the present paper to show that, when certain further types of formal series involving logarithms are taken account of, there will always be a full quota of n such formal series solutions. By way of application of this basic result, certain interesting formal questions are also treated.

In a subsequent paper devoted to the analytic theory of irregular difference equations, also to appear in these pages, I expect to present the extension of my earlier theory to the truly general case. Such an extension would of course be impossible without the result of the present paper. It involves appropriate modifications in the method of contour integration as well as other changes of consequence.

§ 1. The Linear Difference System and the Single Equation.

Without essential restriction it may be assumed that the determinant $|a_{ij}(x)|$ in (1) is not identically zero. In fact in the contrary case there is obviously at least one identical linear homogeneous relation between $y_1(x+1), \dots y_n(x+1)$ with coefficients which are explicitly given in terms of the minors of this determinant. If we replace x by $x-1$ in this relation, we obtain a like relation between $y_1(x), \dots y_n(x)$. On solving for one of the dependent variables, say $y_n(x)$, in this relation, and elimination of $y_n(x)$ in the first $n-1$ equations (1), we obtain a like equivalent system of order $n-1$. Proceeding successively in this way we arrive finally at an equivalent linear system of order $< n$, for which the determinant in question does not vanish identically.

Now for such a system (1) let us write

$$y(x) = \lambda_1(x)\, y_1(x) + \cdots + \lambda_n(x)\, y_n(x) \equiv \sum_{\alpha=1}^{n} \lambda_\alpha(x)\, y_\alpha(x) \tag{3}$$

where $\lambda_1(x), \dots \lambda_n(x)$ are n functions, rational in x but otherwise arbitrary. By use of the equations (1) we obtain successively

$$\tag{3'} \begin{aligned} y(x+1) &= \sum_{\alpha,\beta=1}^{n} \lambda_\alpha(x+1)\, a_{\alpha\beta}(x)\, y_\beta(x), \\ &\quad \cdots\cdots\cdots\cdots \\ y(x+n) &= \sum_{\alpha,\beta,\dots\nu=1}^{n} \lambda_\alpha(x+n)\, a_{\alpha\beta}(x+n-1) \cdots a_{\mu\nu}(x) y_\nu(x). \end{aligned}$$

On the right-hand sides of (3), (3') we have $n+1$ linear homogeneous expressions in $y_1(x), \ldots y_n(x)$. Hence from these equations we obtain at least one linear homogeneous equation of the form

$$L(y) \equiv a_0(x) y(x+n) + a_1(x) y(x+n-1) + \cdots + a_n(x) y(x) = 0, \tag{4}$$

in which not all of the coefficients $a_i(x)$ vanish identically, and furthermore these coefficients are explicitly expressible in terms of the functions λ_i, a_{ij}.

It is possible to choose $\lambda_1, \ldots \lambda_n$ so that neither a_0 nor a_n vanishes identically, i.e. so that the linear difference equation (4) is actually of the nth order. To establish this fact we consider the determinant of the linear homogeneous expressions in $y_1(x), \ldots y_n(x)$ which appear in the first n of the $n+1$ equations (3), (3'). The elements of the first row are $\lambda_1(x), \ldots \lambda_n(x)$ which may clearly be taken at pleasure at an arbitrary point x. The elements of the second row are

$$\sum_{\alpha=1}^{n} \lambda_\alpha(x+1) a_{\alpha 1}(x), \ldots \quad \sum_{\alpha=1}^{n} \lambda_\alpha(x+1) a_{\alpha n}(x)$$

respectively. But $\lambda_1(x+1), \ldots \lambda_n(x+1)$ may clearly be assigned values at pleasure without affecting the values of $\lambda_1(x), \ldots \lambda_n(x)$ already selected. Inasmuch as $|a_{ij}(x)|$ is not identically zero, it is therefore clear that the elements of this second row may be independently selected at will. For the third row a similar consideration leads to the conclusion that these elements too may be independently assigned at the given value of x, since the determinant which enters is

$$\left| \sum_{\beta=1}^{n} a_{i\beta}(x+1) a_{\beta j}(x) \right| \equiv |a_{ij}(x+1)| \, |a_{ij}(x)| \not\equiv 0.$$

By proceeding in this manner it becomes apparent that the determinant of the coefficients of $y_1, \ldots y_n$ in the first equations (3), (3') can be made not to vanish identically, since all of its elements can be taken arbitrarily at any point x_0 such that $|a_{ij}(x)|$ is defined and does not vanish for $x = x_0, x_0+1, \ldots x_0+n-2$. Likewise the similar determinant for the last n of these equations can be made not to vanish. Hence the equation (3), obtained by equating the determinant of the augmented system (3), (3') to zero will have the desired property $a_0(x) \not\equiv 0$, $a_n(x) \not\equiv 0$.

Therefore it appears that to a fundamental set of solutions of the system (1) with $|a_{ij}(x)| \not\equiv 0$ there corresponds a fundamental set of solutions of some single equation (4) of the nth order, and *vice versa.*

If then we can establish that the single linear difference equation (4) possesses a set of n formal series solutions in the usual sense[1], it is evident that the general system (1) will also possess a set of n formal solutions, obtainable from the formal solutions of (4) by means of the equation (3).

These obvious considerations justify us in focussing attention upon the formal series solutions of a single ordinary linear difference equation of the nth order of the type (4) with $a_0(x)$, $a_n(x)$ not identically zero.

§ 2. Statement ot the Formal Problem.

In order to deal conveniently with the questions which arise concerning the formal series solutions, it is desirable to broaden somewhat the initial formulation. Let us demand merely that the coefficients $a_i(x)$ of the equation (4) under consideration are to be formal (i.e. convergent or divergent) power series in descending integral powers of x, or, more generally, of $x^{\frac{1}{p}}$ (p, a positive integer). Here only a finite number of positive integral powers are permitted to enter. Of course the corresponding assumption for the linear difference system (1) is that the coefficients are similar formal power series, and the same integer $p \geqq 1$ is evidently involved in associated equations (1) and (4), even if $\lambda_1(x), \ldots \lambda_n(x)$ in (3) are also such series in $x^{\frac{1}{p}}$. It is obvious then that for the complete specification we must not only give the equation (4), but also the value of the 'basic integer' p which is to be adopted; all possible values of p are evidently positive integral multiples of a least value $p_0 \geqq 1$.

The usual method[2] for the determination of formal series begins with the substitution of a series

[1] We regard a set of n formal solutions as distinct (i.e. linearly independent) in case there is no identical linear homogeneous relation between them in which the coefficients are either constants or of the more general form $ce^{2l\pi\sqrt{-1}x}$ (l, an integer).

[2] Generalized here to the extent that we allow p to exceed 1.

$$s(x) = x^{\frac{\mu x}{p}} e^{P(x)} x^r \left(a + \frac{b}{x^{\frac{1}{p}}} + \cdots\right) \tag{6}$$

$$\left(P(x) = \gamma x + \delta x^{\frac{p-1}{p}} + \cdots + \eta x^{\frac{1}{p}}\right)$$

in the equation and the attempt to determine $\mu, \gamma, \delta, \ldots, a, b, \ldots$ by the method of undetermined coefficients. Here the formal identities:

$$(x+i)^{\frac{\mu(x+i)}{p}} = x^{\frac{\mu x}{p}} x^{\frac{\mu i}{p}} \left(1 + \frac{i}{x}\right)^{\frac{\mu(x+i)}{p}} = x^{\frac{\mu x}{p}} x^{\frac{\mu i}{p}} \left(e^{\frac{\mu i}{p}} + e^{\frac{\mu i}{p}} \frac{\mu i^2}{ax} + \cdots\right);$$

$$e^{\gamma(x+i)} = e^{\gamma x} e^{\gamma i}; \quad e^{\delta(x+i)^{(p-1)/p}} = e^{\delta x^{(p-1)/p}} \left(1 + \delta \frac{(p-1)i}{p x^{\frac{1}{p}}} + \cdots\right); \quad \cdots$$

$$(x+i)^r = x^r \left(1 + \frac{ri}{x} + \cdots\right);$$

for $i = 1, 2, \ldots n$ enable us to remove a factor $x^{\frac{\mu x}{p}} e^{P(x)} x^r$ from the given equation after this substitution $y = s(x)$ has been made. When this factor is removed, the $n+1$ leading terms in the $n+1$ series which appear on the left are precisely

$$a_{0j_0} e^{\frac{\mu n}{p}} x^{\frac{-j_0+\mu n}{p}}, \quad a_{0j_1} e^{\frac{\mu(n-1)}{p}} x^{\frac{-j_1+\mu(n-1)}{p}}, \quad \ldots \quad a_{nj_n} x^{\frac{-j_n}{p}},$$

where we have written a_{ij_i} for the coefficient of the leading term $a_{ij_i} x^{\frac{-j_i}{p}}$ in $a_i(x)$ for $i = 0, 1, \ldots n$. Obviously neither j_0 nor j_n can be infinite since neither $a_0(x)$ nor $a_n(x)$ is identically 0.

Now if such a formal identity is to be possible, there must be two leading terms of the same degree in x, so that

$$\mu = -\frac{j_l - j_m}{l - m}$$

for some l and m $(l \neq m)$, while all other terms are not of higher degree, i.e.

$$-j_i + \mu(n-i) \leqq -j_m + \mu(n-m) \qquad (i = 0, 1, \ldots n)$$

whence

$$j_i - j_m \geqq -\mu(i-m). \qquad (i = 0, 1, \ldots n).$$

But these conditions admit of a very simple geometric interpretation.[1] Let (i, j) represent the cartesian coordinates of a point in the plane for which the i axis is directed horizontally to the right and the j axis is directed upwards. Mark the $n+1$ points (i, j_i) where $i = 0, 1, \ldots n$ (see figure 1). Evidently the conceivable values of μ are given by the negatives of the slopes of all possible lines through two of these points, while the inequality imposed will only be satisfied if all the remaining points lie above or on such a line whose equation is

$$(j - j_m) = -\mu(i - m).$$

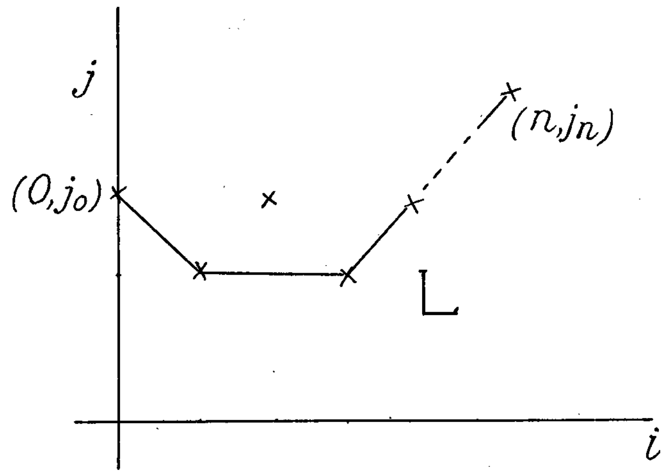

Fig. 1.

This leads us to a unique broken line L, concave upwards, whose vertices fall at certain of these points, while all of the other points lie above or on this line L. Furthermore the points (i, j) (i, j, being integers) which fall below this line can correspond to no terms in the $n+1$ series under consideration.

When, however, μ has one of these values, and the coefficient of the highest power of x on the left is equated to zero, we obtain the 'characteristic equation' in $\varrho = e^{\gamma}$,

$$\left[a_{ij_i}\left(\varrho e^{\frac{\mu}{p}}\right)^{n-i} + \cdots + a_{i'j_{i'}}\left(\varrho e^{\frac{\mu}{p}}\right)^{n-i'}\right] a = 0,$$

in which the first and last terms in parentheses correspond to the extreme vertices (i, j_i), $(i', j_{i'})$ of the line segment of slope μ in the above diagram, and

[1] Cf., for instance, N. E. Nörlund, Differenzenrechnung, pp. 312—313.

in which the intermediate terms correspond to the intermediate marked points on that line segment. Clearly in this manner are obtained as many equations as there are values of μ. If these values of μ are denoted by $\mu_1, \mu_2, \ldots \mu_k$ with $\mu_1 > \mu_2 > \cdots > \mu_k$, the total number of non-zero roots (counted according to multiplicity in the separate equations) is precisely n; for if $i_0 = 0, i_1, \ldots i_k = n$ be the values of i for the successive vertices, these roots are in number precisely $i_1 - i_0, i_2 - i_1, \ldots i_k - i_{k-1}$ respectively. Thus there are in general k characteristic equations, rather than a single characteristic equation as in the regular case. It is the essential advantage of the single equation (4) over the system (1), as basis from the formal point of view, that all of the possible values of μ are immediately obtained for an equation (4).

If now we proceed further in the attempt to obtain a formal series solution by comparison of the terms of successively lower degrees in $x^{\frac{1}{p}}$ we are immediately led to the following results (Cf. Adams, loc. cit., § 1).

If the values $\mu_1, \ldots \mu_k$ are all integral, and none of the characteristic equations have a multiple non-zero root, the first comparison determines γ as indicated above, the remaining comparisons give $\delta, \ldots, a, b, \ldots$ in succession, as specific functions of the previously determined constants, and with $b, c, \ldots$ in particular involving a as a multiplicative factor. Hence in this case there is obtained a full quota of formal series solutions of the type (6) under consideration, and these are evidently determined up to a factor

$$a e^{2k\pi\sqrt{-1}x} \qquad (k, \text{ an integer})$$

corresponding to the arbitrary multiplicative constant a, and the ambiguity in the determination of γ. Inasmuch as the detailed proof is entirely straightforward and of familiar type, and is not necessary for our later purposes, it is omitted here.

More generally, if some of the values $\mu_1, \ldots \mu_k$ are fractional, but no multiple non-zero roots occur in the characteristic equations, let such a fraction be m/l in lowest terms. There exist then corresponding series of the following anormal type

$$(6') \qquad S(x) = x^{\frac{\mu x}{p}} e^{P(x)} x^r \left(a + \frac{b}{x^{\frac{1}{lp}}} + \cdots\right)$$

$$\left(P(x) = \gamma x + \delta x^{\frac{lp-1}{lp}} + \cdots + \eta x^{\frac{1}{lp}}\right).$$

Such a formal series solution has evidently l formally distinct determinations in all cases, since if x makes p positive circuits of $x=0$ in the complex x plane, the factor $x^{\frac{\mu x}{p}}$ is multiplied by a factor $e^{2\pi\mu\sqrt{-1}x}$, and γ is thus augmented by $2\pi\mu\sqrt{-1}$, where $\mu=m/l$ is not an integer. Here again there are always precisely n formal series, when these various determinations are taken into account. But inasmuch as the method is perfectly straightforward and of familiar type, and moreover not necessary for our later purposes, it is omitted here.

As Adams has noted (loc. cit., § 1) the 'general' case in which some of the roots ϱ of the single characteristic equations are of l fold multiplicity while μ is an integer also leads to corresponding anormal series of type (6′), so that again a full quota of series is obtained unless a certain secondary characteristic equation holds. Unfortunately, the method of direct comparison may lead to indefinite algebraic complications if this secondary equation is satisfied.

There is, however, a third type of formal solution which may arise, but whose importance seems to have been largely overlooked.[1] Let $s(x)$ be any formal series of the type (6) or (6′), and let $t(x)$ be a second such formal series with the same coefficient preceding the power series as $s(x)$, save that the constant r may be modified by an integral multiple of $1/lp$. Then, for instance, there may exist two formal series solutions of the form

$$(6'') \qquad s(x),\ s(x)\log x + t(x)$$

in the case $l=1$, or $2l$ such series in the general case $l \geqq 1$. More generally, if $s(x)$, $t(x)$, ... $w(x)$ are k series of the form (6), (6′), all with the same μ, $P(x)$, and if the constants r which enter differ at most by a multiple of $1/lp$, then there may exist kl solutions

$$(6'') \qquad s(x),\ ks(x)\log x + t(x),\ \ldots\ s(x)(\log x)^k + t(x)(\log x)^{k-1} + \cdots + w(x).$$

It will be observed that when $\log x$ is changed to any one of its other determinations, each formal solution is augmented by a linear combination of the preceding ones affected with constant multipliers. We shall regard such series (6″) as of normal type if $l=1$ and as of anormal type if $l>1$.

[1] See, however, N. E. Nörlund, Differenzenrechnung, chap. 11, § 1, where a specialized case (6″) of this logarithmic type is considered for those linear difference equations of 'Fuchsian type', in which the series $a_i(x)/a_0(x)$ begin with a term of not higher than degree $-i$ in x.

It is our primary purpose in Part I of this paper to solve what may be termed the formal problem of linear difference equations of type (4), *and thus of type* (1) *of course, by proving that there always exist precisely n formal series solutions of the three types* (6), (6'), (6'').

The method of direct comparison hitherto employed fails because it does not take due account of the algebraic nature of the difficulties involved. We propose to attack the problem by a method based upon the notion of reducibility, which leads to a successive reduction of these difficulties by a well-defined series of steps, each involving only the solution of linear algebraic equations.

§ 3. Solution of the Converse Problem.

It is an easy matter to solve the converse problem by demonstrating that to every such set of n (linearly independent) formal series there corresponds a uniquely determined linear difference equation of the n-th order.

Suppose, for instance, that $s_1(x), \ldots s_n(x)$ are n series of the simplest type (6). The corresponding difference equation is then essentially given by

$$\begin{vmatrix} y(x+n) & y(x+n-1) & \ldots & y(x) \\ s_1(x+n) & s_1(x+n-1) & \ldots & s_1(x) \\ \cdot & \cdot & \cdot & \cdot \\ s_n(x+n) & s_n(x+n-1) & \ldots & s_n(x) \end{vmatrix} = 0.$$

In this case we have only to divide the $(i+1)$-th row $(i=1, \ldots n)$ by the exponential factor $x^{\mu_i x} e^{P_i(x)} x^{r_i}$ in order to obtain the equation desired. Moreover, even if certain groups of the n series are of the anormal form (6'), a similar conclusion is possible, although the coefficients $a_0(x), a_1(x), \ldots a_n(x)$ obtained are given in the first place as power series in descending powers of some root of $x^{\frac{1}{p}}$. However, it is obvious that the equation written is in reality not altered if the various determinations of these series be permuted, so that actually the coefficients $a_0(x), a_1(x), \ldots a_n(x)$ do not involve fractional powers of $x^{\frac{1}{p}}$ after a suitable factor is removed. To indicate briefly the situation in the case when logarithmic terms enter as in (6''), we consider the simplest possible case, namely the case $n=2, p=1, l=1$ in which there are a pair of solutions,

$$s(x),\ s(x)\log x + t(x),$$

in which $s(x)$, $t(x)$ are of the type (6) with the same μ, γ, r. The corresponding equation is then

$$\begin{vmatrix} y(x+2) & y(x+1) & y(x) \\ s(x+2) & s(x+1) & s(x) \\ s(x+2)\log(x+2) + t(x+2) & s(x+1)\log(x+1) + t(x+1) & s(x)\log x + t(x) \end{vmatrix} = 0.$$

If we multiply the second row by $\log x$, subtract from the third, and make use of the formal identity

$$\log(x+i) = \log x + \frac{i}{x} + \cdots \qquad (i = 1, 2),$$

we may eliminate the logarithms and thus obtain the desired equation of type (4). Obviously a similar manipulation leads to the same result in the most general logarithmic case.

The solution of the converse formal problem thus obtained in all cases is evidently unique, since if we write out equation (4) with $y(x)$ replaced by $s_1(x), \dots s_n(x)$ respectively, we obtain n linear homogeneous equations in the $n+1$ coefficients $a_0(x), a_1(x), \dots a_n(x)$, which determine them up to a multiplicative series factor, just because the formal determinant of the n-th order $|s_i(x+j-1)|$ is not identically zero.

3. Solution of the Formal Problem for $n = 1$.

We will begin with a proof of the following fact:

Every equation (4) *of the first order* ($n = 1$) *has a series solution of the form* (6).

Such an equation may be written in the form

$$y(x+1) = x^{\frac{\mu}{p}} \left(\varrho e^{\frac{\mu}{p}} + \frac{a_1}{x^{\frac{1}{p}}} + \cdots \right) y(x)$$

when μ is an integer and ϱ is a constant not zero, inasmuch as we may divide the equation through by $a_0(x)$ and transpose the term in $y(x)$.

If we change the dependent variable by substituting

$$y(x) = x^{\frac{\mu x}{p}} \varrho^x \bar{y}(x)$$

and divide through by the coefficient of $\bar{y}(x+1)$, the modified equation takes a similar form, simplified to the extent that μ is zero and ϱ is 1:

$$\bar{y}(x+1) = b(x)\bar{y}(x) \quad \left(b(x) = 1 + \frac{b_1}{x^{\frac{1}{p}}} + \cdots\right).$$

Now introduce the dependent variable $z = \log y$, and we obtain at once (formally)

$$z(x+1) - z(x) = \log b(x) = \frac{b_1}{x^{\frac{1}{p}}} + \frac{2b_2 - b_1^2}{2x^{\frac{2}{p}}} + \cdots$$

$$= \frac{c_1}{x^{\frac{1}{p}}} + \frac{c_2}{x^{\frac{2}{p}}} + \cdots.$$

If in this last equation we write

$$z(x) = z_{-p+1}x^{1-\frac{1}{p}} + z_{-p+2}x^{1-\frac{2}{p}} + \cdots + z_{-1}x^{\frac{1}{p}} + z_0 \log x + z_1 x^{\frac{-1}{p}} + \cdots{}^{1},$$

it is immediately found that $z_{-p+1}, z_{-p+2}, \ldots z_0$ are uniquely determined with

$$z_{-p+1} = \frac{c_1}{1 - \frac{1}{p}}, \; z_{-p+2} = \frac{c_2}{1 - \frac{2}{p}}, \ldots z_{-1} = \frac{c_{p-1}}{1 - \frac{p-1}{p}}, \; z_0 = c_p.$$

These results emerge by direct comparison of the terms in $x^{\frac{-1}{p}}, x^{\frac{-2}{p}}, \ldots x^{-1}$ on both sides. Now the comparison of terms in $x^{\frac{-k}{p}}$ $(k > p)$ leads to equations of the form

$$\frac{-k+p}{p} z_{k-p} + \varphi_k = c_k \qquad (k = p+1, p+2, \ldots),$$

in which φ_k is a known polynomial in the coefficients z_i which precede z_{k-p}. Thus $z_1, z_2, \ldots$ are determined in succession and uniquely.

Evidently the formal series for $z(x)$ so obtained leads to a formal series of type (6) for y, so that the proof of the italicized statement is completed.

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[1] In the case $p = 1$, this is to be written as

$$z(x) = z_0 \log x + z_1 x^{-1} + \cdots.$$

§ 4. Simplification of the Formal Problem.

We propose to simplify the formal problem by establishing the following fact:

There will necessarily exist always a complete set of formal series solutions of types (6), (6′), (6″) *if only every equation* (4) *admits at least one formal series solution of the non-logarithmic types* (6), (6′).

The truth of this statement may be argued as follows:

Suppose if possible that the statement is false. In this case, even though such a formal series solution (6), (6′) always exists, yet there are equations (4) for which a complete set of series solutions does not exist. There will then be a least value of n for which a complete set does not exist, and, according to the result of § 3, we must have $n > 1$.

Let $s(x)$ be one of the formal series solutions (6), (6′) of such an equation (4) of least order $n > 1$ for which the theorem fails, and write $y = s(x)\bar{y}$. After division through by a suitable factor (for instance, $s(x+n)$), we obtain an equation of the form (4) in $\bar{y}$, and of order n, in which, however, the basic integer p is perhaps replaced by some integral multiple lp. But this new equation in $\bar{y}$ admits of the obvious formal solution $\bar{y} = 1$, so that if we write the equation in terms of $\bar{y}$, $\Delta\bar{y}$, $\Delta^n\bar{y}$, the term in $\bar{y}$ disappears; in other words we have to deal with an equation (4) of order $n-k$ in $\Delta^k\bar{y}$ where $\Delta^k\bar{y}$ is the lowest order difference to appear explicitly. But this is an equation of the form (4) in $\Delta^k\bar{y}$ of order $n-k<n$, and hence our hypothesis ensures that it admits a complete set of $n-k$ formal series solutions $\bar{s}(x)$.

Thus we are led to consider the formal difference equation

$$\Delta^k\bar{y} = \bar{s}(x).$$

It is clear that if we can show that this equation admits $n-k$ formal solutions corresponding to the $n-k$ known series $\bar{s}(x)$, we are led to a contradiction and the italicized statement must be true. In fact the equation in $\bar{y}$ is satisfied by the $n-k$ series so obtained and in addition by the k distinct series forms, $1, x, \ldots x^{k-1}$ for which $\Delta^k\bar{y}$ is 0, so that the n series exist and are obviously linearly independent.

Consequently it is clear that the italicized statement will hold if the following lemma can be established:

Lemma. *If $s(x)$ is a formal series of type* (6), (6′), (6″), *then the equation*

$$\varDelta y = s(x)$$

admits of a formal series solution of type (6), (6′), (6″).

In fact we need only apply the lemma repeatedly to the equation $\varDelta^k y = \bar{s}(x)$ written above, when we find after k steps a series solution for each series $\bar{s}(x)$.

Let us first prove that the equation of the lemma admits such a solution if $s(x)$ is of the form (6) or (6′). We have then to consider an equation

$$\varDelta y = x^{\frac{\mu x}{p}} e^{P(x)} x^r \left(s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots\right) \qquad (s_0 \neq 0),$$

where μ is an integer, since we may replace p by the integral multiple lp in (6′).

In the case $\mu < 0$ we substitute

$$y(x) = x^{\frac{\mu x}{p}} e^{P(x)} x^r \bar{y}(x),$$

and, upon division through by the coefficient of $\bar{y}(x)$ in this equation, obtain

$$\left[e^{\mu(x+1)\log\left(1+\frac{1}{x}\right)} x^{\frac{\mu}{p}} e^{P(x+1)-P(x)} \left(1 + \frac{1}{x}\right)^r\right] \bar{y}(x+1) - \bar{y}(x) = s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots,$$

where the factor in brackets on the left has the form

$$x^{\frac{\mu}{p}}\left(e^{\gamma+\frac{\mu}{p}} + \frac{f_1}{x^{\frac{1}{p}}} + \frac{f_2}{x^{\frac{2}{p}}} + \cdots\right) \qquad (\mu < 0).$$

If now we substitute in this modified equation for $\bar{y}$,

$$\bar{y}(x) = \bar{y}_0 + \frac{\bar{y}_1}{x^{\frac{1}{p}}} + \cdots,$$

it is obvious that $\bar{y}_0, \bar{y}_1, \ldots$ are determined in succession by direct comparison, with $\bar{y}_0 = -s_0$ for instance. More precisely, the terms in $x^{-\frac{k}{p}}$ lead to an equation of the form

$$\varphi_k - \bar{y}_k = s_k \qquad (k \geqq 1),$$

where φ_k is a polynomial in $\bar{y}_0, \bar{y}_1, \ldots \bar{y}_{k-1}$. Thus $\bar{y}_1, \bar{y}_2, \ldots$ are determined in succession by taking $k = 1, 2, \ldots$, and the desired series solution $y(x)$ is found.

When $\mu > 0$, we substitute

$$y(x) = x^{\frac{\mu x}{p}} e^{P(x)} x^{r - \frac{\mu}{p}} \bar{y}(x)$$

but divide through by the same factor as before. We obtain

$$\left[e^{\frac{\mu(x+1)}{p}\log\left(1+\frac{1}{x}\right)} e^{P(x+1)-P(x)} \left(1 + \frac{1}{x}\right)^{r-\frac{\mu}{p}}\right] \bar{y}(x+1) - x^{\frac{-\mu}{p}} \bar{y}(x) = s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots \qquad (\mu > 0).$$

If now we replace $\bar{y}$ by the same series as before, and note that the term in brackets is a power series in negative powers of $x^{\frac{1}{p}}$, starting off with a constant term not zero, we see at once that $\bar{y}_0, \bar{y}_1, \ldots$ are uniquely determined, with $\bar{y}_0 = s_0 e^{-\gamma - \frac{\mu}{p}}$ for instance, and again the desired solution is obtained.

There remains the possibility that $s(x)$ is of type (6) or (6′) with $\mu = 0$. The same substitution for y as in the first case leads to an equation

$$\left[e^{P(x+1)-P(x)} \left(1 + \frac{1}{x}\right)^r\right] \bar{y}(x+1) - \bar{y}(x) = s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots,$$

where the term in brackets is of the form

$$e^{\gamma} + \frac{f_1}{x^{\frac{1}{p}}} + \frac{f_2}{x^{\frac{2}{p}}} + \cdots.$$

When we substitute the series for $\bar{y}$ as before, we discover at once that $\bar{y}_0, \bar{y}_1, \ldots$ are in general uniquely determined in succession with $\bar{y}_0$, for instance, equal to $s_0/(e^{\gamma} - 1)$. This determination fails when e^{γ} is 1, and then only; in this case we may take $\gamma = 0$.

But when $\gamma = 0$, inspection of the bracket shows that if the leading term in $P(x)$ is $\varkappa x^{\frac{k}{p}}$ $(k < p)$, then the series in brackets begins as follows:

$$1 + \frac{\varkappa k}{p x^{1-\frac{k}{p}}} + \cdots,$$

and the equation in $\bar{y}$ may be written

$$\Delta\bar{y} + \left[\frac{\varkappa k}{px^{1-\frac{k}{p}}} + \cdots\right]\bar{y}(x+1) = s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots \qquad (k < p).$$

Consequently by writing

$$\bar{y}(x) = x^{1-\frac{k}{p}}\left[\bar{y}_0 + \frac{\bar{y}_1}{x^{\frac{1}{p}}} + \cdots\right],$$

we find that $\bar{y}_0 = s_0 p/\varkappa k$, $\bar{y}_1, \ldots$ are determined in succession and a solution is reached, unless indeed k is 0, so that $P(x)$ reduces identically to 0. But in this case the original equation in y becomes

$$\Delta y = x^r\left(s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots\right),$$

which can be at once satisfied if we write

$$y(x) = x^{r+1}\left(y_0 + \frac{y_1}{x^{\frac{1}{p}}} + \cdots\right)$$

with $y_0 = s_0/(r+1)$, $y_1, \ldots$ determined, unless indeed r is a multiple of $1/p$ with $r \geqq -1$, and a term in x^{-1} actually appears on the right-hand side. In such a case we may eliminate this term by writing

$$y(x) = \bar{y}(x) + s_{(r+1)p}\log x,$$

and proceed as before, except that the constant term in the series for y is arbitrary.

Consequently in all cases whatsoever when $s(x)$ is of type (6) or (6′), a solution is obtained. It will be noted that this solution is of the same type unless it involves a linear logarithmic term $c\log x$. This last case of type (6″) arises only when $\mu = 0$, $P(x) = 0$, and a term in x^{-1} appears in $s(x)$.

Thus the lemma will be proved if we can deal with the case when $s(x)$ is of type (6″). We shall dispose of this case by showing how the solution of the equation $\Delta y = s_l(x)$, in which $s_l(x)$ is of the type (6″) with logarithms which enter to the l-th degree, can always be reduced to that of a similar equation in which l is reduced by unity and so finally to an equation of the type $l=0$ already disposed of.

To achieve this reduction, suppose that we have to consider

$$\Delta y = s_l(x) = s_0(x)(\log x)^l + s_{l-1}(x)$$

where $s_0(x)$ is of type (6), (6'). Write

$$y(x) = \bar{s}(x)(\log x)^l + \bar{y}(x),$$

where $\bar{s}(x)$ is a solution of

$$\Delta \bar{s}(x) = s_0(x),$$

which we know to exist of course, according to what has been proved above.

Upon substitution we find that $\bar{y}$ has to satisfy the corresponding equation

$$\begin{aligned}\Delta \bar{y} = s_0(x)(\log x)^l - (s_0(x) + \bar{s}(x))(\log(x+1))^l \\ + \bar{s}(x)(\log x)^l + s_{l-1}(x).\end{aligned}$$

But the right-hand member may be written as a sum of three terms

$$-s_0(x)\,[(\log(x+1))^l - (\log x)^l] - \bar{s}(x)\,[(\log(x+1))^l - (\log x)^l] + s_{l-1}(x).$$

Inasmuch as the identity

$$\begin{aligned}(\log(x+1))^l - (\log x)^l &\equiv (\log x + \Delta(\log x))^l - (\log x)^l \\ &\equiv l(\log x)^{l-1}\Delta \log x + \cdots + (\Delta \log x)^l\end{aligned}$$

obtains, where

$$\Delta \log x = \log\left(1 + \frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \cdots,$$

it is clear that the first term in this sum, as well as the last, involves $\log x$ to a power not exceeding $l-1$. Moreover, on account of the presence of the factor $\Delta \log x$, the middle term will have the same property unless $\bar{s}(x)$ involves $\log x$. Hence it is only necessary to examine into this last possibility.

But it has already been noted that the solution of an equation of the type $\Delta \bar{y} = s_0(x)$, where $s_0(x)$ is of the form (6), (6'), can only involve logarithms linearly, and that this can only happen in the case $\mu = P(x) = 0$ when a term in x^{-1}

appears in $s_0(x)$. Therefore the only case requiring further consideration is that in which the equation takes the form

$$\Delta y = s_0(x)(\log x)^l + s_{l-1}(x)$$

where

$$s_0(x) = x^r\left(s_0 + \frac{s_1}{x^{\frac{1}{p}}} + \cdots\right)$$

and where $s_0(x)$ contains a term in x^{-1}. It is clear that all the difficulty arises from this single term, and that if we can find a solution for the simple equation

$$\Delta y = (\Delta \log x)(\log x)^l,$$

the difficulty is disposed of. But if we substitute

$$y(x) = \frac{(\log x)^{l+1}}{l+1} + \bar{y}(x)$$

we find

$$\Delta \bar{y} = (\Delta \log x)(\log x)^l - \frac{1}{l+1}[(\log(x+1))^{l+1} - (\log x)^{l+1}]$$

$$= -\frac{l}{1.2}(\log x)^{l-1}(\Delta \log x)^2 - \cdots - \frac{1}{l+1}(\Delta \log x)^{l+1}.$$

Now on the right-hand side there appears a sum involving $\log x$ to the $(l-1)$-st power at most. Consequently in every case we can reduce the problem to one of a similar type but with l decreased by at least one, as we desired to prove.

Thus, whatever be the series $s(x)$ in the equation of the lemma, we are led to a corresponding solution of type (6), (6′), (6″), and the lemma is fully established.

§ 6. The Formal Problem and Formal Reducibility.

Let us term the equation (4), namely $L(y) = 0$, 'reducible' in case we may write symbolically

$$L(y) \equiv M_{n-d}(L_d(y))$$

where M_{n-d} and L_d are difference expressions of the same type as L, of orders $n-d$ and d respectively.

Now if there exists no equation $L(y)=0$ without a formal series solution (6), (6′), we have seen that a full quota of formal series necessarily exists (§ 5), and the formal problem is solved. In the contrary case there exists an equation $L(y)=0$ of least order $n>1$ for which no solution (6), (6′) exists.

Such an equation E must necessarily be irreducible for any admissible basic integer p.

In fact if we could factor L symbolically as indicated above, then every solution of $L_d(y)=0$ would also satisfy $L(y)=0$. But the equation $L_d(y)=0$ is of lower order than $L(y)=0$, and so possesses at least one series solution (6), (6′), by hypothesis. Hence $L(y)=0$ would possess the same solution, which would be absurd.

Moreover, if we effect any change of variables in E of the general form

$$y(x)=x^{\frac{\mu x}{p}}e^{P(x)}x^r\bar{y}(x),$$

the new equation in y must obviously also be irreducible in the same sense, no matter how the integer μ, the polynomial $P(x)$, and the constant r be chosen.[1]

In consequence we proceed to consider several cases in which specific types of reducibility are established (§§ 7—10). It will then be easy to prove that such an equation E cannot exist, so that a complete set of formal solutions is thereby proved to be present always.

§ 7. First Reducible Case.

An equation $L(y)=0$ with $k>1$ values of μ, say $\mu_1, \mu_2, \ldots \mu_k$, with $\mu_1=0$ and $\mu_i<0$ $(i=2, \ldots k)$, is necessarily reducible with a symbolic factor of order d equal to the number d of non-zero roots ϱ corresponding to $\mu_1=0$.

In the first place we observe that by definition of μ_i, the hypothesis made ensures that the first segment of the broken line in the (i, j) diagram (Fig. 1) is horizontal while the later segments have positive slope. Hence, after the leading coefficient of $L(y)=0$ is made equal to 1 by division through by $a_0(x)$, we may write (4) in the form

[1] Note that this change of variables leaves the equation of the same general form (4), although the basic integer p may be altered.

$$L(y) \equiv y(x+n) + a_1(x)y(x+n-1) + \cdots + a_n(x)y(x) = 0, \tag{7}$$

where the series $a_1(x)$, $a_2(x)$, ... $a_n(x)$ in the equation so obtained can contain no positive powers of $x^{\frac{1}{p}}$, while $a_d(x)$ has a constant term not zero, corresponding to the second vertex of the horizontal segment. Furthermore it is apparent that the series $a_i(x)$ for $i > d$ are series of the same type but lacking constant terms.

We propose to search for a symbolic factor of the form

$$L_d(x, y) \equiv y(x+d) + p_1(x)y(x+d-1) + \cdots + p_d(x)y(x), \tag{8}$$

where $p_1(x)$, ... $p_d(x)$ are power series in negative integral powers of $x^{\frac{1}{p}}$, containing constant terms identical with those in $a_1(x)$, ... $a_d(x)$ respectively, but otherwise arbitrary. The more explicit notation $L_d(x, y)$ is used instead of $L_d(y)$, since we shall have occasion to change the x in the expression to $x+i$. Now if L_d is such a factor we must have

$$L(y) \equiv L_d(x+n-d, y) + \lambda_1(x)L_d(x+n-d-1, y) + \cdots + \lambda_{n-d}(x)L_d(x, y). \tag{9}$$

Here we shall restrict $\lambda_1(x)$, ... $\lambda_{n-d}(x)$ to be similar power series but lacking constant terms; this restriction is actually required if such an identity is to hold.

In the first place we observe that, in the coefficients of $y(x+n)$, ... $y(x)$ which enter, at least the constant terms agree on both sides, in virtue of the particular choice made of the constant terms in $p_i(x)$ and $\lambda_i(x)$. Let us consider next the terms in $x^{-\frac{1}{p}}$ in these coefficients. The comparison under consideration is more easily effected if we write the above identity in the form of n explicit equations between these coefficients:

$$\begin{aligned}
a_1(x) &\equiv p_1(x+n-d) + \lambda_1(x), \\
a_2(x) &\equiv p_2(x+n-d) + \lambda_1(x)p_1(x+n-d-1) + \lambda_2(x), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n-1}(x) &\equiv \lambda_{n-d-1}(x)p_d(x+1) + \lambda_{n-d}(x)p_{d-1}(x), \\
a_n(x) &\equiv \lambda_{n-d}(x)p_d(x),
\end{aligned} \tag{10}$$

where the law of formation is obvious.

Since the constant terms λ_{i0}, p_{i0} in λ_i, p_i have been specified as stated, the comparison of the terms in $x^{-\frac{1}{p}}$ gives the following n equations:

$$
(11_1)\qquad
\begin{aligned}
a_{11} &= p_{11} + \lambda_{11}, \\
a_{21} &= p_{21} + \lambda_{11} p_{10} + \lambda_{21}, \\
&\cdots\cdots\cdots\cdots \\
a_{n-1,1} &= \lambda_{n-d-1,1} p_{d0} + \lambda_{n-d,1} p_{d-1,0}, \\
a_{n1} &= \lambda_{n-d,1} p_{d0},
\end{aligned}
$$

where we employ the notation f_{ij}, to denote the coefficient of $x^{-\frac{j}{p}}$ in any series $f_i(x)$. Hence the last written equation determines $\lambda_{n-d,1}$ since $p_{d0} \neq 0$; the next to the last equation determines $\lambda_{n-d-1,1}$, and so on, until the $(d+1)$-th equation determines λ_{11}. But then if we turn to the remaining first d equations we see that these in order determine $p_{11}, p_{21}, \ldots p_{d1}$. Thus the coefficients of $x^{-\frac{1}{p}}$ in $\lambda_i(x)$ and $p_i(x)$ are uniquely determined by this comparison.

Next we may proceed to the determination of the second and higher order coefficients λ_{ik} and p_{ik} by comparison of the coefficients of $x^{-\frac{k}{p}}$ for $k=2, 3, \ldots$. Equations are obtained of a similar form

$$
(11_k)\qquad
\begin{aligned}
a_{1k} &= p_{1k} + \lambda_{1k} + A_{1k}, \\
a_{2k} &= p_{2k} + \lambda_{1k} p_{10} + A_{2k}, \\
&\cdots\cdots\cdots\cdots \\
a_{n-1,k} &= \lambda_{n-d-1,k} p_{d0} + \lambda_{n-d,k} p_{d-1,0} + A_{n-1,k}, \\
a_{nk} &= \lambda_{n-d,k} p_{d0} + A_{nk},
\end{aligned}
$$

where A_{ik} are known functions of the coefficients p_{ij}, λ_{ij} $(j<k)$.

Thus for $k=2$, for instance, we see that λ_{i2} and p_{i2} are again uniquely determined, and clearly this process may be indefinitely continued, so that $L(y)$ can be symbolically factored as stated, and the statement under consideration is proved.

§ 8. Second Reducible Case.

An equation $L(y)=0$ with a single value of $\mu=0$, and so with a characteristic equation in ϱ of degree n with no zero roots (§ 2), *is necessarily reducible unless the n values of ϱ are all equal.*

The equation $L(y)=0$ under consideration is of course still taken in the allowable form (7) above. In the case before us the coefficients $a_i(x)$ in it are evidently power series in negative powers of $x^{\frac{1}{p}}$, of which at least the last, $a_n(x)$, contains a constant term a_{n0}.

If the n values of ϱ are not equal, let ϱ_1 denote some particular root of multiplicity $d<n$. We now attempt to find a symbolic factor $L_d(x,y)$ of order $d<n$, corresponding to this root, which shall be of the same form (8) as in the preceding paragraph, except that $p_i(x)$, $(i=1, 2, \ldots d)$ will here be taken as power series in negative powers of $x^{\frac{1}{p}}$ with constant terms as follows:

$$p_{10}=-d\varrho_1, p_{20}=\frac{d(d-1)}{1 \,.\, 2}\varrho_1^2, \ldots p_{d0}=(-1)^d\varrho_1^d. \tag{12}$$

Hence the single characteristic equation of $L_d(x,y)=0$ is necessarily $(\varrho-\varrho_1)^d=0$. Furthermore we write $L(y)$ once more in the form (9), except that the series $\lambda_i(x)$, $(i=1, 2, \ldots n-d)$, are power series in negative powers of $x^{\frac{1}{p}}$ whose constant terms we proceed to specify.

In the first place we observe that the characteristic equation of the symbolic product is clearly

$$(\varrho-\varrho_1)^d(\varrho^{n-d}+\lambda_{10}\varrho^{n-d-1}+\cdots+\lambda_{n-d,0})=0.$$

Hence this product will have the same characteristic equation as $L(y)=0$ if $\lambda_{10}, \ldots \lambda_{n-d,0}$ are properly determined, and this determination is obviously unique, with

$$\varrho_1^{n-d}+\lambda_{10}\varrho_1^{n-d-1}+\cdots+\lambda_{n-d,0}\neq 0,\ \lambda_{n-d,0}\neq 0.$$

At this stage then, p_{i0} and λ_{i0} are uniquely determined, and the constant terms in the coefficients on both sides of (9) agree. The equations (10) simply state the equality of corresponding coefficients of course.

We have next to consider the terms in $x^{-\frac{1}{p}}$ on both sides of (10). The equations obtained differ somewhat in form from (11_1) inasmuch as the constants λ_{i0} need not be 0 in the case before us. These equations are in fact

$$(13_1)\qquad \begin{aligned} a_{11} &= p_{11} + \lambda_{11}, \\ a_{21} &= p_{21} + (\lambda_{10} p_{11} + \lambda_{11} p_{10}) + \lambda_{21}, \\ &\cdots\cdots\cdots\cdots \\ a_{n-1,1} &= (\lambda_{n-d-1,0}\, p_{d1} + \lambda_{n-d-1,1}\, p_{d0}) \\ &\qquad\qquad + (\lambda_{n-d,0}\, p_{d1} + \lambda_{n-d,1}\, p_{d0}), \\ a_{n1} &= (\lambda_{n-d,0}\, p_{d1} + \lambda_{n-d,1}\, p_{d0}). \end{aligned}$$

But the equations (13_1) are the same equations as result from the algebraic identity in u,

$$(13_1')\qquad \sum_{j=1}^{n-1} a_{j1}\, u^{n-j} \equiv A_d(u) \varDelta_{n-d-1}(u) + B_{n-d}(u) \varPi_{d-1}(u),$$

where A_d and B_{n-d} are the known polynomials

$$\begin{aligned} A_d(u) &\equiv (u-\varrho_1)^d \equiv u^d + p_{10} u^{d-1} + \cdots + p_{d0}, \\ B_{n-d}(u) &\equiv u^{n-d} + \lambda_{10} u^{n-d-1} + \cdots + \lambda_{n-d,0}, \end{aligned}$$

and $\varDelta_{n-d-1}$, $\varPi_{d-1}$ are to be determined from the equations

$$\begin{aligned} \varDelta_{n-d-1}(u) &\equiv \lambda_{11} u^{n-d-1} + \lambda_{21} u^{n-d-2} + \cdots + \lambda_{n-d,1}, \\ \varPi_{d-1}(u) &\equiv p_{11} u^{d-1} + p_{21} u^{d-2} + \cdots + p_{d-1,1}, \end{aligned}$$

in which the coefficients are arbitrary.

This is readily seen if we write the j-th equation (13_1) in the form

$$a_{j1} = \sum_i (\lambda_{i0}\, p_{j-i,1} + \lambda_{i1}\, p_{j-i,0}) \qquad (j = 1, \ldots n),$$

where we take $\lambda_{00} = 1$, $\lambda_{01} = 0$, $p_{00} = 1$, $p_{01} = 0$, and all the subscripts are to be positive or zero of course.

But the polynomials A_d, B_{n-d} are relatively prime, while $\varDelta_{n-d-1}$ and $\varPi_{d-1}$ are entirely arbitrary polynomials of degrees $n-d-1$ and $d-1$ at most respectively. On the left of $(13_1')$ stands a known polynomial of degree $n-1$ at most in u. It is then a familiar and fundamental theorem of algebra that $\varDelta_{n-d-1}$ and $\varPi_{d-1}$ can be uniquely determined so that this identity holds. In consequence λ_{i1} and p_{i1} are uniquely determined by the stated conditions.

More generally if we compare the coefficients of $x^{-\frac{k}{p}}$ on both sides of (10), we obtain a set of equations (13_k) related to (13_1) as the equations (11_k) are to (11_1). More precisely, the second subscripts 1 are changed to k, and in addition there appears on the right of the j-th equation a further term A_{jk} which is a known polynomial in the coefficients $\lambda_{i1}, \dots \lambda_{i,k-1}, p_{i1}, \dots p_{i,k-1}$. Thus we see at once that $\lambda_{i2}, p_{i2}, \lambda_{i3}, p_{i3}, \dots$ are uniquely determined in succession, and the statement under consideration is established.

§ 9. Third Reducible Case.

Suppose that the difference equation $L(y)=0$ has a single value of μ, namely $\mu=0$, and that the n roots of the characteristic equation are equal to 1, so that the difference equation may be written

$$L(y) \equiv \Delta^n y + b_1(x)\Delta^{n-1}y + \cdots + b_n(x)y = 0, \tag{14}$$

where $b_1(x), \dots b_n(x)$ are formal power series in negative powers of $x^{\frac{1}{p}}$ without constant terms. Suppose, however, that the n series

$$x b_1(x),\ x^2 b_2(x),\ \dots\ x^n b_n(x)$$

do not all contain only negative powers of $x^{\frac{1}{p}}$ with or without constant terms. Then, unless the modified equation in z,

$$z(x+n) + b_1(x)z(x+n-1) + \cdots + b_n(x)z(x) = 0,$$

has a single (negative) value of μ and all the roots of its characteristic equation are equal, the equation (14) is necessarily reducible, at least after the basic integer p is replaced by lp.

Before entering upon the proof, it is interesting to remark that, from a formal point of view, a difference equation of type (14) is much more closely allied to an ordinary linear differential equation than is the most general difference equation (4). This fact explains the usefulness of the difference notation employed. We note indeed that the above difference equation is to be considered as analogous to an ordinary linear differential equation with an irregular singular point of rank at most 1 at $x=\infty$. On this account we may expect a new type of characteristic equation, analogous to that of the corresponding differential equation, to play a rôle. It will be our first step to introduce appropriate definitions.

In analogy with the definitions of § 2, let us denote by b_{ij_i} the coefficient of the leading term in $x^{\frac{-j_i}{p}}$ of the power series for $b_i(x)$, with $b_0(x) \equiv 1$ and so $j_0 = 0$. Furthermore, if $b_n(x)$ were to vanish identically, the equation (14) would be obviously reducible, so that we may take j_n to be finite.

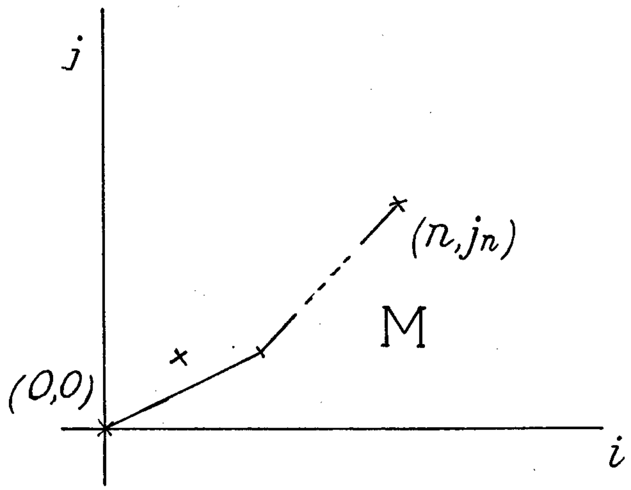

Fig. 2.

Now mark the $n+1$ points (i, j_i) as in Fig. 1, § 2, and draw the corresponding broken line, concave upwards, whose vertices are chosen from among these points, while all of the remaining points lie on or above the line (Fig. 2). Evidently this broken line M starts at the origin with a positive slope, which is less than p, however, according to the hypothesis of the italicized statement.

It is obvious that M is precisely the line L of § 2 for the modified equation in z. In accordance with the italicized statement, we have to prove that when this line consists of more than a single segment, and even when it consists of a single segment but the roots of the characteristic equation are not all equal, the equation (14) is reducible.

If the broken line M is made up of more than a single segment, let $d < n$ be the integer which gives the coordinate i of the right-hand end point of the first segment. From the geometric construction it is clear that if we write

$\varkappa = j_d/d$ so that $\varkappa$ is the slope of the first segment, the following inequalities must obtain

$$j_i \geqq \varkappa i, \quad (i \leqq d); \quad j_i > \varkappa i, \quad (i > d).$$

We may write $\varkappa$ in the form m/l where the positive integers $l \geqq 1$ and m are relatively prime.

Thus the equation under consideration is essentially of the form

$$(14') \qquad x^{\frac{n\varkappa}{p}} L(y) \equiv x^{\frac{n\varkappa}{p}} \Delta^n y + x^{\frac{(n-1)\varkappa}{p}} c_1(x) \Delta^{n-1} y + \cdots + c_n(x) y = 0 \quad (0 < \varkappa < p),$$

where $c_1(x), \ldots c_n(x)$ are power series in negative powers of $x^{\frac{1}{lp}}$, among which $c_d(x)$ at least contains a constant term while the series $c_i(x)$ for $i > d$ contain no such term.

In this case when the modified equation of the lemma in z has $k > 1$ values of μ of which the first is $\mu_1 = -\varkappa$ so that $\mu_1 > -p$, the corresponding characteristic equation is

$$\left(\varrho e^{\frac{-\varkappa}{p}}\right)^d + c_{10} \left(\varrho e^{\frac{-\varkappa}{p}}\right)^{d-1} + \cdots + c_{d0} = 0,$$

so far as the non-zero roots are concerned.

We shall show that such an equation is reducible if the basic integer is taken to be lp.[1] With this change of basic integer, $\varkappa$ is replaced by lj_d/d, an integer. Of course the relation $\varkappa < p$ continues to hold after this modification, where p and $\varkappa$ refer to the modified basic integer and modified $\varkappa$.

Now let us introduce the difference operator $\bar{\Delta}$ such that

$$\bar{\Delta} u = x^{\frac{\varkappa}{p}} \Delta u.$$

We find then

$$\Delta u = x^{\frac{-\varkappa}{p}} \bar{\Delta} u$$

and furthermore

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[1] As a matter of fact the symbolic factorization accomplished only involves powers of $x^{\frac{1}{p}}$ in the coefficients, so that the stated reducibility is effective for the original basic integer. We omit, however, the proof of this fact, which is easily made directly.

$$\Delta^2 u = (x+1)^{\frac{-\varkappa}{p}} \Delta(\overline{\Delta} u) + \Delta\left(x^{\frac{-\varkappa}{p}}\right)\overline{\Delta} u$$

$$= x^{\frac{-2\varkappa}{p}}\left[\left(1+\frac{1}{x}\right)^{\frac{-\varkappa}{p}} \overline{\Delta}^2 u + \left(-\frac{\varkappa}{p} x^{\frac{\varkappa}{p}-1} + \cdots\right)\overline{\Delta} u\right].$$

This gives immediately

$$x^{\frac{2\varkappa}{p}} \Delta^2 u = \overline{\Delta}^2 u + \theta_1^{(2)}(x)\overline{\Delta}^2 u + \theta_2^{(2)}(x)\overline{\Delta} u,$$

where $\theta_1^{(2)}(x)$, $\theta_2^{(2)}(x)$ are power series in negative powers of $x^{\frac{1}{p}}$ without constant terms.

Moreover we find that in general for $m \geqq 1$,

$$x^{\frac{m\varkappa}{p}} \Delta^m u = \overline{\Delta}^m u + \sum_{j=0}^{m-1} \theta_j^{(m)}(x)\overline{\Delta}^{m-j} u \tag{15}$$

where $\theta_j^{(m)}(x)$ are definite power series in negative powers of $x^{\frac{1}{p}}$ without constant terms. This has already been demonstrated above for $m = 1, 2$, and is readily proved in general by induction. We have merely to observe that we have

$$x^{\frac{(m+1)\varkappa}{p}} \Delta^{m+1} u = x^{\frac{\varkappa}{p}}\left[x^{\frac{m\varkappa}{p}} \Delta^m(\Delta u)\right]$$

$$= x^{\frac{\varkappa}{p}}\left[\overline{\Delta}^m\left(x^{\frac{-\varkappa}{p}}\overline{\Delta} u\right) + \sum_{j=0}^{m-1}\theta_j^{(m)}(x)\overline{\Delta}^{m-j}\left(x^{\frac{-\varkappa}{p}}\overline{\Delta} u\right)\right],$$

if the formula holds for the particular m under consideration, and to expand the terms involved. To this end we note that since

$$\Delta(uv) = u(x+1)\Delta v + (\Delta u)v,$$

we have

$$\overline{\Delta}(uv) = u(x+1)\overline{\Delta} v + (\overline{\Delta} u)v.$$

Likewise we find

$$\overline{\Delta}^2(uv) = \overline{\Delta}(u(x+1)\overline{\Delta} v + (\overline{\Delta} u)v)$$

$$= u(x+2)\overline{\Delta}^2 v + 2\overline{\Delta} u(x+1)\overline{\Delta} v + (\overline{\Delta}^2 u)v$$

by another application of the same formula. Thus in general we readily deduce the identity

$$(16) \qquad \bar{\Delta}^k(uv) = u(x+k)\bar{\Delta}^k v + k\bar{\Delta} u(x+k-1)\bar{\Delta}^{k-1} v + \cdots + (\bar{\Delta}^k u)v,$$

which is of familiar form. Applying this expansion, the formula (15) is seen to hold for $m+1$; the fact that the operator $\bar{\Delta}$ applied to a power series actually lowers the degree of each term is to be borne in mind. Thus (15) is established by induction.

By use of (15), the difference equation (14′) may evidently be given the form (14″):

$$(14'') \qquad x^{\frac{nx}{p}} L(y) \equiv \bar{\Delta}^n y + \bar{c}_1(x)\bar{\Delta}^{n-1} y + \cdots + \bar{c}_n(x) y = 0$$

where $\bar{c}_1(x), \ldots \bar{c}_n(x)$ are power series in negative powers of $x^{\frac{1}{p}}$ in which a constant term certainly appears in $\bar{c}_d(x)$ but not in any $\bar{c}_i(x)$ for $i > d$. More precisely, it is apparent that the constant terms in $\bar{c}_1(x), \ldots \bar{c}_d(x)$ are the same as in $c_1(x), \ldots c_d(x)$ respectively.

We propose to demonstrate that this expression admits of a symbolic factor of the form

$$L_d(y) \equiv \bar{\Delta}^d y + p_1(x)\bar{\Delta}^{d-1} y + \cdots + p_d(x) y$$

where $p_1(x), \ldots p_d(x)$ are power series in negative powers of $x^{\frac{1}{p}}$ with the same constant terms as $\bar{c}_1(x), \ldots \bar{c}_d(x)$ respectively. To achieve this factorization, we shall set up an identity of the form

$$(17) \qquad x^{\frac{nx}{p}} L(y) \equiv \bar{\Delta}^{n-d}(L_d(y)) + \lambda_1(x)\bar{\Delta}^{n-d-1}(L_d(y)) + \cdots + \lambda_{n-d}(x) L_d(y)$$

where $\lambda_1(x), \ldots \lambda_{n-d}(x)$ are to be power series in $x^{\frac{1}{p}}$ without constant terms but not otherwise specified at the outset.[1] In order to effect the comparison of both sides we need to expand terms of the type

$$\bar{\Delta}^{n-d-i}(p_j(x)\bar{\Delta}^{d-j} y) \qquad (i \geqq 0, j > 0),$$

[1] Note the formal analogy here and later with the method used in the preceding paragraphs.

and to do so we employ the identity (16) which gives the expansion

$$p_j(x+n-d-i)\bar{\Delta}^{n-i-j}y + (n-d-i)\bar{\Delta}p_j(x+n-d-i-1)\bar{\Delta}^{n-i-j-1}y + \cdots \\ + (\bar{\Delta}^{n-d-i}p_j(x))\bar{\Delta}^{d-j}y.$$

This leads to the explicit identities

$$(18)\quad \begin{aligned} \bar{c}_1(x) &\equiv p_1(x+n-d) + \lambda_1(x), \\ \bar{c}_2(x) &\equiv p_2(x+n-d) + (n-d)\bar{\Delta}p_1(x+n-d-1) + \lambda_1(x)p_1(x+n-d-1) + \lambda_2(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{c}_{n-1}(x) &\equiv (n-d)\bar{\Delta}^{n-d-1}p_d(x+1) + (n-d-1)\lambda_1(x)\bar{\Delta}^{n-d-2}p_d(x+1) + \cdots \\ &\quad + \lambda_{n-d-1}(x)p_d(x+1) + \bar{\Delta}^{n-d}p_{d-1}(x) + \lambda_1(x)\bar{\Delta}^{n-d-1}p_{d-1}(x) + \cdots \\ &\quad + \lambda_{n-d}(x)p_{d-1}(x), \\ \bar{c}_n(x) &= \bar{\Delta}^{n-d}p_d(x) + \lambda_1(x)\bar{\Delta}^{n-d-1}p_d(x) + \cdots + \lambda_{n-d}(x)p_d(x). \end{aligned}$$

The close analogy of these equations with the equations (10) of § 7 should be noted. Since the operation $\bar{\Delta}$ applied to any power series in negative powers of $x^{\frac{1}{p}}$ lowers the degree of the initial term, it is clear that the coefficients of $x^{\frac{-k}{p}}$ on the right-hand sides can only involve the constants λ_{ij}, p_{ij} for $j \leqq k$, and that the terms involving the operator $\bar{\Delta}$ can never give rise to terms involving λ_{ik}, p_{ik}. But, when these terms are omitted, the expressions on the right are identical with those on the right in (10).

It is apparent that, because of the way in which λ_{i0}, p_{i0} were chosen, the constant terms on both sides agree. Also, according to the preceding remark, the comparison for the terms in $x^{\frac{-1}{p}}, x^{\frac{-2}{p}}, \ldots$ will determine $\lambda_{i1}, p_{i1}, \lambda_{i2}, p_{i2}, \ldots$ uniquely, the equations of determination being precisely like (11_1) for $k=1$ and (11_k) for $k=2, 3, \ldots$, except that a_{ij} is replaced by $\bar{c}_{ij}$ of course.

The stated reducibility is thereby proved in the case when the broken line M consists of more than a segment so that the modified equation in z has more than a single value of μ.

There remains then the case when there is but one such value of $\mu > -p$, in which case it is necessary to establish that so long as the roots of the single characteristic equation for z are not all equal, the equation (14) is still reducible, at least if lp is taken as the basic integer.

If we write $\varrho = \bar{\varrho} e^{\overset{x}{\varrho}}$ in this characteristic equation, it takes the form

$$\varrho^n + c_{10}\varrho^{n-1} + \cdots + \bar{c}_{n0} = 0, \qquad (\bar{c}_{n0} \neq 0),$$

and the equation in $\bar{\varrho}$ also has not all equal roots.

Now in this case we can still adopt the preceding form $(14'')$ although here the last coefficient $\bar{c}_n(x)$ has a constant term not zero. Furthermore if $\bar{\varrho}_1$ is a root of the equation above of d fold multiplicity $(d < n)$, we may again propose a symbolic factorization (17) where we choose

$$p_{10} = -d\bar{\varrho}_1, \; p_{20} = \frac{d(d-1)}{2}\varrho_1^2, \; \ldots \; p_{d0} = (-1)^d \bar{\varrho}_1^d,$$

and then $\lambda_{10}, \ldots \lambda_{n-d,0}$ so that the following identity holds:

$$(u - \varrho_1)^d (u^{n-d} + \lambda_{10} u^{n-d-1} + \cdots + \lambda_{n-d,0}) \equiv$$
$$u^n + \bar{c}_{10} u^{n-1} + \cdots + \bar{c}_{n0}.$$

In this way the constant terms on the two sides in the coefficients of (17) are made to agree. Likewise the comparison of the terms in $x^{\frac{-1}{p}}$ clearly yields equations of the form (13_1) which determine λ_{i1}, p_{i1} uniquely; more generally, the comparison of terms in $x^{\frac{-k}{p}}$ $(k > 1)$ leads to equations related to (13_1) just as the equations (11_k) are related to (11_1), and so determine $\lambda_{i2}, p_{i2}, \lambda_{i3}, p_{i3}, \ldots$ in succession uniquely.

Thus here again the desired reducibility is established.

§ 10. **Fourth Reducible Case.**

If the equation $L(y) = 0$ may be written in the form

$$x^n \varDelta^n y + x^{n-1} b_1(x) \varDelta^{n-1} y + \cdots + b_n(x) y = 0,$$

where $b_1(x), \ldots b_n(x)$ are power series in negative powers of $x^{\frac{1}{p}}$ with or without constant terms, then the equation is reducible with a symbolic factor of degree 1.

This case is analogous to that of an ordinary linear differential equation of Fuchsian type and the fact stated above is easily proved.[1]

If we substitute in this equation

$$y = x^l\left[y_0 + y_1 x^{\frac{-1}{p}} + y_2 x^{\frac{-2}{p}} + \cdots\right]$$

and compare coefficients of like powers of x, we find a set of equations

$$f(l)y_0 = 0,$$
$$f\left(l - \frac{1}{p}\right)y_1 + F_1 = 0,$$
$$f\left(l - \frac{2}{p}\right)y_2 + F_2 = 0,$$
$$\cdots\cdots\cdots\cdots$$

where we have written for brevity

$$f(l) = l(l-1)\cdots(l-n+1) + b_{10}l(l-1)\cdots(l-n+2) + \cdots + b_{n0}$$

and where F_i is a known polynomial in $l, y_0, y_1, \ldots y_{i-1}$ for any i.

But the polynomial f in l is clearly of the n-th degree and has obviously at least one root l such that $l - \frac{1}{p}, l - \frac{2}{p}, \ldots$ are not roots. For such a value of l, it is evident that the first equation is satisfied for any y_0, while $y_1, y_2, \ldots$ are successively determined uniquely by the following equations, as soon as y_0 is assigned, so that a formal series solution of this type is determined. However, if this series be denoted by $g(x)$, and if we write

$$\Delta_1(x, y) = y(x+1) - \frac{g(x+1)}{g(x)}y(x) \qquad \left(\frac{g(x+1)}{g(x)} = 1 + \frac{g_1}{x^{\frac{1}{p}}} + \cdots\right),$$

it is clear that we may express $L(y)$ in the form

$$L(y) \equiv M_{n-1}(\Delta_1(x, y)) + c(x)y$$

where M_{n-1} is a linear difference operator of order $n-1$.[2] But if we substitute

[1] See N. E. Nörlund, Differenzenrechnung, Chap. 11.

[2] We have merely to employ successively the relations

$$y(x+n) = \frac{g(x+n)}{g(x+n-1)}y(x+n-1) + \Delta_1(x+n-1, y)$$
$$y(x+n-1) = \frac{g(x+n-1)}{g(x+n-2)}y(x+n-2) + \Delta_1(x+n-2, y)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

in order to express $L(y)$ in this form.

for y the series $g(x)$ we conclude $c(x) g(x) \equiv 0$, so that $c(x)$ is identically zero. Thus it is clear that $L(y)$ is certainly reducible with symbolic factor $\Delta_1(x, y)$, and the proof of the italicized statement is thereby completed.

§ 11. Completion of Solution of Formal Problem.

We are now prepared to resume the argument begun in § 6.

If an equation E of the type there specified exists, and if $\mu_1, \ldots \mu_k$ are the corresponding values of μ, we can at once prove that k must be 1. In fact if $k > 1$, let μ_1 be the greatest value of μ. Make the transformation

$$y = x^{\frac{\mu_1 x}{p}} \bar{y},$$

which is at once verified to yield a new equation of the same form with values of μ

$$0, \mu_2 - \mu_1, \ldots \mu_k - \mu_1$$

of which one is zero and the others negative. By the result of § 7 this equation would be reducible, and admit a symbolic factor $L_d(y)$ of lower order, having at least one series solution (6), (6′) shared of course by the equation E. This would be absurd.

Hence the equation E has only one value of μ, say μ_1, which, by the transformation already employed leads to an equation E for which this value of μ is 0.

Now this modified equation E will fulfill all the conditions prescribed in the italicized statement of § 8 and so be reducible unless all of the roots of its characteristic equation are equal, say to ϱ_1. The further transformation

$$y = \varrho_1^x \bar{y}$$

will then lead to a like equation E in which these equal values of ϱ are replaced by 1, since the effect of the last transformation is to divide the roots of the characteristic equation by ϱ_1.

Consequently we are led to an equation E of the form (14) which would fulfill the hypotheses of the italicized statement of § 9 or of § 10 and so be reducible, unless the equation in z has only a single negative value of μ, say $-\varkappa$, with $\varkappa < p$, and all of the roots of its characteristic equation are equal.

However, in this remaining case, since the characteristic equation must contain a complete set of terms[1], a first conclusion is that $\varkappa$ must be a positive integer and in addition that the equation under discussion has the form (14′) where the coefficients $c_i(x)$ are power series in negative powers of $x^{\frac{1}{p}}$, with constant terms as follows:

$$c_{10} = -n\bar{\varrho}_1,\ c_{20} = \frac{n(n-1)}{1\,.\,2}\bar{\varrho}_1^2,\ \dots\ c_{n0} = (-1)^n\bar{\varrho}_1^n,$$

where $\bar{\varrho}_1$ is the single value of $\varrho e^{\frac{-\varkappa}{p}}$.

Suppose now we make the transformation

$$y = e^{fx^{1-\varkappa/p}}\bar{y} \equiv \varphi(x)\bar{y} \qquad \left(f = \frac{\bar{\varrho}_1}{1-\frac{\varkappa}{p}}\right)$$

in (14′). Evidently this transformation takes the equation E into another of the same type E. In order to characterize it in more detail, we substitute

$$\varDelta^i y = \varphi(x+i)\varDelta^i\bar{y} + i\varDelta\varphi(x+i-1)\varDelta^{i-1}\bar{y} + \cdots + (\varDelta^i\varphi(x))\bar{y}$$

for $i = 1, 2, \dots n$ and divide through by $\varphi(x+n)$. The equation obtained may then be written

$$\varDelta^n\bar{y} + \left[\frac{n\varDelta\varphi(x+n-1)}{\varphi(x+n)} + x^{\frac{-\varkappa}{p}}c_1(x)\frac{\varphi(x+n-1)}{\varphi(x+n)}\right]\varDelta^{n-1}\bar{y} + \cdots$$
$$+ \frac{1}{\varphi(x+n)}\left[\varDelta^n\varphi(x) + x^{\frac{-\varkappa}{p}}c_1(x)\varDelta^{n-1}\varphi(x) + \cdots + x^{\frac{-n\varkappa}{p}}c_n(x)\,\varphi(x)\right]\bar{y} = 0.$$

It suffices for our present purposes to observe that every coefficient so obtained can contain no positive power of $x^{\frac{1}{p}}$ or even a constant term, and also that the coefficient of $\varDelta^{n-1}\bar{y}$ is given by a series in negative powers which starts off with a term of lower degree than $-\varkappa/p$. In fact we find readily

$$\frac{n\varDelta\varphi(x+n-1)}{\varphi(x+n)} = n\left(\varrho_1 x^{\frac{-\varkappa}{p}} + \cdots\right);\ x^{\frac{-\varkappa}{p}}c_1(x)\frac{\varphi(x+n-1)}{\varphi(x+n)} = -n\varrho_1 x^{\frac{-\varkappa}{p}} + \cdots,$$

so that this is the case.

[1] All of these terms must appear when all of the roots are equal.

But the equation E thus obtained has of course, for reasons already presented, but a single value of $\mu = -\bar{\varkappa}$ and all the roots of its characteristic equation equal. It will therefore be reducible unless it is of the exceptional type (14) in which the modified equation in z has the properties stated. Of course it cannot be of the type treated in § 10, which is always reducible. Moreover, by the argument given above, the value of $\bar{\varkappa}$ for the equation in y must again be a positive integer, and, on account of the nature of the coefficient of $\Delta^{n-1}\bar{y}$, $\varkappa$ will be at least one more than in the first equation.

Thus, step by step, we are able to increase the integer $\varkappa$ in the equation (14'), until at last we are led to the case $\varkappa \geqq p$. But this has been proved to be reducible in all cases (§ 10).

Hence such an equation E cannot exist. Thus we have proved the following fundamental result:

Any difference equation (4) (*or difference system* (1)) *in which the coefficients are formal power series in negative descending powers of* $x^{\frac{1}{p}}$ $(p \geqq 1)$, *such that* $a_0(x) \not\equiv 0$, $a_n(x) \not\equiv 0$ $(|a_{ij}(x)| \not\equiv 0)$ *admits of n linearly independent formal series solutions with elements of the types* (6), (6'), (6'').

§ 12. A Theorem on Reducibility.

Let $s_1(x), \ldots s_n(x)$ be a set of n linearly independent formal solutions of an equation (4). It is apparent that $s_1(x), \ldots s_n(x)$ are not determined at least up to a constant multiple of $e^{2l\pi\sqrt{-1}x}$ where l is an integer, and that if any sum such as

$$\bar{s}_1(x) = c_1 e^{2\pi l_1\sqrt{-1}x} s_1(x) + \cdots + c_n e^{2\pi l_n\sqrt{-1}x} s_n(x)$$

is of the type (6), (6'), or (6'') the set $s_1(x), \ldots s_n(x)$ admits of still further modification. We shall refer to the complete collection $\bar{s}(x)$ so obtained as the family of formal solutions. Evidently it has the further property that if $\bar{s}(x)$ is in the family of formal solutions, so also is every determination of $\bar{s}(x)$ for the given basic integer p. Any subset of formal series of types (6), (6'), (6'') with this double property will be termed a 'natural family' of formal series, and the number of linearly independent elements in the family will be designated as its 'order'.

We propose to prove the following

Theorem: *To any decomposition of $L(y)$ of order n into irreducible symbolic factors $L_1, L_2, \ldots L_k$ so that*

$$L \equiv L_k L_{k-1} \ldots L_1,$$

there corresponds a sequence of natural families of formal solutions $F_1, \ldots F_k$ each containing the preceding as a sub-family, but such that there exists no further intermediate natural families, and such that the general solution of $L_1(y)=0$ is furnished by F_1, of $L_2(L_1(y))=0$ by F_2, etc.

Conversely to a set of families of formal solutions $F_1, F_2, \ldots F_k$, each containing the preceding, but in such wise that there exist no further intermediate families, there is a corresponding irreducible factorization which is essentially unique.

Let us establish the first part of this theorem.

Evidently $L_1(y)$, $L_2L_1(y)=0, \ldots L(y)=0$ define a sequence of expanding families $F_1, F_2, \ldots F_k$. But if there exists any intermediate family F^* between F_{i-1} and F_i for instance, the equation $L^*(y)=0$ admits $L_{i-1}\ldots L_1(y)=0$ as a symbolic factor. In fact we can write

$$L^*(y) \equiv ML_{i-1} \ldots L_1(y) + Q(y),$$

where Q is of lower order than $L_{i-1}\ldots L_1(y)$. But if we substitute any member y of F_{i-1} for y in this identity we conclude $Q(y)=0$, so that Q must vanish identically. Similarly we can prove

$$L_i L_{i-1} \ldots L_1(y) \equiv NL^*(y).$$

Therefore we have

$$\begin{aligned} L(y) &\equiv L_k L_{k-1} \ldots L_{i+1} NL^*(y) \\ &\equiv L_k L_{k-1} \ldots L_{i+1} NML_{i-1} \ldots L_1(y), \end{aligned}$$

and it follows at once that

$$L_k L_{k-1} \ldots L_{i+1}(NM-L_i)L_{i-1} \ldots L_1(y) \equiv 0.$$

Hence we prove successively that the factors $L_k, L_{k-1}, \ldots L_{i+1}$ on the left may be removed so that

$$(NM-L_i)L_{i-1} \ldots L_1(y)=0.$$

If $NM - L_i$ were not identically zero, we could continue this process and finally conclude $y \equiv 0$ which is absurd. Consequently the factor L_i must be identical with the product NM and so not irreducible, contrary to our hypothesis.

This establishes the first part of the theorem.

To prove the second part of the theorem we set up the equation $L_1(y) = 0$ with the family F_1 as its general formal solution. Obviously L_1 is determined up to a multiplication power series in $x^{-\frac{1}{p}}$. Likewise we let $P_i(y) = 0$ be the equation with general solution F_i for $i = 2, \ldots k$. We can then readily establish that L_1 is a factor of P_2, P_2 of P_3, and so on, with $P_n \equiv L$:

$$P_2 \equiv L_2 L_1, \; P_3 \equiv L_3 P_2, \ldots L \equiv L_k P_{k-1}.$$

It follows that L is expressible as a symbolic product

$$L \equiv L_k P_{k-1} \equiv L_k L_{k-1} P_{k-2} \equiv \cdots \equiv L_k L_{k-1} \cdots L_1.$$

To complete the proof that to such a sequence of natural families correspond irreducible factors $L_1, L_2, \ldots L_k$ it is only necessary to observe that if any L_i is not reducible we obtain further intermediate families, contrary to hypothesis.

This theorem may be regarded as fundamental for all questions concerning reducibility of a single equation (4) with a given basic integer p. As an illustration of this fact we cite the following two special results:

A necessary and sufficient condition that $L(y) = 0$ of order n be completely reducible (i. e. expressible as a product of n symbolic factors) is that there exists no anormal series solution (6′) with $l > 1$. Every irreducible factorization will then involve n factors, each of the first order.

A necessary and sufficient condition that $L(y) = 0$ of order $n > 1$ be irreducible is that the complete set of formal solutions consists of a single solution (6′) with $l = n$, and its various determinations.

The necessity of the condition of the first statement is obvious, since in the expanding set of families $F_1, \ldots F_k$, we necessarily add all of the determinations of series of anormal type when we add a single one of course. The sufficiency may be seen as follows. At the i-th step the family F_{i-1} is expanded by the addition of at least one new member of normal type. This may be expressed in the form (see (6″))

$$s(x)(\log x)^m + t(x)(\log x)^{m-1} + \cdots + w(x)$$

with $m \geqq 0$. If $m = 0$ (i. e. the solution does not involve logarithms), it is evident that the addition of this new member leads to a larger natural family of order one greater, which must then be F_i. But if $m > 0$, the solution

$$m s(x)(\log x)^{m-1} + (m-1) t(x)(\log x)^{m-2} + \cdots + v(x)$$

must be in F_{i-1}; otherwise it would suffice to add this last solution only with its various determinations to form a smaller F_i. But if these are in F_{i-1}, since $l = 1$ we obtain a natural family of order one greater only, which must of course be F_i. Thus the irreducible factors will always be of the first order if there are no anormal series.

The truth of the second statement is obvious. In fact the family of all formal solutions not containing logarithms yields a factor of $L(y) = 0$ unless there are no solutions containing logarithms. Thus for the completely irreducible case we are restricted to the types (6), (6′) of series. But each such type and its various determinations yields a natural family and a corresponding factor. We conclude that there can exist one single series solution of anormal type with $l = n$. But in this case $L(y) = 0$ is evidently irreducible.

§ 13. Equivalence, Normal Forms and Invariants.

There is a second method of decomposition of natural families of series solutions which is of importance in dealing with the notion of equivalence, namely that in which it is sought to break up the given natural family into a maximum number of entirely distinct natural families. Inasmuch as the formal significance of this process is most apparent when we use a linear system (1) instead of a single equation (4), we introduce the matrix notation at this point.

In matrix notation we may write the system (1) in the form

$$Y(x+1) = A(x)\, Y(x) \qquad (|A(x)| \not\equiv 0),$$

where $Y = S(x)$ will be taken to denote the matrix of formal solutions, one in each column of $S(x)$, which are such that $|S(x)| \not\equiv 0$.

Now suppose that we have a second equation

$$\overline{Y}(x+1) = \overline{A}(x)\, \overline{Y}(x),$$

which is obtained from the first by a linear transformation

$$Y(x) = B(x)\,\bar{Y}(x),$$

so that

$$\bar{A}(x) = B^{-1}(x+1)\,A(x)\,B(x) \qquad (|\bar{A}(x)| \not\equiv 0).$$

If the elements of $B(x)$ are power series in $x^{-\frac{1}{p}}$ (p, the given basic integer) we shall say that the equations in Y and $\bar{Y}$ are 'properly equivalent'. But if the equations in Y and $\bar{Y}$ are only 'properly equivalent' after p has been replaced by a suitable multiple of p, we shall merely speak of the equations as 'improperly equivalent'. We propose to deal here only with the latter type of equivalence. Evidently the relation of equivalence in either sense is transitive.

It is clear that equivalent systems possess formal solutions which are of essentially similar type.

For the purpose before us we group the formal solutions $s_i(x)$ according to the exponential factors

$$x^{\frac{\mu x}{p}} e^{P(x)}$$

which occur in the series for each particular solution. Evidently all the solutions then fall into a set of entirely distinct families, provided that we regard two factors as essentially the same if the constant μ is the same in each and the two polynomials $P(x)$ differ at most by a term $2l\pi\sqrt{-1}\,x$ (l, an integer).

Aside from this factor, we have in any particular set of this kind a linear family of precisely the same type as presents itself in a linear differential system with a regular singular point at $x=\infty$.

On this account, we can list the formal solutions in entirely distinct families, each of which has the following form

$$\begin{array}{l} s_{i1}(x), \ldots s_{il}(x), \\ s_{i1}(x) m \log x + t_{i1}(x), \ldots s_{il}(x) m \log x + t_{il}(x), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_{i1}(\log x)^m + \cdots + w_{i1}(x), \ldots s_{il}(x)(\log x)^m + \cdots + w_{il}(x), \end{array}$$

in which $s_{i1}(x), \ldots s_{il}(x)$, $t_{i1}(x) \ldots t_{il}(x), \ldots$ all involve precisely the same exponential factor and $m \geqq 0$ indicates the degree to which logarithms enter. Furthermore for all of these elements the constant r may be regarded as the same.

Suppose that there are $k \geqq 1$ such families with complete exponential factors

$$x^{\frac{\mu x}{p}} e^{P(x)} x^r,$$

which we designate by $e_1(x), \ldots e_k(x)$ respectively. If we denote by $\bar{s}_{ij}, \bar{t}_{ij}, \ldots \bar{w}_{ij}$ the parts of the respective series $s_{ij}, t_{ij}, \ldots w_{ij}$ in such a family aside from the corresponding exponential factors, say $e_j(x)$, it is clear that the matrix of formal solutions can be expressed as a product matrix of the form

$$\left\| \begin{matrix} \bar{s}_{11} & \bar{t}_{11} & \ldots & \bar{s}_{12} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{s}_{n1} & \bar{t}_{n1} & \ldots & \bar{s}_{n2} & \ldots \end{matrix} \right\| \left\| \begin{matrix} e_1(x) & e_1(x) m \log x & \ldots & e_1(x)(\log x)^{m_1} & \ldots \\ 0 & e_1(x) & \ldots & e_1(x)(\log x)^{m_1-1} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & & e_k(x) \end{matrix} \right\|$$

which may be written $B(x)E(x)$, where $B(x)$ is made up of elements such that the linear transformation $Y(x) = B(x)\,\breve{Y}(x)$ takes the given equation into an (improperly) equivalent equation. But this equation in $\bar{Y}$ has $E(x)$ as matrix solution so that we have

$$A(x) = E(x+1)E^{-1}(x).$$

Hence we are led to the following result:

Theorem. *An arbitrary system* (1) *of this type is (improperly) equivalent to a normal system*

$$Y(x+1) = \bar{A}(x)\,Y(x) \qquad (\bar{A}(x) = E(x+1)E^{-1}(x)),$$

in which the matrix $E(x)$ has elements 0 *except for square blocks along the diagonal of the form*

$$\begin{matrix} e(x) & e(x) m \log x & \ldots & e(x)(\log x)^m \\ 0 & e(x) & \ldots & e(x)(\log x)^{m-1} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & e(x) \end{matrix}$$

where

$$e(x) = x^{\frac{\mu x}{p}} e^{P(x)} x^r$$

$$\left(P(x) = \gamma x + \delta x^{\frac{lp-1}{lp}} + \cdots \eta x^{\frac{1}{lp}}\right)$$

and $l\mu$ is an integer. Furthermore with any $e(x)$ and m there are also $l-1$ blocks in the other $l-1$ determinations of $e(x)$ and with the same m in case $l>1$.

Evidently the functions $e(x)$ and the integers m are to be regarded as invariantive characteristics. In other words in order that two systems (1) be (improperly) equivalent it is necessary and sufficient that they possess the same set of functions $e(x)$ and corresponding integers m, each with the same multiplicity.

This theorem enables us to deal with most questions arising out of the notion of equivalence. We cite merely the following two special results, which are readily proven:

There exists a set of n distinct natural families of solutions if and only if the formal series are of normal type and no logarithms enter. In this case the normal form is as follows:

$$y_i(x+1) = \left(1+\frac{1}{x}\right)^{\frac{\mu_i x}{p}+r_i} x^{\frac{\mu_i}{p}} e^{P_i(x+1)-P_i(x)} y_i(x) \quad (i=1, \ldots n),$$

in which $\mu_1, \ldots \mu_n$ are integers.

No two distinct natural families of solutions can be found when and only when there is a single exponential factor $e(x)$ which is not repeated, together with its various determinations.

§ 14. Difference and Differential Systems.

It has been stated above that in certain cases difference and differential systems are closely related from a formal point of view. If we bear in mind that the formal solutions of linear differential systems

$$Y'(x) = B(x)\, Y(x),$$

in which of course the elements of $B(x)$ are formal power series in descending powers of $x^{\frac{1}{p}}$, are the same in type as those of linear difference systems (1) for which $\mu=0$, we conclude that the matrix $S(x)$ of formal solutions of a linear difference system is also the solution of a linear differential system if every μ_i is 0. If we fix the arbitrary exponential factors $e^{2l\pi\sqrt{-1}x}$ in these formal solutions, this corresponding linear differential system is uniquely determined, since we have $B(x) = S'(x)S^{-1}(x)$.

The fact that the polynomials $P_i(x)$ which enter are of degree not exceeding 1 in x indicates that the corresponding differential system has formal solutions which are of rank at most 1.

Thus we obtain the following theorem:

Theorem: *A necessary and sufficient condition that a linear difference system* (1)

$$Y(x+1) = A(x)\,Y(x),$$

in which the elements of $A(x)$ *are descending formal power series in* $x^{\frac{1}{p}}$, *be formally compatible with some linear differential system*

$$Y'(x) = B(x)\,Y(x)$$

(i.e. that both admit a common formal matrix solution $S(x)$*) is that the values of* μ *in the formal solutions of the difference system are all* 0. *Here the elements of* $B(x)$ *are also power series in descending powers of* $x^{\frac{1}{p}}$, *and the following formal identity holds:*

$$A'(x) = B(x+1)A(x) - A(x)B(x).$$

Conversely, if a linear differential system of this type admits of formal solutions which are all of rank at most 1, *it will be compatible with such a linear difference system, and the same formal identity will hold.*

The formal identity is obtained by noting the relations

$$S'(x+1) = B(x+1)S(x+1) = B(x+1)A(x)S(x)$$

and at the same time

$$\begin{aligned} S'(x+1) &= A(x)S'(x) + A'(x)S(x) \\ &= [A(x)B(x) + A'(x)]\,S(x). \end{aligned}$$

As we shall show in our second paper on irregular difference equations, an actual linear differential system of this type is not only formally but also actually compatible with a difference system, so that the solutions of the differential system are also solutions of the difference system.

Hitherto we have taken the given difference system in a form involving x and $x+1$, rather than x and $x+d$. The question arises as to when two linear difference systems with distinct values of d, say d_1 and d_2, can be compatible in the formal sense specified above. Since the formal series in such equations are at once seen to involve exponential factors $x^{\frac{\mu x}{dp}}$, it is clear that at least when the ratio d_1/d_2 is not a rational number, every value of μ must be 0; note that μ itself is either integral or rational. Hence in this case the two difference systems are both formally compatible with a linear differential system. On the other hand if the ratio d_1/d_2 is rational, and if d denotes the greatest common submultiple of d_1 and d_2, so that $d_1 = k_1 d$, $d_2 = k_2 d$ where k_1 and k_2 are relatively prime integers, we may find integers l_1, l_2 such that $k_1 l_1 + k_2 l_2 = 1$. Consequently we may express $Y(x+d)$ as $A(x)\,Y(x)$ by repeated use of the given difference equations. Thus we are led to the following results:

Two linear difference systems

$$Y(x+d_1) = A_1(x)\,Y(x), \quad Y(x+d_2) = A_2(x)\,Y(x)$$

are formally compatible in the case when d_1/d_2 is not a rational number if and only if they are both compatible with a linear differential system

$$Y'(x) = B(x)\,Y(x)$$

of the type referred to above. Here the following formal identity holds:

$$A_1(x+d_2)A_2(x) = A_2(x+d_1)A_1(x).$$

In case d_1/d_2 is rational and d is the greatest common submultiple of d_1 and d_2, these systems are formally compatible if and only if we have

$$A_1(x) \equiv A(x+d_1-d)A(x+d_1-2d)\cdots A(x),$$
$$A_2(x) \equiv A(x+d_2-d)A(x+d_2-2d)\cdots A(x).$$

We defer to our second paper all discussion as to the relation between the formal situations considered in the last three paragraphs, and the corresponding analytic situations.