ON APPELL'S DECOMPOSITION OF A DOUBLY PERIODIC FUNCTION OF THE THIRD KIND.

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1. Introduction.

Hermite[1] has defined a function $\varphi(z)$ to be doubly periodic of the third kind if it is meromorphic and satisfies two periodicity relations of the form

$$\text{(1)} \qquad \begin{cases} \varphi(z + 2\omega) = e^{az+b} \cdot \varphi(z), \\ \varphi(z + 2\omega') = e^{cz+d} \cdot \varphi(z), \end{cases}$$

where $a, b, c, d, \omega, \omega'$ are constants and ω'/ω is a complex number $\alpha + i\beta$, $\beta \neq 0$. It can be shown that the properties of $\varphi(z)$ are deducible from those of a suitably defined function $F(z)$ which is likewise meromorphic and satisfies the simpler periodicity relations

$$\text{(2)} \qquad \begin{cases} F(z + \pi) \quad = F(z), \\ F(z + \pi\tau) = e^{-2miz} F(z), \quad i = \sqrt{-1}, \quad m \neq 0, \end{cases}$$

where τ is a complex number with non-vanishing imaginary part, and m an integer, positive or negative. It can be proved that m is the excess of the number of zeros over the number of poles of the function in a period cell.

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[1] Hermite, Comptes Rendus, 1861, 1862; Journal für die reine und angewandte Mathematik; Band 100; Œuvres, tome II, p. 109; tome IV. p. 223.

In a series of memoirs, Appell[1] has studied the problem of expanding functions of the above type in trigonometric series; his fundamental contribution in this connection is the discovery of a certain function $\chi_m(x, y)$ of two complex variables which in his theory plays a rôle analogous to that of the Zeta function in Hermite's decomposition of an elliptic function into simple elements.

In what follows we shall prove the results of Appell for the case where $F(z)$ has more poles that zeros in a period cell (i. e. $m < 0$) *without assuming á priori the existence and properties of the function* $\chi_m(x, y)$. Indeed, we shall be led, in a very natural manner, to the consideration of this function as the fundamental element of decomposition of the functions under discussion. The method adopted for the proof of Appell's theorem was suggested by that used in the proof of the well known[2] theorem relative to the expansion of a class of singly periodic functions in a series of cotangents.

2. Case where the Poles of $F(z)$ are Simple.

Let us suppose that the function $F(z)$ has only simple poles and that it satisfies the periodicity relations (2), with $\tau = \alpha + i\beta, \beta > 0$, and $m < 0$, say $m = -\mu$. Let the poles of $F(z)$ in the fundamental period cell have affixes $a_1, a_2, \ldots, a_p$, the corresponding residues being $R_1, R_2, R_3, \ldots R_p$. We may assume that none of these singularities lie on the boundaries of the cell.

From the second of equations (2) it follows that

$$(3) \qquad \begin{cases} F(z + n\pi\tau) = e^{2\mu n i z} \cdot q^{\mu n(n-1)} \cdot F(z), \\ \qquad q = e^{\pi i \tau}, \quad |q| < 1, \end{cases}$$

where n is any positive or negative integer. Consider a parallelogram $ABCD$ composed of $(n+1)$ cells above the real axis and n below. The poles of the function $F(z)$ in this parallelogram are the points with affixes

$$z = a_k + r\pi\tau, \quad \begin{pmatrix} k = 1, 2, 3, \ldots, p. \\ r = 0, \pm 1, \pm 2, \ldots, \pm n, \end{pmatrix},$$

the corresponding residues being

[1] Appell, Annales Scientifiques de l'École Normale Supérieure, Série III, tomes 1, 2, 3, 5; Acta Mathématica, 42, 1920; Mémorial des Sciences Mathématiques, fascicule XXXVI, Paris, 1929.

[2] Whittaker and Watson, Modern Analysis, Fourth Edition, p. 139.

(4) $$R_{(r,k)} = e^{2\mu r i a_k} \cdot q^{\mu r(r-1)} \cdot R_k.$$

Now consider the function

(5) $$\Phi(t) \equiv F(t) \operatorname{ctg}(t-z);$$

the poles of which, in $ABCD$, are $t=z$ and $t=a_k+r\pi\tau$, the corresponding residues being $F(z)$ and $R_{(r,k)} \operatorname{ctg}(a_k+r\pi\tau-z)$. If we apply Cauchy's theorem to the function $\Phi(t)$, and take into account its periodicity, it is seen that

(6) $$I_1+I_2 = F(z) + \sum_{(k,r)} R_{(r,k)} \operatorname{ctg}(a_k+r\pi\tau-z),$$

$$\begin{pmatrix} k = 1, 2, 3, \ldots, p \\ r = 0, \pm 1, \pm 2, \ldots \pm n, \end{pmatrix}$$

where,

(7) $$I_1 = \frac{1}{2\pi i} \cdot \int_{(AB)} F(t) \operatorname{ctg}(t-z)\, dt,$$

and I_2 is the corresponding integral along CD.

We shall now show that I_1 and I_2 tend to zero as n tends to infinity. Thus, we may write, x being real,

(8) $$\pi I_1 = q^{\mu n(n+1)} \cdot \int_0^\pi \frac{e^{i(x-z)} \cdot q^{-n} + e^{-i(x-z)} \cdot q^n}{e^{i(x-z)} \cdot q^{-n} - e^{-i(x-z)} \cdot q^n} \cdot e^{2\mu n i x} \cdot F(x)\, dx.$$

Since, by hypothesis, $F(z)$ is free of poles in the interval from 0 to π, we may write

(9) $$|I_1| < M \left| q^{\mu n(n+1)} \right| \left| \frac{1 + |e^{2iz}|\,|q^{2n}|}{1 - |e^{2iz}|\,|q^{2n}|} \right|,$$

M being some constant, selected so that $|F(x)| \leq M$ for all x in the range of integration. Since $|q| < 1$ and μ and $n(n+1)$ are positive it follows that

(10) $$\operatorname*{Limit}_{n\to\infty} |I_1| = 0.$$

In a similar way, we may show that $\operatorname*{Limit}_{n\to\infty} |I_2| = 0$. Hence we deduce from (4) and (6) that

(11) $$F(z) = \sum_{k=1}^{p} \sum_{n=-\infty}^{\infty} R_k e^{2\mu n i a_k} \cdot q^{\mu n(n-1)} \operatorname{ctg}(z - a_k - n\pi\tau),$$

which, using Appell's notation, may be written in the form

(12) $$F(z) = \sum_{k=1}^{p} R_k \chi_\mu(z, a_k),$$

where,

(13) $$\chi_\mu(z, a_k) = \sum_{n=-\infty}^{\infty} e^{2\mu n i a_k} q^{\mu n(n-1)} \operatorname{ctg}(z - a_k - n\pi\tau).$$

This is Appell's result for the case where the only singularities of $F(z)$ are simple poles. We have, therefore, arrived at his theorem and at his fundamental function in a perfectly straightforward manner. The convergence of the series for $\chi_\mu(z, a_k)$ for all values of z, with the exception of those values which are congruent to a_k modulus $r\pi + s\pi\tau$ (r, s being integers) may be easily established; we shall not, however, stop for a discussion of this point.

3. Case where the Poles of $F(z)$ are Multiple.

Now consider the case in which some of the poles of $F(z)$ are multiple. For concreteness let us suppose that there is only one pole in a period cell and that it is of order k; in the fundamental period cell, let this pole have the affix a, so that the principal part of the Laurent expansion of $F(t)$ relative to it has the form

(14) $$\frac{A_k}{(t-a)^k} + \frac{A_{k-1}}{(t-a)^{k-1}} + \cdots + \frac{A_2}{(t-a)^2} + \frac{A_1}{t-a}.$$

In the parallelogram $ABCD$, already considered, the points $t = a + r\pi\tau$, $r = 0, \pm 1, \pm 2, \ldots, \pm n$, are likewise poles of order k. The residues of $F(t)$ relative to these poles may be calculated as follows:

Let

(15) $$G_r(t) \equiv F(t + r\pi\tau) = q^{\mu r(r-1)} \lambda_r(t) F(t), \quad \lambda_r(t) = e^{2\mu r i t};$$

then $t = a$ is a pole of order k of $G(t)$. In the neighborhood of $t = a$ we may write

$$G_r(t) = q^{\mu r(r-1)} \sum_{\substack{j=1 \\ (j \geqq s)}}^{k} \sum_{s=1}^{k-1} \frac{A_j \lambda_r^{(s-1)}(a)}{(s-1)!} \cdot \frac{1}{(t-a)^{j-s+1}} + P(t-a), \tag{16}$$

where $P(t-a)$ is a power series in $(t-a)$.

Next, let

$$\psi(t) = \operatorname{ctg}(t-z); \quad a_r \equiv a + r\pi\tau; \tag{17}$$

in the neighborhood of $t = a_r$, we have

$$\psi(t) = \sum_{\sigma=1}^{\infty} \frac{\psi^{(\sigma-1)}(a_r)}{(\sigma-1)!} (t-a_r)^{\sigma-1}. \tag{18}$$

Hence, relative to $t = a_r$, the function $\Phi(t) \equiv G_r(t)\psi(t)$ has a principal part which is

$$q^{\mu r(r-1)} \cdot \sum_{\substack{j=1 \\ (j \geqq s)}}^{k} \sum_{s=1}^{k-1} \sum_{\sigma=1}^{k-1} \frac{A_j \lambda_r^{(s-1)}(a)\, \psi^{(\sigma-1)}(a_r)}{(s-1)!\,(\sigma-1)!} \cdot \frac{1}{(t-a)^{j-s-\sigma+2}}. \tag{19}$$

It follows that the residue of $\Phi(t)$ relative to $t = a_r \equiv a + r\pi\tau$ has the value

$$q^{\mu r(r-1)} \cdot \sum_{\substack{j=1 \\ (j \geqq s)}}^{k} \sum_{s=1}^{k-1} \frac{A_j \lambda_r^{(s-1)}(a)\, \psi^{(j-s)}(a + r\pi\tau)}{(s-1)!\,(j-s)!}, \tag{20}$$

which is obtained from (19) on setting $\sigma = j - s + 1$.

Recalling Leibniz's theorem for the nth derivative of the product of two functions, the expression (20) may be written in the form

$$q^{\mu r(r-1)} \cdot \sum_{j=1}^{k} \frac{A_j}{(j-1)!} D_a^{(j-1)} [\lambda_r(a)\, \psi(a + r\pi\tau)]. \tag{21}$$

It follows, as in § 2, that on applying Cauchy's theorem to the function $\Phi(t)$ and letting n tend to infinity, we get

$$F(z) = \sum_{j=1}^{k} \frac{A_j}{(j-1)!} \left[\sum_{n=-\infty}^{\infty} q^{\mu n(n-1)} \cdot D_a^{(j-1)} [e^{2\mu n i a} \operatorname{ctg}(z - a - n\pi\tau)] \right], \tag{22}$$

so that from (13) we finally have

$$F(z) = \sum_{j=1}^{k} \frac{A_j}{(j-1)!} D_a^{(j-1)} \chi_\mu(z, a), \tag{23}$$

which is Appell's formula of decomposition for the case under discussion.

4. Combining the results of §§ 2, 3 it is seen that in case some of the poles a_j of $F(z)$ in § 2 are of order k_j, the expansion of $F(z)$ may be obtained by replacing $R_j \chi_\mu(z, a_j)$ in (12) by the sums $\sum_{r=1}^{k_j} \frac{A_r}{(r-1)!} D_{a_j}^{(r-1)} \chi_\mu(z, a_j)$.

This completes the discussion for the case where m in (2) is negative. It may be noted that the method of the present article fails to yield a decomposition when m is positive, for then we encounter certain indetermination when n is allowed to approach infinity. However, Appell has also solved this question with the aid of the function $\chi_\mu(a, z)$, obtained from (13) by interchanging the rôles of z and a.