ANALYSIS OF CONDITIONS OF GENERALISED ALMOST PERIODICITY.

BY

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In the paper »Almost Periodicity and General Trigonometric Series» by A. S. Besicovitch and H. Bohr[1], devoted to the study of various types of almost periodicity, the type of B-almost periodicity was considered which included all the other types there studied.

We shall quote the definition of this type. But first we give some auxiliary definitions.

We call a set E of real numbers a relatively dense (r. d.) set if there exists a number $l > 0$ such that any interval of length l includes at least one number of the set. Such a number l is called an inclusion interval of the set.

We say that a set E is satisfactorily uniform if there exists a number $b > 0$ such that the maximum value $\nu(b)$ of the number of numbers of E included in an interval of length b is less than twice the minimum value $\mu(b)$ of the same number, i. e., if

$$\nu(b) < 2\mu(b). \tag{1}$$

Obviously we may always assume b an integer.

Definition of B a.p. functions. We say that a function $f(t)$ (real or complex) of a real variable t is B-almost periodic (B a.p.) if corresponding to any positive number ε, exists a satisfactorily uniform set of numbers

$$\ldots \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 \ldots$$

[1] Acta mathematica Vol. 57.

such that

(2) $$\overline{M}_x \overline{M}_i \left\{ \frac{1}{c} \int_x^{x+c} |f(t+\tau_i) - f(t)|\, dt \right\} < \varepsilon \quad \text{for all } c > 0$$

and

(3) $$\overline{M}_x \{ |f(x+\tau_i) - f(x)| \} < \varepsilon \quad \text{for all } i$$

(M_i, $\overline{M}_i$, M_x, $\overline{M}_x$ denote respectively the mean value or the upper mean value with respect to all integral values of i, or all real values of x.)

Denote by A the class of all exponential polynomials $s(x) = \Sigma a e^{i\lambda x}$, where all a are arbitrary real or complex numbers, and all λ arbitrary real numbers. We say that a function $f(x)$ is a B-limit function of the class A, if given any $\varepsilon > 0$ there exists a function $s(x)$ such that

$$\overline{M}_x \{ |f(x) - s(x)| \} < \varepsilon .$$

The class of all B-limit functions of the class A is called the B-closure of the class A and is denoted $C_B(A)$. The main result of the quoted paper, concerning $B\,a.p.$ functions is that the class of $B\,a.p.$ functions is identical with $C_B(A)$. It was considered there whether the conditions (2) could be replaced by the following simpler one:

(4) $$\overline{M}_x \overline{M}_i \{ |f(x+\tau_i) - f(x)| \} < \varepsilon .$$

But it was proved that the new type of almost periodic functions defined in this way ($\overline{B}\,a.p.$ functions) is different from the type of $B\,a.p.$ functions. In fact the class of all $B\,a.p.$ functions includes the class of $\overline{B}\,a.p.$ functions and is wider than the latter.

We shall now introduce a new definition:

Definition of B^* a.p. functions. We say that an integrable (L) *function* $f(t)$ *(real or complex) of a real variable t is B^*-almost periodic* (B^* *a.p.*) *if corresponding to any positive ε there exists a satisfactorily uniform set of numbers*

$$\ldots \tau_{-2} < \tau_{-1} < \tau_0 = 0 < \tau_1 < \tau_2 \ldots$$

such that

(5) $$\overline{M}_x \overline{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\, dt < \varepsilon .$$

Thus the condition (2) is replaced by its particular case when $c = 1$, and the condition (3) is dropped. Nevertheless it will be shown that the type of B^*-almost periodicity is identical with that of B-almost periodicity. Thus the definition of B^*-almost periodicity does not introduce a new type of almost periodicity, but gives a new and simplified definition of $B\,a.p.$ functions.

In connection with the quoted result on the type of $\bar{B}\,a.p.$ functions, in whose definition »the smoothing integration» of (2) is completely eliminated, it may be said that the new definition of $B\,a.p.$ functions reaches the extreme bound of a possible simplification.

Obviously any $B\,a.p.$ function is a $B^*\,a.p.$ function. In order to prove the converse we shall prove that any $B^*\,a.p.$ function belongs to $C_B(A)$.

We shall first prove a number of lemmas.

Lemma 1. *For any satisfactorily uniform set*

$$\ldots \tau_{-2} < \tau_{-1} < \tau_0 = 0 < \tau_1 < \tau_2 \ldots$$

and for any non negative function $\Phi(x)$ we have

$$\frac{1}{4}\,\overline{M}_x\{\Phi(x)\} \leqq \overline{M}_i \frac{1}{b}\int_0^b \Phi(x+\tau_i)\,dx < 4\,\overline{M}_x\{\Phi(x)\} \tag{6}$$

where

$$\nu(b) < 2\,\mu(b).$$

Proof. Denoting

$$A(j_0) = \frac{1}{2j_0+1}\sum_{i=-j_0}^{i=+j_0}\frac{1}{c}\int_0^c \Phi(x+\tau_i)\,dx$$

we have

$$A(j_0) = \frac{1}{(2j_0+1)c}\int_{\tau_{-j_0}}^{\tau_{j_0}+c} \lambda(x)\,\Phi(x)\,dx \tag{7}$$

where $\lambda(x)$ denotes the number of intervals $(\tau_i,\ \tau_i + c)$ $(-j_0 \leqq i \leqq +j_0)$ including the point x. We have

$$\begin{cases} 0 < \lambda(x) \leqq \nu(c) \text{ in the intervals } (\tau_{-j_0},\ \tau_{-j_0}+c) \text{ and } (\tau_{j_0},\ \tau_{j_0}+c) \\ \mu(c) \leqq \lambda(x) \leqq \nu(c) \text{ in the interval } (\tau_{-j_0}+c,\ \tau_{j_0}) \end{cases}$$

so that we conclude from (7)

$$(8) \qquad \frac{\mu(c)}{(2j_0+1)c}\int\limits_{\tau_{-j_0}+c}^{\tau_{j_0}} \Phi(x)\,dx \leqq A(j_0) \leqq \frac{\nu(c)}{(2j_0+1)c}\int\limits_{\tau_{-j_0}}^{\tau_{j_0}+c} \Phi(x)\,dx.$$

We shall consider (8) for large values of j_0. Denote

$$(9) \qquad \begin{aligned} &\min(-\tau_{-j_0}-c,\ \tau_{j_0}) = T_1, \\ &\max(\tau_{j_0}+c,\ -\tau_{-j_0}) = T_2. \end{aligned}$$

We conclude from the satisfactory uniformity of the set of τ_i that for large values of j_0

$$\frac{1}{2} < \frac{-\tau_{-j_0}}{\tau_{j_0}} < 2$$

whence by the definition of numbers T_1, T_2 we have also for large values of j_0,

$$(10) \qquad \frac{1}{2} < \frac{T_1}{T_2} < 2.$$

Observe now that

$$(11) \qquad \overline{\lim_{j_0\to\infty}}\, \frac{1}{2T_1}\int\limits_{-T_1}^{T_1} \Phi(x)\,dx = \overline{\lim_{j_0\to\infty}}\, \frac{1}{2T_2}\int\limits_{-T_2}^{T_2} \Phi(x)\,dx = \overline{M}\{\Phi(x)\}.$$

By (8), (9)

$$(12) \qquad \frac{\mu(c)\,2T_1}{c(2j_0+1)}\,\frac{1}{2T_1}\int\limits_{-T_1}^{T_1} \Phi(x)\,dx \leqq A(j_0) \leqq \frac{\nu(c)\,2T_2}{c(2j_0+1)}\,\frac{1}{2T_2}\int\limits_{-T_2}^{T_2} \Phi(x)\,dx.$$

Denoting by $[a]$ the largest integer $\leqq a$ we write

$$(13) \qquad \frac{\mu(c)\,2T_1}{c} > \frac{1}{2}\,\frac{T_1}{T_2}\,\frac{\nu(c)\,2T_2}{c} \geqq \frac{1}{2}\,\frac{T_1}{T_2}\,\nu(c)\left[\frac{2T_2}{c}\right].$$

By (9)

$$(14) \qquad 2T_2 \geqq \tau_{j_0} - \tau_{-j_0} + c$$

whence

$$\frac{\mu(c)\,2\,T_1}{c} > \frac{1}{2}\frac{T_1}{T_2}\nu(c)\left[\frac{\tau_{j_0}-\tau_{-j_0}+c}{c}\right]. \tag{15}$$

From the definition of $\nu(c)$ we conclude that $\nu(c)\left[\frac{\tau_{j_0}-\tau_{-j_0}+c}{c}\right]$ is greater than or equal to the number of τ_i in any interval of length $\tau_{j_0}-\tau_{-j_0}$ and consequently in the interval $(\tau_{-j_0}, \tau_{j_0})$, i. e.

$$\nu(c)\left[\frac{\tau_{j_0}-\tau_{-j_0}+c}{c}\right] \geqq 2j_0+1 \tag{16}$$

whence by (15)

$$\frac{\mu(c)\,2\,T_1}{c} > \frac{1}{2}\frac{T_1}{T_2}(2j_0+1). \tag{17}$$

Similarly we write

$$\frac{\nu(c)\,2\,T_2}{c} < \frac{2\mu(c)\,2\,T_1}{c}\frac{T_2}{T_1}. \tag{18}$$

By (9)

$$2\,T_1 \leqq \tau_{j_0}-\tau_{-j_0}-c$$

whence

$$\frac{\nu(c)\,2\,T_2}{c} < \frac{2\mu(c)(\tau_{j_0}-\tau_{-j_0}-c)}{c}\frac{T_2}{T_1} < 2\mu(c)\left[\frac{\tau_{j_0}-\tau_{-j_0}}{c}\right]\frac{T_2}{T_1}.$$

Observing, as in (16),

$$\mu(c)\left[\frac{\tau_{j_0}-\tau_{-j_0}}{c}\right] \leqq 2j_0+1$$

we conclude

$$\frac{\nu(c)\,2\,T_2}{c} < 2(2j_0+1)\frac{T_2}{T_1}. \tag{19}$$

By (12), (17), (19)

$$\frac{1}{2}\frac{T_1}{T_2}\frac{1}{2\,T_1}\int_{-T_1}^{T_1}\Phi(x)\,dx < A(j_0) < 2\frac{T_2}{T_1}\frac{1}{2\,T_2}\int_{-T_2}^{T_2}\Phi(x)\,dx$$

whence by (10) we have for large values of j_0

$$(20) \qquad \frac{1}{4}\frac{1}{2T_1}\int\limits_{-T_1}^{+T_1}\Phi(x)dx < A(j_0) < 4\frac{1}{2T_2}\int\limits_{-T_2}^{+T_2}\Phi(x)dx.$$

Taking the upper limit of all terms of this inequality, as $j_0 \to \infty$, we conclude on account of (11)

$$\frac{1}{4}\overline{M}_x\{\Phi(x)\} \leqq \overline{M}_i\frac{1}{c}\int\limits_0^c \Phi(x+\tau_i)dx \leqq 4\overline{M}_x\{\Phi(x)\}$$

which proves the lemma.

Remark. Obviously the lemma holds also when τ_0 is different from zero.

Lemma 2. *If $f(t)$ is a B^* a.p. function then for any $\varepsilon > 0$ the set of all the values of τ satisfying the inequality*

$$(21) \qquad \overline{M}_t\{|f(t+\tau)-f(t)|\} < \varepsilon$$

is relatively dense.

Proof. The function $f(t)$ being B^* *a.p.* there exists a satisfactorily uniform set

$$\ldots \tau_{-2} < \tau_{-1} < \tau_0 = 0 < \tau_1 < \tau_2 \ldots$$

such that

$$(22) \qquad \overline{M}_x\overline{M}_i\int\limits_x^{x+1}|f(t+\tau_i)-f(t)|\,dt < \frac{\varepsilon}{584},$$

from which we immediately conclude that

$$(23) \qquad \overline{M}_x\overline{M}_i\frac{1}{c}\int\limits_x^{x+c}|f(t+\tau_i)-f(t)|\,dt < \frac{\varepsilon}{584}$$

for any integer $c > 0$.

On account of the satisfactory uniformity of the set of τ_i's we can choose an integer c such that

$$(24) \qquad \nu(c) < 2\mu(c).$$

From (23) it follows that the inequality

$$M_i \frac{1}{c} \int\limits_{x_0}^{x_0+c} |f(t+\tau_i) - f(t)|\, dt < \frac{\varepsilon}{584}$$

is satisfied for some real values of x_0. Assume that $x_0 = 0$ (for we can always come to this case by the change of variable $t = x_0 + t'$) so that

$$M_i \frac{1}{c} \int\limits_{0}^{c} |f(t+\tau_i) - f(t)|\, dt < \frac{\varepsilon}{584}. \tag{25}$$

We shall now prove another inequality which together with the above inequality will lead to the proof of the lemma.

We have by Lemma 1

$$M_j \frac{1}{c} \int\limits_{\tau_j - c/2}^{\tau_j + c/2} \left\{ M_i \frac{1}{c} \int\limits_{x}^{x+c} |f(t+\tau_i) - f(t)|\, dt \right\} dx$$

$$< 4\, M_x\, M_i \frac{1}{c} \int\limits_{x}^{x+c} |f(t+\tau_i) - f(t)|\, dt < \frac{4\varepsilon}{584}. \tag{26}$$

Hence by Fatou's theorem

$$M_j \overline{M}_i \frac{1}{c} \int\limits_{\tau_j - c/2}^{\tau_j + c/2} \left\{ \frac{1}{c} \int\limits_{x}^{x+c} |f(t+\tau_i) - f(t)|\, dt \right\} dx < \frac{4\varepsilon}{584}. \tag{27}$$

Observing now

$$\int\limits_{\tau_j - c/2}^{\tau_j + c/2} \left\{ \int\limits_{x}^{x+c} |f(t+\tau_i) - f(t)|\, dt \right\} dx = \int\limits_{\tau_j - c/2}^{\tau_j + 3c/2} \left\{ \int\limits_{x_1(t)}^{x_2(t)} |f(t+\tau_i) - f(t)|\, dx \right\} dt \tag{28}$$

$$\geqq \frac{c}{2} \int\limits_{\tau_j}^{\tau_j + c} |f(t+\tau_i) - f(t)|\, dt$$

we obtain

$$\overline{M}_j \overline{M}_i \frac{1}{c} \int_{\tau_j}^{\tau_j+c} |f(t + \tau_i) - f(t)| \, dt < \frac{8\varepsilon}{584}$$

or

(29)
$$\overline{M}_j \overline{M}_i \frac{1}{c} \int_0^c |f(t + \tau_j + \tau_i) - f(t + \tau_j)| \, dt < \frac{8\varepsilon}{584}.$$

From (25) and (29) we conclude that there exists an integer $I_0 > 0$ such that for all $I \geqq I_0$

(30)
$$\frac{1}{2I + 1} \sum_{-I \leqq k \leqq +I} \frac{1}{c} \int_0^c |f(t + \tau_k) - f(t)| \, dt < \frac{\varepsilon}{584}$$

(31)
$$\frac{1}{2I + 1} \sum_{-I \leqq k \leqq +I} \overline{M}_i \frac{1}{c} \int_0^c |f(t + \tau_k + \tau_i) - f(t + \tau_k)| \, dt < \frac{8\varepsilon}{584}.$$

It follows from (30) that the number of values of k in the interval $(-I, +I)$ satisfying the inequality

(32)
$$\frac{1}{c} \int_0^c |f(t + \tau_k) - f(t)| \, dt > \frac{36\varepsilon}{584}$$

is less than $\frac{2I + 1}{36}$.

Similarly the number of values of k in the same interval satisfying the inequality

(33)
$$M_i \frac{1}{c} \int_0^c |f(t + \tau_k + \tau_i) - f(t + \tau_k)| \, dt > \frac{36\varepsilon}{584}$$

is less than $\frac{8(2I + 1)}{36}$ and consequently the number of values of k for which one of the inequalities (32), (33) is satisfied is less than $\frac{1}{4}(2I + 1)$.

Thus the number n of values of k for which the inequalities

$$\frac{1}{c}\int_0^c |f(t+\tau_k)-f(t)|\,dt \leqq \frac{36\varepsilon}{584} \tag{34}$$

$$M_i \frac{1}{c}\int_0^c |f(t+\tau_k+\tau_i)-f(t+\tau_k)|\,dt \leqq \frac{36\varepsilon}{584} \tag{35}$$

are satisfied simultaneously, is greater than $\frac{3}{4}(2I+1)$, i. e.

$$n > \frac{3}{4}(2I+1). \tag{36}$$

For any such value of k we have on account of (25)

$$\overline{M}_i \frac{1}{c}\int_0^c |f(t+\tau_k+\tau_i)-f(t+\tau_i)|\,dt < \frac{73\varepsilon}{584} \tag{37}$$

and thus by Lemma 1

$$M_t\{|f(t+\tau_k)-f(t)|\} < \frac{292\varepsilon}{584} = \frac{\varepsilon}{2}.$$

Let k', k'' be two values of k satisfying (34), (35). Writing the above inequality for each of them we deduce

$$M_t\{|f(t+\tau_{k'}-\tau_{k''})-f(t)|\} < \varepsilon. \tag{38}$$

The lemma will be proved if we prove that the set of all numbers $\tau_{k'}-\tau_{k''}$ is relatively dense. We shall indeed prove that every interval of length c $(r-c/2,\ r+c/2)$ contains at least one of the numbers $\tau_{k'}-\tau_{k''}$. Assume the contrary. Let

$$k_1 < k_2 < \cdots < k_n$$

be those integers of $(-I,\ +I)$ which satisfy (34), (35).

If there are intervals of length greater than l between consecutive numbers of the set

$$\tau_{k_1},\quad \tau_{k_2},\ \ldots,\quad \tau_{k_n} \tag{39}$$

then we divide it (the set) by these intervals into groups of consecutive terms distant from one another not more than l. If there is no interval of length greater than l, then we consider the whole set (39) as one group.

To each group

$$\tau_{k_p}, \quad \tau_{k_{p+1}}, \quad \ldots, \quad \tau_{k_q}$$

corresponds the interval $(\tau_{k_p} + r - c/2,\ \tau_{k_q} + r + c/2)$ which does not contain any τ_k satisfying (34), (35).

The number of all the τ_i in this interval is greater than or equal to

$$\mu(c)\left[\frac{\tau_{k_q} - \tau_{k_p} + c}{c}\right] > \frac{1}{2}\nu(c)\left[\frac{\tau_{k_q} - \tau_{k_p} + c}{c}\right]$$

and in the interval $(\tau_{k_p},\ \tau_{k_q})$ is less than or equal to

$$\nu(c)\left[\frac{\tau_{k_q} - \tau_{k_p} + c}{c}\right]$$

and thus the first number is greater than the half of the second one.

The intervals $(\tau_{k_p} + r - c/2,\ \tau_{k_q} + r + c/2)$ corresponding to all the groups of the numbers of (39) do not overlap and thus the number of τ_i belonging to all these intervals is greater than $\frac{1}{2}n$. None of them being substituted for τ_k in the inequalities (34), (35) satisfies either of them. They all belong to the interval $(\tau_{-I} + r - c/2,\ \tau_{+I} + r + c/2)$. The number of those of them which do not belong to $(\tau_{-I},\ \tau_{+I})$ is less than or equal to $\nu(c)\left[\frac{|r| + 3c}{c}\right]$, and thus of those which do belong is greater than $\frac{1}{2}n - \nu(c)\left[\frac{|r| + 3c}{c}\right]$. Thus the number of all τ_i belonging to the interval $(\tau_{-I},\ \tau_{+I})$ is greater than

$$n + \frac{1}{2}n - \nu(c)\left[\frac{|r| + 3c}{c}\right]$$

so that we can write

$$2I + 1 > \frac{3}{2}n - \nu(c)\left[\frac{|r| + 3c}{c}\right].$$

and by (36)

$$2I+1>\frac{9}{8}(2I+1)-\nu(c)\left[\frac{|r|+3c}{c}\right]$$

for all $I \geqq I_0$, which obviously cannot be true. Thus the lemma is proved.

Lemma 3. *If a function $f(x)$ is B^* a.p. and if*

$$f_\delta(x)=\frac{1}{\delta}\int_x^{x+\delta} f(t)\,dt$$

then

$$\overline{M}_x\{|f(x)-f_\delta(x)|\}\to 0,\ \text{as}\ \delta\to 0.$$

Proof. Given any $\varepsilon>0$ there exists a satisfactorily uniform set of numbers τ_i such that

$$\overline{M}_x M_i \int_x^{x+1} |f(t+\tau_i)-f(t)|\,dt<\frac{\varepsilon}{20}.$$

Denoting, as before, by c a positive integer satisfying the inequality $\nu(c)<2\mu(c)$ we shall have

(40) $$M_x \overline{M}_i \frac{1}{2c}\int_x^{x+2c} |f(t+\tau_i)-f(t)|\,dt<\frac{\varepsilon}{20},$$

whence there exists an a such that

(41) $$\overline{M}_i \frac{1}{c}\int_a^{a+2c} |f(t+\tau_i)-f(t)|\,dt<\frac{\varepsilon}{10}.$$

Choose a positive $\delta<c$ such that

(42) $$\frac{1}{c}\int_a^{a+c} |f_\delta(t)-f(t)|\,dt<\frac{\varepsilon}{20}.$$

We write

$$\int_a^{a+c} |f_\delta(t+\tau_i) - f_\delta(t)|\, dt \leqq \int_a^{a+c+\delta} |f(t+\tau_i) - f(t)|\, dt$$

(43)

$$\leqq \int_a^{a+2c} |f(t+\tau_i) - f(t)|\, dt$$

and thus by (41)

$$M_i \frac{1}{c} \int_a^{a+c} |f_\delta(t+\tau_i) - f(t+\tau_i)|\, dt$$

$$\leqq \overline{M}_i \frac{1}{c} \int_a^{a+c} |f_\delta(t+\tau_i) - f_\delta(t)|\, dt + \overline{M}_i \frac{1}{c} \int_a^{a+c} |f(t+\tau_i) - f(t)|\, dt +$$

$$+ \frac{1}{c} \int_a^{a+c} |f_\delta(t) - f(t)|\, dt$$

$$< \overline{M}_i \frac{1}{c} \int_a^{a+2c} |f(t+\tau_i) - f(t)|\, dt + \overline{M}_i \frac{1}{c} \int_a^{a+c} |f(t+\tau_i) - f(t)|\, dt + \frac{\varepsilon}{20}$$

$$< 2\, \overline{M}_i \frac{1}{c} \int_a^{a+2c} |f(t+\tau_i) - f(t)|\, dt + \frac{\varepsilon}{20} < \frac{\varepsilon}{4}.$$

Hence by Lemma 1

$$\overline{M}_x \{ |f_\delta(x) - f(x)| \} < \varepsilon$$

which proves the lemma.

We now use all these preliminary lemmas to prove the main result.

Theorem. *If a function $f(x)$ is B^* a. p. then given any $\varepsilon > 0$ we can find an exponential polynomial $s(x)$ such that*

$$\overline{M}_x \{ |f(x) - s(x)| \} < \varepsilon.$$

Proof. For proving this theorem it is sufficient to prove that there exists a uniformly almost periodic function $\varphi(x)$ satisfying the inequality

(44) $$\overline{M}_x \{ |f(x) - \varphi(x)| \} < \varepsilon$$

since uniformly $a.p.$ functions can be approximated uniformly by exponential polynomials.

By Lemma 3 there exists a $\delta > 0$ such that

$$\overline{M}_x\{|f(x) - f_\delta(x)|\} < \varepsilon/2. \tag{45}$$

The function $f(x)$ being B^* $a.p.$ there exists a satisfactorily uniform set of numbers τ_i such that

$$\overline{M}_x \overline{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\,dt < \frac{\varepsilon\delta}{2}. \tag{46}$$

Define a function $\varphi(x)$ by the equation

$$\varphi(x) = \overline{M}_i \frac{1}{\delta} \int_x^{x+\delta} f(t+\tau_i)\,dt.$$

We shall have

$$\begin{aligned} |f_\delta(x) - \varphi(x)| &\leqq \overline{M}_i \frac{1}{\delta} \int_x^{x+\delta} |f(t+\tau_i) - f(t)|\,dt \\ &\leqq \frac{1}{\delta} \overline{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\,dt. \end{aligned} \tag{47}$$

Hence by (46)

$$\overline{M}_x\{|f_\delta(x) - \varphi(x)|\} \leqq \frac{1}{\delta} \overline{M}_x \overline{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\,dt < \frac{\varepsilon}{2} \tag{48}$$

and by (45),

$$\overline{M}_x\{|f(x) - \varphi(x)|\} < \varepsilon. \tag{49}$$

To complete the proof we shall prove that $\varphi(x)$ is uniformly $a.p.$ We write

$$\begin{aligned} |\varphi(x+\tau) - \varphi(x)| &\leqq \overline{M}_i \frac{1}{\delta} \int_{x+\tau_i}^{x+\tau_i+\delta} |f(t+\tau) - f(t)|\,dt \\ &\leqq \frac{c}{\delta} \overline{M}_i \int_{x+\tau_i}^{x+\tau_i+c} |f(t+\tau) - f(t)|\,dt \end{aligned}$$

and by Lemma 1

$$|\varphi(x+\tau)-\varphi(x)| < \frac{4c}{\delta}\overline{M}_x\{|f(x+\tau)-f(x)|\}. \tag{50}$$

Thus any τ satisfying the inequality

$$\overline{M}_x\{|f(x+\tau)-f(x)|\} < \frac{\eta\delta}{4c}, \tag{51}$$

where η is an arbitrary positive number, is a translation number of $\varphi(x)$ belonging to η. But by Lemma 2 the set of all the values of τ satisfying (51) is relatively dense and, thus corresponding to any $\eta > 0$ the set of uniform translation numbers of $\varphi(x)$ belonging to η is relatively dense, i. e., $\varphi(x)$ is uniformly *a. p.*, which proves the theorem.