ON A CLASS OF PERFECT SETS.

BY

S. VERBLUNSKY

of MANCHESTER.

1. By an M-set, or a set of multiplicity, we mean a set E in $(0, 2\pi)$ such that there exists a trigonometric series

$$\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

whose coefficients tend to zero but are not all zero, and which converges to 0 in CE. This paper is concerned with perfect M-sets.

A perfect set P may be supposed to be constructed by subtracting its contiguous intervals $d_1, d_2, \ldots$, in this order from $(0, 2\pi)$. When $d_1, \ldots d_n$ have been subtracted, there remain certain closed intervals $\varrho_1, \ldots \varrho_m$, and from one of these, ϱ_i say, d_{n+1} is to be subtracted. If

$$\lim_{n \to \infty} \frac{d_{n+1}}{\varrho_i} = 0,$$

then P is an M-set. This is the theorem we prove.

Some years ago, Nina Bary[1] constructed a class of perfect M-sets. This class was subjected to two conditions. Bary enunciated the hypothesis that the second condition was superflous. The class of perfect sets subject to the first condition alone, is apparently wider than the class mentioned above. But we show that they are identical, and thus verify Bary's hypothesis.

[1] N. Bary, *Fund. Math.* IX 1927 (62—115) 62, 3.

2. In what follows, when we speak of subtracting open intervals (α_i, β_i) $i = 1, \dots \nu$, from a closed interval (a, b), it is to be understood that

$$a < \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_\nu < b.$$

Consider the following process of forming a perfect set P in the closed interval $(0, 2\pi) = \varrho^0$.

(i) Subtract from ϱ^0 the l^0 open intervals $d_1^1, \dots d_{l^0}^1$, where l^0 is any assigned integer. There remain $k_1 = 1 + l^0$ closed intervals $\varrho_1^1, \dots \varrho_{k_1}^1$.

(ii) Subtract from each closed interval ϱ_i^1 $(i = 1, \dots k_1)$ the l_i^1 open intervals $d_{i1}^2, \dots d_{il_i^1}^2$, where l_i^1 is any assigned positive integer. There remain $k_2 = \sum_{i=1}^{k_1} (1 + l_i^1)$ closed intervals $\varrho_1^2 \dots \varrho_{k_2}^2$.

(iii) Generally, let $\varrho_1^{m-1}, \dots \varrho_{k_{m-1}}^{m-1}$ denote the closed intervals which remain at the $(m-1)^{\text{th}}$ stage. From each interval ϱ_i^{m-1} $(i = 1, \dots k_{m-1})$ subtract the l_i^{m-1} open intervals $d_{i1}^m, \dots d_{il_i^{m-1}}^m$, where l_i^{m-1} is any assigned positive integer. There remain $k_m = \sum_{i=1}^{k_{m-1}} (1 + l_i^{m-1})$ closed intervals $\varrho_1^m, \dots \varrho_{k_m}^m$.

Let D_i^m denote the sum of the open intervals $d_{i1}^m, \dots d_{il_i^{m-1}}^m$. Let $D_m = \sum_{i=1}^{k_{m-1}} D_i^m$. Then $\sum_{m=1}^{\infty} D_m$ is complementary to a perfect set P. Let $\max \varrho_i^m$, $\min \varrho_i^m$ denote the greatest and least lengths of the intervals which remain in ϱ_i^{m-1} when D_i^m is subtracted. It has been proved by N. Bary (*loc. cit.*), that P is an M-set if these two conditions are satisfied.

Condition I. There is a sequence ε_m of positive numbers such that

$$\lim \varepsilon_m = 0; \quad \frac{D_i^m}{\varrho_i^{m-1}} \leq \varepsilon_m \quad (i = 1, 2, \dots k_{m-1};\ m = 1, 2, \dots).$$

Condition II. There is an absolute constant C such that

$$\left| \frac{\max \varrho_i^m}{\min \varrho_i^m} \right| < C \qquad \begin{pmatrix} m = 1, 2, \dots \\ i = 1, \dots k_{m-1} \end{pmatrix}.$$

It was conjectured by Bary that the Condition II is superflous. We shall prove that this conjecture is valid. The proof of Bary consists in constructing a periodic function $F(x)$ which is constant in each contiguous interval of P, but not constant on CP, such that

$$\lim n \int_0^{2\pi} F(x) \cos nx \, dx = \lim n \int_0^{2\pi} F(x) \sin nx \, dx = 0.$$

Then the series obtained by formal differentiation of the Fourier series of $F(x)$ converges to zero in CP. Thus P is an M-set.

The exposition of Bary was devised so as to avoid, wherever possible, an appeal to Condition II; and this condition is used only at one point of the proof. In order to dispense with this condition, what is necessary is a more detailed examination of the structure of P. This however is hardly possible so long as we imagine the perfect set to be constructed as above. If, however, all the numbers l which enter into the above construction equal 1, the problem becomes manageable. The reader will suppose that this involves a restriction on the class of perfect sets. By no means. An essential part of our proof consists in showing that if P satisfies Condition I, then the contiguous intervals can be subtracted from $(0, 2\pi)$ in such a fashion, that, with a suitable notation all the numbers l equal 1, and Condition I is satisfied for the new method of construction. This result naturally enables us to simplify the rest of Bary's proof, and we have thought it best to give a complete demonstration of the theorem.

3. In the closed interval $d = (\alpha, \beta)$, let $d_i = (\alpha_i, \beta_i)$ $i = 1, \ldots n$ be n open intervals such that

$$\alpha < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < \beta.$$

Let

$$\sum_{i=1}^{n} d_i < \theta d \tag{1}$$

where $0 < \theta < 1$. The set $d - \Sigma d_i$ may be regarded as obtained from (α, β) by subtracting the intervals d_i successively. This subtraction can be effected in $n!$ ways according to the order in which we subtract the d_i. Let $d_{r_1}, \ldots d_{r_n}$ be a permutation of $d_1, \ldots d_n$, and let them be subtracted in that order from d. We define »the index of d_{r_1}» (Ind. d_{r_1}) for that order of subtraction, to be d_{r_1}/d. Suppose that the indices of $d_{r_1}, \ldots d_{r_i}$ $(i < n)$ have been defined. When

$d_{r_1}, \ldots d_{r_i}$ have been subtracted from d, there remain $i+1$ intervals, and from one of these, δ say, the interval $d_{r_{i+1}}$ is to be subtracted. We define the index of $d_{r_{i+1}}$ for the given order of subtraction to be $d_{r_{i+1}}/\delta$.

The following lemma is of fundamental importance.

Lemma I. *If* (1) *is satisfied, then there is a permutation* $d_{r_1}, \ldots d_{r_n}$, *such that if the intervals* d_i *are subtracted in this order, then the index of each is* $< \theta$.

The proof is by induction. The lemma is true for $n = 1$. Assume it for $n - 1$. There are two cases.

(i) If

$$d_1 \geq \theta(\alpha_2 - \alpha),$$

then

$$\begin{aligned} d_2 + \cdots + d_n &< \theta(\beta - \alpha) - \theta(\alpha_2 - \alpha) \\ &< \theta(\beta - \alpha_2). \end{aligned} \tag{2}$$

Subtract d_1 first. Its index is less than θ by (1). The intervals $d_2, \ldots d_n$ must now be subtracted in some order from (β_1, β). By (2),

$$d_2 + \cdots + d_n < \theta(\beta - \beta_1).$$

By the lemma for $n-1$, there is a permutation $d_{r_2}, \ldots d_{r_n}$ of $d_2, \ldots d_n$ such that the index of d_{r_j} $(j = 2, \ldots n)$ is less than θ, when the intervals are subtracted from (β_1, β) in that order. Then $d_1, d_{r_2}, \ldots d_{r_n}$ is a permutation with the required properties.

(ii) If

$$d_1 < \theta(\alpha_2 - \alpha), \tag{3}$$

then since

$$d_2 + \cdots + d_n < \theta(\beta - \alpha),$$

there is by the lemma for $n-1$, a permutation $d_{r_1}, \ldots d_{r_{n-1}}$ of $d_2, \ldots d_n$ such that on subtracting the $(n-1)$ intervals in that order, the index of each is $< \theta$. We finally subtract d_1 from (α, α_2). Then its index is less than θ by (3). Hence $d_{r_1}, \ldots d_{r_{n-1}}, d_1$ is a permutation with the required properties.

4. Consider now the construction of P in 2. Suppose that the Condition I is satisfied. Take $m = 1$. Then

$$d_1^1 + \cdots + d_p^1 < \varepsilon_1 \cdot 2\pi.$$

By lemma 1, these intervals d_i^1 can be subtracted in a certain order so that the index of each is $< \varepsilon_1$. Let us denote the intervals in the new order by

$$d_1, d_2, \ldots d_{l^0}. \tag{4}$$

Write

$$\eta_n = \varepsilon_1. \qquad (n = 1, \ldots, l^0) \tag{5}$$

Then if the d_i for $1 \leq i \leq l^0$ are subtracted in the order (4), the index of each is $< \eta_i$.

We now consider Condition I for $m = 2$. The set complementary to (4) consists of the intervals $\varrho_1^1, \ldots \varrho_{k_1}^1$. From ϱ_1^1 we subtract the l_1^1 open intervals $d_{11}^2, \ldots d_{1 l_1^1}^2$. The Condition I gives

$$d_{11}^2 + \cdots + d_{1 l_1^1}^2 < \varepsilon_2 \varrho_1^1.$$

By lemma 1, these intervals can be subtracted in a certain order, so that the index of each is $< \varepsilon_2$. Let us denote the intervals in the new order by

$$d_{l^0+1},\ d_{l^0+2}, \ldots\ d_{l^0+l_1^1}. \tag{6}$$

Write

$$\eta_n = \varepsilon_2 \qquad (n = l^0 + 1, \ldots l^0 + l_1^1). \tag{7}$$

From ϱ_2^1 we subtract the l_2^1 open intervals $d_{21}^2, \ldots d_{2 l_2^1}^2$. The Condition I gives

$$d_{21}^2 + \cdots + d_{2 l_2^1}^2 < \varepsilon_2 \varrho_2^1.$$

By lemma 1, these intervals can be subtracted in a certain order so that the index of each is $< \varepsilon_2$. Denote the intervals in the new order by

$$d_{l^0+l_1^1+1}, \ldots,\ d_{l^0+l_1^1+l_2^1}. \tag{8}$$

Write

$$\eta_n = \varepsilon_2 \qquad (n = l^0 + l_1^1 + 1, \ldots l^0 + l_1^1 + l_2^1).$$

We repeat this process till we have considered $\varrho_{k_1}^1$. Then we have defined the sequences

$$d_i,\ \eta_i \qquad (i = 1, \ldots l^0 + l_1^1 + \cdots + l_{k_1}^1). \tag{9}$$

It is clear how the process is continued. We have $\lim \eta_i = 0$. We have thus proved

Lemma 2. *If P be a perfect set constructed as in 2 and which satisfies Condition I, then the contiguous intervals of P can be written as $d_1, d_2, \ldots d_n, \ldots$ so that if they are subtracted from $(0, 2\pi)$ in this order, then* $\lim \operatorname{Ind} d_n = 0$.

In order to prove that a perfect set P which satisfies the Condition I, is an M-set, it is sufficient to prove

Theorem I. *Let P be a perfect set in $(0, 2\pi)$, obtained by subtracting the contiguous intervals $d_1, d_2, \ldots$ in this order. If* $\lim \operatorname{Ind} d_n = 0$, *then P is an M-set.*

5. Before we can prove Theorem I, we must consider another method of constructing P. From the interval $(0, 2\pi)$, subtract the closed interval δ_0. There remain two closed intervals, which we denote from left to right by ϱ_1, ϱ_2. From ϱ_1 we subtract the open interval δ_1 and from ϱ_2 we subtract the open interval δ_2. There remain four closed intervals which we denote from left to right by $\varrho_{11}, \varrho_{12}, \varrho_{21}, \varrho_{22}$. From ϱ_{11} we subtract the open interval δ_{11}, from ϱ_{12} we subtract δ_{12}, from ϱ_{21} we subtract δ_{21} and from ϱ_{22} we subtract δ_{22}. There remain eight closed intervals ϱ_{ijk} $(i, j, k = 1, 2)$. It is clear how the process is continued. The intervals δ with the same number of suffixes are subtracted in lexicographical order; and the intervals with $\nu + 1$ suffixes after the intervals with ν suffixes:

$$\delta_0,\ \delta_1,\ \delta_2,\ \delta_{11},\ \delta_{12},\ \delta_{21},\ \delta_{22},\ \delta_{121},\ \delta_{112},\ \delta_{121},\ \delta_{122},\ \ldots \tag{10}$$

The set complementary to the sum of the intervals δ is a perfect set P. We wish to prove,

Lemma 3. *Let P be a perfect set formed by subtracting its contiguous intervals $d_1, d_2, \ldots$ in this order, such that* $\lim \operatorname{Ind} d_n = 0$. *Then there is a way of writing the intervals d_i in the form (10), such that if the intervals δ are subtracted in the order (10), then* $\operatorname{Ind} \delta_{p_1 p_2 \ldots p_n}$ $(p_i = 1, 2)$ *tends to zero as $n \to \infty$.*

We write

$$\delta_0 = d_1. \tag{11}$$

Let the members of

$$d_1,\ d_2,\ d_3,\ \ldots \tag{12}$$

which are contained in ϱ_1 be written as

$$d_{\lambda_1},\ d_{\lambda_2},\ \ldots \tag{13}$$

where the suffixes form an increasing sequence. Let the members of (12) which are contained in ϱ_2 be written as

$$d_{\mu_1},\ d_{\mu_2},\ \ldots \tag{14}$$

where the suffixes form an increasing sequence. Then d_{λ_1} is the first d_i which is subtracted from ϱ_1 in the order (12), and d_{μ_1} is the first d_i which is subtracted from ϱ_2 in the order (12). Write

$$\delta_1 = d_{\lambda_1},\quad \delta_2 = d_{\mu_1}.$$

Then as far as the first three terms of (10) are concerned, Ind δ_0, Ind δ_1, Ind δ_2 for (10) are equal respectively to Ind d_1, Ind d_{λ_1}, Ind d_{μ_1} for (12).

We now consider the four intervals $\varrho_{11}, \varrho_{12}, \varrho_{21}, \varrho_{22}$. Let the members of (12) which occur in them be written respectively

$$d_{x_1},\quad d_{x_2},\ \ldots \tag{15}$$

$$d_{y_1},\quad d_{y_2},\ \ldots \tag{16}$$

$$d_{z_1},\quad d_{z_2},\ \ldots \tag{17}$$

$$d_{w_1},\quad d_{w_2},\ \ldots \tag{18}$$

where the suffixes in each sequence are in ascending order. Then d_{x_1} is the first d_i which is subtracted from ϱ_{11} in the order (12); ... d_{w_1} is the first d_i which is subtracted from ϱ_{22} in the order (12). Write

$$\delta_{11} = d_{x_1},\ \delta_{12} = d_{y_1},\ \delta_{21} = d_{z_1},\ \delta_{22} = d_{w_1}.$$

Then Ind δ_{11}, ... Ind δ_{22} for (10) are equal respectively to Ind d_{x_1}, ... Ind d_{w_1} for (12).

It is clear how this process is continued. Further, this process exhausts the d_i. Now given $\varepsilon > 0$, we have

$$\text{Ind } d_i < \varepsilon,\quad i \geq n(\varepsilon)$$

the index referring to the order (12). Let $d_1, \ldots d_{n(\varepsilon)}$ occur respectively in the $r_1^{\text{th}}, \ldots r_{n(\varepsilon)}^{\text{th}}$ place in (10). Then in (10), any δ which occurs after the N^{th} place, where $N = \text{Max}\ (r_1, \ldots r_{n(\varepsilon)})$ is a d whose suffix exceeds $n(\varepsilon)$. But by construction, the index of this δ in the order (10) equals the index of the identical d

in the order (12). Hence $\delta_{p_1 \ldots p_n} < \varepsilon$ provided that $\delta_{p_2 \ldots p_n}$ occurs in (10) after the N^{th} place; i.e. provided that n is sufficiently large. This proves the lemma.

We can now enunciate Theorem I in the form mentioned in 2, with all the numbers l equal to 1. For by lemma 3, we can enumerate the intervals $d_1, d_2, \ldots$ as in (10), and if ε_m denote the greatest of the indices of the 2^{m-1} intervals δ with $m-1$ suffixes, then $\varepsilon_m \to 0$. We have then to prove

Theorem II. *From $\varrho_1^0 = (0, 2\pi)$ subtract the open interval d_1^1. There remain 2 closed intervals ϱ_1^1, ϱ_2^1. From each closed interval ϱ_i^1 $(i = 1, 2)$ subtract the open interval d_i^2. There remain 2^2 closed intervals $\varrho_1^2, \ldots \varrho_4^2$. Generally, let $\varrho_1^{m-1}, \ldots \varrho_{2^{m-1}}^{m-1}$ denote the closed intervals which remain at the $(m-1)^{\text{th}}$ stage. From each ϱ_i^{m-1} subtract the open interval d_i^m. There remain 2^m closed intervals $\varrho_1^m, \ldots \varrho_{2^m}^m$.*

Let $D_m = \sum_{i=1}^{2^{m-1}} d_i^m$. Then $\sum_1^{\infty} D_m$ is complementary to a perfect set P. If there is a sequence ε_m of positive numbers such that

$$\lim \varepsilon_m = 0, \quad d_i^m / \varrho_i^{m-1} \leq \varepsilon_m, \quad (i = 1, \ldots 2^{m-1}, m = 1, 2, \ldots)$$

then P is an M-set.

6. We define a sequence $F_m(x)$ of continuous periodic functions by induction. Let

(i) $F_1(0) = F_1(2\pi) = 0$; (ii) $F_1(x) = 1$ in d_1^1; (iii) $F_1(x)$ is linear in ϱ_1^1, ϱ_2^1.

Suppose that $F_1(x), \ldots F_m(x)$ have been defined, so that $F_m(x)$ is constant in each interval of $S_m = \sum_{i=1}^{m} D_i$, and is linear in each interval of R_m, the complement of S_m.

We define

$$F_{m+1}(x) = F_m(x) \text{ on } S_m.$$

Let ϱ_i^m be an interval of R_m. From it, the interval d_i^{m+1} is subtracted, leaving the intervals $\varrho_{2i-1}^{m+1}, \varrho_{2i}^{m+1}$. We denote by σ_i^{m+1} the larger of these intervals if they are unequal, and the first (the left hand one), if they are equal. We denote by τ_i^{m+1} the other of these intervals.

We complete the definition of $F_{m+1}(x)$ uniquely, by the condition of continuity, and by

$$\begin{aligned} F_{m+1}(x) &= F_m(x) && \text{in } \sigma_i^{m+1}, \\ &= \text{const.} && \text{in } d_i^{m+1}, \\ &= \text{a linear function} && \text{in } \tau_i^{m+1}. \end{aligned}$$

Let Δ_i^m denote the variation of $F_m(x)$ on ϱ_i^m, and let $\Delta_{\tau_i}^{m+1}$ denote the variation of $F_{m+1}(x)$ on τ_i^{m+1}. Then

$$\begin{aligned} \left|\frac{\Delta_{\tau_i}^{m+1}}{\Delta_i^m}\right| &= \frac{\tau_i^{m+1} + d_i^{m+1}}{\tau_i^{m+1} + d_i^{m+1} + \sigma_i^{m+1}} \\ &\leq \frac{\frac{1}{2}(\tau + \sigma) + d}{\tau + \sigma}. \end{aligned}$$

But

$$d_i^{m+1} \leq \varepsilon_{m+1}(\tau + d + \sigma),$$

so that

$$d \leq \frac{\varepsilon_{m+1}(\tau + \sigma)}{1 - \varepsilon_{m+1}}.$$

Hence

$$\begin{aligned} \left|\frac{\Delta_{\tau_i}^{m+1}}{\Delta_i^m}\right| &\leq \frac{1}{2} + \frac{\varepsilon_{m+1}}{1 - \varepsilon_{m+1}} \\ &\leq \frac{3}{4} \end{aligned} \tag{19}$$

for $\varepsilon_{m+1} < \frac{1}{5}$, i. e. for $m \geq M$.

For an assigned m, any point x belongs either to S_m or to R_m. If $x < S_m$, then

$$F_{\mu+1}(x) - F_\mu(x) = 0 \qquad (\mu = m, m+1, \ldots).$$

If $x < R_m$, then $x < \varrho_i^m$, where $i = i(x)$. If $x < \sigma_i^{m+1}$, then

$$F_{m+1}(x) - F_m(x) = 0.$$

If $x < \tau_i^{m+1}$, or $x < d_i^{m+1}$, then

$$\begin{aligned} |F_{m+1}(x) - F_m(x)| &\leq \frac{d_i^{m+1}}{\varrho_i^m} |\Delta_i^m| \\ &\leq \varepsilon_{m+1} |\Delta_i^m|. \end{aligned} \tag{20}$$

Every point $x < P$ belongs to an infinite sequence $\varrho_1^0 > \varrho_{i_1}^1 > \varrho_{i_2}^2 > \dots > \varrho_{i_m}^m > \dots$, where $i_1, i_2, \dots$ is determined by x. Every point $x < CP$ belongs to a finite sequence $\varrho_1^0 > \dots > \varrho_{i_\mu}^\mu > d_{i_\mu}^{\mu+1}$.

If $x < \varrho_{i_{m+1}}^{m+1}$, then, either $\varrho_{i_{m+1}}^{m+1} = \sigma_{i_m}^{m+1}$, in which case $F_{m+1}(x) = F_m(x)$, or else $\varrho_{i_{m+1}}^{m+1} = \tau_{i_m}^{m+1}$, in which case we can apply (20).

Consider the remainder

$$r_m(x) = \sum_{q=m}^{\infty} [F_{q+1}(x) - F_q(x)] \qquad (m \geq M) \tag{21}$$

of the series

$$F_1(x) + [F_2(x) - F_1(x)] + \cdots \tag{22}$$

Then

$$|r_m(x)| \leq \sum_{q=m}^{\infty} |F_{q+1}(x) - F_q(x)| = \sum_{q=m}^{\infty}{}' |F_{q+1}(x) - F_q(x)|, \tag{23}$$

where if $x < P$, the accent denotes that the sum is taken for such q for which $\varrho_{i_{q+1}}^{q+1} \neq \sigma_{i_q}^{q+1}$; while for $x < CP$, the accented sum denotes 0 if $m > \mu$, and the non-zero terms of

$$\sum_{q=m}^{\mu} |F_{q+1}(x) - F_q(x)|$$

if $m \leq \mu$. Here μ has the meaning given above; i. e. $x < d_{i_\mu}^{\mu+1}$. To evaluate $|F_{\mu+1}(x) - F_\mu(x)|$, we can apply (20).

Let

$$\eta_n = \operatorname{Max}(\varepsilon_{n+1}, \varepsilon_{n+2}, \dots).$$

Then $\lim \eta_n = 0$. By (23) and (20), we have

$$|r_m(x)| \leq \eta_m \sum_{q=m}^{\infty}{}' |\Delta_{i_q}^q|. \tag{24}$$

If for a particular value of q, $|\Delta_{i_q}^q|$ is a term and is not the last term in (24), then the next term $|\Delta_{i_{q+s}}^{q+s}|$, $s \geq 1$, arises from an interval $\varrho_{i_{q+s}}^{q+s}$. Now the occurrence of $|\Delta_{i_q}^q|$ in the accented sum means that $F_{q+1}(x) \neq F_q(x)$. Also, the fact that $|\Delta_{i_q}^q|$ is not the last term means that x is not contained in $d_{i_q}^{q+1}$. Hence $x < \varrho_{i_{q+1}}^{q+1}$ and $\varrho_{i_{q+1}}^{q+1} = \tau_{i_q}^{q+1}$.

Also, since $\varrho^{q+1}_{i_{q+1}} > \varrho^{q+s}_{i_{q+s}}$, we have

$$\left|\Delta^{q+s}_{i_{q+s}}\right| \leq \left|\Delta^{q+1}_{i_{q+1}}\right|.$$

By (19),

$$\left|\Delta^{q+1}_{\tau i_q}\right| \leq \frac{3}{4}\left|\Delta^{q}_{i_q}\right|,$$

so that

$$\left|\Delta^{q+s}_{i_{q+s}}\right| \leq \frac{3}{4}\left|\Delta^{q}_{i_q}\right|.$$

Hence for all x and $m \geq M$,

$$\begin{aligned} |r_m(x)| &\leq \eta_m \left|\Delta^m_{i_m}\right| \sum_{q=0}^{\infty}\left(\frac{3}{4}\right)^q \\ &\leq 3\,\eta_m\left|\Delta^m_{i_m}\right|. \end{aligned}$$

Hence the series (22) is uniformly convergent; $F(x)$ is continuous, and is constant in each interval of CP, and

$$|F(x) - F_m(x)| \leq 3\,\eta_m \left|\Delta^m_i\right|. \qquad (x < \varrho^m_i) \qquad (25)$$

7. Let λ be a number which satisfies $1 < \lambda < 2$. Choose M so large that

$$\frac{2\pi}{\lambda^M} < \varrho^1_1,\ \varrho^1_2.$$

Let $x < P$. Then x is the limit of the sequence $\varrho^0_1 > \varrho^1_{i_1} > \varrho^2_{i_2} > \cdots$ Consider the sequence

$$\sigma^1_1,\ \sigma^2_{i_1},\ \sigma^3_{i_2},\ \ldots \qquad (26)$$

The numbers σ have the meaning previously assigned, so that $\sigma^{r+1}_{i_r} = \mathrm{Max}\,(\varrho^{r+1}_{2i_r-1}, \varrho^{r+1}_{2i_r})$. The numbers (26) form a diminishing sequence which tends to zero.[1] Hence given $m \geq M$, there is a unique $k = k(x, m)$ such that

$$\sigma^k_{i_{k-1}} \geq \frac{2\pi}{\lambda^m},\quad \sigma^{k+1}_{i_k} < \frac{2\pi}{\lambda^m}.$$

Clearly,

$$k(x, m) \leq k(x, m+1). \qquad (27)$$

We have $x < \varrho^k_{i_k}$, where $k = k(x, m)$. By the Heine-Borel theorem, P is contained

[1] Supposing, as we may, that P is non-dense.

in the sum of a finite number of such intervals $\varrho_{i_k}^k$. On the other hand, the number $k = k(x, m)$ is the same for every x of $P\varrho_{i_k}^k$. The intervals $\varrho_{i_k}^k$, $k = k(x, m)$, are therefore non-overlapping, and there is a finite number of them. These intervals contain P and constitute a set which we denote by R'_m. The intervals of R'_m are separated by contiguous intervals of P.

Let μ_m denote the least k for which an interval of the form ϱ_i^k belongs to R'_m. Then $k(x, m) \geq \mu_m$ for all $x \subset P$.

By (27),

$$\mu_m \leq \mu_{m+1}. \tag{28}$$

If $\varrho_i^{\mu_m} < R'_m$, then $\varrho_{2i-1}^{\mu_m+1}$, $\varrho_{2i}^{\mu_m+1} < \frac{2\pi}{\lambda^m}$. Given a natural number N, every ϱ_i^k, for $k \leq N$, is greater than $\frac{2\pi}{\lambda^m}$, for all sufficiently large m. Hence $\mu_{m+1} > N$ for $m \geq m(N)$; i. e.

$$\lim_{m\to\infty} \mu_m = \infty. \tag{29}$$

By (28) and (29) every positive integer n determines uniquely an $m = m(n)$ such that

$$\frac{\lambda^{m-1}}{\sqrt{\eta_{\mu_{m-1}}}} \leq n < \frac{\lambda^m}{\sqrt{\eta_{\mu_m}}}. \tag{30}$$

We define a sequence $\{\varphi_m(x)\}$ by

(i) $\varphi_m(x) = F(x)$ for $x < CR'_m$; (ii) if $x < \varrho_i^k < R'_m$, then $\varphi_m(x) = F_k(x)$.

We define a sequence $\{f_m(x)\}$ by

(i) $f_m(x) = F(x)$ for $x < CR'_m$; (ii) if $x < \varrho_i^k < R'_m$, then $f_m(x) = F_{k+1}(x)$.

Then each of $\varphi_m(x), f_m(x)$ is continuous in $(0, 2\pi)$, increases from 0 at $x = 0$ to 1 at the left end of d_1^1, and diminishes from 1 at the right end of d_1^1 to 0 at $x = 2\pi$.

8. To prove that P is an M-set, it is sufficient to show that

$$\lim_{n\to\infty} n \int_0^{2\pi} F(\alpha)\cos n(\alpha - x)\, d\alpha = 0$$

for all x.

We have

$$I = I(n) = n\int_0^{2\pi} F(\alpha)\cos n(\alpha - x)\,d\alpha$$

$$= n\int_0^{2\pi} (F - f_m)\cos n(\alpha - x)\,d\alpha + n\int_0^{2\pi} f_m \cos n(\alpha - x)\,d\alpha$$

$$= I_1 + I_2,$$

where $m = m(n)$ is defined above.

Since $f_m = F$ in CR'_m,

$$|I_1| \leq n\int_0^{2\pi} |F - f_m|\,d\alpha$$

$$\leq n\int_{R'_m} |F - f_m|\,d\alpha.$$

Now R'_m consists of a number of separated intervals ϱ_i^k with $k \geq \mu_m$. On an interval ϱ_i^k, $f_m = F_{k+1}$. Hence

$$|F - f_m| = |F - F_{k+1}|.$$

Now $\varrho_i^k = \varrho_{2i-1}^{k+1} + d_i^{k+1} + \varrho_{2i}^{k+1}$, and $F = F_{k+1}$ on d_i^{k+1}. Thus

$$\int_{\varrho_i^k} |F - f_m| = \int_{\varrho_{2i-1}^{k+1}} + \int_{\varrho_{2i}^{k+1}} |F - F_{k+1}|$$

$$\leq 3\eta_{\mu_m}\left[|\varDelta_{2i-1}^{k+1}|\varrho_{2i-1}^{k+1} + |\varDelta_{2i}^{k+1}|\varrho_{2i}^{k+1}\right]$$

by (25). Since $\varrho_i^k < R'_m$, we have

$$\varrho_{2i-1}^{k+1},\ \varrho_{2i}^{k+1} < \frac{2\pi}{\lambda^m}.$$

Further,

$$|\varDelta_{2i-1}^{k+1}| + |\varDelta_{2i}^{k+1}| = \text{absolute variation of } F_{k+1}(x) \text{ on } \varrho_i^k$$

$$= \text{absolute variation of } f_m \text{ on } \varrho_i^k.$$

Hence

$$|I_1| \leq n \cdot 3\eta_{\mu_m} \cdot \frac{2\pi}{\lambda^m} \cdot [\text{total variation of } f_m \text{ in } (0, 2\pi)]$$

$$\leq \frac{\lambda^m}{\sqrt{\eta_{\mu_m}}} \cdot \frac{6\pi\eta_{\mu_m}}{\lambda^m} \cdot 2$$

by (30),

$$\leq 12\pi\sqrt{\eta_{\mu_m}}. \tag{31}$$

Next

$$I_2 = n\int_0^{2\pi} f_m \cos n(\alpha - x)\, d\alpha = -\int_0^{2\pi} f'_m \sin n(\alpha - x) d\alpha$$

$$= -\int_0^{2\pi} (f'_m - \varphi'_m) \sin n(\alpha - x) d\alpha - \int_0^{2\pi} \varphi'_m \sin n(\alpha - x) d\alpha$$

$$= I_3 + I_4.$$

Since $f_m = \varphi_m$ on CR'_m, we have

$$|I_3| \leq \int_0^{2\pi} |f'_m - \varphi'_m|$$

$$\leq \int_{R'_m} |f'_m - \varphi'_m|.$$

On $\varrho_i^k < R'_m$, we have $f'_m = F'_{k+1}$, $\varphi'_m = F'_k$. But F_{k+1} is constant on d_i^{k+1}. Hence

$$\int_{\varrho_i^k} |f'_m - \varphi'_m| = \int_{d_i^{k+1}} |F'_k| + \int_{\varrho_{2i-1}^{k+1}} + \int_{\varrho_{2i}^{k+1}} |F'_{k+1} - F'_k|. \tag{32}$$

We have,

$$\int_{d_i^{k+1}} |F'_k| = \text{absolute variation of } F_k \text{ on } d_i^{k+1}.$$

Also on that one of the intervals ϱ_{2i}^{k+1}, ϱ_{2i}^{k+1} which is σ_i^{k+1}, we have $F_{k+1} = F_k$; so that the last two integrals in (32) equal

$$\int\limits_{\tau_i^{k+1}} |F'_{k+1} - F'_k|. \tag{33}$$

Now F'_{k+1}, F'_k are of the same sign, and

$$F'_{k+1} = \frac{\text{varn. of } F_k \text{ in } \tau_i^{k+1} + d_i^{k+1}}{\tau_i^{k+1}}.$$

Hence (33) equals [abs. varn. of F_k on d_i^{k+1}]. Thus

$$\begin{aligned}\int\limits_{\varrho_i^k} |f'_m - \varphi'_m| &= \quad 2 \cdot [\text{abs. varn. of } F_k \text{ on } d_i^{k+1}] \\ &\leq 2\eta_{\mu_m} \cdot [\text{abs. varn. of } \varphi_m \text{ on } \varrho_i^k].\end{aligned}$$

Hence

$$\begin{aligned}|I_3| &\leq 2\eta_{\mu_m} \cdot [\text{Total varn. of } \varphi_m \text{ on } (0, 2\pi)] \\ &\leq 4\eta_{\mu_m}.\end{aligned} \tag{34}$$

9. We must now evaluate I_4, and this is the critical part of the proof. Since $\varphi_m = F$ in CR'_m, and F is constant in each of the intervals of which CR'_m consists, (they are contiguous intervals of P), we have

$$I_4 = -\int\limits_{R'_m} \varphi'_m \sin n(\alpha - x)\, d\alpha. \tag{35}$$

We shall use the abbreviation AV (f, δ) for »the absolute variation of f on δ», i. e. if $\delta = (\alpha, \beta)$, for $|f(\beta) - f(\alpha)|$.

We have

$$\left|\int\limits_{\varrho_i^k} \varphi'_m \sin n(\alpha - x)\, d\alpha\right| \leq \mathrm{AV}(\varphi_m, \varrho_i^k), \tag{36}$$

since φ_m is monotone on ϱ_i^k. Also, if $\varrho_i^k < R'_m$,

$$\left|\int\limits_{\varrho_i^k} \varphi'_m \sin n(\alpha - x)\, d\alpha\right| = \left|\varphi'_m \int\limits_{\varrho_i^k} \sin n(\alpha - x)\, d\alpha\right|$$

since $\varphi_m = F_k$ in ϱ_i^k and F_k is linear in the interval. The last expression does not exceed

$$|\varphi'_m| \cdot \frac{2}{n} \leq |\varphi'_m| \cdot 2\sqrt{\eta_{\mu_{m-1}}}/\lambda^{m-1}.$$

But $\varphi'_m = F'_k$, and this equals

$$[\text{varn. of } F_k \text{ on } \varrho_i^k]/\varrho_i^k = [\text{varn. of } \varphi_m \text{ on } \varrho_i^k]/\varrho_i^k.$$

Hence

$$\left|\int\limits_{\varrho_i^k} \varphi'_m \sin n(\alpha - x)\, d\alpha\right| \leq \frac{2\sqrt{\eta_{\mu_{m-1}}}}{\varrho_i^k \lambda^{m-1}} \cdot \mathrm{AV}(\varphi_m, \varrho_i^k). \qquad (\varrho_i^k < R'_m) \quad (37)$$

We now require the following lemma.

Lemma 4. *Let ϱ_{2j-1}^k be an interval of R'_m which lies to the left of d_1^1. Let*

$$\varrho_{2j-1}^k < \frac{2\pi}{\lambda^m}(\lambda - 1). \tag{38}$$

Then ϱ_{2j}^k can be expressed as the sum of

(i) *an interval $\varrho_u^s < R'_m$ of length $\geq \frac{2\pi}{\lambda^m}$;*

(ii) *the sum of pairs of abutting intervals ϱ_v^t, d_w^t, such that*

$$\Sigma(\varrho_v^t + d_w^t) < (\lambda - 1)\varrho_u^s. \qquad (\varrho_v^t < R'_m)$$

Further, $\mathrm{AV}(\varphi_m, \varrho_v^t + d_v^t) = \mathrm{AV}$ *of φ_m on an interval of length $\varrho_v^t + d_v^t$ in ϱ_u^s. The pairs of intervals in* (ii) *may be absent.*

A similar lemma holds for the case in which ϱ_{2j}^k is an interval of R'_m to the left of d_1^1 and

$$\varrho_{2j}^k < \frac{2\pi}{\lambda^m}(\lambda - 1).$$

Then ϱ_{2j-1}^k can be expressed in the way stated in the lemma. In the enunciation of the lemma, it is not implied that d_w^t is necessarily on the right of ϱ_v^t. It may be on the left. Finally, similar lemmas are true when we consider intervals $\varrho_{2j-1}^k, \varrho_{2j}^k$ on the right of d_1^1.

As we are considering intervals on the left of d_1^1, all functions F, F_i, φ_m are non-diminishing, and we can replace »absolute variation» (AV) by »simple variation» (V).

10. We now proceed with the proof of the lemma. By the definition of λ, we have $1 < \lambda < 2$. By (38), $\varrho_{2j-1}^k < \frac{2\pi}{\lambda^m}$. But $\varrho_{2j-1}^k < R'_m$. The definition of R'_m requires $\sigma_j^k \geq \frac{2\pi}{\lambda^m}$. Hence $\varrho_{2j}^k \geq \frac{2\pi}{\lambda^m}$.

If $\varrho_{2j}^k < R'_m$, we take ϱ_{2j}^k for ϱ_u^s of the lemma, and the intervals in (ii) are absent. Suppose, now, that ϱ_{2j}^k is not an interval of R'_m. Then since $\sigma_j^k \geq \frac{2\pi}{\lambda^m}$, we must have

$$\operatorname{Max} \varrho_{4j-1}^{k+1},\ \varrho_{4j}^{k+1} \geq \frac{2\pi}{\lambda^m},$$

since otherwise, ϱ_{2j}^k would belong to R'_m. But we cannot have

$$\operatorname{Min} \varrho_{4j-1}^{k+1},\ \varrho_{4j}^{k+1} \geq \frac{2\pi}{\lambda^m},$$

for then, since ϱ_{2j}^k contains both these intervals, we would have

$$\varrho_{2j}^k > \frac{4\pi}{\lambda^m} > \frac{2\pi}{\lambda^{m-1}}$$

i. e. $\sigma_j^k > \frac{2\pi}{\lambda^{m-1}}$; and $\sigma_{2j-1}^{k+1} < \varrho_{2j-1}^k < \frac{2\pi}{\lambda^m} < \frac{2\pi}{\lambda^{m-1}}$, which implies that $\varrho_{2j-1}^k < R'_{m-1}$ a contradiction. Thus,

$$\sigma_{2j}^{k+1} \geq \frac{2\pi}{\lambda^m},\quad \tau_{2j}^{k+1} < \frac{2\pi}{\lambda^m}.$$

Then $\tau_{2j}^{k+1} < R'_m$; also

$$\tau_{2j}^{k+1} + d_{2j}^{k+1} < (\lambda - 1)\,\sigma_{2j}^{k+1}.$$

For if not, then

$$\tau_{2j}^{k+1} + d_{2j}^{k+1} \geq (\lambda - 1)\frac{2\pi}{\lambda^m},$$

and

$$\sigma_{2j}^{k+1} \geq \frac{2\pi}{\lambda^m}.$$

By addition,

$$\varrho_{2j}^{k} \geq \frac{2\pi}{\lambda^{m-1}},$$

which, as before, implies that $\varrho_{2j-1}^{k} < R'_{m-1}$, a contradiction.

If now $\sigma_{2j}^{k+1} < R'_m$, we take σ_{2j}^{k+1} for ϱ_u^s; τ_{2j}^{k+1} for ϱ_v^t and d_{2j}^{k+1} for d_w^t. Then (i) and (ii) of the lemma are satisfied. Also, $\varphi_m = F_{k+1}$ in τ_{2j}^{k+1}; $\varphi_m = F$ in d_{2j}^{k+1}; i. e. $\varphi_m = F_{k+1}$ in d_{2j}^{k+1}. Hence

$$\mathrm{V}(\varphi_m, \tau_{2j}^{k+1} + d_{2j}^{k+1}) = \mathrm{V}(F_k, \tau_{2j}^{k+1} + d_{2j}^{k+1}).$$

But $\varphi_m = F_{k+1}$ in σ_{2j}^{k+1}; i. e. $\varphi_m = F_k$ in σ_{2j}^{k+1}, and F_k is linear in $\varrho_{2j}^{k} = \sigma_{2j}^{k+1} + \tau_{2j}^{k+1} + d_{2j}^{k+1}$. Hence

$$\mathrm{V}(\varphi_m, \tau_{2j}^{k+1} + d_{2j}^{k+1}) = \text{Varn. of } \varphi_m \text{ in an equal interval in } \varrho_u^s. \tag{40}$$

Suppose, however, that σ_{2j}^{k+1} is not an interval of R'_m. The interval σ_{2j}^{k+1} was the larger of ϱ_{4j-1}^{k+1}, ϱ_{4j}^{k+1}. It will be convenient to introduce a new suffix i, and to write

$$\sigma_{2j}^{k+1} = \varrho_i^{k+1}.$$

If ϱ_i^{k+1} is not interval of R'_m, then

$$\text{Max } \varrho_{2i-1}^{k+2},\ \varrho_{2i}^{k+2} \geq \frac{2\pi}{\lambda^m}.$$

For if not, we would have $\sigma_i^{k+2} < \frac{2\pi}{\lambda^m}$, which together with $\sigma_{2j}^{k+1} \geq \frac{2\pi}{\lambda^m}$ implies that $\sigma_{2j}^{k+1} < R'_m$, a contradiction. But we cannot have

$$\text{Min } \varrho_{2i-1}^{k+2},\ \varrho_{2i}^{k+2} \geq \frac{2\pi}{\lambda^m}.$$

For then, since ϱ_i^{k+1} contains both these intervals, we would have

$$\sigma_{2j}^{k+1} = \varrho_i^{k+1} > \frac{2\pi}{\lambda^{m-1}},$$

and, a fortiori, $\varrho_{2j}^{k} > \frac{2\pi}{\lambda^{m-1}}$, which, as proved above, is false. Hence

$$\sigma_i^{k+2} \geq \frac{2\pi}{\lambda^m},\ \tau_i^{k+2} < \frac{2\pi}{\lambda^m}.$$

Then $\tau_i^{k+2} < R'_m$. Also

$$\tau_{2j}^{k+1} + d_{2j}^{k+1} + \tau_i^{k+2} + d_i^{k+2} < (\lambda - 1)\,\sigma_i^{k+2}. \tag{41}$$

For if not, then $\tau_{2j}^{k+1} + d_{2j}^{k+1} + \tau_i^{k+2} + d_i^{k+2} \geq (\lambda - 1)\frac{2\pi}{\lambda^m}$,

and

$$\sigma_i^{k+2} \geq \frac{2\pi}{\lambda^m}.$$

By addition,

$$\varrho_{2j}^k \geq \frac{2\pi}{\lambda^{m-1}}$$

which we know to be false.

If now $\sigma_i^{k+2} < R'_m$, we take σ_i^{k+2} for ϱ_u^s; we have two pairs of abutting intervals τ_{2j}^{k+1}, d_{2j}^{k+1} and τ_i^{k+2}, d_i^{k+2}. Then (i) and (ii) of the lemma are satisfied. Further, $\varphi_m = F_{k+2} = F_{k+1} = F_k$ in σ_i^{k+2}, so that (40) is true. Now $\varphi_m = F = F_{k+2}$ in d_i^{k+2}, $\varphi_m = F_{k+2}$ in τ_i^{k+2}. Hence

$$\begin{aligned} \mathrm{V}(\varphi_m,\, \tau_i^{k+2} + d_i^{k+2}) &= \mathrm{V}(F_{k+1},\, \tau_i^{k+2} + d_i^{k+2}) \\ &= \mathrm{V}(F_k,\, \tau_i^{k+2} + d_i^{k+3}) \end{aligned}$$

since $F_{k+1} = F_k$ in $\sigma_{2j}^{k+1} < \tau_i^{k+2} + d_i^{k+2}$. Hence

$$\mathrm{V}(\varphi_m, \tau_i^{k+2} + d_i^{k+2}) = \text{Varn. of } \varphi_m \text{ in an equal interval in } \sigma_i^{k+2}.$$

If, however, σ_i^{k+2} is not an interval of R'_m, then writing

$$\sigma_i^{k+2} = \varrho_h^{k+2},$$

we apply the above argument again. It is clear that since R'_m contains only a finite number of intervals ϱ_v^t, we arrive at the decomposition of the lemma after a finite number of steps.

It should be noticed that in ϱ_u^s, $\varphi_m = F_{k-1}$. For $\varrho_{2j}^k = \sigma_j^k$, $\varrho_i^{k+1} = \sigma_{2j}^{k+1}$, $\varrho_h^{k+2} = \sigma_i^{k+2}, \ldots$; and by construction, given F_r in ϱ_i^r, we have $F_{r+1} = F_r$ in σ_i^{r+1}.

11. We can now evaluate I_4. The intervals which constitute R'_m can be divided into two classes. Those which lie to the left of d_1^1 form the set L, and those which lie to the right of d_1^1 form the set R. Then (35) becomes

$$I_4 = -\int_L \varphi_m' \sin n(\alpha - x)\, d\alpha - \int_R \varphi_m' \sin n(\alpha - x)\, d\alpha$$
$$= \quad I_5 + I_5'.$$

The intervals ϱ_i^k which constitute L are of two kinds. Either $i = 2j - 1$ is odd, or $i = 2j$ is even. The intervals of the first kind form a set L_0, the intervals of the second kind form a set L_e. Thus

$$I_5 = -\int_{L_0} \varphi_m' \sin n(\alpha - x)\, d\alpha - \int_{L_e} \varphi_m' \sin n(\alpha - x)\, d\alpha$$
$$= \quad I_6 + I_7.$$

Let the intervals ϱ_{2j-1}^k which constitute L_0 be denoted from left to right by

$$\delta_1,\ \delta_2,\ \ldots\ \delta_r.$$

Then

$$|I_6| \leq \sum_{p=1}^{r} \left| \int_{\delta_p} \varphi_m' \sin n(\alpha - x)\, d\alpha \right|$$
$$\leq \sum_{p=1}^{r} J_p.$$

We have $\delta_1 = \varrho_{2j-1}^k$ say, for some j and for some $k \geq \mu_m$. (α) If $\delta_1 \geq \frac{2\pi}{\lambda^m}(\lambda - 1)$, then by (37),

$$J_1 \leq \frac{\lambda}{\pi(\lambda - 1)} \sqrt{\eta_{\mu_m - 1}}\, \mathrm{V}(\varphi_m, \delta_1). \tag{42}$$

(β) If $\delta_1 < \frac{2\pi}{\lambda_m}(\lambda - 1)$, then ϱ_{2j}^k can be decomposed as in lemma 4. If any of the intervals ϱ_u^s, ϱ_v^t in that lemma belong to L_0, they are the intervals

$$\delta_2,\ \ldots\ \delta_r \tag{43}$$

say. If ϱ_u^s is one of these intervals, say δ_j, then by (36),

$$J_2 + \cdots + J_{j-1} + J_{j+1} + \cdots + J_r \leq \mathrm{V}(\varphi_m, \delta_2 + \cdots + \delta_{j-1} + \delta_{j+1} + \cdots + \delta_r).$$

By lemma 4, the last expression does not exceed the variation of φ_m on an interval of length $< (\lambda - 1)\varrho_u^s$, contained in ϱ_u^s, so that

$$J_2 + \cdots + J_{j-1} + J_{j+1} + \cdots + J_r \leq (\lambda - 1)\,\mathrm{V}(\varphi_m, \varrho_u^s).$$

Further $\delta_j = \varrho_u^s \geq \frac{2\pi}{\lambda^m}$ by (i) of lemma 4, so that by (37),

$$J_j \leq \frac{\lambda \sqrt{\eta_{\mu_{m-1}}}}{\pi}\,\mathrm{V}(\varphi_m, \varrho_u^s),$$

and

$$\sum_2^r J_p \leq \left[(\lambda - 1) + \frac{\lambda}{\pi}\sqrt{\eta_{\mu_{m-1}}}\right]\mathrm{V}(\varphi_m, \varrho_u^s). \tag{44}$$

If ϱ_u^s is not one of the intervals (43), we have by the above argument,

$$\sum_2^r J_p \leq (\lambda - 1)\,\mathrm{V}(\varphi_m, \varrho_u^s),$$

so that (44) is true in any case.

Also, by (36),

$$\begin{aligned} J_1 &\leq \mathrm{V}(\varphi_m, \delta_1) \\ &\leq \mathrm{V}(F_{k-1}, \varrho_{2j-1}^k + d_j^k). \end{aligned}$$

Now F_{k-1} is linear in ϱ_j^{k-1} and $\varphi_m = F_{k-1}$ in ϱ_u^s. Hence

$$J_1 \leq \frac{\varrho_{2j-1}^k + d_j^k}{\varrho_u^s} \cdot \mathrm{V}(\varphi_m, \varrho_u^s).$$

Since $\varrho_u^s \geq \frac{2\pi}{\lambda^m}$, $\varrho_{2j-1}^k < \frac{2\pi}{\lambda^m}(\lambda - 1)$, we have

$$J_1 \leq \left[(\lambda - 1) + \frac{d_j^k}{\varrho_u^s}\right]\mathrm{V}(\varphi_m, \varrho_u^s).$$

But

$$\begin{aligned} d_j^k &\leq \varepsilon_k\left[\varrho_{2j-1}^k + d_j^k + \varrho_{2j}^k\right] \\ &\leq \eta_{\mu_{m-1}}\left[\varrho_{2j-1}^k + d_j^k + \varrho_{2j}^k\right]. \end{aligned}$$

For relevant n we have $m \geq m_0$ say and $\eta_{\mu_{m-1}} < \frac{1}{2}$. Thus

$$d_j^k \leq 2\eta_{\mu_{m-1}}[\varrho_{2j-1}^k + \varrho_{2j}^k].$$

Now we have just seen that

$$\varrho_{2j-1}^k < (\lambda - 1)\varrho_u^s,$$

and

$$\varrho_{2j}^k < \varrho_u^s + (\lambda - 1)\varrho_u^s$$

by lemma 4. Hence

$$d_j^k \leq 4\lambda\eta_{\mu_{m-1}}\varrho_u^s$$

and

$$J_1 \leq [(\lambda - 1) + 4\lambda\eta_{\mu_{m-1}}] \mathrm{V}(\varphi_m, \varrho_u^s).$$

Hence, and by (44),

$$\sum_1^r J_p \leq [2(\lambda - 1) + 5\lambda\sqrt{\eta_{\mu_{m-1}}}] \mathrm{V}(\varphi_m, \varrho_u^s). \tag{45}$$

The intervals $\delta_{r+1}, \delta_{r+2}, \ldots \delta_\nu$ all lie to the right of ϱ_{2j}^k; and $\varrho_{2j}^k > \varrho_u^s$. Hence

$$\sum_1^r J_p \leq [2(\lambda - 1) + 5\lambda\sqrt{\eta_{\mu_{m-1}}}] \mathrm{V}(\varphi_m, \varrho_{2j}^k). \tag{46}$$

We now consider the interval $\delta_{r+1} = \varrho_{2i-1}^l$ say. If $\delta_{r+1} \geq \frac{2\pi}{\lambda^m}(\lambda - 1)$, then we have as for (42),

$$J_{r+1} \leq \frac{\lambda}{\pi(\lambda - 1)}\sqrt{\eta_{\mu_{m-1}}}\,\mathrm{V}(\varphi_m, \delta_{r+1}).$$

If $\delta_{r+1} < \frac{2\pi}{\lambda^m}(\lambda - 1)$, we have a relation of the form

$$\sum_{r+1}^{r+q} J_p \leq [2(\lambda - 1) + 5\lambda\sqrt{\eta_{\mu_{m-1}}}] \mathrm{V}(\varphi_m, \varrho_{2i}^l),$$

and so on. All the intervals

$$\delta_1,\ \varrho_{2j}^k,\ \delta_{r+1},\ \varrho_{2i}^l, \ldots$$

are separated and lie in ϱ_1^1. After a finite number of steps, we shall have considered every $J_p, p = 1, \ldots \nu$. Hence

$$|I_6| \leq \left[\frac{\lambda}{\pi(\lambda-1)}\sqrt{\eta_{\mu_{m-1}}} + 2(\lambda-1) + 5\lambda\sqrt{\eta_{\mu_{m-1}}}\right] \mathrm{V}(\varphi_m, \varrho_1^1),$$

and $\mathrm{V}(\varphi_m, \varrho_1^1) = 1$. Similarly for I_7. A similar evaluation applies to I_5', and so

$$|I_4| \leq 4\left[\frac{\lambda}{\pi(\lambda-1)}\sqrt{\eta_{\mu_{m-1}}} + 2(\lambda-1) + 5\lambda\sqrt{\eta_{\mu_{m-1}}}\right]. \tag{47}$$

By (31), (34), and (47),

$$\begin{aligned}\overline{\lim_{n\to\infty}}\,|I(n)| &= \lim_{\lambda\to 1}\overline{\lim_{n\to\infty}}\,|I_1 + I_3 + I_4| \\ &\leq \lim_{\lambda\to 1}\overline{\lim_{n\to\infty}}\left[12\pi\sqrt{\eta_{\mu_m}} + 4\eta_{\mu_m} + \frac{4\lambda}{\pi(\lambda-1)}\sqrt{\eta_{\mu_{m-1}}} + 8(\lambda-1) + 20\lambda\sqrt{\eta_{\mu_{m-1}}}\right] \\ &= 0.\end{aligned}$$