NON-HOMOGENEOUS TERNARY QUADRATIC FORMS.

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1. This work has arisen from the consideration of possible extensions of Minkowski's theorem on the product of two non-homogeneous linear forms. If

$$L_1 = \alpha x + \beta y, \qquad L_2 = \gamma x + \delta y$$

are two linear forms with real coefficients, and c_1, c_2 are any two real numbers, Minkowski's theorem asserts that there exist integers x, y such that

$$(1) \qquad |(L_1 + c_1)(L_2 + c_2)| \leqq \tfrac{1}{4} \Delta,$$

where $\Delta = |\alpha\delta - \beta\gamma|$, and we suppose $\Delta \neq 0$. It is conjectured that a similar result holds for the product of n non-homogeneous linear forms in n variables, with 2^{-n} in place of $\frac{1}{4}$. So far this conjecture has been proved only for $n = 3$, by Remak, and for $n = 4$, by Dyson.

Minkowski's theorem can be stated in another form, which suggests other possible extensions. Write

$$L_1 L_2 = a x^2 + b x y + c y^2 = Q(x, y);$$

then $Q(x, y)$ is an indefinite binary quadratic form with discriminant

$$b^2 - 4ac = \Delta^2.$$

Determine real numbers x_0, y_0 so that

$$c_1 = \alpha x_0 + \beta y_0, \qquad c_2 = \gamma x_0 + \delta y_0.$$

Then Minkowski's theorem asserts that for any indefinite binary quadratic form $Q(x, y)$, and any real x_0, y_0, there exist integers x, y such that

$$(2) \qquad |Q(x + x_0,\ y + y_0)| \leqq \tfrac{1}{4} \Delta.$$

The extension which now suggests itself is one to indefinite quadratic forms in more than two variables.

In particular, let $Q(x, y, z)$ be an indefinite ternary quadratic form with real coefficients, of determinant $D \neq 0$. The problem is whether there exist constants k such that, for any real x_0, y_0, z_0, there are integers x, y, z satisfying

$$|Q(x+x_0,\ y+y_0,\ z+z_0)| \leqq k\,|D|^{\frac{1}{3}};$$

and if so, what is the least k for which this is true? The exponent $\frac{1}{3}$ is dictated by considerations of homogeneity.

The existence of *some* such k, though not immediately obvious, is fairly easy to prove. I have succeeded in determining the best possible value of k, but the result has not the same simple and natural appearance as Minkowski's original theorem. I prove:

Theorem 1. *Let $Q(x, y, z)$ be an indefinite ternary quadratic form, with real coefficients, of determinant $D \neq 0$. Then, for any real x_0, y_0, z_0, there exist integers x, y, z such that*

$$|Q(x+x_0,\ y+y_0,\ z+z_0)| \leqq \left(\tfrac{27}{100}|D|\right)^{\frac{1}{3}}. \tag{3}$$

This is true with strict inequality unless Q is equivalent[1] *to a multiple of*

$$x^2 + 5y^2 - z^2 + 5yz + zx, \tag{4}$$

in which case it is not.

One of the lemmas (Lemma 3) which I use in the proof of Theorem 1 has a certain intrinsic interest, since it forms a simple generalization of Minkowski's theorem which seems to have escaped notice. It asserts that we can satisfy, instead of (1), the inequality

$$-\nu\Delta \leqq (L_1+c_1)(L_2+c_2) \leqq \mu\Delta, \tag{5}$$

provided μ, ν are positive numbers satisfying

$$\mu\nu \geqq \tfrac{1}{16}.$$

Minkowski's theorem is the particular case $\mu = \nu = \frac{1}{4}$. An interesting feature of the result is that there are other values of μ, ν with $\mu\nu = \frac{1}{16}$, for which (5) is the best possible inequality of its kind.

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[1] Equivalence refers here to linear substitutions with integral coefficients and determinant ± 1. An assertion such as the preceding one is unaffected by such a substitution, since variables of the form $x+x_0$, $y+y_0$, $z+z_0$ are transformed into variables of a similar kind.

If Q is a *null form*, *i. e.* if $Q(x, y, z) = 0$ for some set of integers x, y, z not all zero, the problem can be treated by a rather simpler method, and a more precise inequality can be proved. This takes the form:

Theorem 2. *If Q is a null form, then for any real x_0, y_0, z_0 there exist integers x, y, z such that*

$$|Q(x + x_0, y + y_0, z + z_0)| \leqq (\tfrac{1}{4}|D|)^{\frac{1}{3}}. \tag{6}$$

There exist null forms for which this is not true with strict inequality.

Finally, I prove that the minimum established in Theorem 1 is 'isolated'. The precise meaning of this term will be clear from the following enunciation:

Theorem 3. *There exists a positive absolute constant δ such that, if $Q(x, y, z)$ is not equivalent to a multiple of the special form (4), then for any real x_0, y_0, z_0 there exist integers x, y, z satisfying*

$$|Q(x + x_0, y + y_0, z + z_0)| \leqq (1 - \delta)(\tfrac{27}{100}|D|)^{\frac{1}{3}}. \tag{7}$$

This is a remarkable result in that it has no analogue for Minkowski's original theorem.[1] The proof is naturally rather difficult.

2. Lemma 1. *Let $Q(x, y, z)$ be an indefinite ternary quadratic form of determinant $D < 0$. Then there exist integers x_1, y_1, z_1 such that*

$$0 < Q(x_1, y_1, z_1) \leqq (4|D|)^{\frac{1}{3}}. \tag{8}$$

Proof. This is Theorem 2 of my paper "On indefinite ternary quadratic forms", Proc. London Math. Soc. (in course of publication).

Lemma 2. *Let $Q(x, y, z)$ be an indefinite ternary quadratic form of determinant $D < 0$, and let a be any positive value of Q arising from integral values of x, y, z whose highest common factor is 1. Then Q is equivalent to a multiple of*

$$(x + hy + gz)^2 + \phi(y, z), \tag{9}$$

where $\phi(y, z)$ is an indefinite binary quadratic form, whose discriminant Δ^2 satisfies

$$\Delta^2 = \frac{4|D|}{a^3}. \tag{10}$$

[1] For a proof that there is no such analogous theorem, see Theorem 2 of my paper "Non-homogeneous binary quadratic forms", *Proc. K. Akad. Wet. Amsterdam*, 49 (1946), 815—821.

Proof. After applying a suitable linear substitution with integral coefficients and determinant ± 1 to the variables, we can suppose that $Q(1, 0, 0) = a$. Then, if h, g are suitably chosen,

$$Q(x, y, z) = a\{(x + hy + gz)^2 + \phi(y, z)\},$$

where $\phi(y, z)$ is a binary quadratic form. By comparison of determinants,

$$D = a^3(-\tfrac{1}{4}\Delta_1),$$

where Δ_1 is the discriminant of $\phi(y, z)$. Thus $\Delta_1 > 0$, which implies that ϕ is indefinite, and on writing $\Delta_1 = \Delta^2$, we have (10). This proves the lemma.

Lemma 3. *Let*

$$L_1 = \alpha_{11}x + \alpha_{12}y, \qquad L_2 = \alpha_{21}x + \alpha_{22}y$$

be linear forms with real coefficients, and let $\Delta = |\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}| \neq 0$. *Let* μ, ν *be positive numbers satisfying*

(11) $$\mu\nu \geqq \tfrac{1}{16}.$$

Then, for any real c_1, c_2 *there exist integers* x, y *such that*

(12) $$-\nu\Delta \leqq (L_1 + c_1)(L_2 + c_2) \leqq \mu\Delta.$$

If $\nu = 2\mu$, *this is true with strict inequality in both parts of* (12) *unless the quadratic form* L_1L_2 *is equivalent to a positive multiple of*

(13) $$x^2 + xy - \tfrac{1}{4}y^2.$$

Proof. We can suppose without loss of generality that $\Delta = 1$. We can also suppose that $\nu \geqq \mu$, and since the result reduces to Minkowski's theorem when $\nu = \mu$ we can suppose $\nu > \mu$, whence $\nu > \frac{1}{4}$.

Let M denote the lower bound of

$$|(L_1 + c_1)(L_2 + c_2)|$$

for all integral x, y. By Minkowski's theorem, $M \leqq \frac{1}{4} < \nu$. If M is attained, or approached, by *negative* values of $(L_1 + c_1)(L_2 + c_2)$, then (12) is satisfied with strict inequality in both parts, and there is nothing to prove. Hence we can suppose that

(14) $$M \geqq \mu,$$

and that, for an arbitrarily small positive number ε_0, there exist integers x^*, y^* for which

$$(15) \qquad (L_1^* + c_1)(L_2^* + c_2) = \frac{M}{1-\varepsilon}, \quad \text{where} \quad 0 \leqq \varepsilon < \varepsilon_0.$$

We suppose that (12) has no solution with strict inequality in both parts.

Write

$$X = \frac{L_1 - L_1^*}{L_1^* + c_1}, \qquad Y = \frac{L_2 - L_2^*}{L_2^* + c_2},$$

then X, Y are linear forms in the integral variables $x - x^*$, $y - y^*$. The determinant of these forms has absolute value

$$(16) \qquad \frac{1-\varepsilon}{M},$$

by (15). Since

$$(X+1)(Y+1) = \frac{(L_1 + c_1)(L_2 + c_2)}{(L_1^* + c_1)(L_2^* + c_2)},$$

the hypothesis that (12) has no solution, together with (15), tells that

$$(X+1)(Y+1) \geqq \frac{\mu(1-\varepsilon)}{M} \quad \text{or} \quad \leqq -\frac{\nu(1-\varepsilon)}{M}$$

for all integral values of the variables. The definition of M tells us that

$$|(X+1)(Y+1)| \geqq 1 - \varepsilon$$

for all integral values of the variables. Writing

$$(17) \qquad K^2 = \frac{\nu(1-\varepsilon)}{M} \qquad (K > 0),$$

we can say that

$$(18) \qquad (X+1)(Y+1) \geqq 1 - \varepsilon \quad \text{or} \quad \leqq -K^2$$

for all integral values of the variables.

The points (X, Y) which correspond to integral values of the variables form a lattice, whose determinant is given by (16), and every point of which satisfies (18). We proceed to prove that there is no lattice point, except the origin, in the rectangle

$$(19) \qquad |X + Y| < 2\{1 + \sqrt{1-\varepsilon}\}, \qquad |X - Y| < 2K.$$

It will suffice (by reflection in the origin) to consider points satisfying $X + Y \leqq 0$, and therefore satisfying

$$(20) \qquad -2\sqrt{1-\varepsilon} < X + Y + 2 \leqq 2.$$

Since

$$4(X+1)(Y+1) = (X+Y+2)^2 - (X-Y)^2 \geqq -(X-Y)^2 > -4K^2$$

by (19), the condition (18) implies

$$(21) \qquad (X+Y+2)^2 - (X-Y)^2 \geqq 4(1-\varepsilon).$$

This implies $|X+Y+2| \geqq 2\sqrt{1-\varepsilon}$, and so, by (20),

$$0 \geqq X + Y \geqq -2\{1-\sqrt{1-\varepsilon}\}.$$

Also, by (20) and (21), $(X-Y)^2 \leqq 4\varepsilon$. Hence we must have

$$(22) \qquad |X| \leqq \varepsilon_1, \qquad |Y| \leqq \varepsilon_1,$$

where ε_1 depends only on ε and tends to zero with ε. But if X, Y are not both zero, we can find an integer m such that the point (mX, mY) lies in the rectangle (19) but does not satisfy (22). This gives a contradiction, and therefore there is no lattice point except the origin in the rectangle (19).

The area of the rectangle is

$$8K\{1+\sqrt{1-\varepsilon}\},$$

hence, by Minkowski's theorem, since the determinant of the lattice is given by (16), we have

$$(23) \qquad \frac{1-\varepsilon}{M} \geqq 2K\{1+\sqrt{1-\varepsilon}\}.$$

By (17), this is

$$\frac{1-\varepsilon}{M} \geqq 4\nu\{1+\sqrt{1-\varepsilon}\}^2.$$

Using (14), we obtain

$$\mu\nu \leqq \frac{1-\varepsilon}{4\{1+\sqrt{1-\varepsilon}\}^2}.$$

If $\varepsilon > 0$, this contradicts (11), and so in this case, (12) must have a solution with strict inequality in both parts. If $\varepsilon = 0$, we still have a contradiction unless equality occurs in the last step of the argument, i. e. unless $M = \mu$. In this case we still have a solution of (12), by (15), though not with strict inequality. This proves the main assertion of the Lemma.

We have now to investigate the case when $\nu = 2\mu$, and (12) has no solution with strict inequality. As we have just seen, this requires that

$$\varepsilon = 0, \qquad \mu\nu = \tfrac{1}{16}, \qquad M = \mu.$$

There must also be no lattice point, other than the origin, in the rectangle (19), which is now

$$|X + Y| < 4, \qquad |X - Y| < 2\sqrt{2}.$$

Since equality occurs in (23), the lattice must be a critical lattice for the rectangle, and so must have two generating lattice points on its boundary. By (18), every lattice point satisfies

$$(X + 1)(Y + 1) \geqq 1 \quad \text{or} \quad \leqq -2.$$

By the same argument as before, we find that the only points on the boundary of the rectangle which satisfy this condition and the same condition for $-X, -Y$ are

$$\pm(2, 2), \quad \pm(1 + \sqrt{2},\ 1 - \sqrt{2}), \quad \pm(1 - \sqrt{2},\ 1 + \sqrt{2}).$$

The lattice generated by any two of these (not images of one another in the origin) is the lattice given by

$$X = 2u + (1 + \sqrt{2})v, \qquad Y = 2u + (1 - \sqrt{2})v.$$

Here u, v take all integral values, and so are related to the variables $x - x^*, y - y^*$ by an integral unimodular substitution. Since $(L_1 + c_1)(L_2 + c_2)$ is a positive multiple of $(X + 1)(Y + 1)$, it follows that $L_1 L_2$ is a positive multiple of XY, and so is equivalent to a positive multiple of

$$\{2x + (1 + \sqrt{2})y\}\{2x + (1 - \sqrt{2})y\} = 4x^2 + 4xy - y^2.$$

This completes the proof of Lemma 3.

Lemma 4. *Let $\phi(y, z)$ be an indefinite binary quadratic form with real coefficients, of discriminant Δ^2 (where $\Delta > 0$). Let μ, ν be positive numbers satisfying $\mu\nu \geqq \frac{1}{16}$. Then for any real y_0, z_0 there exist real y, z satisfying*

(24) $$y \equiv y_0 \pmod 1, \qquad z \equiv z_0 \pmod 1,$$

(25) $$-\nu\Delta \leqq \phi(y, z) \leqq \mu\Delta.$$

If $\nu = 2\mu$ this is true with strict inequality in both parts of (25), unless $\phi(y, z)$ is equivalent to a positive multiple of

(26) $$y^2 + yz - \tfrac{1}{4}z^2.$$

Proof. This is merely a restatement of Lemma 3 in other terms.

Lemma 5. *Let β, B be real numbers with $B > \frac{1}{4}$, and suppose that*

$$\beta^2 \leqq B + \tfrac{1}{4}[2B]^2, \tag{27}$$

where $[2B]$ *denotes the largest integer which does not exceed* $2B$. *Then for any real* x_0 *there exists an* x *with* $x \equiv x_0 \pmod{1}$ *such that*

$$|x^2 - \beta^2| \leqq B. \tag{28}$$

Provided that $2B$ *is not an integer, strict inequality in* (27) *implies strict inequality in* (28). *If* $2B$ *is an integer, a sufficient condition for the validity of* (28) *with strict inequality is that*

$$\beta^2 < B + \tfrac{1}{4}(2B - 1)^2. \tag{29}$$

Proof.[1] Suppose first that $\beta^2 \leqq \frac{1}{4}$. There exists an x with $x \equiv x_0 \pmod{1}$ such that $|x| \leqq \frac{1}{2}$. We have

$$|x^2 - \beta^2| \leqq \max(\tfrac{1}{4} - \beta^2, \beta^2) \leqq \tfrac{1}{4} < B.$$

Suppose next that $\beta^2 > \frac{1}{4}$. Write $m = [2B]$, so that

$$\tfrac{1}{2}m \leqq B < \tfrac{1}{2}(m+1), \qquad \beta^2 \leqq B + \tfrac{1}{4}m^2. \tag{30}$$

Determine an integer $l \geqq 0$ such that

$$l^2 \leqq 4\beta^2 - 1 < (l+1)^2. \tag{31}$$

By (30) and (31),

$$l^2 \leqq 4\beta^2 - 1 \leqq 4B + m^2 - 1 < 2(m+1) + m^2 - 1 = (m+1)^2,$$

whence

$$l \leqq m. \tag{32}$$

There exists an x such that $x \equiv x_0 \pmod{1}$ and

$$\tfrac{1}{2}l \leqq |x| \leqq \tfrac{1}{2}(l+1);$$

for the intervals from $\frac{1}{2}l$ to $\frac{1}{2}(l+1)$ and from $-\frac{1}{2}(l+1)$ to $-\frac{1}{2}l$ include all values of $x \pmod{1}$. We have

$$\tfrac{1}{4}l^2 - \beta^2 \leqq x^2 - \beta^2 \leqq \tfrac{1}{4}(l+1)^2 - \beta^2.$$

Now, by (30), (31), (32),

$$\tfrac{1}{4}(l+1)^2 - \beta^2 \leqq \tfrac{1}{4}(l+1)^2 - \tfrac{1}{4}(l^2+1) = \tfrac{1}{2}l \leqq \tfrac{1}{2}m \leqq B.$$

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[1] An imperfect form of this lemma occurred as Lemma 1 in my paper "Non-homogeneous binary quadratic forms", *Proc. K. Akad. Wet. Amsterdam*, 49 (1946), 815—821.

Also, if $l = m$,

$$\beta^2 - \tfrac{1}{4} l^2 = \beta^2 - \tfrac{1}{4} m^2 \leqq B,$$

and if $l < m$,

$$\beta^2 - \tfrac{1}{4} l^2 < \tfrac{1}{4}(l+1)^2 + \tfrac{1}{4} - \tfrac{1}{4} l^2 = \tfrac{1}{2}(l+1) \leqq \tfrac{1}{2} m \leqq B.$$

Hence this value of x satisfies (28).

The final clauses of the Lemma follow at once from the main part, on replacing B by B', where B' is slightly less than B.

Lemma 6. *If* $d > \frac{1}{5}$, *then*

$$(33) \qquad (2d - 1)(2d + [d]^2) > \tfrac{50}{27} d^3.$$

Also, if d is a positive integer other than 3,

$$(34) \qquad (2d - 1)(2d + (d-1)^2) > \tfrac{50}{27} d^3.$$

Proof. Suppose first that $\frac{1}{5} < d < 1$. The inequality (33) is then

$$(2d - 1)\, 2d > \tfrac{50}{27} d^3,$$

or

$$25 d^2 - 54 d + 27 < 0,$$

$$(5d - \tfrac{27}{5})^2 < \tfrac{54}{25}.$$

This is satisfied, since

$$-\tfrac{7}{5} < 5d - \tfrac{27}{5} < -\tfrac{2}{5}.$$

Suppose next that $d \geqq 1$. We note that (33) follows from (34), since $[d] > d - 1$, and (34) would be valid with equality in place of inequality when $d = 3$. Now (34) can be written

$$27(2d^3 - d^2 + 2d - 1) > 50 d^3,$$

which is the same as

$$(d-3)^2(4d - 3) > 0,$$

and so is valid for $d \geqq 1$, $d \neq 3$.

3. Proof of Theorem 1. Let $Q(x, y, z)$ be an indefinite ternary quadratic form of determinant $D \neq 0$. We have to prove that for any real x_0, y_0, z_0 there exist real x, y, z satisfying

$$(35) \qquad x \equiv x_0 \pmod 1, \quad y \equiv y_0 \pmod 1, \quad z \equiv z_0 \pmod 1,$$

$$(36) \qquad |Q(x, y, z)| \leqq (\tfrac{27}{100} |D|)^{\frac{1}{3}}.$$

By considering the form $-Q$ instead of Q, we can suppose without loss of generality that $D < 0$.

By Lemma 2, $Q(x, y, z)$ is equivalent to a multiple of

$$Q_1(x, y, z) = (x + hy + gz)^2 + \phi(y, z), \tag{37}$$

where $\phi(y, z)$ is an indefinite binary form of discriminant Δ^2, and Δ^2 satisfies (10). By Lemma 1 we can suppose that a satisfies $0 < a \leqq (4 |D|)^{\frac{1}{3}}$, whence

$$\Delta \geqq 1. \tag{38}$$

It will suffice to prove the result for the form (37).

Let d be defined by

$$d = (\tfrac{27}{50} \Delta^2)^{\frac{1}{3}}; \tag{39}$$

then

$$d \geqq (\tfrac{27}{50})^{\frac{1}{3}} > \tfrac{4}{5}. \tag{40}$$

Define μ, ν by

$$\mu \Delta = \tfrac{1}{2} d - \tfrac{1}{4}, \qquad \nu \Delta = \tfrac{1}{2} d + \tfrac{1}{4} [d]^2. \tag{41}$$

Then μ, ν are positive, and

$$\mu \nu = \tfrac{1}{16} \Delta^{-2} (2d - 1)(2d + [d]^2) > \tfrac{1}{16} \tag{42}$$

by Lemma 6 and (40) and (39). Hence, by Lemma 4, there exist y, z satisfying $y \equiv y_0 \pmod 1$ and $z \equiv z_0 \pmod 1$ such that

$$-(\tfrac{1}{2} d + \tfrac{1}{4} [d]^2) < \phi(y, z) < \tfrac{1}{2} d - \tfrac{1}{4}. \tag{43}$$

If $\phi(y, z) \geqq 0$, we choose x so that $x \equiv x_0 \pmod 1$ and $|x + hy + gz| \leqq \frac{1}{2}$. Then

$$0 \leqq (x + hy + gz)^2 + \phi(y, z) < \tfrac{1}{4} + (\tfrac{1}{2} d - \tfrac{1}{4}) = (\tfrac{27}{100} \Delta^2)^{\frac{1}{3}},$$

by (39). Since the determinant of the ternary form (37) is $-\frac{1}{4}\Delta^2$, this proves that, in the present case, (36) is valid for the form (37), with strict inequality. If $\phi(y, z) < 0$, we apply Lemma 5, with

$$\beta^2 = -\phi(y, z), \qquad B = \tfrac{1}{2} d > \tfrac{1}{4}.$$

The condition (27) is satisfied, by (43). Hence there exists x with $x \equiv x_0 \pmod 1$ such that

$$|(x + hy + gz)^2 - (-\phi(y, z))| \leqq B = \tfrac{1}{2} d = (\tfrac{27}{100} \Delta^2)^{\frac{1}{3}},$$

and the conclusion follows as before, but not necessarily with strict inequality in (36).

We have now to investigate the case in which (36) is not valid with strict inequality. As we have seen in the preceding proof, this possibility can only arise when $\phi(y, z) < 0$. Since (43) has strict inequality, it follows from Lemma 5 that $2B = d$ must be an integer. It follows further from Lemma 5 that the inequality

$$0 < -\phi(y, z) < \tfrac{1}{2} d + \tfrac{1}{4}(d-1)^2$$

cannot be satisfied. Thus, if we define ν' by

$$\nu' \Delta = \tfrac{1}{2} d + \tfrac{1}{4}(d-1)^2,$$

it is impossible that $\mu \nu' > \frac{1}{16}$. This, however, is the same as the inequality (34) of Lemma 6, and is satisfied unless $d = 3$. Hence we must have $d = 3$, whence $\Delta^2 = 50$ by (39).

With $d = 3$, we have $\mu \Delta = \frac{1}{2} d - \frac{1}{4} = \frac{5}{4}$, and $\nu' \Delta = \frac{1}{2} d + \frac{1}{4}(d-1)^2 = \frac{5}{2}$, and the preceding argument has shown us that it is impossible that

$$-\tfrac{5}{2} < \phi(y, z) < \tfrac{5}{4}$$

for $y \equiv y_0 \pmod{1}$, $z \equiv z_0 \pmod{1}$. It follows from the last clause of Lemma 4 that $\phi(y, z)$ is equivalent to a positive multiple of the form (26). Since the discriminant of $\phi(y, z)$ is 50, and of the form (26) is 2, the multiple must be 5. Hence, after an integral unimodular substitution on y, z, we can write

$$(44) \qquad Q_1(x, y, z) = (x + hy + gz)^2 + 5y^2 + 5yz - \tfrac{5}{4} z^2.$$

The argument of this paragraph has been based on any representation of a multiple of the original form as (37), derived from any positive value a of the form satisfying $a \leqq (4 |D|)^{\frac{1}{3}}$. It has been proved that if (36) is not valid with strict inequality, the positive value a must be such that $\Delta^2 = 4|D|/a^3$ has the value 50. Applying the same argument to the form (44), whose determinant is $-\frac{1}{2}(25)$, it follows that any positive value a of the form (44), satisfying

$$a \leqq (4 \cdot \tfrac{25}{2})^{\frac{1}{3}},$$

must be such that

$$\frac{4\left(\frac{25}{2}\right)}{a^3} = 50.$$

Hence the only positive value a of the form (44), with integers x, y, z whose highest common factor is 1, which satisfies $a < \sqrt[3]{50}$ is $a = 1$.

By applying a substitution $x = \pm x' + my + nz$, where m, n are integers, we can suppose that in (44) we have

$$0 \leqq g \leqq \tfrac{1}{2}, \qquad |h| \leqq \tfrac{1}{2}.$$

Now

$$Q_1(2, 0, -1) = (2-g)^2 - \tfrac{5}{4},$$

which is positive and does not exceed $\frac{11}{4}$. Hence it is 1, and consequently $g = \frac{1}{2}$. Also

$$Q_1(x, 1, -1) = (x - \tfrac{1}{2} + h)^2 - \tfrac{5}{4},$$

and on choosing $x = 2$ if $h \geqq 0$ and $x = -1$ if $h < 0$ we obtain $h = 0$ by a similar argument. Hence, if we cannot satisfy (36) with strict inequality subject to (35), then $Q(x, y, z)$ is equivalent to a multiple of the form (4).

Finally, we have to prove that for the form (4) there exist x_0, y_0, z_0 such that (36) has no solution with strict inequality. We take

$$x_0 = \tfrac{1}{2}, \quad y_0 = \tfrac{1}{2}, \quad z_0 = 0,$$

and write the form as

$$\tfrac{1}{4}\{(2x+z)^2 + 5(2y+z)^2 - 10z^2\}.$$

We have to show that this has absolute value at least $\frac{3}{2}$ when x, y, z satisfy (35). This is the same as saying that

$$|X^2 + 5Y^2 - 10Z^2| \geqq 6$$

when X, Y, Z are integers with $X - Z$ and $Y - Z$ both odd. If Z is odd, this follows from the two congruences

$$X^2 + 5Y^2 - 10Z^2 \equiv 2 \pmod 4$$
$$\equiv 0 \text{ or } \pm 1 \pmod 5.$$

If Z is even, the same congruences are still valid, with the same conclusion.

This completes the proof of Theorem 1.

4. Lemma 7. *If $Q(x, y, z)$ is a null form, then either* (i) *for any real x_0, y_0, z_0 there are real x, y, z satisfying $x \equiv x_0 \pmod 1$, $y \equiv y_0 \pmod 1$, $z \equiv z_0 \pmod 1$ which make $Q(x, y, z)$ arbitrarily small, or* (ii) *$Q(x, y, z)$ is equivalent to a multiple of*

$$(x + hy)^2 - \lambda y(z + ly) \qquad (\lambda > 0), \tag{45}$$

for certain values of h, λ, l.

Proof. Since Q represents zero for integral values of x, y, z not all zero, we can suppose without loss of generality that $Q(0, 0, 1) = 0$. Then we can write

$$Q = ax^2 + 2hxy + by^2 + 2z(fy + gx).$$

Suppose first that f/g is irrational. By a well known theorem on Diophantine approximation we can find, for any $\varepsilon > 0$, x and y to satisfy $x \equiv x_0 \pmod{1}$, $y \equiv y_0 \pmod{1}$, and

$$0 < |gx + fy| < \varepsilon.$$

For this is equivalent to satisfying

$$0 < |gu + fv + k| < \varepsilon$$

in integers u, v. Having chosen x and y, Q is of the form

$$A + Bz,$$

where $0 < |B| < 2\varepsilon$, and we can find $z \equiv z_0 \pmod{1}$ such that

$$|A + Bz| \leqq \tfrac{1}{2}|B| < \varepsilon.$$

Thus in this case the assertion (i) is true.

Now suppose that f/g is rational. After multiplying Q by a suitable factor, we can suppose that f and g are relatively prime integers. There exist integers f_1, g_1 such that $fg_1 - f_1 g = 1$. The integral unimodular substitution

$$x' = g_1 x + f_1 y, \quad y' = gx + fy, \quad z' = z$$

transforms Q into

$$a_1 x'^2 + 2h_1 x'y' + b_1 y'^2 + 2z'y'.$$

Here $a_1 \neq 0$, since we suppose that the determinant D of the form is not zero. On completing the square, we obtain a multiple of a form of the type (45). The condition $\lambda > 0$ can be satisfied by changing y into $-y$ if necessary; that $\lambda \neq 0$ follows from the hypothesis that $D \neq 0$.

Lemma 8. *Let Q be the form (45). Suppose there exists y with $y \equiv y_0 \pmod{1}$ such that*

(46) $$0 < |y|^3 \leqq (2\lambda)^{-1}.$$

Then there exist x, y, z such that

(47) $$x \equiv x_0 \pmod{1}, \quad y \equiv y_0 \pmod{1}, \quad z \equiv z_0 \pmod{1}$$

and

(48) $$|Q(x, y, z)| \leqq (\tfrac{1}{4}|D|)^{\frac{1}{3}}.$$

Proof. Choosing y as in the enunciation, Q has the form

$$A - Bz,$$

where $0 < |B| \leqq \lambda (2\lambda)^{-\frac{1}{3}}$. For *any* x we can choose z with $z \equiv z_0 \pmod 1$ such that

$$|A - Bz| \leqq \tfrac{1}{2}|B| \leqq (\tfrac{1}{16}\lambda^2)^{\frac{1}{3}} = (\tfrac{1}{4}|D|)^{\frac{1}{3}}.$$

Lemma 9. *Let Q be the form* (45), *and suppose* $\lambda \leqq 4$. *Then, for any* x_0, y_0, z_0 *there exist* x, y, z *to satisfy* (47) *and* (48).

Proof. Suppose first that $0 < \lambda \leqq \frac{1}{2}$. Then $(2\lambda)^{-1} \geqq 1$, and it is plain that there exists $y \equiv y_0 \pmod 1$ to satisfy (46). Thus the result follows from Lemma 8.

Suppose next that $\frac{1}{2} < \lambda \leqq 4$. Then $(2\lambda)^{-1} \geqq \frac{1}{8}$, and there exists y with $y \equiv y_0 \pmod 1$ such that $|y|^3 \leqq (2\lambda)^{-1}$. If $y \neq 0$, the result follows again from Lemma 8. If $y = 0$ we have

$$Q(x, 0, z) = x^2.$$

Choosing x to satisfy $|x| \leqq \frac{1}{2}$, we have

$$|Q(x, 0, z)| \leqq \tfrac{1}{4} < (\tfrac{1}{16}\lambda^2)^{\frac{1}{3}} = (\tfrac{1}{4}|D|)^{\frac{1}{3}}.$$

Lemma 10. *For any real* x_0 *there exists* $x \equiv x_0 \pmod 1$ *such that*

$$|x^2 - \beta^2| \leqq \begin{cases} \frac{1}{2} & \text{if } \beta^2 \leqq \frac{1}{2}, \\ \sqrt{\beta^2 - \frac{1}{4}} & \text{if } \beta^2 \geqq \frac{1}{2}. \end{cases} \tag{49}$$

Proof. By Lemma 5, it suffices to verify that (27) is satisfied with $B = \frac{1}{2}$ or $B = \sqrt{\beta^2 - \frac{1}{4}}$, as the case may be. The former is immediate, and the latter follows from the fact that the stronger inequality

$$\beta^2 \leqq B + \tfrac{1}{4}(2B - 1)^2$$

is satisfied (with equality) when $B = \sqrt{\beta^2 - \frac{1}{4}}$.

Lemma 11. *Let Q be the form* (45), *and suppose that* $\lambda > 4$. *Then for any* x_0, y_0, z_0 *there exist* x, y, z *to satisfy* (47) *and* (48).

Proof. We choose $y \equiv y_0 \pmod 1$ to satisfy $|y| \leqq \frac{1}{2}$, and then choose $z \equiv z_0 \pmod 1$ to satisfy

$$0 \leqq z + ly < 1 \quad \text{if} \quad y \geqq 0,$$

$$0 \geqq z + ly > -1 \quad \text{if} \quad y < 0.$$

Writing

$$\lambda y(z + ly) = \beta^2, \tag{50}$$

we have

$$0 \leqq \beta^2 \leq \lambda |y|. \tag{51}$$

By Lemma 10, if $\beta^2 \leqq \frac{1}{2}$, we can choose $x \equiv x_0 \pmod 1$ so that

$$|Q| = |(x + hy)^2 - \beta^2| \leqq \tfrac{1}{2} < 1 < \left(\frac{\lambda^2}{16}\right)^{\frac{1}{3}} = (\tfrac{1}{4}|D|)^{\frac{1}{3}}.$$

Again, if $\beta^2 > \frac{1}{2}$, we can still choose x to satisfy the final inequality, provided

$$\sqrt{\beta^2 - \tfrac{1}{4}} \leqq \left(\frac{\lambda^2}{16}\right)^{\frac{1}{3}}.$$

Hence we may suppose that

$$\beta^2 > \tfrac{1}{4} + \left(\frac{\lambda^2}{16}\right)^{\frac{2}{3}}. \tag{52}$$

We now make a different choice of z, keeping the same y. We put

$$z' = z - 1 \quad \text{or} \quad z + 1 \quad \text{according as} \quad y \geq 0 \quad \text{or} \quad y < 0.$$

Then

$$-\lambda y (z' + ly) = \lambda |y| - \beta^2 \geq 0$$

by (50) and (51). We have

$$Q(x, y, z') = (x + hy)^2 - \lambda y (z' + ly) = (x + hy)^2 + \lambda |y| - \beta^2.$$

Choosing $x \equiv x_0 \pmod 1$ so that $|x + hy| \leqq \frac{1}{2}$, we obtain

$$|Q| \leqq \tfrac{1}{4} + \lambda |y| - \beta^2 \leqq \tfrac{1}{4} + \tfrac{1}{2}\lambda - \beta^2 < \tfrac{1}{2}\lambda - \left(\frac{\lambda^2}{16}\right)^{\frac{2}{3}}.$$

by (52). Writing $\lambda = 4\mu^3$, we have

$$|Q| \leqq 2\mu^3 - \mu^4 = \mu^2 - (\mu - \mu^2)^2 \leqq \mu^2 = \left(\frac{\lambda^2}{16}\right)^{\frac{1}{3}}.$$

Again (48) is satisfied, and the proof of Lemma 11 is complete.

5. Proof of Theorem 2. The main assertion of Theorem 2 follows from Lemmas 7, 9 and 11. That the constant $\frac{1}{4}$ is the best possible follows from the example

$$Q(x, y, z) = 4x^2 + y^2 - z^2, \qquad x_0 = y_0 = z_0 = \tfrac{1}{2}.$$

This is plainly a null form, and since $D = -4$, it suffices to prove that $|Q| \geqq 1$ for $x \equiv y \equiv z \equiv \frac{1}{2} \pmod 1$. This is the same as saying that

$$|4X^2 + Y^2 - Z^2| \geqq 4$$

if X, Y, Z are odd integers. Since $Y^2 - Z^2 \equiv 0 \pmod 8$, and $4X^2 \equiv 4 \pmod 8$, the result is immediate.

6. We now prepare for the proof of Theorem 3. The vital weapon is Lemma 14, an extension of Lemma 3, which asserts that the result of that Lemma can be substantially improved when $\frac{\nu}{\mu}$ is approximately 2, provided that $L_1 L_2$ is not equivalent to a positive multiple of the special form (13). This is a remarkable result, for, as we have already observed[1], such a situation does *not* arise in the case of Minkowski's original theorem, where $\nu = \mu$. As might be expected, the arguments which lead to the proof of Lemma 14, though elementary, are of a delicate nature.

Lemma 12. *Suppose that R, S are real numbers satisfying*

(53) $$5 < R < 6, \qquad \tfrac{1}{2} < S < 2.$$

Let α, β be real numbers, satisfying

(54) $$|\alpha| < 10^{-6}, \qquad |\beta| < 10^{-6}.$$

Suppose that neither of the inequalities

(55) $$10^{-5} < \alpha R^{2n} - \beta R^n < 1,$$

(56) $$10^{-5} < -\alpha R^{2n} + \beta R^n S < 1$$

is satisfied by any positive integer n. Then

(57) $$\alpha = \beta = 0.$$

Proof. The result is immediate if $\beta = 0$. For suppose $\alpha > 0$. Since $\alpha R^2 < 1$, there exists a positive integer n such that

$$\alpha R^{2n} < 1 \leqq \alpha R^{2n+2}.$$

Then $\alpha R^{2n} \geqq R^{-2} > 10^{-5}$, and (55) is satisfied, contrary to hypothesis. Similarly if $\alpha < 0$, we get a contradiction on using (56).

We may therefore suppose that $\beta \neq 0$. In fact, we may suppose that

(58) $$\beta > 0.$$

For if we replace α, β, S by α', β', S', defined by

$$\alpha = -\alpha', \qquad \beta = -\frac{\beta'}{S}, \qquad S = \frac{1}{S'},$$

the hypotheses are unaltered, except for a slight change in the second half of (54). In fact, this is never used in anything approaching its full strength.

[1] See footnote on p. 67.

Our first step is to deduce from (58) that $\alpha > 0$. Suppose that $\alpha \leqq 0$. Then

$$R \leqq \frac{(-\alpha) R^{2n+2} + \beta R^{n+1} S}{(-\alpha) R^{2n} + \beta R^n S} < R^2.$$

Hence there exists a positive integer n such that

$$R^{-2} < (-\alpha) R^{2n} + \beta R^n S < 1,$$

and (56) is satisfied, contrary to hypothesis.

We now have $\alpha > 0$, $\beta > 0$. The inequality

$$10^{-5} < \alpha x^2 - \beta x < 1, \tag{59}$$

corresponding to (55), is satisfied if x lies between the larger root of $\alpha x^2 - \beta x - 10^{-5} = 0$ and the larger root of $\alpha x^2 - \beta x - 1 = 0$, i. e. if

$$\frac{1}{2\alpha}\{\beta + \sqrt{\beta^2 + 4\alpha(10^{-5})}\} < x < \frac{1}{2\alpha}\{\beta + \sqrt{\beta^2 + 4\alpha}\}. \tag{60}$$

Note that the upper bound here is greater than $\alpha^{-\frac{1}{2}}$, and so greater than 1000 by (54). If (59) has no solution of the form $x = R^n$, the ratio of the two bounds in (60) cannot exceed R. We have, therefore

$$\frac{\beta + \sqrt{\beta^2 + 4\alpha}}{\beta + \sqrt{\beta^2 + 4\alpha(10^{-5})}} \leqq R < 6,$$

$$\sqrt{\beta^2 + 4\alpha} - 6\sqrt{\beta^2 + 4\alpha(10^{-5})} < 5\beta < 5\sqrt{\beta^2 + 4\alpha(10^{-5})},$$

$$\beta^2 + 4\alpha < 121(\beta^2 + 4\alpha(10^{-5})),$$

$$\alpha\{1 - 121(10^{-5})\} < 30\beta^2,$$

$$\alpha < 31\beta^2. \tag{61}$$

Since $\beta R S < 10^{-4}$, there exists a positive integer n such that

$$10^{-4} R^{-1} \leqq \beta R^n S < 10^{-4}. \tag{62}$$

By (61) and (62),

$$\alpha R^{2n} < 31\beta^2 R^{2n} < 31(10^{-4} S^{-1})^2 < 124(10^{-8}).$$

Hence

$$10^{-4} > -\alpha R^{2n} + \beta R^n S > 10^{-4} 6^{-1} - 124(10^{-8}) > 10^{-5}.$$

Thus (56) is satisfied, contrary to hypothesis. This proves Lemma 12.

Lemma 13. *Let $\xi_1, \xi_2, \eta_1, \eta_2$ be real numbers, each having absolute value less than 10^{-7}. Suppose that, for all integers p, q, the product*

(63) $$\Pi = \{p + q + 1 + (p-q)\sqrt{2} + p\xi_1 + q\xi_2\}\{p + q + 1 - (p-q)\sqrt{2} + p\eta_1 + q\eta_2\}$$

satisfies

(64) $$\textit{either}\quad \Pi > 1 - 10^{-7} \quad \textit{or} \quad \Pi < -2(1-10^{-7}).$$

Then

(65) $$\xi_1 = \xi_2 = \eta_1 = \eta_2 = 0.$$

Proof. The hypothesis is unaltered if we replace $\xi_1, \xi_2, \eta_1, \eta_2$ by $\eta_2, \eta_1, \xi_2, \xi_1$, since this is tantamount to interchanging p and q. Hence it will suffice to prove that $\eta_1 = \eta_2 = 0$.

Let

(66) $$R = 3 + 2\sqrt{2} = 5.82\ldots.$$

We first use the hypothesis with the following choice of p and q:

(67) $$\begin{cases} p = \dfrac{1}{4\sqrt{2}}\{(\sqrt{2}+1)R^n + (\sqrt{2}-1)R^{-n} - 2\sqrt{2}\}, \\[2ex] q = \dfrac{1}{4\sqrt{2}}\{(\sqrt{2}-1)R^n + (\sqrt{2}+1)R^{-n} - 2\sqrt{2}\}, \end{cases}$$

where n is a positive integer. We note that $R^n = u + v\sqrt{2}$, $R^{-n} = u - v\sqrt{2}$, where u is odd and v even. Hence $R^n + R^{-n} - 2$ is a multiple of 4, and $R^n - R^{-n}$ is a multiple of $4\sqrt{2}$. Consequently the above values of p and q are integers. Further, we have

$$p + q + 1 + (p-q)\sqrt{2} = R^n, \qquad p + q + 1 - (p-q)\sqrt{2} = R^{-n}.$$

We substitute in (63), and suppose that both factors are positive (a condition which is certainly satisfied by small values of n). On dividing the first factor by R^n, and the second by R^{-n}, we obtain, by (64),

(68) $$\{1 + R^{-n}(p\xi_1 + q\xi_2)\}\{1 + R^n(p\eta_1 + q\eta_2)\} > 1 - 10^{-7},$$

provided that both factors are positive. By (67), $0 < p < R^n$ and $0 \leqq q < R^n$, hence the first factor in (68) is certainly positive, and does not exceed $1 + 2(10^{-7})$. Hence the second factor, if positive, is greater than

$$(1 - 10^{-7})(1 + 2(10^{-7}))^{-1} > 1 - 10^{-6}.$$

In other words, it is impossible that

$$-1 < R^n(p\eta_1 + q\eta_2) < -10^{-6}$$

for any positive integer n. By (67),

$$4\sqrt{2}R^n(p\eta_1 + q\eta_2) = R^{2n}\{(\sqrt{2}+1)\eta_1 + (\sqrt{2}-1)\eta_2\} - 2\sqrt{2}R^n(\eta_1+\eta_2) + \{(\sqrt{2}-1)\eta_1 + (\sqrt{2}+1)\eta_2\}.$$

The last term on the right is numerically less than $2\sqrt{2}(10^{-7})$. Hence it is impossible that

$$-4\sqrt{2} + 2\sqrt{2}(10^{-7}) < R^{2n}\{(\sqrt{2}+1)\eta_1 + (\sqrt{2}-1)\eta_2\} - 2\sqrt{2}R^n(\eta_1+\eta_2) < -4\sqrt{2}(10^{-6}) - 2\sqrt{2}(10^{-7}).$$

We define α, β by

(69) $$\alpha = (\sqrt{2}+1)\eta_1 + (\sqrt{2}-1)\eta_2, \qquad \beta = 2(\eta_1+\eta_2).$$

It follows from the above that there is no positive integer n for which

(70) $$10^{-5} < -R^{2n}\alpha + \sqrt{2}R^n\beta < 1.$$

We now apply the hypothesis with another choice of p and q. Define p and q by

(71) $$p = \tfrac{1}{4}\{(\sqrt{2}+1)R^n - (\sqrt{2}-1)R^{-n} - 2\},$$
$$q = \tfrac{1}{4}\{(\sqrt{2}-1)R^n - (\sqrt{2}+1)R^{-n} - 2\}.$$

For the same reasons as before, these values of p and q are integers. We have

(72) $$p+q+1+(p-q)\sqrt{2} = \sqrt{2}R^n, \qquad p+q+1-(p-q)\sqrt{2} = -\sqrt{2}R^{-n}.$$

We substitute in (64), and suppose that the two factors have opposite signs (a condition which is certainly satisfied for small values of n). We obtain, on division by the two expressions in (72),

$$\left\{1 + \frac{1}{\sqrt{2}}R^{-n}(p\xi_1 + q\xi_2)\right\}\left\{1 - \frac{1}{\sqrt{2}}R^n(p\eta_1 + q\eta_2)\right\} > 1 - 10^{-7}.$$

By (71), $0 < p < R^n$ and $0 \leqq q < R^n$, hence the first factor is necessarily positive and does not exceed $1 + \sqrt{2}(10^{-7})$. Hence there is no positive integer n for which

$$10^{-6} < \frac{1}{\sqrt{2}}R^n(p\eta_1 + q\eta_2) < 1.$$

By (71),

$$4R^n(p\eta_1+q\eta_2)=R^{2n}\{(\sqrt{2}+1)\eta_1+(\sqrt{2}-1)\eta_2\}-2R^n(\eta_1+\eta_2)$$
$$-\{(\sqrt{2}-1)\eta_1+(\sqrt{2}+1)\eta_2\}.$$

The last term on the right is numerically less than $2\sqrt{2}(10^{-7})$. Hence it is impossible that

$$4\sqrt{2}(10^{-6})+2\sqrt{2}(10^{-7})<R^{2n}\alpha-R^n\beta<4\sqrt{2}-2\sqrt{2}(10^{-7}),$$

where α,β are defined, as before, by (69). It follows that there is no positive integer n for which

$$10^{-5}<R^{2n}\alpha-R^n\beta<1. \tag{73}$$

The hypotheses of Lemma 12 are satisfied, with $S=\sqrt{2}$. For (53) follows from (66); (54) follows from (69) and the initial hypothesis concerning the magnitude of η_1 and η_2; and (55), (56) are identical with (70) and (73). It follows from Lemma 12 that $\alpha=\beta=0$, and so, from (69), that $\eta_1=\eta_2=0$. As we saw at the beginning of the proof, this suffices to establish (65).

Lemma 14. *Suppose that $\gamma<10^{-17}$. Let μ,ν be any positive numbers satisfying*

$$\left|\frac{\nu}{\mu}-2\right|<\gamma, \tag{74}$$

$$\mu\nu>\tfrac{1}{16}(1-\gamma). \tag{75}$$

Then, with the notation of Lemma 3, either there exist integers x, y such that

$$-\nu\Delta\leqq(L_1+c_1)(L_2+c_2)\leqq\mu\Delta, \tag{76}$$

or the quadratic form L_1L_2 is equivalent to a positive multiple of the special form (13).

Proof. We may suppose without loss of generality that $\Delta=1$. We assume that (76) has no solution in integers x, y, and prove that the alternative conclusion must hold.

We proceed as in the proof of Lemma 3. The condition $\nu>\frac{1}{4}$ is satisfied, by (74) and (75). We obtain a lattice in the plane, of determinant $(1-\varepsilon)/M$, where

$$M\geqq\mu, \tag{77}$$

such that every lattice point satisfies

(78) $$(X+1)(Y+1) \geqq 1-\varepsilon \quad \text{or} \quad \leqq -K^2,$$

where K is defined by

(79) $$K^2 = \frac{\nu(1-\varepsilon)}{M} \qquad (K > 0).$$

Again there is no lattice point, except the origin, in the rectangle

(80) $$|X+Y| < 2\{1+\sqrt{1-\varepsilon}\}, \qquad |X-Y| < 2K,$$

and it follows that

$$\frac{1-\varepsilon}{M} \geqq 2K\{1+\sqrt{1-\varepsilon}\}.$$

Hence, as before,

$$M \leqq \frac{1-\varepsilon}{4\nu\{1+\sqrt{1-\varepsilon}\}^2} \leqq \frac{1}{16\nu}.$$

By (75) and (77), we have

(81) $$\mu \leqq M < \frac{\mu}{1-\gamma}.$$

We note, for future reference, that

(82) $$K^2 = \frac{\nu(1-\varepsilon)}{M} > \frac{\nu(1-\varepsilon)(1-\gamma)}{\mu} > (1-\varepsilon)(1-\gamma)(2-\gamma) > 2-4\gamma,$$

by (79), (81), (74).

The rectangle defined by

(83) $$|X+Y| < H, \qquad |X-Y| < 2K$$

has area $4HK$, and so must contain a lattice point other than the origin, provided H satisfies

$$HK > \frac{1-\varepsilon}{M}.$$

By (79), (81), (75),

$$\frac{1-\varepsilon}{KM} = \sqrt{\frac{1-\varepsilon}{M\nu}} \leqq \frac{1}{\sqrt{\mu\nu}} < \frac{4}{\sqrt{1-\gamma}}.$$

Hence, if we take $H = 4(1-\gamma)^{-\frac{1}{2}}$, there is a lattice point other than the origin satisfying (83). Since such a lattice point cannot satisfy (80), there must exist a lattice point satisfying

(84) $$2\{1+\sqrt{1-\varepsilon}\} \leqq X+Y < \frac{4}{\sqrt{1-\gamma}}, \qquad |X-Y| < 2K.$$

We have

$$4(X-1)(Y-1)=(X+Y-2)^2-(X-Y)^2>-4K^2,$$

and consequently, by (78), applied to $-X, -Y$ we must have

$$(X+Y-2)^2-(X-Y)^2 \geqq 4(1-\varepsilon).$$

This implies

$$(X-Y)^2<\left(\frac{4}{\sqrt{1-\gamma}}-2\right)^2-4(1-\varepsilon)<10\gamma.$$

Hence, writing $X=2+\xi$, $Y=2+\eta$, we have

$$|\xi+\eta|<\frac{4}{\sqrt{1-\gamma}}-4<3\gamma<\sqrt{10\gamma},$$

$$|\xi-\eta|<\sqrt{10\gamma}.$$

So we have proved that there exists a lattice point

(85) $$U=(2+\xi,\ 2+\eta) \quad \text{with} \quad |\xi|<\sqrt{10\gamma},\ |\eta|<\sqrt{10\gamma}.$$

Next we consider the rectangle defined by

$$|X+Y|<2\{1+\sqrt{1-\varepsilon}\}, \qquad |X-Y|<2L,$$

where L is so chosen that the area exceeds $4(1-\varepsilon)/M$. For this it suffices that

$$2\{1+\sqrt{1-\varepsilon}\}L>\frac{1-\varepsilon}{M}.$$

Now

$$\frac{1-\varepsilon}{2M\{1+\sqrt{1-\varepsilon}\}} \leqq \frac{1}{4M} \leqq \frac{1}{4\mu}=\frac{1}{4\sqrt{\mu\nu}}\sqrt{\frac{\nu}{\mu}}<\sqrt{\frac{2+\gamma}{1-\gamma}}<\sqrt{2}(1+\gamma),$$

by (81), (74), (75). Hence it will suffice to take $L=\sqrt{2}(1+\gamma)$, and we obtain the existence of a lattice point, other than the origin satisfying

(86) $$0 \leqq X+Y<2\{1+\sqrt{1-\varepsilon}\}, \qquad 2K \leqq |X-Y|<2\sqrt{2}(1+\gamma).$$

Now

$$4(X-1)(Y-1)=(X+Y-2)^2-(X-Y)^2 \leqq 4-4K^2<0.$$

Hence (78) gives

$$(X+Y-2)^2-(X-Y)^2 \leqq -4K^2,$$

and, by (82),

$$(X+Y-2)^2<8(1+\gamma)^2-4(2-4\gamma)<40\gamma.$$

We suppose that $X \geqq Y$, and refer later to the possibility $X < Y$. Writing

$$X = 1 + \sqrt{2} + \xi_1, \qquad Y = 1 - \sqrt{2} + \eta_1,$$

we have

(87) $$|\xi_1 + \eta_1| < \sqrt{40\gamma},$$

and, by (86),

$$2K - 2\sqrt{2} \leqq \xi_1 - \eta_1 < 2\sqrt{2}\,\gamma.$$

By (82), $2K > 2\sqrt{2 - 4\gamma} > 2\sqrt{2} - 4\gamma$, so that

(88) $$|\xi_1 - \eta_1| < 4\gamma.$$

It follows from (87) and (88) that $|\xi_1| < 10\sqrt{\gamma}$, $|\eta_1| < 10\sqrt{\gamma}$. We have now established the existence of a lattice point

(89) $$P_1 = (1 + \sqrt{2} + \xi_1,\ 1 - \sqrt{2} + \eta_1) \quad \text{with} \quad |\xi_1| < 10\sqrt{\gamma},\ |\eta_1| < 10\sqrt{\gamma}.$$

The point $P_2 = U - P_1$ is of the form

(90) $$P_2 = (1 - \sqrt{2} + \xi_2,\ 1 + \sqrt{2} + \eta_2) \quad \text{with} \quad |\xi_2| < 20\sqrt{\gamma},\ |\eta_2| < 20\sqrt{\gamma}$$

from (85) and (89). A pair of points satisfying the same conditions is obtained if we adopt the possibility $X < Y$ above, except that $20\sqrt{\gamma}$ and $10\sqrt{\gamma}$ would be interchanged in (89) and (90).

The points P_1 and P_2 generate the lattice. For the determinant of their coordinates is nearly $4\sqrt{2}$, and the determinant of the lattice is

$$\frac{1-\varepsilon}{M} \geqq 2K\{1 + \sqrt{1-\varepsilon}\} > 4\sqrt{2}(1 - 3\gamma),$$

by (82) and the inequality following (80). Hence it is impossible that the former should be a multiple of the latter by an integral factor greater than 1.

The general point of the lattice is (X, Y), where

$$X = p(1 + \sqrt{2} + \xi_1) + q(1 - \sqrt{2} + \xi_2), \quad Y = p(1 - \sqrt{2} + \eta_1) + q(1 + \sqrt{2} + \eta_2),$$

where p, q are arbitrary integers. By (78) and (82), the hypothesis (64) of Lemma 13 is satisfied, provided that

$$\varepsilon < 10^{-7} \quad \text{and} \quad 2\gamma < 10^{-7},$$

which is so. The initial hypothesis of Lemma 13 is satisfied, by (89) and (90), provided that

$$20\sqrt{\gamma} < 10^{-7}, \quad \text{i.e.} \quad \gamma < \tfrac{1}{4}(10^{-16}),$$

which is so. Hence, by Lemma 13,

$$\xi_1 = \xi_2 = \eta_1 = \eta_2 = 0.$$

and the lattice is given by

$$X = p(1+\sqrt{2}) + q(1-\sqrt{2}) = 2u + (1+\sqrt{2})v,$$

$$Y = p(1-\sqrt{2}) + q(1+\sqrt{2}) = 2u + (1-\sqrt{2})v,$$

where u, v take all integral values. It follows, as in the proof of Lemma 3, that $L_1 L_2$ is equivalent to a positive multiple of the quadratic form (13).

Lemma 15. *Suppose that* $0 < \delta_1 < 10^{-3}$. *Let*

$$Q(x, y, z) = (x + hy + gz)^2 + 5M(y^2 + yz - \tfrac{1}{4}z^2), \tag{91}$$

where

$$1 - \delta_1 < M < 1 + \delta_1. \tag{92}$$

Suppose that every value of $Q(x, y, z)$, *arising from integers* x, y, z *whose highest common factor is* 1, *which satisfies*

$$0 < Q < 3, \tag{93}$$

necessarily satisfies

$$1 - \delta_1 < Q < 1 + \delta_1. \tag{94}$$

Then

$$h \equiv 0 \pmod 1) \quad \textit{and} \quad g \equiv \tfrac{1}{2} \pmod 1). \tag{95}$$

Proof. On writing $x = \pm x' + my + nz$, where m and n are integers, it is clear that there is no loss of generality in supposing that

$$0 \leqq g \leqq \tfrac{1}{2}, \qquad |h| \leqq \tfrac{1}{2}. \tag{96}$$

We have

$$Q(2, 0, -1) = (2-g)^2 - \tfrac{5}{4}M > \tfrac{9}{4} - \tfrac{5}{4}(1 + \delta_1) > 0,$$

and

$$(2-g)^2 - \tfrac{5}{4}M < 4 - \tfrac{5}{4}(1 - \delta_1) < 3.$$

Hence, by the hypothesis,

$$1 - \delta_1 < Q(2, 0, -1) < 1 + \delta_1,$$

which implies

$$(2-g)^2 < \tfrac{5}{4}M + 1 + \delta_1 < \tfrac{9}{4}(1 + \delta_1) < (\tfrac{3}{2} + \delta_1)^2.$$

Hence

(97) $$\tfrac{1}{2} - \delta_1 < g \leqq \tfrac{1}{2}.$$

If $h > 0$, we consider

$$Q(2, 1, -1) = (2 + h - g)^2 - \tfrac{5}{4} M,$$

which satisfies

$$Q > \tfrac{9}{4} - \tfrac{5}{4}(1 + \delta_1) > 0,$$

$$Q < (2 + \tfrac{1}{2} - \tfrac{1}{2} + \delta_1)^2 - \tfrac{5}{4}(1 - \delta_1) < 3.$$

Hence

$$(2 + h - g)^2 - \tfrac{5}{4} M < 1 + \delta_1,$$

whence

$$2 + h - g < \tfrac{3}{2} + \delta_1,$$

$$h < \delta_1.$$

If $h < 0$, we consider

$$Q(1, -1, 1) = (1 - h + g)^2 - \tfrac{5}{4} M,$$

which again satisfies $0 < Q < 3$, and obtain

$$1 - h + g < \tfrac{3}{2} + \delta_1,$$

$$-h < 2\delta_1.$$

Thus, in either case, we have

(98) $$|h| < 2\delta_1.$$

We write

(99) $$g = \tfrac{1}{2} - \bar{g}, \qquad 0 \leqq \bar{g} < \delta_1.$$

Let y, z be any integers for which

(100) $$y^2 + yz - \tfrac{1}{4} z^2 = -\tfrac{1}{4},$$

and for which z is odd. Let $x = \frac{1}{2}(3 - z)$, and consider

$$Q(x, y, z) = (\tfrac{3}{2} + hy - \bar{g}z)^2 - \tfrac{5}{4} M.$$

If y, z are such that

$$\sqrt{\tfrac{5}{4}(1 + \delta_1)} - \tfrac{3}{2} < hy - \bar{g}z < \sqrt{\tfrac{5}{4}(1 - \delta_1) + 3} - \tfrac{3}{2},$$

then $0 < Q < 3$. These conditions are certainly satisfied if

(101) $$|hy - \bar{g}z| < \tfrac{1}{4}.$$

The hypothesis tells us that, in this case,

$$1 - \delta_1 < (\tfrac{3}{2} + hy - \bar{g}z)^2 - \tfrac{5}{4} M < 1 + \delta_1,$$

whence

$$(102) \qquad |hy - \bar{g}z| < \delta_1.$$

Thus (101) implies (102).

We define y, z by

$$(103) \quad z = \frac{1}{2\sqrt{2}}\{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}\}, \quad 2y + z = \tfrac{1}{2}\{(\sqrt{2}+1)^{2n+1} - (\sqrt{2}-1)^{2n+1}\},$$

where n is any integer, positive or negative. Since $(\sqrt{2}+1)^{2n+1} = u + v\sqrt{2}$, where u, v are odd integers, and $(\sqrt{2}-1)^{2n+1} = -u + v\sqrt{2}$, the definitions make z an odd integer, and y an integer. Moreover,

$$(2y+z)^2 - 2z^2 = -1,$$

so that (100) is satisfied. It follows that, with these values of y and z, (101) implies (102). We have

$$hy - \bar{g}z = H(\sqrt{2}+1)^{2n+1} + G(\sqrt{2}-1)^{2n+1},$$

where

$$(104) \qquad H = \tfrac{1}{4}\left(1 - \frac{1}{\sqrt{2}}\right)h - \frac{1}{2\sqrt{2}}\bar{g}, \qquad G = -\tfrac{1}{4}\left(1 + \frac{1}{\sqrt{2}}\right)h - \frac{1}{2\sqrt{2}}\bar{g}.$$

We have proved that there is no integer n for which

$$(105) \qquad \delta_1 \leqq |H(\sqrt{2}+1)^{2n+1} + G(\sqrt{2}-1)^{2n+1}| < \tfrac{1}{4}.$$

By (98), (99), (104), we have

$$(106) \qquad |H| < 2\delta_1, \qquad |G| < 2\delta_1.$$

Suppose first that $|H| \geqq |G|$ and $H \neq 0$. We can find a positive integer n such that

$$(107) \qquad \frac{1}{8(\sqrt{2}+1)^2} \leqq |H|(\sqrt{2}+1)^{2n+1} < \tfrac{1}{8}.$$

For this value of n, we have

$$\frac{|G|(\sqrt{2}-1)^{2n+1}}{|H|(\sqrt{2}+1)^{2n+1}} \leqq \frac{1}{(\sqrt{2}+1)^{4n+2}} \leqq \{8(\sqrt{2}+1)^2|H|\}^2 < (100\,\delta_1)^2 < \tfrac{1}{100},$$

by (107) and (106). Hence (107) implies

$$\frac{99}{800(\sqrt{2}+1)^2} < |H(\sqrt{2}+1)^{2n+1} + G(\sqrt{2}-1)^{2n+1}| < \tfrac{101}{800}.$$

Hence (105) is satisfied, contrary to what had been proved. Similarly, if $|G| \geqq |H|$ and $G \neq 0$ we reach a contradiction, by giving n a suitable negative value. Hence $H = G = 0$, whence $h = g = 0$ from (104), and the Lemma is established.

Lemma 16. *With the same hypotheses as in Lemma 15, we have*

(108) $$M = 1.$$

Proof. By Lemma 15, it suffices to consider the form

$$Q(x, y, z) = (x + \tfrac{1}{2} z)^2 + 5 M (y^2 + yz - \tfrac{1}{4} z^2),$$

where M satisfies (92). We have

$$Q(x, 0, z) = x^2 + xz + \frac{1 - 5M}{4} z^2$$
$$= x^2 + xz - Nz^2,$$

where

(109) $$|N - 1| < \tfrac{5}{4} \delta_1.$$

Let $F_1, F_2, \ldots$ denote the Fibonacci numbers, defined by

$$F_1 = 1, \quad F_2 = 2, \quad F_{n+1} = F_n + F_{n-1} \qquad (n = 1, 2, \ldots).$$

Then $(F_n, F_{n+1}) = 1$ and

$$F_{n+1}^2 - F_{n+1} F_n - F_n^2 = (-1)^{n-1}.$$

Hence

$$Q(F_{2n}, 0, -F_{2n-1}) = F_{2n}^2 - F_{2n} F_{2n-1} - N F_{2n-1}^2$$

(110) $$= 1 - (N - 1) F_{2n-1}^2.$$

Since $(F_{2n+1}/F_{2n-1})^2 < 10$, we can, by (109), if $N \neq 1$, determine n so that

$$\tfrac{1}{10} < |N - 1| F_{2n-1}^2 < 1.$$

Then (110) gives a value of $Q(x, y, z)$, arising from integral values of x, y, z whose highest common factor is 1, satisfying $0 < Q < \frac{9}{10}$ or $\frac{11}{10} < Q < 2$. This is contrary to the hypothesis, and so $N = 1$, whence $M = 1$.

Lemma 17. *Suppose that* $0 < \delta_2 < 10^{-6}$, *and that* $\vartheta > 4/5$. *Then the inequality*

(111) $$(2\vartheta - 1)(2\vartheta + [\vartheta]^2) \leqq \tfrac{50}{27} \left(\frac{\vartheta}{1 - \delta_2}\right)^3$$

implies

(112) $$3(1 - 15\sqrt{\delta_2}) < \vartheta < 3.$$

Proof. If $\frac{1}{5} < \vartheta < 1$, the proof is the same as in Lemma 6, with trivial modifications. Now suppose that $\vartheta \geqq 1$. Then (111) implies

$$(2\vartheta - 1)(2\vartheta + (\vartheta - 1)^2) < \tfrac{50}{27}\left(\frac{\vartheta}{1-\delta_2}\right)^3,$$

or

$$54\vartheta^3 - 27\vartheta^2 + 54\vartheta - 27 < 50\left(\frac{\vartheta}{1-\delta_2}\right)^3,$$

whence

$$4\vartheta^3 - 27\vartheta^2 + 54\vartheta - 27 < 50\vartheta^3\{(1-\delta_2)^{-3} - 1\} < 200\,\delta_2\vartheta^3.$$

Thus

$$(1 - 3\vartheta^{-1})^2(4 - 3\vartheta^{-1}) < 200\,\delta_2.$$

Since $\vartheta^{-1} \leq 1$, this gives $|1 - 3\vartheta^{-1}| < \sqrt{200\,\delta_2}$, whence

$$1 - \sqrt{200\,\delta_2} < 3\vartheta^{-1} < 1 + \sqrt{200\,\delta_2},$$

and

$$3(1 - 15\sqrt{\delta_2}) < \vartheta < 3(1 + 15\sqrt{\delta_2}).$$

It is impossible that $3 \leqq \vartheta < 3(1 + 15\sqrt{\delta_2})$, for then (111) would give

$$(2\vartheta - 1)(2\vartheta + 9) \leqq \tfrac{50}{27}\left(\frac{\vartheta}{1-\delta_2}\right)^3,$$

which is obviously false, since the left hand side is at least 75 and the right hand side is less than

$$50\left(\frac{1 + 15\sqrt{\delta_2}}{1-\delta_2}\right)^3 < 50\left(\frac{1.015}{1 - 10^{-6}}\right)^3.$$

Hence (112) holds.

7. Proof of Theorem 3. We shall prove that Theorem 3 is true with

$$\delta = 10^{-40}. \tag{113}$$

We may suppose that we have a form $Q(x, y, z)$ and numbers x_0, y_0, z_0 such that, for all x, y, z satisfying

$$x \equiv x_0 \pmod{1}, \qquad y \equiv y_0 \pmod{1}, \qquad z \equiv z_0 \pmod{1}, \tag{114}$$

we have

$$|Q(x, y, z)| > (1 - \delta)\left(\tfrac{27}{100}|D|\right)^{\frac{1}{3}}. \tag{115}$$

We have to prove that Q is equivalent to a multiple of the special form (4). We may suppose that $D < 0$.

By Lemma 1, there exists a positive value a of $Q(x, y, z)$, arising from integral values of x, y, z whose highest common factor is 1, such that

$$a \leqq (4\,|D|)^{\frac{1}{3}}. \tag{116}$$

By Lemma 2, $Q(x, y, z)$ is equivalent to a multiple of

$$Q_1(x, y, z) = (x + hy + gz)^2 + \phi(y, z), \tag{117}$$

where $\phi(y, z)$ is an indefinite binary quadratic form, whose discriminant Δ^2 satisfies

$$\Delta^2 = \frac{4\,|D|}{a^3} \geqq 1. \tag{118}$$

Define d and ϑ by

$$d = (\tfrac{27}{50}\Delta^2)^{\frac{1}{3}}, \qquad \vartheta = d(1 - \delta); \tag{119}$$

and define positive numbers μ, ν by

$$\mu\Delta = \tfrac{1}{2}\vartheta - \tfrac{1}{4}, \qquad \nu\Delta = \tfrac{1}{2}\vartheta + \tfrac{1}{4}[\vartheta]^2. \tag{120}$$

It is impossible that there should exist y, z satisfying (114) such that

$$-\nu\Delta \leqq \phi(y, z) \leqq \mu\Delta. \tag{121}$$

For if they existed, we could determine x, as in the proof of Theorem 1, to satisfy (114) (with the transformed values of x_0, y_0, z_0) and would have

$$|(x + hy + gz)^2 + \phi(y, z)| \leqq \tfrac{1}{2}\vartheta = (1 - \delta)\tfrac{1}{2}d = (1 - \delta)(\tfrac{27}{400}\Delta^2)^{\frac{1}{3}}.$$

This contradicts (115) for the form Q_1 of determinant $-\frac{1}{4}\Delta^2$, and so contradicts (115) for the original form Q.

Since (121) has no solution, Lemma 4 tells us that $\mu\nu < \frac{1}{16}$, i.e.

$$(2\vartheta - 1)(2\vartheta + [\vartheta]^2) < \Delta^2 = \tfrac{50}{27}\left(\frac{\vartheta}{1 - \delta}\right)^3.$$

The condition $\vartheta > \frac{4}{5}$ of Lemma 17 is satisfied, by (119) and (118). Consequently Lemma 17, with $\delta_2 = \delta$, implies that

$$3(1 - 15\sqrt{\delta}) < \vartheta < 3. \tag{122}$$

From (119) we obtain

$$3(1 - 15\sqrt{\delta}) < d < 3(1 + 2\delta), \tag{123}$$

and

$$\Delta^2 = 50\left(\frac{d}{3}\right)^3 < 50(1 + 2\delta)^3 < 50(1 + 10\delta),$$

$$\Delta^2 = 50\left(\frac{d}{3}\right)^3 > 50(1 - 15\sqrt{\delta})^3 > 50(1 - 50\sqrt{\delta}).$$

Hence

(124) $$1 - 50\sqrt{\delta} < \frac{\Delta^2}{50} < 1 + 10\delta.$$

By (120) and (122),

$$\frac{\nu}{\mu} = \frac{2\vartheta + 4}{2\vartheta - 1},$$

(125) $$\left|\frac{\nu}{\mu} - 2\right| = \frac{2\,|\vartheta - 3|}{2\vartheta - 1} < \tfrac{1}{2}\,|\vartheta - 3| < 25\sqrt{\delta}.$$

Also, by (120) and (122),

$$16\mu\nu\Delta^2 = (2\vartheta - 1)(2\vartheta + 4) = 50 - (3 - \vartheta)(4\vartheta + 18)$$
$$> 50 - 30(45\sqrt{\delta}).$$

By (124),

(126) $$16\mu\nu > 50(1 - 27\sqrt{\delta})\Delta^{-2} > (1 - 27\sqrt{\delta})(1 + 10\delta)^{-1} > 1 - 30\sqrt{\delta}.$$

By (125) and (126), the hypotheses (74) and (75) of Lemma 14 are satisfied, if we take

$$\gamma = 30\sqrt{\delta}.$$

The condition $\gamma < 10^{-17}$ is satisfied, by (113). The statement that (121) cannot be satisfied is the same, apart from a slight change of notation, as the statement that (76) cannot be satisfied. Hence Lemma 14 tells us that $\phi(y, z)$ is equivalent to a positive multiple of $y^2 + yz - \frac{1}{4}z^2$. By (117), after an integral unimodular substitution on y, z, we can write

(127) $$Q_1(x, y, z) = (x + hy + gz)^2 + 5M(y^2 + yz - \tfrac{1}{4}z^2),$$

where $M > 0$. Comparing the determinant of this form with its known value $-\frac{1}{4}\Delta^2$, we have

(128) $$M = \left(\frac{\Delta^2}{50}\right)^{\frac{1}{2}} = \left(\frac{d}{3}\right)^{\frac{3}{2}}.$$

By (123),

$$(1 - 15\sqrt{\delta})^{\frac{3}{2}} < M < (1 + 2\delta)^{\frac{3}{2}},$$

hence

(129) $$|M - 1| < 30\sqrt{\delta}.$$

The preceding argument has been based on expressing the given form $Q(x, y, z)$ as equivalent to a positive multiple of a form $Q_1(x, y, z)$ of the type (117), derived by using any positive value a of $Q(x, y, z)$ which satisfied (116). We have seen that such a positive value must be such that Δ^2, defined by (118), satisfies (124). Since Q is equivalent to a positive multiple of Q_1, the corresponding result must hold for Q_1. The determinant of Q_1 being $-\frac{1}{4}\Delta^2$, it follows that any positive value a_1 of Q_1, which satisfies

$$a_1 \leqq (\Delta^2)^{\frac{1}{3}}, \tag{130}$$

must be such that Δ_1, defined by

$$\Delta_1^2 = \frac{\Delta^2}{a_1^3}, \tag{131}$$

satisfies

$$50(1 - 50\sqrt{\delta}) < \Delta_1^2 < 50(1 + 10\delta). \tag{132}$$

The condition (130) is certainly satisfied if $a_1 < 3$. Hence with this condition, we must have, by (124) and (132),

$$a_1^3 = \frac{\Delta^2}{\Delta_1^2} < \frac{1 + 10\delta}{1 - 50\sqrt{\delta}} < 1 + 60\sqrt{\delta},$$

$$a_1^3 = \frac{\Delta^2}{\Delta_1^2} > \frac{1 - 50\sqrt{\delta}}{1 + 10\delta} > 1 - 60\sqrt{\delta}.$$

Hence, any positive value a_1 of $Q_1(x, y, z)$, arising from integers x, y, z whose highest common factor is 1, which satisfies $a_1 < 3$, must satisfy

$$1 - 30\sqrt{\delta} < a_1 < 1 + 30\sqrt{\delta}.$$

Taking $\delta_1 = 30\sqrt{\delta}$, the hypotheses of Lemma 15 are all satisfied. It follows that $Q_1(x, y, z)$ is equivalent to

$$(x + \tfrac{1}{2}z)^2 + 5M(y^2 + yz - \tfrac{1}{4}z^2).$$

The hypotheses of Lemma 16 are satisfied, and hence $M = 1$.

Hence we have finally proved that $Q(x, y, z)$ is equivalent to a positive multiple of

$$x^2 + 5y^2 - z^2 + xz + 5yz,$$

and this completes the proof of Theorem 3.