PERTURBATIONS OF DISCONTINUOUS SOLUTIONS OF NON-LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS.

By

NORMAN LEVINSON[1])

of CAMBRIDGE, U. S. A.

1. The study of the solutions of the system

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, t, \varepsilon), \qquad i = 1, 2, \ldots n, \tag{1.0}$$

where the X_i are regular functions of ε for small ε is classical. More recently non-linear systems like (1.0) have been studied when one or more of the X_i has a pole at $\varepsilon = 0$, or what is equivalent, where ε or some power of ε occurs as the coefficient of the left member of one or more of the equations (1.0), [1, 2, 3, 4, 5, 6]. In this case the system when $\varepsilon = 0$ is of lower order than when $\varepsilon \neq 0$. In the studies [2, 4, 6] it is assumed that the system has a solution with a continuous derivative in case $\varepsilon = 0$ and conditions are given for this to be the case when $\varepsilon \neq 0$.

In applied mathematics there are cases where the system has only discontinuous "solutions" when $\varepsilon = 0$ and yet is known empirically that when $\varepsilon > 0$ the system has a continuous solution which approaches the discontinuous one as $\varepsilon \to 0$. This fact has been exploited by the Russian school of non-linear mechanics. Here a rigorous treatment will be given for a case where the system has a discontinuous "solution" when $\varepsilon = 0$. The main result has already been announced, without proof, [3]. Since [3] has appeared, a system with a discontinuous solution has been treated [5] by Tihonov. In Tihonov's treatment the "jump arcs" instead of being solutions of (2.2) must be straight lines. Also the existence of derivatives with respect to initial values is not considered.

The specific system we shall consider here is

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[1] John Simon Guggenheim Memorial Fellow on leave from the Massachusetts Institute of Technology.

$$(1.1) \qquad \frac{dx_i}{dt} = f_i \frac{du}{dt} + \varphi_i\,, \qquad i = 1, 2, \ldots, n\,, \qquad \varepsilon \frac{d^2u}{dt^2} + g\frac{du}{dt} + h = 0\,.$$

Here f_i, φ_i, g and h are functions of $x_1, x_2, \ldots, x_n, u, t$ and ε. They are continuous in ε for small $\varepsilon \geqq 0$. It will be convenient to use a vector notation and to denote the vector with components x_i by x, the vector with components f_i by f, and φ_i by φ. Thus we can write (1.1) as

$$(1.2) \qquad \frac{dx}{dt} = f\frac{du}{dt} + \varphi\,, \quad \varepsilon\frac{d^2u}{dt^2} + g\frac{du}{dt} + h = 0$$

where the first equation of (1.2) is a vector equation. The vector f is $f(x, u, t, \varepsilon)$ and similarly for the vector φ and the scalars g and h.

By the degenerate system we shall mean (1.2) with $\varepsilon = 0$. We observe that the degenerate system is of lower order than (1.2). We shall write the degenerate system as

$$(1.3) \qquad \frac{dy}{dt} = f\frac{dv}{dt} + \varphi\,, \qquad g\frac{dv}{dt} + h = 0$$

where y is a vector with components $y_1, \ldots, y_n$ and v is a scalar. In (1.3) f is $f(y, v, t, 0)$ and similarly for φ, g and h. Let us now consider a solution of (1.3) as a curve in the $n+2$ dimensional space (y, v, t). We assume that such a solution starts at a point, A. We observe that when the solution reaches a point on the hypersurface $g(y, v, t, 0) = 0$, a singular situation may prevail with regard to dv/dt.

This situation can be seen clearly by taking a very simple special case. We take the van der Pol equation with a change of time scale which may be written as

$$(1.4) \qquad \varepsilon\frac{d^2u}{dt^2} + (u^2-1)\frac{du}{dt} + u = 0$$

and we consider the related degenerate equation

$$(1.5) \qquad (v^2-1)\frac{dv}{dt} + v = 0\,.$$

The solution $v(t)$ of (1.5) which at $t = 0$ satisfies $v(0) > 1$ is readily obtainable. However all we need observe is that since

$$\frac{dv}{dt} = -\frac{v}{v^2-1} < 0\,,$$

v is decreasing. Thus for $v \geqq 1$

$$\frac{dv}{dt} < -\frac{1}{v^2(0)-1}$$

and after a finite elapse of t we have for some $t = t_1$, $v(t_1) = 1$. As $t \to t_1 - 0$, $\frac{dv}{dt} \to -\infty$. That is for $v = 1+0$, $dv/dt = -\infty$. Any attempt to continue the solution, as a continuous function of t, beyond $t = t_1$, fails since for $v = 1-0$, $dv/dt = +\infty$. Thus the solution cannot pass continuously from above $v = 1$ to below $v = 1$. Moreover since $v = 1$ is obviously not a solution of (1.5) the solution cannot be continued as $v = 1$. We note that $v = 1$ here corresponds to $g = 0$ in (1.3).

If we turn to (1.4) with $\varepsilon > 0$ we see that the line $u = 1$ offers no special difficulty. Thus a solution $u(t)$ of (1.4) obviously can be continued beyond a point where $u = 1+0$. Let $u = 1+0$ when $t = t_1 - 0$ and let us integrate (1.4) from $t_1 - \delta$ to $t_1 + \delta$ where $\delta > 0$ is small. We find

$$\varepsilon \frac{du}{dt}\Big]_{t_1-\delta}^{t_1+\delta} + \left(\frac{u^3}{3} - u\right)\Big]_{t_1-\delta}^{t_1+\delta} + \int_{t_1-\delta}^{t_1+\delta} u\, dt = 0 .$$

Now let us proceed heuristically. If as $\varepsilon \to 0$ we assume $\frac{du(t_1 \pm \delta)}{dt}$ approach finite limiting values and if we assume that $|u|$ remains uniformly bounded in the range $(t_1 - \delta, t_1 + \delta)$ then we get

$$\left(\frac{u^3}{3} - u\right)\Big]_{t_1-\delta}^{t_1+\delta} = O(\delta) .$$

Now letting $\delta \to 0$ and recalling that $u(t_1 - 0) = 1$, we get

$$\tfrac{1}{3}u^3(t_1+0) - u(t_1+0) + \tfrac{2}{3} = 0 .$$

Solving this last equation we get either $u(t_1+0) = 1$ or $u(t_1+0) = -2$. The value 1 we discard on the basis of our experience with (1.5) and we are led to investigate further the possibility $u(t_1+0) = -2$. Actually (1.4) can be investigated directly [1] and it is indeed found that as $\varepsilon \to 0$, solutions of (1.4) on reaching $u = 1+0$ tend to jump to $u = -2$. We shall not pursue this intuitive discussion further but rather proceed in § 2 to give a definition of a solution of (1.3) which may be discontinuous. The definition will be justified because we shall show that as $\varepsilon \to 0$ solutions of (1.2) tend to solutions, as we define them here, of (1.3).

As was observed in [3] the system (1.2) includes as a special case the system

$$(1.6) \qquad \frac{dx}{dt} = H(x, w, t, \varepsilon) , \qquad \varepsilon \frac{dw}{dt} = G(x, w, t, \varepsilon)$$

where x and H are vectors and H and G are continuous in ε for small $\varepsilon \geqq 0$. G and w are scalars. With ε not appearing on the right this is the system treated by Tihonov [5]. In case the right members of (1.6) are not linear in w the system can be brought to the form (1.2) (with $f \equiv 0$) simply by differentiating the last equation with respect to t. On the other hand if the right members of (1.6) are linear in w then, if we introduce the variable u given by $\frac{du}{dt} = w$, the system (1.6) assumes the form

$$\frac{dx}{dt} = f(x, t, \varepsilon)\frac{du}{dt} + \varphi(x, t, \varepsilon) \qquad \varepsilon\frac{d^2u}{dt^2} + g(x, t, \varepsilon)\frac{du}{dt} + h(x, t, \varepsilon) = 0$$

which is a special case of (1.2).

2. We shall now give the definition of a solution of (1.3). We consider the solution as a curve, S_0, in the $n+2$ dimensional space (y, v, t). At the point A let $t = \alpha$ and $g > 0$. For $\alpha \leqq t < \tau_1$, let $\big(y(t), v(t)\big)$ be a solution of (1.3) and let

$$g\big(y(t), v(t), t, 0\big) > 0\,, \quad \alpha \leqq t < \tau_1\,. \tag{2.0}$$

As $t \to \tau_1 - 0$ let $g \to 0$. The point $\big(y(\tau_1-0), v(\tau_1-0), \tau_1\big)$ we denote by B_1. We shall denote $v(\tau_1-0)$ by v_B and $y(\tau_1-0)$ by y_B. AB_1 is an arc of S_0. We assume that at B_1

$$I = \sum_{i=1}^{n} \frac{\partial g}{\partial y_i} f_i + \frac{\partial g}{\partial v} \neq 0\,. \tag{2.1}$$

Here $I = I_B$. The next arc of S_0 is B_1C_1 where B_1C_1 is a curve $y(v)$ in the hyperplane $t = \tau_1$ which satisfies the vector equations

$$\frac{dy}{dv} = f(y, v, \tau_1, 0)\,. \tag{2.2}$$

(The system (2.2) is (1.3) with t held constant and with the last equation of (1.3) omitted). The solution of (2.2) starts at B_1. We then consider (2.2) for increasing or decreasing v according as h at B is respectively negative or positive. We assume that the solution of (2.2) can be continued for increasing or decreasing v, depending as we have seen on the sign of h at B, until we reach the first value of $v \neq v_B$ for which

$$\int_{v_B}^{v} g\big(y(v), v, \tau_1, 0\big)\, dv = 0\,. \tag{2.3}$$

This value of v we call v_C and the point C_1 is given by $\big(y(v_C), v_C, \tau_1\big)$. At C_1 we assume

$$g\big(y(v_C), v_C, \tau_1, 0\big) = g_C > 0\,. \tag{2.4}$$

On the basis of remarks already made we have

$$(v_C - v_B)h_B < 0 \tag{2.5}$$

where $h_B = h(y_B, v_B, \tau_1, 0)$. (The integral corresponding to (2.3) in the case of the van der Pol equation we considered above is simply

$$\int_1^v (v^2-1)dv = 0$$

and since h at B_1 (where $v = 1$) is positive we must take v decreasing. Thus we see that here $v_C = -2$.)

At C_1 we return to the system (1.3) and consider the solution with initial values at C_1 and with t increasing. We assume that this solution can be continued with $g > 0$ until $t \to \tau_2 - 0$ where $g \to 0$. From this point which we denote by B_2 we assume we can proceed in the manner already indicated at B_1 with v increasing or decreasing depending on the sign of h at B_2. Proceeding in this way the solution S_0 is defined geometrically as $AB_1C_1B_2C_2 \ldots B_NC_NA'$ where A' is an ordinary point (i. e. one where $g > 0$). We assume that (1.3) has a solution S_0 as just defined for $\alpha \leqq t \leqq \beta$. We assume further that there exists an open set R in the $n+2$ dimensional space (y, v, t) containing the curve S_0 such that $f(y, v, t, \varepsilon)$, φ, g and h and their first order partial derivatives with respect to y_i, v and t are uniformly continuous and bounded as functions of y, v, t and ε when (y, v, t) is in R and $\varepsilon \geqq 0$ is small. We shall also assume that g and h have second order partial derivatives continuous in R and for small ε although this assumption can be avoided.

We see that S_0 considered as a curve (but not as a function of t) is continuous but that at the points C_1, C_2, etc. it has a discontinuous tangent. We see further that S_0 is the sum of two kinds of arcs, the arcs AB_1, C_1B_2, C_2B_3, etc. which are solutions of (1.3) and might be called regular arcs, and the arcs B_1C_1, B_2C_2, etc. which are solutions of (2.2) lying in planes $t = \tau_j$ and which might be called jump arcs since the arcs are traversed in a zero elapse of t. As a function of t the solution is discontinuous and jumps from B_j to C_j at $t = \tau_j$, $j = 1, 2, \ldots, N$.

The condition (2.1) can be weakened by allowing I to vanish on $g = 0$ but requiring that I be different from zero and of the same sign off $g = 0$ in the neighborhood of B. A somewhat simpler situation where there will be no jump at all arises when I changes sign in passing through $g = 0$. These cases will not be pursued further.

In what follows the norm of a vector is defined as $|x| = \Sigma|x_i|$.

The basic results for (1.2) are given in the following theorems.

Theorem 1. *Let the degenerate system* (1.3) *have a solution* S_0 *in the sense defined above, for* $\alpha \leqq t \leqq \beta$. *Let* $\varepsilon > 0$. *If* ε, δ_1 *and* δ_2 *are small enough there is a solution* $x(t)$, $u(t)$ *of* (1.2) *over* (α, β) *for any set of initial values satisfying*

$$|x(\alpha)-y(\alpha)|+|u(\alpha)-v(\alpha)| \leqq \delta_1$$

$$\left|\frac{du(\alpha)}{dt}-\frac{dv(\alpha)}{dt}\right| \leqq \frac{\delta_2}{\varepsilon}.$$

Moreover as ε, δ_1 *and* δ_2 *tend to zero, the curve representing the solution* $x(t)$, $u(t)$ *in* (x, u, t) *tends to* S_0. *In particular for any fixed* $\delta > 0$

$$|x(t)-y(t)|+|u(t)-v(t)| \tag{2.6}$$

tends uniformly to zero over the intervals, $\alpha \leqq t \leqq \tau_1-\delta$, $\tau_1+\delta \leqq t \leqq \tau_2-\delta, \ldots,$ $\tau_N+\delta \leqq t \leqq \beta$, *as* ε, δ_1 *and* $\delta_2 \to 0$. *Also*

$$\frac{du}{dt}-\frac{dv}{dt} \tag{2.7}$$

tends uniformly to zero over the intervals $\alpha+\delta \leqq t \leqq \tau_1-\delta$, $\tau_1+\delta \leqq t \leqq \tau_2-\delta, \ldots,$ $\tau_N+\delta \leqq t \leqq \beta$ *as* ε, δ_1 *and* $\delta_2 \to 0$. *The same is true for* $\frac{d^2u}{dt^2}-\frac{d^2v}{dt^2}$.

Theorem 2. *It is also the case that if* S_0 *is a solution of* (1.3) *for* $\alpha \leqq t \leqq \beta$ *then corresponding to any set of initial values sufficiently near* $y(\alpha)$, $v(\alpha)$ *there is a solution of* (1.3) *which tends to* S_0 *as the initial values tend to those of* S_0. *Moreover the convergence is uniform in t if the portions of* S_0 *between* $\tau_j \pm \delta$ *are omitted.*

Theorem 2 is in a sense Theorem 1 for the case $\varepsilon = +0$.

In what follows let us denote by $\frac{\partial}{\partial a}$ differentiation with respect to one of the $n+1$ initial coordinates $(x(\alpha), u(\alpha))$ or with respect to the corresponding initial coordinate of $(y(\alpha), v(\alpha))$. Then we have

Theorem 3. *Subject to the same hypothesis as Theorem 1 we have*

$$\left|\frac{\partial x(t)}{\partial a}-\frac{\partial y(t)}{\partial a}\right|+\left|\frac{\partial u(t)}{\partial a}-\frac{\partial v(t)}{\partial a}\right| \to 0\,, \tag{2.8}$$

$$\left|\frac{\partial}{\partial a}\frac{du}{dt}-\frac{\partial}{\partial a}\frac{dv}{dt}\right| \to 0 \tag{2.9}$$

for ε, δ_1, *and* $\delta_2 \to 0$, *the convergence being uniform for* (2.8) *and* (2.9) *over the same* *intervals of t as for* (2.6) *and* (2.7) *respectively. Moreover denoting the initial value of* du/dt *at* α *by* b *we have*

$$\left|\frac{\partial x(t)}{\partial b}\right|+\left|\frac{\partial u(t)}{\partial b}\right|\to 0$$

and

$$\left|\frac{\partial}{\partial b}\frac{du}{dt}\right|\to 0$$

uniformly over the same sets of intervals as for (2.6) *and* (2.7) *respectively as* ε, δ_1 *and* $\delta_2\to 0$.

Theorem 4. *The functions* $\dfrac{\partial y(t)}{\partial a}$ *and* $\dfrac{\partial v(t)}{\partial a}$ *are uniformly continuous with respect to changes in the initial values of* y *and* v *at* α *over the same set of intervals of* t *as in Theorem* 2.

Theorem 4 is in a sense Theorem 3 for the case $\varepsilon=+0$.

As an application of these theorems in case the right members do not contain t or in case they are periodic in t we have the fact that if the degenerate system (1.3) has a periodic solution and if the Jacobian associated with the determination of this solution by varying initial coordinates is different from zero, then it follows by Theorem 3 that (1.2) will also have a periodic solution. We shall show this in § 8.

3. We shall require several lemmas. The first is well known.

Lemma 1. *Let* $\zeta(t)$ *be a vector with an integrable derivative and let*

$$\left|\frac{d\zeta}{dt}\right|\leqq a(t)+b(t)|\zeta|\,,\quad t_0\leqq t\leqq t_1\,,\tag{3.0}$$

where $b(t)\geqq 0$. *Then for* $t_0\leqq t\leqq t_1$,

$$|\zeta(t)-\zeta(t_0)|\leqq|\zeta(t_0)|(e^{\int_{t_0}^t b(s)ds}-1)+\int_{t_0}^t a(\tau)e^{\int_\tau^t b(s)ds}d\tau\,.\tag{3.1}$$

Clearly (3.1) *implies*

$$|\zeta(t)|\leqq|\zeta(t_0)|e^{\int_{t_0}^t b(s)ds}+\int_{t_0}^t a(\tau)e^{\int_\tau^t b(s)ds}d\tau\,.\tag{3.2}$$

Corresponding results hold if $t_1<t_0$.

Proof. Let

$$Z(t) = \int_{t_0}^{t} \left| \frac{d\zeta}{dt} \right| dt .$$

Then $|\zeta(t)-\zeta(t_0)| \leqq Z(t)$ and (3.0) becomes

$$\frac{dZ}{dt} \leqq a(t)+b(t)Z+b(t)|\zeta(t_0)| .$$

Thus

$$\left(\frac{dZ}{dt}-b(t)Z\right) e^{-\int_{t_0}^{t} b(s)ds} \leqq (a(t)+b(t)|\zeta(t_0)|)e^{-\int_{t_0}^{t} b(s)ds} .$$

Thus integrating we find

$$Z(t)e^{-\int_{t_0}^{t} b(s)ds} \leqq \int_{t_0}^{t} (a(\tau)+b(\tau)|\zeta(t_0)|)e^{-\int_{t_0}^{\tau} b(s)ds}\, d\tau$$

from which (3.1) follows.

The next lemma is very similar to one Friedrichs and Wasow [2]. Here z is a vector and w is a scalar and $L(z, w, t)$ and $M(z, w, t)$ are vectors while $H(z, w, t)$ and $J(z, w, t)$ are scalars.

Lemma 2. *Let $z(t)$, $w(t)$ satisfy*

$$\frac{dz}{dt} = L\frac{dw}{dt}+M , \qquad \varepsilon\frac{d^2w}{dt^2}+H\frac{dw}{dt} = J \tag{3.3}$$

for $\alpha \leqq t \leqq \gamma$ where L, M, H and J are continuous in the region given by $\alpha \leqq t \leqq \gamma$ and $|z|+|w| \leqq \lambda$ for some $\lambda > 0$. In this region let

$$|L| \leqq k,\ |M| \leqq k(|z|+|w|)+\varepsilon_1,\ |J| \leqq k(|z|+|w|)+\varepsilon_1,\ H \geqq m > 0 . \tag{3.4}$$

Moreover at $t = \alpha$ let

$$|z(\alpha)|+|w(\alpha)| \leqq \delta_1 , \quad \left|\frac{dw(\alpha)}{dt}\right| \leqq \frac{\delta_2}{\varepsilon}. \tag{3.5}$$

Let $k_1 = k(k+m+1)/m$ and let

$$\Delta = \frac{4(k+1)^2(m+1)}{mk}(\delta_1+\delta_2+\varepsilon_1) .$$

Then for $\alpha \leqq i \leqq \gamma$, and if ε_1, δ_1, and δ_2 are small enough

$$|z(t)|+|w(t)| \leqq \Delta e^{k_1(t-\alpha)}$$

and

$$\left|\frac{dw}{dt}\right| \leqq \frac{\delta_2}{\varepsilon} e^{-m(t-\alpha)/\varepsilon} + \frac{k\Delta}{m} e^{k_1(t-\alpha)} .$$

Proof of Lemma 2. From (3.3) we have if we set $\frac{dw}{dt} = \theta$,

$$\frac{dz}{dt} = L\theta + M, \quad \frac{dw}{dt} = \theta, \quad \varepsilon\frac{d\theta}{dt} + H\theta = J . \tag{3.6}$$

Or

$$\left|\frac{dz}{dt}\right| + \left|\frac{dw}{dt}\right| \leqq (k+1)|\theta| + k(|z|+|w|) + \varepsilon_1 .$$

If ζ denotes the vector (z, w) then Lemma 1 yields

$$|z(t)|+|w(t)| \leqq \left(\frac{\varepsilon_1}{k}+\delta_1\right) e^{k(t-\alpha)} + (k+1)\int_\alpha^t |\theta(\tau)| e^{k(t-\tau)} d\tau . \tag{3.7}$$

From the last equation of (3.6)

$$\theta e^{\frac{1}{\varepsilon}\int_\alpha^t H(s)ds} = \theta(\alpha) + \frac{1}{\varepsilon}\int_\alpha^t J e^{\frac{1}{\varepsilon}\int_\alpha^\tau H(s)ds} d\tau .$$

Or since $H \geqq m > 0$

$$|\theta(t)| \leqq |\theta(\alpha)| e^{-m(t-\alpha)/\varepsilon} + \frac{1}{\varepsilon}\int_\alpha^t |J| e^{-m(t-\tau)/\varepsilon} d\tau .$$

Thus

$$|\theta(t)| \leqq \frac{\delta_2}{\varepsilon} e^{-m(t-\alpha)/\varepsilon} + \frac{k}{\varepsilon}\int_\alpha^t (|z(\tau)|+|w(\tau)|) e^{-m(t-\tau)/\varepsilon} d\tau + \frac{\varepsilon_1}{m}. \tag{3.8}$$

From (3.7)

$$|\theta(t)| \leqq \frac{\delta_2}{\varepsilon} e^{-m(t-\alpha)/\varepsilon} + \frac{\varepsilon_1}{m} + \frac{k}{m} e^{k(t-\alpha)}\left(\frac{\varepsilon_1}{k}+\delta_1\right) + \frac{k(k+1)}{\varepsilon}\int_\alpha^t e^{-m(t-\tau)/\varepsilon} d\tau \int_\alpha^\tau |\theta(\sigma)| e^{k(\tau-\sigma)} d\sigma .$$

From this

$$|\theta(t)| \leqq \frac{\delta_2}{\varepsilon} e^{-m(t-\alpha)/\varepsilon} + \frac{k}{m} e^{k(t-\alpha)}\left(\frac{2\varepsilon_1}{k}+\delta_1\right) + \frac{k(k+1)}{m} e^{k(t-\alpha)}\int_\alpha^t |\theta(\sigma)| e^{-k(\sigma-\alpha)} d\sigma . \tag{3.9}$$

Applying Lemma 1 with $\zeta = \int_\alpha^t |\theta(\sigma)| e^{-k(\sigma-\alpha)} d\sigma$ we get

$$\int_\alpha^t |\theta(\sigma)| e^{-k(\sigma-\alpha)} d\sigma \leqq e^{k(k+1)(t-\alpha)/m}\left(\delta_1 + \frac{\delta_2}{m} + \frac{2\varepsilon_1}{k}\right).$$

In (3.7) and (3.9) this proves the lemma providing ε_1, δ_1 and δ_2 are small enough so that $\Delta e^{k_1(\gamma-\alpha)} < \lambda$.

Before turning to the proof of Theorem 1 we make the following observation which is valid for Theorem 1, 2, 3 and 4. It suffices to take the case where there is only one jump arc on S_0 since by repeated use of the theorem for the case of one jump arc, the theorem for N jump arcs follows at once. Therefore we shall assume that S_0 is of the form AB_1C_1A'. Clearly there will be no confusion if we call S_0, $ABCA'$.

The proof of Theorem 1 is divided into four parts. In the first part we proceed from A to a point short of B; the second part involves the immediate vicinity of B; the third part takes the arc BC with the two small portions at ends B and C omitted; the fourth part takes the rest of S_0 to A'. We shall use K throughout the paper to represent finite constants which depend on the bounds of $|f|$, $|\varphi|$, $|g|$, $|h|$ and their first order partial derivatives in R or part of R for small ε, on the distance from S_0 to the nearest point on the boundary of R, and on the length of S_0. In particular the constants K will remain finite as $\varepsilon+\delta_1+\delta_2 \to +0$. Any deviation from this use of K will be noted.

Proof of Theorem 1, Part 1.

Here we prove Theorem 1 for the interval $\alpha \leqq t \leqq \gamma$ where $\gamma < \tau_1$. If we denote $x-y$ by z and $u-v$ by w we have from (1.2) and (1.3)

$$\frac{dz}{dt} = f(x, u, t, \varepsilon)\frac{dw}{dt}+F_1 \tag{3.10}$$

$$\varepsilon\frac{d^2w}{dt^2}+g(x, u, t, \varepsilon)\frac{dw}{dt} = F_2 \tag{3.11}$$

where F_1 is a vector and

$$F_1 = -\left[f(x, u, t, \varepsilon)-f(y, v, t, 0)\right]\frac{h(y, v, t, 0)}{g(y, v, t, 0)}+\varphi(x, u, t, \varepsilon)-\varphi(y, v, t, 0)$$

and

$$F_2 = \left[g(x, u, t, \varepsilon)-g(y, v, t, 0)\right]\frac{h(y, v, t, 0)}{g(y, v, t, 0)}-h(x, u, t, \varepsilon)$$
$$+h(y, v, t, 0)+\frac{\varepsilon}{g^2}\left[[g(y, v, t, 0)\frac{dh}{dt}-h(y, v, t, 0)\frac{dg}{dt}\right].$$

Let the minimum of $g(y(t), v(t), t, 0)$ over (α, γ) be denoted by $2m$ unless this minimum exceeds 1 in which case we take $m = 1$. Then so long as (x, u, t) is in R we have using the mean value theorem,

$$|F_1| \leqq \frac{K}{m}\left(|z|+|w|+|f(y, v, t, \varepsilon)-f(y, v, t, 0)|+|\varphi(y, v, t, \varepsilon)-\varphi(y, v, t, 0)|\right).$$

Since f and φ are uniformly continuous for small ε there must be a continuous function $\psi(\varepsilon)$ such that $\psi(0) = 0$ and such that

$$|f(y, v, t, \varepsilon) - f(y, v, t, 0)| \leqq \psi(\varepsilon)$$

with similar results for φ, g and h so long as (y, v, t) is in R.

Clearly we can choose $\psi(\varepsilon) > \varepsilon$. We get

$$|F_1| \leqq \frac{K}{m}\left(|z| + |w| + \psi(\varepsilon)\right) \tag{3.12}$$

and similarly computing $\frac{dh}{dt}$ and $\frac{dg}{dt}$ we find

$$|F_2| \leqq \frac{K}{m^3}\left(|z| + |w| + \psi(\varepsilon)\right). \tag{3.13}$$

We have $f(x, u, t, \varepsilon) = f(y(t)+z, v(t)+w, t, \varepsilon)$ and similarly for φ, g, and h. If we now apply Lemma 2 to (3.10) and (3.11) and make use of (3.12) and (3.13) we see that if ε, δ_1, and δ_2 as defined in the statement of Theorem 1 are small enough, (x, u, t) is in R for $\alpha \leqq t \leqq \gamma$, and indeed for $\alpha \leqq t \leqq \gamma$, at least so long as $g(x, u, t, \varepsilon) \geqq m$,

$$|x(t) - y(t)| + |u(t) - v(t)| \leqq e^{K/m^4}\left(\delta_1 + \delta_2 + \psi(\varepsilon)\right) \tag{3.14}$$

and

$$\left|\frac{du}{dt} - \frac{dv}{dt}\right| \leqq \frac{\delta_2}{\varepsilon} e^{-m(t-\alpha)/\varepsilon} + e^{K/m^4}\left(\delta_1 + \delta_2 + \psi(\varepsilon)\right). \tag{3.15}$$

From (3.14) and the continuity of g we see that we will indeed have $g(x, u, t, \varepsilon) \geqq m$ if ε, δ_1 and δ_2 are small enough. We see from (3.14) and (3.15) that for $t < \tau_1$, Theorem 1 is established except for the difference of the second derivatives of u and v. This we shall show in Lemma 4. The discontinuity at τ_1 is precisely the point of interest here and we begin to handle it in § 4.

4. In the next part of the proof of Theorem 1 we shall show that for small ε, δ_1 and δ_2 the solution of (1.2) intersects the hypersurface $g = 0$ at a point which tends to B as ε, δ_1 and $\delta_2 \to 0$. Moreover as ε, δ_1, $\delta_2 \to 0$, $\frac{du}{dt} \to \infty$ or $-\infty$ at the point of intersection.

Proof of Theorem 1, Part 2.

We shall consider here the case where $h(y, v, t, 0) < 0$ at B. The case where $h > 0$ is treated in exactly the same manner. At B we have $t = \tau_1$ and we shall designate y at B by y_B and v by v_B. As $t \to \tau_1 - 0$ we have, since $h < 0$, $\frac{dv}{dt} =$

$-\frac{h}{g} \to +\infty$. Let $v_1 < v_B$ be near enough to v_B so that as t increases from α toward τ_1 there is a value of $t = t_1$ near to τ_1 such that $v(t_1) = v_1$ and dv/dt is large for $t_1 \leqq t < \tau_1$. We shall denote the point $(y(t_1), v(t_1), t_1)$ by the letter Q. If we choose γ so that $t_1 < \gamma < \tau_1$, and apply the results (3.14) and (3.15) we see that if ε, δ_1 and δ_2 are small enough then for some t (which tends to t_1 as ε, δ_1, $\delta_2 \to 0$) we have $u(t) = v_1$. Let us denote this point by P. Then at P we have $t = t_P$, $x(t_P) = x_P$ and $u = u_P = u(t_P) = v_1$. Also as ε, δ_1, $\delta_2 \to 0$, $P \to Q$. Clearly we can choose Q as near to B as we wish.

We now change from t to v (and u) as the independent variable. Since when $v = u$ the values of t for the points on S_0 and the solution of (1.2) are not in general equal we will reserve t for the system (1.2) and in this section designate the variable t for (1.3) by the letter s. We have then that (1.2) can be written as

$$\frac{dx}{du} = f + \varphi p,\ p = \frac{dt}{du},\ \varepsilon\frac{dp}{du} = p^2 g + p^3 h \tag{4.0}$$

where $f = f(x, u, t, \varepsilon)$, etc. while (1.3) becomes

$$\frac{dy}{dv} = f + \varphi q,\ q = \frac{ds}{dv},\quad 0 = g + qh\,. \tag{4.1}$$

where $f = f(y, v, s, 0)$, etc. Since u and v are the independent variables here we can with no confusion use them interchangeably. We consider (4.0) for $u \geqq v_1$. The solution of (1.2) can now be regarded at least in certain range of u as a solution of (4.0).

Suppose $v_2 > v_B$. Let us choose

$$K_1 > 2(|f| + |\varphi| + 1) \tag{4.2}$$

for all (x, u, t) in R. (Clearly K_1 is a K). Let R_1 denote the region of (x, u, t) bounded by the planes $u = v_1$, and $u = v_2$ and by

$$|x - x_P| + |t - t_P| \leqq K_1(v_2 - v_1)\,.$$

Clearly if v_1 and v_2 are chosen near enough to v_B we have R_1 contained in R for small ε. If we consider the change in g as we follow a solution of (4.0) we have

$$\frac{dg}{du} = \sum \frac{\partial g}{\partial x_i}\frac{dx_i}{du} + \frac{\partial g}{\partial u} + \frac{\partial g}{\partial t}\frac{dt}{du}.$$

Or

$$\frac{dg}{du} = J_1(x, u, t, \varepsilon)+pJ_2(x, u, t, \varepsilon)$$

where

$$J_1 = \sum \frac{\partial g}{\partial x_i} f_i + \frac{\partial g}{\partial u}, \quad J_2 = \sum \frac{\partial g}{\partial x_i} \varphi_i + \frac{\partial g}{\partial t}.$$

We recall the definition of I in (2.1) and our assumption that $I \neq 0$. Since $g(y(v), v, s(v), 0) > 0$ for $v < v_B$ and zero at $v = v_B$ we see that $dg/dv \leqq 0$. This fact, the fact that $I \neq 0$ and the fact that dv/dt is large implies that $I < 0$. Let K_2 be chosen so that in R and for small ε, $|J_2| \leqq K_2$. If R_1 is small enough, that is if v_1 and v_2 are near enough to v_B, we certainly have $J_1 < \frac{1}{2}I$ and $h < \frac{1}{2}h_B$ for (x, u, t) in R_1 and for small ε.

Let v_1 be near enough to v_B so that

$$q(v_1) = -\frac{g_Q}{h_Q} < -\frac{I}{10K_2}.$$

This is possible since $g_Q \to 0$ as $v_1 \to v_B$. Let ε, δ_1 and δ_2 be so small that P is near enough to Q and p near enough to q at $u = v_1$ so that

$$0 < p(v_1) < -\frac{2g_P}{h_B} < -\frac{I}{4K_2}.$$

Now let us suppose that for our solution of (4.0) there is a v_3, $v_1 < v_3 < v_2$ such that for $v_1 \leqq u < v_3$, $p(u) < -2g_P/h_B$ but that for $u = v_3$ we have either $p = -2g_P/h_B$ or (x, u, t) reaches the boundary of R_1. We shall show that this is impossible. By integrating

$$\varepsilon \frac{dp}{p^2} = (g+ph)du \tag{4.3}$$

from v_1 to v_3 we see that $p > 0$ since if $p = 0$ the left side diverges. For $v_1 \leqq u < v_3$, since $0 < p < -2g_P/h_B$, we can assume $p < 1$ since we can take Q and therefore P near B where g vanishes. From (4.0)

$$|x-x_P|+|t-t_P| \leqq \int_{v_1}^{v_3} (|f|+|\varphi|+1)du < K_1(v_2-v_1) .$$

Thus $(x(u), t(u))$ is in R_1 for $v_1 \leqq u \leqq v_3$ and therefore we must have $p = -2g_P/h_B$ at $u = v_3$. Since $p < -2g_P/h_B < -I/4K_2$ for $v_1 \leqq u < v_3$, we have

$$\frac{dg}{du} = J_1+pJ_2 < \tfrac{1}{2}I+pK_2 \leqq \tfrac{1}{2}I-\tfrac{1}{4}I = \tfrac{1}{4}I < 0 . \tag{4.4}$$

Thus g is decreasing as u increases up to v_3. Since g is decreasing we have at v_3

$$g+ph < g_P+\tfrac{1}{2}ph_B \leqq g_P-g_P = 0 .$$

Thus by (4.3), $dp/du < 0$ at v_3. That is p is decreasing and therefore we cannot have $p = -2g_P/h_B$ for the first time at v_3.

We see then that our solution of (4.0) remains in R_1 and can be extended to $u = v_2$ and that $0 < p < -2g_P/h_B$ for $v_1 \leqq u \leqq v_2$. We can take v_1 as close to v_B as we wish. Thus Q can be as near B as we wish. By taking ε, δ_1 and δ_2 small enough we can bring P as close to Q, and therefore to B, as we wish. The nearer we take P to B the smaller is g_P and therefore the smaller is p for $v_1 \leqq u \leqq v_2$. Since $q = -g/h$ until $g = 0$ after which q is identically zero on S_0, until $v = v_C$, we see that q for $v_1 \leqq v \leqq v_2$ also gets smaller as we take v_1 nearer to v_B. Since u and v are independent variables for the respective systems we can identify u with v. Thus we can write our differential equations as

$$(4.5)\qquad \begin{aligned} \frac{dx}{dv} &= f(x, v, t, 0)+\omega_1 , \quad & \frac{dt}{dv} &= p ,\\ \frac{dy}{dv} &= f(y, v, s, 0)+\omega_2 , \quad & \frac{ds}{dv} &= q , \end{aligned}$$

where $\omega_1 = f(x, v, t, \varepsilon)-f(x, v, t, 0)+p\varphi(x, v, t, \varepsilon)$ and $\omega_2 = q\varphi(y, v, s, 0)$. Using the facts just enumerated we have, for $v_1 \leqq v \leqq v_2$

$$|\omega_1|+|\omega_2| \leqq K\big(g_P+g_Q+\psi(\varepsilon)\big)$$

where $\psi(\varepsilon)$ has the same properties as in § 3. Applying a standard theorem to (4.5) relating two approximate solutions of a system or else setting $\zeta = (x-y, t-s)$ and making use of Lemma 1 we have

$$|x(v)-y(v)|+|t(v)-s(v)| \leqq K\big[\psi(\varepsilon)+g_P+g_Q+|x(v_1)-y(v_1)|+|t(v_1)-s(v_1)|\big]$$

for $v_1 \leqq v \leqq v_2$. Since the right member can be made as small as we wish we have demonstrated Theorem 1 up to the intersection of the hyper-plane $v = v_2$ with S_0. We observe that v_2 can be kept fixed in the latter part of our argument as ε, δ_1, and $\delta_2 \to 0$ while v_1 must approach v_B.

Proof of Theorem 1, Part 3. Here we demonstrate Theorem 1 over the part of the jump arc BC beginning with $v = v_2$ and ending short of the point C.

Since $I < 0$ and $g_P > 0$ and is small, we see that (4.4) implies that the solution of (4.0) crosses the hypersurface $g = 0$ in exactly one point which is near B. Let us designate the point where the solution crosses $g = 0$ by the subscript 4. We have by integrating (4.3)

$$\varepsilon\left(\frac{1}{p_4}-\frac{1}{p}\right) = \int_{u_4}^{u} (g+ph)\,du \,.$$

Or for $u \geqq u_4$ (and so long as x and p remains finite)

$$\frac{\varepsilon}{p(u)} = \frac{\varepsilon}{p_4} - \int_{u_4}^{u} g\,du - \int_{u_4}^{u} ph\,du \,. \tag{4.6}$$

In (4.6) g is $g(x(u), u, t(u), \varepsilon)$ and similarly for h.

We have already seen that $p(u) < -2g_P/h_B$ for $v_1 \leqq v \leqq v_2$. Thus given any $\delta_3 > 0$, by taking P near enough to B, we can make $p(u) \leqq \delta_3$, $v_1 \leqq u \leqq v_2$. Given any $\delta_4 > 0$ we can choose ε, δ_1, and δ_2 small enough so that

$$|x(v_2)-y(v_2)|+|t(v_2)-s(v_2)| \leqq \delta_4 \,. \tag{4.7}$$

Now so long as $p(u) \leqq \delta_3$ we have from comparing the two systems

$$\frac{dx}{dv} = f(x, v, t, \varepsilon)+p\varphi(x, v, t, \varepsilon)\,, \quad \frac{dt}{dv} = p$$

$$\frac{dy}{dv} = f(y, v, s, 0), \quad \frac{ds}{dv} = 0$$

for $v_2 \leqq v < v_C$, just as in the argument following (4.5),

$$|x-y|+|t-s| \leqq K(\delta_3+\delta_4+\psi(\varepsilon)) \,. \tag{4.8}$$

We recall incidentally that on BC, $s(v) = \tau_1$. We shall show that $p(u) \leqq \delta_3$ almost up to $u = v_C$. We choose $\delta_5 > 0$ as small as we wish and then choose $\delta_6 > 0$ small enough so that

$$\int_{v_B}^{v} g(y(v), v, s(v), 0)\,dv < -4\delta_6,\ v_2 \leqq v \leqq v_C-\delta_5 \,. \tag{4.9}$$

This is possible since $g < 0$ for $v_B < v \leqq v_2$ and since the continuous function of v

$$\int_{v_B}^{v} g\,dv < 0\,,\ v_B < v < v_C \,.$$

The logical procedure in this section is to first choose δ_5 and then δ_6. We then observe that we can require δ_3 and δ_4 to be as small as we wish if we make ε, δ_1, and δ_2 small enough. By taking δ_3, δ_4 and ε small enough we have from (4.8) that for $v_C > v \geqq v_2$ and so long as $p \leqq \delta_3$

$$|g(x(v), v, t(v), \varepsilon)-g(y(v), v, s(v), 0)| < \delta_6/(v_C-v_B+1) \, . \tag{4.10}$$

We can also satisfy (4.10) for $v_1 \leqq v \leqq v_2$ on the basis of Part 2 simply by taking ε, δ_1, and δ_2 small enough. Turning to (4.6) we have

$$p(u) = \frac{\varepsilon}{\dfrac{\varepsilon}{p_4} - \displaystyle\int_{u_4}^{u} g(x(v), v, t(v), \varepsilon) dv - \int_{u_4}^{u} phdv} \, . \tag{4.11}$$

But by (4.10) followed by (4.9) we have for $v_2 < u < v_C - \delta_5$

$$-\int_{u_4}^{u} g(x(v), v, t(v), \varepsilon) dv \geqq -\int_{u_4}^{u} g(y(v), v, s(v), 0) dv - \delta_6$$

$$\geqq 3\delta_6 - \left| \int_{v_B}^{u_4} g(y(v), v, s(v), 0) dv \right| \, .$$

Since we can bring u_4 as close to v_B as we wish by taking ε, δ_1, and δ_2 small enough, we can make

$$\left| \int_{v_B}^{u_4} g dv \right| < \delta_6 \, .$$

Thus

$$-\int_{u_4}^{u} g(x(v), v, t(v), \varepsilon) dv > 2\delta_6 \, .$$

Also so long as $p \leqq \delta_3$,

$$\left| \int_{u_1}^{u} phdv \right| \leqq K_3 \delta_3 \, .$$

If we choose δ_3 small enough so $K_3\delta_3 < \delta_6$ then certainly the last two inequalities used in (4.11) yield

$$p(u) < \varepsilon/\delta_6 \tag{4.12}$$

up to the point where we first have either $p(u) = \delta_3$ or $u = v_C - \delta_5$. But if ε is small enough so that $\varepsilon/\delta_6 < \delta_3$ we certainly cannot have $p(u) = \delta_3$ in $u_4 \leqq u \leqq v_C - \delta_5$. Thus (4.8) holds in this range and we have established Theorem 1 on S_0 up to any point just short of C.

Theorem 1, Part 4. Here we prove Theorem 1 in the neighborhood of C and beyond. Given any $\delta_5 > 0$ we saw that for $v_B \leqq v \leqq v_C - \delta_5$ we have $|x-y|+|t-\tau_1|+$

$\frac{dt}{du} \to 0$ as ε, δ_1, $\delta_2 \to 0$. We now show that in the neighborhood of C, p grows quickly. As we have already seen from (4.3) $p > 0$ so long as p remains small. We shall show that p must exceed min $[g_C/(2K), 1]$ in the neighborhood of C where now K is the max of $|h|$ in R.

Let us denote a value of $v < v_C$ by v_5 and a value $v > v_C$ by v_6. We choose v_5 and v_6 near v_C. So long as $p < 1$ we can construct R_2, much like R_1, so that if v_5 and v_6 are chosen close enough together the solution of (4.0) can be continued up to v_6 and will lie in R_2 which in turn will lie in R. This is certainly the case then if p is small. We can also choose v_5 and v_6 near enough to v_C so that in R_2, $g > \frac{1}{2}g_C > 0$.

Integrating (4.3) between v_1 and u_4 we have

$$\frac{\varepsilon}{p_4} = \frac{\varepsilon}{p_1} - \int_{v_1}^{u_4} (g+ph)\,du\,.$$

Since we can choose v_1 as close to v_B as we want and since as ε, δ_1, and $\delta_2 \to 0$, $u_4 \to v_B$ we see that we can require that

$$\left| \int_{v_1}^{u_4} (g+ph)\,du \right| < \delta_7$$

for any $\delta_7 > 0$. Thus

$$\frac{\varepsilon}{p_4} < \frac{\varepsilon}{p_1} + \delta_7\,.$$

Now since we can choose ε, δ_1, and δ_2 as small as we wish after having selected v_1 we can make ε/q_1 and therefore also ε/p_1 as small as we want. Thus we can require

$$\frac{\varepsilon}{p_4} < 2\delta_7\,.$$

Thus (4.6) gives

$$p(u) > \frac{\varepsilon}{2\delta_7 - \int_{u_4}^{u} g\,du - \int_{u_4}^{u} ph\,du}\,.$$

From the result of part 3 it follows that if we take v_5 close enough to v_C and ε, δ_1, and δ_2 small enough we have

$$\left| \int_{u_4}^{v_5} (g+ph)\,du \right| < \delta_7\,.$$

We see that the choice of δ_7 affects the choice of v_5 but not v_6. We have

$$p(u) > \frac{\varepsilon}{3\delta_7 - \int_{v_5}^{u} (g+ph)\, du}. \tag{4.13}$$

Now let us assume that

$$g+ph \geqq \frac{5\delta_7}{v_6 - v_5}, \quad v_5 \leqq u \leqq v_6 . \tag{4.14}$$

Then we find from (4.13) that $p(u) \to \infty$, contrary to our assumption that $p < 1$. Thus (4.14) cannot hold and we have

$$ph < \frac{5\delta_7}{v_6 - v_5} - g < \frac{5\delta_7}{v_6 - v_C} - g$$

for some u, $v_5 < u < v_6$. If δ_7 is small enough this can be replaced by

$$ph < -\tfrac{1}{2} g_C .$$

Since $g_C > 0$ we must have $h \neq 0$. If $h > 0$ we have $p < 0$ which is impossible. If $h < 0$ and $|h| < K$ we have $p > g_C/(2K)$. Thus if $p < 1$ we certainly have $p > g_C/(2K)$ for some u, $v_5 < u < v_6$.

Since we can choose v_6 as close to C as we wish we see that indeed we can enclose C in a sphere in (x, u, t) with center at C and of radius arbitrarily small and that having chosen the sphere we can, by taking ε, δ_1, and δ_2 small enough be sure that R_2 lies in the sphere and that the solution of (4.0) enters the sphere with p very small but at some point in the sphere $p = \min\,(g_C/(2K), 1)$. Let us denote this point by D. At D we transform back to the original independent variables. To show that the solution (1.3) changes but little from τ_1 to t_D, irrespective of whether $\tau_1 - t_D$ is positive or negative we have only to note that $t_D - \tau_1 \to 0$ as $\varepsilon, \delta_1, \delta_2 \to 0$. We can now apply Lemma 2 at D as we did at the point A and we get Theorem 1 for the range of t given by $\max\,(\tau_1, t_D) \leqq t \leqq \beta$.

The proof of Theorem 2 is very much simpler and shorter than that of Theorem 1. It is quite direct except near B where we must use an argument similar to that used in the case of Theorem 1, Part. 2.

5. The proofs of Theorems 3 and 4 proceed along somewhat different lines. We shall first prove Theorem 4. We use the letter a to designate an initial value of x_i and y_i or of u and v. The initial value of $\dfrac{du(\alpha)}{dt}$ which has no counterpart in the degenerate system we shall denote by b. We observe that at $t = \alpha$, $\partial y_i/\partial a$ is zero

unless a is the intial value of y_i in which case $\partial y_i/\partial a = 1$ at $t = \alpha$. Similarly with $\partial v/\partial a$. We have $\frac{\partial y}{\partial a} = \frac{\partial x}{\partial a}$ and $\frac{\partial v}{\partial a} = \frac{\partial u}{\partial a}$ at $t = \alpha$ since a represents the same initial coordinate in (y, v) as in (x, u). We always have $\frac{\partial}{\partial a}\left(\frac{du}{dt}\right) = 0$ at $t = \alpha$.

Proof of Theorem 4. We have on differentiating with respect to a

$$\begin{aligned} \frac{d}{dt}\frac{\partial y}{\partial a} &= f\frac{d}{dt}\frac{\partial v}{\partial a} + \frac{\partial f}{\partial a}\frac{dv}{dt} + \frac{\partial \varphi}{\partial a} \\ \frac{d}{dt}\frac{\partial v}{\partial a} &= \frac{h\frac{\partial g}{\partial a} - g\frac{\partial h}{\partial a}}{g^2}. \end{aligned} \tag{5.0}$$

Here $f = f(y, v, t, 0)$, etc. and

$$\frac{\partial f}{\partial a} = \sum \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial a} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial a}$$

and similarly for φ, g and h. The system (5.0) is linear in $\partial y/\partial a$ and $\partial v/\partial a$ and the coefficients are continuous for $\alpha \leqq t < \tau_1$. The initial values are known and thus $\partial y/\partial a$ and $\partial v/\partial a$ are determined for $\alpha \leqq t < \tau_1$. Moreover for $\alpha \leqq t \leqq \gamma < \tau_1$ the functions $\partial y/\partial a$ and $\partial v/\partial a$ are uniformly continuous with respect to t and with respect to changes in the initial values of y and v at $t = \alpha$.

At $v = v_0 < v_B$ but with v_0 near v_B we change from t to v as the independent variable. We observe that $v = v_0$ determines a point on S_0 near B. As before it is convenient to replace t by s and reserve t for the system (1.2). Let s at $v = v_0$ be s_0 where clearly $s_0 < \tau_1$. We denote $\partial y/\partial a$ and $\partial v/\partial a$ at $s = s_0$, or more precisely at $s = s_0 - 0$, by $\frac{\partial y(s_0)}{\partial a}$ and $\frac{\partial v(s_0)}{\partial a}$. We denote $\partial y/\partial a$ and $\partial s/\partial a$ at $v = v_0 + 0$ by $\frac{\partial y(v_0)}{\partial a}$ and $\frac{\partial s(v_0)}{\partial a}$. Clearly since $v(s_0) = v_0$ we have from $v(s, a) = v_0$,

$$\frac{dv}{ds}\frac{\partial s(v_0)}{\partial a} + \frac{\partial v(s_0)}{\partial a} = 0\,.$$

Thus

$$\frac{\partial s(v_0)}{\partial a} = -\frac{\partial v(s_0)}{\partial a}\frac{ds(v_0)}{dv}. \tag{5.1}$$

Also

$$\frac{\partial y(v_0)}{\partial a} = \frac{dy}{ds}\frac{\partial s(v_0)}{\partial a} + \frac{\partial y(s_0)}{\partial a} = -\frac{dy(v_0)}{dv}\frac{\partial v(s_0)}{\partial a} + \frac{\partial y(s_0)}{\partial a}. \tag{5.2}$$

For $v_0 \leqq v \leqq v_B$ we have

$$\frac{d}{dv}\frac{\partial y}{\partial a} = \frac{\partial f}{\partial a} + \frac{\partial \varphi}{\partial a}\frac{ds}{dv} + \varphi\frac{d}{dv}\frac{\partial s}{\partial a}, \qquad \frac{d}{dv}\frac{\partial s}{\partial a} = \frac{g\dfrac{\partial h}{\partial a} - h\dfrac{\partial g}{\partial a}}{h^2} \tag{5.3}$$

where the intitial values are taken from (5.1) and (5.2). For $v_B \leqq v \leqq v_C$ we have

$$\frac{d}{dv}\frac{\partial y}{\partial a} = \frac{\partial f}{\partial a}, \qquad \frac{d}{dv}\frac{\partial s}{\partial a} = 0 . \tag{5.4}$$

Thus $\partial y(v)/\partial a$ and $\partial s(v)/\partial a$ are determined up to the point C.

We have $g(y_B, v_B, \tau_1, 0) = 0$. Thus

$$\sum \frac{\partial g}{\partial y_i}\frac{\partial y_{Bi}}{\partial a} + \frac{\partial g}{\partial v}\frac{\partial v_B}{\partial a} + \frac{\partial g}{\partial t}\frac{\partial \tau_1}{\partial a} = 0 .$$

Since

$$\frac{\partial y_B}{\partial a} = \frac{dy}{dv}\frac{\partial v_B}{\partial a} + \frac{\partial y(v_B)}{\partial a}$$

we have, recalling the definition of $I = I_B$,

$$I\frac{\partial v_B}{\partial a} + \sum \frac{\partial g}{\partial y_i}\frac{\partial y_i(v_B)}{\partial a} + \frac{\partial g}{\partial t}\frac{\partial \tau_1}{\partial a} = 0 .$$

Since $I \neq 0$ we see that $\partial v_B/\partial a$ is determined in terms of $\partial \tau_1/\partial a$.

The coordinate v_C is given by

$$\int_{v_B}^{v_C} g(y(v), v, \tau_1, 0)dv = 0 .$$

Differentiating with respect to a and recalling that $g = 0$ at B we have

$$g(y_C, v_C, \tau_1, 0)\frac{\partial v_C}{\partial a} + \int_{v_B}^{v_C}\left(\sum \frac{\partial g}{\partial y_i}\frac{\partial y_i}{\partial a} + \frac{\partial g}{\partial s}\frac{\partial \tau_1}{\partial a}\right)dv = 0 . \tag{5.5}$$

We have

$$\frac{\partial \tau_1}{\partial a} = \frac{\partial s(v_B)}{\partial a} + \frac{ds(v_B)}{dv}\frac{\partial v_B}{\partial a}.$$

Since $ds(v_B)/dv = 0$ we see[1] that $\partial \tau_1/\partial a = \partial s(v_B)/\partial a$. Thus (5.5) determines $\partial v_C/\partial a$. Also

[1] Strictly speaking we find $\partial v_B/\partial a$ in terms of $\partial \tau_1/\partial a$ and then $\partial \tau_1/\partial a$ in terms of $\partial v_B/\partial a$ which is not rigorous. Actually we should proceed with finite differences corresponding to an increment in a, Δa, and then take limits as $\Delta a \to 0$ to show the existence of $\partial v_B/\partial a$ from a single formula obtained by eliminating the term $\Delta \tau_1/\Delta a$ from the equation involving I and the equation following (5.5).

$$\frac{\partial y_C}{\partial a} = \frac{dy}{dv}\frac{\partial v_C}{\partial a} + \frac{\partial y(v_C)}{\partial a} \tag{5.6}$$

where $\partial y(v_C)/\partial a$ is $\partial y/\partial a$ at $v = v_C - 0$. As we approach C from $s > s_C$ we have

$$\frac{\partial v_C}{\partial a} = \frac{\partial v(s_C)}{\partial a} + \frac{dv}{ds}\frac{\partial s_C}{\partial a} \tag{5.7}$$

$$\frac{\partial y_C}{\partial a} = \frac{\partial y(s_C)}{\partial a} + \frac{dy}{dt}\frac{\partial s_C}{\partial a} \tag{5.8}$$

where $\partial y(s_C)/\partial a$ is $\partial y(s)/\partial a$ at $s = s_C + 0$ and similary for $\partial v(s_C)/\partial a$. However $\partial s_C/\partial a = \partial \tau_1/\partial a$. Using (5.5) and (5.7) we have

$$-\frac{1}{g_c}\int_{v_B}^{v_C}\left(\sum \frac{\partial g}{\partial y_i}\frac{\partial y_i}{\partial a} + \frac{\partial g}{\partial s}\frac{\partial \tau_1}{\partial a}\right)dv = \frac{\partial v(s_C)}{\partial a} + \frac{dv}{ds}\frac{\partial \tau_1}{\partial a}. \tag{5.9}$$

That is $\partial v(s_C)/\partial a$ is determined. Likewise $\partial y(s_C)/\partial a$ is determined. Indeed from (5.8) and (5.6)

$$\frac{\partial y(s_C)}{\partial a} = \frac{dy(v_C)}{dv}\frac{\partial v_C}{\partial a} + \frac{\partial y(v_C)}{\partial a} - \frac{dy(s_C)}{ds}\frac{\partial \tau_1}{\partial a}.$$

From (5.7) and (1.3) therefore

$$\frac{\partial y(s_C)}{\partial a} = f_C\frac{\partial v(s_C)}{\partial a} + \frac{\partial y(v_C)}{\partial a} - \varphi_C\frac{\partial \tau_1}{\partial a}. \tag{5.10}$$

We now use (5.0) again from $s_C \leqq s \leqq \beta$ and thereby determine $\partial y(\beta)/\partial a$ and $\partial v(\beta)/\partial a$. The functions $\partial y(s)/\partial a$ and $\partial v(s)/\partial a$ are clearly uniformly continuous with respect to the initial values of y and v at $s=\alpha$ and with respect to s for $\alpha \leqq s \leqq \tau_1 - \delta$, $\tau_1 + \delta \leqq t \leqq \beta$ for any given $\delta > 0$.

This completes the proof of Theorem 4.

6. We proceed now to the proof of Theorem 3. In the course of the proof we shall make use of Theorem 4. We recall the remarks made at the beginning of § 5 concerning the meaning of a and b. We require the following result which is a slightly modified form of Lemma 2 and is proved in the same way.

Lemma 3. *In Lemma 2 let ε_1 in* (3.4) *be replaced by*

$$\varepsilon_1 + \frac{\varepsilon_2}{\varepsilon} e^{-m(t-\alpha)/\varepsilon}$$

and let δ_1 and δ_2 be replaced by δ_3 and δ_4 respectively. Then the conclusion of Lemma 2 becomes

$$|z(t)|+|w(t)| \leqq \Delta_1 e^{k_1(t-\alpha)}$$

$$\left|\frac{dw}{dt}\right| \leqq \frac{\delta_4}{\varepsilon} e^{-m(t-\alpha)/\varepsilon} + \frac{\varepsilon_2(t-\alpha)}{\varepsilon^2} e^{-m(t-\alpha)/\varepsilon} + \frac{k\Delta_1}{m} e^{k_1(t-\alpha)}$$

for $\alpha \leqq t \leqq \gamma$ where Δ_1 replaces Δ in Lemma 2 and

$$\Delta_1 = \frac{4(k+1)^2(m+1)^2}{km^2}(\varepsilon_1+\varepsilon_2+\delta_3+\delta_4) ,$$

and k_1 is the same as in Lemma 2.

We begin by considering $\alpha \leqq t \leqq \gamma < \tau_1$.

Proof of Theorem 3, Part 1. Here we have

$$\text{(6.0)} \qquad \begin{aligned} \frac{d}{dt}\frac{\partial x}{\partial a} &= f\frac{d}{dt}\frac{\partial u}{\partial a}+\frac{\partial f}{\partial a}\frac{du}{dt}+\frac{\partial \varphi}{\partial a} \\ \varepsilon\frac{d^2}{dt^2}\frac{\partial u}{\partial a}+g\frac{d}{dt}\frac{\partial u}{\partial a}+\frac{\partial g}{\partial a}\frac{du}{dt}+\frac{\partial h}{\partial a} &= 0 \end{aligned}$$

where $f = f(x(t), u(t), t, \varepsilon)$, etc. and where

$$\frac{\partial f}{\partial a} = \sum \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial a}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial a}$$

etc. The system is linear in $\partial x/\partial a$ and $\partial u/\partial a$. The coefficients are all function of t (and of ε and the initial values of x, u, du/dt at $t = \alpha$).

For the degenerate system we have (5.0) where the last equation can be written as

$$g\frac{d}{dt}\frac{\partial v}{\partial a}+\frac{\partial g}{\partial a}\frac{dv}{dt}+\frac{\partial h}{\partial a} = 0 .$$

Now let $z = \frac{\partial x}{\partial a}-\frac{\partial y}{\partial a}$ and $w = \frac{\partial u}{\partial a}-\frac{\partial v}{\partial a}$. The initial values of z and w are zero. From (6.0) and (5.0) we have

$$\text{(6.1)} \qquad \frac{dz}{dt} = f(x(t), u(t), t, \varepsilon)\frac{dw}{dt}+\left(\sum\frac{\partial f}{\partial x_i}z_i+\frac{\partial f}{\partial u}w\right)\frac{dv}{dt}+\sum\frac{\partial \varphi}{\partial x_i}z_i+\frac{\partial \varphi}{\partial u}w+F .$$

$$\text{(6.2)} \qquad \varepsilon\frac{d^2w}{dt^2}+g(x(t), u(t), t, \varepsilon)\frac{dw}{dt}+\left(\sum\frac{\partial g}{\partial x_i}z_i+\frac{\partial g}{\partial u}w\right)\frac{dv}{dt}+\sum\frac{\partial h}{\partial x_i}z_i+\frac{\partial h}{\partial u}w+G = 0$$

where

$$F = \left(\frac{du}{dt}-\frac{dv}{dt}\right)\left[\sum\frac{\partial f}{\partial x_i}z_i+\frac{\partial f}{\partial u}w+\sum\frac{\partial f}{\partial x_i}\frac{\partial y_i}{\partial a}+\frac{\partial f}{\partial u}\frac{\partial v}{\partial a}\right]$$
$$+\frac{dv}{dt}\left[\sum\left(\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial y_i}\right)\frac{\partial y_i}{\partial a}+\left(\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial a}\right]$$
$$+\sum\left(\frac{\partial \varphi}{\partial x_i}-\frac{\partial \varphi}{\partial y_i}\right)\frac{\partial y_i}{\partial a}+\left(\frac{\partial \varphi}{\partial u}-\frac{\partial \varphi}{\partial v}\right)\frac{\partial v}{\partial a}+\frac{d}{dt}\frac{\partial v}{\partial a}\left[f(x, u, t, \varepsilon)-f(y, v, t, 0)\right].$$

As ε, δ_1, and $\delta_2 \to 0$ we have for $\alpha \leqq t \leqq \gamma < \tau_1$ from Theorem 1, part 1, that

$$\sum\left|\frac{\partial f(x, u, t, \varepsilon)}{\partial x_i}-\frac{\partial f(y, v, t, 0)}{\partial y_i}\right| \to 0$$

uniformly where x is $x(t)$, y is $y(t)$ etc. Similar results hold for the other differences that occur in F except for the difference $\frac{du}{dt}-\frac{dv}{dt}$. For the latter we have by (3.15)

$$\left|\frac{du}{dt}-\frac{dv}{dt}\right| \leqq \frac{\delta_2}{\varepsilon}e^{-m(t-\alpha)/\varepsilon}+e^{K/m^4}(\delta_1+\delta_2+\psi(\varepsilon)).$$

The term $\frac{d}{dt}\frac{\partial v}{\partial a}=\frac{\partial}{\partial a}\frac{dv}{dt}$ is replaced by use of $\frac{dv}{dt}=-\frac{h}{g}$. We have easily from (5.0) and Lemma 1 that the terms $\partial y_i/\partial a$ and $\partial v/\partial a$ are bounded over $\alpha \leqq t \leqq \gamma$, by a bound that depends only on K and m. Thus so long as $|z|+|w| \leqq 1$ there is an E, a function of K and m, which becomes large when m gets small, such that

$$|F| \leqq E\left[\psi(\varepsilon, \delta_1, \delta_2)+\frac{\delta_2}{\varepsilon}e^{-m(t-\alpha)/\varepsilon}\right] \tag{6.3}$$

where $\psi(\varepsilon, \delta_1, \delta_2)$ is a continuous function of ε, δ_1, δ_2, and m which tends to zero when ε, δ_1 and $\delta_2 \to 0$. A similar result holds for G which has terms like those of F and the additional term $\varepsilon\frac{\partial}{\partial a}\frac{d^2v}{dt^2}$. It is here that we use the existence of the second order partial derivatives of g and h. We find that G satisfies an inequality of the same form as (6.3). From the last equation of (5.0) $\frac{\partial}{\partial a}\frac{dv}{dt}$ at $t=\alpha$ can be computed. We have $\frac{\partial}{\partial a}\frac{du}{dt}=0$ at $t=\alpha$. Thus $\frac{dw}{dt}$ at $t=\alpha$ is bounded and therefore δ_4 is of the form $K\varepsilon$. With this we see that (6.1) and (6.2) satisfy the hypothesis of Lemma 3 where $\varepsilon_1 = E\psi$ and $\varepsilon_2 = E\delta_2$. Using Lemma 3 we see that Theorem 3 is valid for $\alpha \leqq t \leqq \gamma < \tau_1$ in so far as the derivatives with respect to a are concerned.

As regards the derivatives with respect to b we use (6.0) without (5.0). We set $z = \partial x/\partial b$ and $w = \partial u/\partial b$ and apply Lemma 3. Since at $t = \alpha$, $z = w = 0$ we have here that $\delta_3 = 0$. Since $\frac{\partial}{\partial b}\frac{du}{dt} = 1$ at $t = \alpha$ we have $\delta_4 = \varepsilon$. Clearly for $|z|+|w| \leqq 1$ we have

$$\left|\frac{\partial f}{\partial b}\frac{du}{dt}+\frac{\partial \varphi}{\partial b}\right| \leqq \left|\frac{\partial f}{\partial b}\right| \left|\frac{du}{dt}-\frac{dv}{dt}\right| + \left|\frac{\partial f}{\partial b}\right| \left|\frac{dv}{dt}\right| + \left|\frac{\partial \varphi}{\partial b}\right|$$

$$\leqq E\left[\psi(\varepsilon, \delta_1, \delta_2)+\frac{\delta_2}{\varepsilon}e^{-m(t-\alpha)/\varepsilon}+|z|+|w|\right]$$

where E is the same kind of function as appeared in (6.3). A similar result holds for $\frac{\partial g}{\partial b}\frac{du}{dt}+\frac{\partial h}{\partial b}$. If we now use Lemma 3 we find that $\left|\frac{\partial x(t)}{\partial b}\right| + \left|\frac{\partial u(t)}{\partial b}\right| \to 0$ uniformly over $\alpha \leqq t \leqq \gamma < \tau_1$ as ε, δ_1 and $\delta_2 \to 0$ while $\left|\frac{d}{dt}\frac{\partial u}{\partial b}\right| \to 0$ uniformly over $\alpha+\delta \leqq t \leqq \gamma < \tau_1$ for any fixed $\delta > 0$. This completes the proof of Theorem 3 over $\alpha \leqq t \leqq \gamma < \tau_1$.

Theorem 3, Part 2.

At $v = v_0 < v_B$ we change from t to u (or v) as the independent variable. We have from (4.0)

$$\frac{d}{du}\frac{\partial x}{\partial a} = \frac{\partial f}{\partial a}+\frac{\partial \varphi}{\partial a}p+\varphi\frac{\partial p}{\partial a}, \quad \frac{d}{du}\frac{\partial t}{\partial a} = \frac{\partial p}{\partial a}$$

(6.4)

$$\varepsilon\frac{d}{du}\frac{\partial p}{\partial a} = 2p\frac{\partial p}{\partial a}g+p^2\frac{\partial g}{\partial a}+3p^2\frac{\partial p}{\partial a}h+p^3\frac{\partial h}{\partial a}.$$

We also have (5.3) for the degenerate system. The initial values at $u = v = v_0$ of $\partial x(v_0)/\partial a$ and $\partial t(v_0)/\partial a$ may be found in the same way as $\partial y(v_0)/\partial a$ and $\partial s(v_0)/\partial a$ in (5.1) and (5.2). Here $\partial x(v_0)/\partial a$ means $\partial x(v_0+0)/\partial a$ etc. From Part 1 we know that $\frac{\partial x(t_0)}{\partial a}-\frac{\partial y(s_0)}{\partial a}$ and $\frac{\partial u(t_0)}{\partial a}-\frac{\partial v(s_0)}{\partial a}$ can be made as small as we want by taking ε, δ_1, and δ_2 small enough. Thus $\frac{\partial x(v_0)}{\partial a}-\frac{\partial y(v_0)}{\partial a}$ and $\frac{\partial t(v_0)}{\partial a}-\frac{\partial s(v_0)}{\partial a}$ can be made as small as we wish.

We have also to consider $\partial p(v_0)/\partial a$. We have since $p = 1/\frac{du}{dt}$

(6.5)
$$\frac{\partial p(v_0)}{\partial a} = -\frac{1}{\left(\frac{du}{dt}\right)^2}\left[\frac{d^2u}{dt^2}\frac{\partial t(v_0)}{\partial a}+\frac{d}{dt}\frac{\partial u(t_0)}{\partial a}\right].$$

A similar formula is valid for $\partial q(v_0)/\partial a$. The terms in (6.5) tend to the corresponding terms in $\partial q(v_0)/\partial a$, with the possible exception of $\frac{d^2u}{dt^2}$, as ε, δ_1, and $\delta_2 \to 0$ by Theorem 1 and Part 1 of Theorem 3. Here we have the following lemma.

Lemma 4. *For $\alpha+\delta \leqq t \leqq \gamma < \tau_1$, where $\delta > 0$, $\frac{d^2u}{dt^2}-\frac{d^2v}{dt^2} \to 0$ uniformly as ε, δ_1, and $\delta_2 \to 0$ under the hypothesis of Theorem 1. The result also holds for $\tau_1+\delta \leqq t \leqq \beta$.*

We shall prove this lemma at the end of the section. We see now that indeed $\frac{\partial p(v_0)}{\partial a}-\frac{\partial q(v_0)}{\partial a} \to 0$ as ε, δ_1, and $\delta_2 \to 0$. It is easy to see that $\frac{\partial p(v)}{\partial a}-\frac{\partial q(v)}{\partial a} \to 0$ uniformly for $v_0 \leqq v \leqq v_B-\delta$ for any $\delta > 0$ as ε, δ_1, and $\delta_2 \to 0$. Since $q = \frac{ds}{dv} = -\frac{g}{h}$ we also see that $\frac{\partial q(v)}{\partial a}$ exists and is a continuous function of v for $v < v_B$ and moreover $\frac{\partial q(v_B-0)}{\partial a}$ exists. If we now let z represent $\left(\frac{\partial x(v)}{\partial a}-\frac{\partial y(v)}{\partial a}, \frac{\partial t(v)}{\partial a}-\frac{\partial s(v)}{\partial a}\right)$ then from (5.3) and (6.4)

$$\left|\frac{dz}{dv}\right| \leqq \left|\frac{\partial f(x, v, t, \varepsilon)}{\partial a}-\frac{\partial f(y, v, s, 0)}{\partial a}\right|+K\left|\frac{\partial p}{\partial a}-\frac{\partial q}{\partial a}\right|$$

$$\left|p\frac{\partial \varphi(x, v, t, \varepsilon)}{\partial a}-q\frac{\partial \varphi(y, v, s, 0)}{\partial a}\right|+\frac{\partial q}{\partial a}|\varphi(x, v, t, \varepsilon)-\varphi(y, v, s, 0|.$$

So long as $|p|+|q| \leqq 1$ and as $|z| \leqq 1$,

$$\left|\frac{dz}{dv}\right| \leqq K|z|+\sum\left|\frac{\partial f}{\partial x_i}-\frac{\partial f}{\partial y_i}\right|\left|\frac{\partial y_i}{\partial a}\right|+\left|\frac{\partial f}{\partial t}-\frac{\partial f}{\partial s}\right|\left|\frac{\partial s}{\partial a}\right|$$

$$+\sum\left|\frac{\partial \varphi}{\partial x_i}-\frac{\partial \varphi}{\partial y_i}\right|\left|\frac{\partial y_i}{\partial a}\right|+\left|\frac{\partial \varphi}{\partial t}-\frac{\partial \varphi}{\partial s}\right|\left|\frac{\partial s}{\partial a}\right|+K|p-q|+K\left|\frac{\partial p}{\partial a}-\frac{\partial q}{\partial a}\right|+K|\varphi(x, v, t, \varepsilon)-\varphi(y, v, t, 0)|.$$

Let $v_3 < v_C$. We observe that v_3 is unrelated to v_3 of Theorem 1. If $\psi(\varepsilon, \delta_1, \delta_2, v_3)$ represents a function which tends to zero as ε, δ_1, and $\delta_2 \to 0$ then for $v_0 \leqq v \leqq v_3$

$$\left|\frac{dz}{dv}\right| \leqq K|z|+K|p-q|+K\left|\frac{\partial p}{\partial a}-\frac{\partial q}{\partial a}\right|+\psi(\varepsilon, \delta_1, \delta_2, v_3).$$

From Lemma 1

$$(6.6)\quad |z(v)| \leqq e^{K(v-v_0)}\left[|z(v_0)|+K\int_{v_0}^{v}|p-q|\,dv+K\int_{v_0}^{v}\left|\frac{\partial p}{\partial a}-\frac{\partial q}{\partial a}\right|dv+(v-v_0)\psi\right].$$

It follows easily from Theorem 1 that

$$\int_{v_0}^{v_3}|p-q|\,dv \to 0$$

as ε, δ_1, and $\delta_2 \to 0$. We can also make $|z(v_0)|$ as small as we wish. Thus the terms on the right side of (6.6) all go to zero as ε, δ_1, $\delta_2 \to 0$ with the possible exception of

$$J = \int_{v_0}^{v_3}\left|\frac{\partial p}{\partial a}-\frac{\partial q}{\partial a}\right|dv\,.$$

Let $v_0 < v_1 < v_B$. Then since $q = 0$ for $v_B < v < v_C$,

$$J \leqq \int_{v_0}^{v_1}\left|\frac{\partial p}{\partial a}-\frac{\partial q}{\partial a}\right|dv+\int_{v_1}^{v_B}\left|\frac{\partial q}{\partial a}\right|dv+\int_{v_1}^{v_3}\left|\frac{\partial p}{\partial a}\right|dv\,.$$

By taking v_1 close enough to v_B and then taking ε, δ_1, and δ_2 small enough the first two terms on the right above can be made as small as we wish. Thus given any $\varepsilon_3 > 0$ we can make, by (6.6).

$$(6.7)\qquad |z(v)| \leqq \varepsilon_3+K\int_{v_1}^{v_3}\left|\frac{\partial p}{\partial a}\right|dv$$

providing we have chosen v_0 near enough to v_B so $|p|+|q| \leqq 1$ for $v_0 \leqq v \leqq v_3$ and so long as $|z| \leqq 1$ and $v_0 \leqq v \leqq v_3$. From $\varepsilon dp/du = p^2g+p^3h$ we find

$$\varepsilon\frac{d}{du}\left(\frac{1}{p^2}\frac{\partial p}{\partial a}\right)-p^2h\left(\frac{1}{p^2}\frac{\partial p}{\partial a}\right) = \frac{\partial g}{\partial a}+p\frac{\partial h}{\partial a}.$$

Thus

$$(6.8)\quad \frac{\partial p(v)}{\partial a} = p^2(v)\left(\frac{1}{p^2(v_1)}\frac{\partial p(v_1)}{\partial a}\right)e^{\frac{1}{\varepsilon}\int_{v_1}^{v}p^2h\,du}+\frac{p^2}{\varepsilon}\int_{v_1}^{v}\left(\frac{\partial g}{\partial a}+p\frac{\partial h}{\partial a}\right)e^{\frac{1}{\varepsilon}\int_{\sigma}^{v}p^2h\,du}d\sigma\,.$$

Let $v_2 > v_B$ and let $\min|h| \geqq \mu > 0$ near B. By (4.12), $p \leqq \varepsilon/\delta_6$ for $v_2 \leqq v \leqq v_3$. By taking ε, δ_1, and δ_2 small enough we have then $p < \varepsilon^{\frac{3}{4}}$ for $v_2 \leqq v \leqq v_3$. Since $h < 0$ near B we have for $v_2 \leqq v \leqq v_3$,

$$(6.9)\qquad \int_{v_1}^{v}p^2e^{\frac{1}{\varepsilon}\int_{v_1}^{\sigma}p^2h\,du}d\sigma \leqq \frac{\varepsilon}{\mu}+\int_{v_2}^{v}p^2e^{\frac{1}{\varepsilon}\int_{v_1}^{\sigma}p^2h\,du}d\sigma \leqq \frac{\varepsilon}{\mu}+K\varepsilon^{3/2}\,.$$

Also in much the same way

$$\left| \int_{v_1}^{v} \frac{p^2}{\varepsilon} du \int_{v_1}^{u} \left(\frac{\partial g}{\partial a} + p\frac{\partial h}{\partial a}\right) e^{\frac{1}{\varepsilon}\int_\sigma^u p^2 h du} d\sigma \right| = \left| \int_{v_1}^{v} \left(\frac{\partial g}{\partial a} + p\frac{\partial h}{\partial a}\right) d\sigma \int_\sigma^v \frac{p^2}{\varepsilon} e^{\frac{1}{\varepsilon}\int_\sigma^u p^2 h du} du \right|$$

(6.10)

$$\leqq K \int_{v_1}^{v_2} (1+|z(\sigma)|) d\sigma \left(\frac{1}{\mu} + \varepsilon^{\frac{1}{2}}\right) + K\varepsilon^{\frac{1}{2}} \int_{v_2}^{v} (1+|z(\sigma)|) d\sigma\ .$$

From (6.8), (6.9) and (6.10)

$$\int_{v_1}^{v_3} \left| \frac{\partial p(v)}{\partial a} \right| dv \leqq \varepsilon_4 + \frac{K}{\mu} \int_{v_1}^{v_2} (1+|z(\sigma)|) d\sigma + K\varepsilon^{\frac{1}{2}} \int_{v_1}^{v_3} (1+|z(\sigma)|) d\sigma \tag{6.11}$$

where for any choice of v_1, ε_4 can be made as small as we wish by taking ε, δ_1, and δ_2 small enough. Thus (6.7) yeilds, for arbitrary ε_5,

$$|z(v)| \leqq \varepsilon_5 + \frac{K(v_2-v_1)}{\mu} + \frac{K}{\mu} \int_{v_1}^{v_2} |z(\sigma)| d\sigma + K\varepsilon^{\frac{1}{2}} \int_{v_1}^{v_3} |z(\sigma)| d\sigma\ . \tag{6.12}$$

Let $v_2 - v_1$ be small enough so that $K(v_2-v_1)/\mu < \varepsilon_6$ where ε_6 is a preassigned positive quantity. Taking v_1 and v_2 closer to v_B affects the previous argument only in so far as it may be necessary to decrease ε, δ_1 and δ_2. Let max $|z(v)| = M$, $v_1 \leqq v \leqq v_3$. (If $M \geqq 1$ decrease v_3 so that we get $M < 1$). Then from

$$M \leqq \varepsilon_5 + \varepsilon_6 + M\varepsilon_6 + K\varepsilon^{\frac{1}{2}} M \tag{6.12}$$

or

$$M(1-\varepsilon_6-K\varepsilon^{\frac{1}{2}}) \leqq \varepsilon_5 + \varepsilon_6\ .$$

Thus $2(\varepsilon_5+\varepsilon_6) \geqq M \geqq |z(v)|$. (In particular then it is unecessary to decrease v_3 to achieve $M < 1$). We see then that (6.11) becomes, for any preassigned $\varepsilon_7 > 0$,

$$\int_{v_1}^{v_3} \left| \frac{\partial p(v)}{\partial a} \right| dv \leqq \varepsilon_7 \tag{6.13}$$

and thus from (6.7)

$$|z(v)| \leqq \varepsilon_8\ , \quad v_0 \leqq v \leqq v_3\ , \tag{6.14}$$

for any preassigned ε_8. Thus $\frac{\partial x}{\partial a} - \frac{\partial y}{\partial a}$ and $\frac{\partial t}{\partial a} - \frac{\partial s}{\partial a}$ tend to zero uniformly over $v_0 \leqq v \leqq v_3$ as ε, δ_1, and δ_2 tend to zero. That is we have established Theorem 3 up to any point short of C.

The case $\partial x/\partial b$, $\partial t/\partial b$ and $\partial p/\partial b$ is handled with the usual modification. That is we now set $z = \left(\frac{\partial x}{\partial b}, \frac{\partial t}{\partial b}\right)$ and use the fact that at v_0 the values of $|z|$ and $\partial p/\partial b$ tend to zero as $\varepsilon \to 0$.

We now turn to the proof of Lemma 4.

Proof of Lemma 4. Actually the result of this lemma is a consequence of Part 1 of Theorem 3 which we have already proved. There we have shown that

$$\frac{d}{dt}\frac{\partial u}{\partial a}-\frac{d}{dt}\frac{\partial v}{\partial a}\to 0\,, \quad \alpha+\delta \leqq t \leqq \gamma\,, \tag{6.15}$$

uniformly as ε, δ_1 and $\delta_2 \to 0$. By a well known artifice $\partial/\partial a$ can be changed to d/dt. Indeed replace t by $\sigma+\alpha$ in (1.2). Then we have

$$\begin{gathered}\frac{dt}{d\sigma}=1\,,\quad \frac{dx}{d\sigma}=f(x,u,t,\varepsilon)\frac{du}{d\sigma}+\varphi\,,\\ \varepsilon\frac{d^2u}{d\sigma^2}+g\frac{du}{d\sigma}+h=0\,.\end{gathered} \tag{6.16}$$

Here we have one more dependent variable than in (1.2) namely t. The independent variable σ is assigned the intitial value $\sigma=0$. Clearly then $t(0)=\alpha$ and α is an a for (6.16). Also $u=u\big(\sigma+\alpha, \varepsilon, x(0), u(0), u'(0)\big)$ where $x(0)$ etc. are the values of x etc. at $\sigma=0$. Clearly $\frac{\partial u}{\partial \alpha}=\frac{du}{d\sigma}=\frac{du}{dt}$ and similarly for v. Using Part 1 of Theorem 1 for the system (6.16) we have then the proof of Lemma 4 as a consequence of (6.15).

The proof of Lemma 4 for $\tau_1+\delta \leqq t \leqq \beta$ follows in the same way once (6.15) is demonstrated over this range of t.

7. Theorem 3, Part 3.

Before discussing the behavior of $\partial x/\partial a$, $\partial t/\partial a$ and $\partial p/\partial a$ near C it is convenient to obtain the following result

$$\frac{\varepsilon}{p^2}\frac{\partial p}{\partial a}-\int_{v_B}^{v}\frac{\partial g}{\partial a}dv\to 0 \tag{7.0}$$

uniformly for $v_B \leqq v \leqq v_C-\delta$ for $\delta>0$, as ε, δ_1, and $\delta_2\to 0$. In (7.0)

$$\frac{\partial g}{\partial a}=\sum\frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial a}+\frac{\partial g}{\partial t}\frac{\partial t}{\partial a}.$$

The proof is a consequence of integrating

$$\frac{d}{du}\left(\frac{\varepsilon}{p^2}\frac{\partial p}{\partial a}\right)=h\frac{\partial p}{\partial a}+\frac{\partial g}{\partial a}+p\frac{\partial h}{\partial a}$$

over $v_1 \leqq u \leqq v \leqq v_C-\delta$ and using (6.13) and the fact that $p\to 0$ uniformly in the interval $v_B \leqq u \leqq v_C-\delta$ as ε, δ_1, $\delta_2 \to 0$.

We take $v_3 < v_C$ and near v_C and choose the point $P_4(x_4, u_4, t_4)$ so that P_4 is the first point where we have $p = p_4 = \varepsilon^{\frac{1}{4}}$ and $t_4 > t_3$. P_4 here is unrelated to the point P_4 used in the proof of Theorem 1. Clearly from (4.12) of Theorem 1, $P_4 \to C$ as $\varepsilon, \delta_1, \delta_2 \to 0$.

From (6.4), letting $z = \left(\frac{\partial x}{\partial a}, \frac{\partial t}{\partial a}\right)$ we have

$$\left|\frac{dz}{du}\right| \leqq K|z|+K|zp|+K\left|\frac{\partial p}{\partial a}\right| \leqq K|z|+K\left|\frac{\partial p}{\partial a}\right|, \; v_3 \leqq u \leqq v_4 , \tag{7.1}$$

and therefore by Lemma 1

$$|z(u)| \leqq K|z_3|+K\int_{v_3}^{u}\left|\frac{\partial p}{\partial a}\right| du, \; v_3 \leqq u \leqq v_4 .$$

We have since $g > 0$ near C, $g+ph > \frac{1}{2}g$, $v_3 \leqq u \leqq v_4$, if we have chosen ε small enough. Thus from $\varepsilon dp/du = gp^2+hp^3$, p is an increasing function of u. Also $dp > \frac{1}{2}gp^2du/\varepsilon$ so that

$$\frac{1}{\varepsilon}\int_{v_3}^{v_4} p^2 du < K(p_4-p_3) < Kp_4 . \tag{7.2}$$

From (6.8)

$$\frac{\partial p(u)}{\partial a} = \frac{p^2(u)}{\varepsilon}\left(\frac{\varepsilon}{p_3^2}\frac{\partial p(v_3)}{\partial a}\right) e^{\frac{1}{\varepsilon}\int_{v_3}^{u} p^2 h du} + \frac{p^2(u)}{\varepsilon}\int_{v_3}^{u}\left(\frac{\partial g}{\partial a}+p\frac{\partial h}{\partial a}\right) e^{\frac{1}{\varepsilon}\int_{\sigma}^{u} p^2 h dv} d\sigma . \tag{7.3}$$

Using the fact that $\frac{\varepsilon}{p_3^2}\frac{\partial p(v_3)}{\partial a}$ is bounded by (7.0) and also using (7.2) we find easily now that for $v_3 \leqq v \leqq v_4$

$$\int_{v_3}^{v}\left|\frac{\partial p}{\partial a}\right| du \leqq Kp_4 + Kp_4\int_{v_3}^{v}|z|du, \quad \varepsilon+\delta_1+\delta_2 < \Psi(v_3) \tag{7.4}$$

where $\Psi(v_3)$ is a continuous function which tends to zero as $v_3 \to v_C$. With the formula below (7.1) this yields

$$|z(u)| \leqq M \leqq K|z_3|+Kp_4+Kp_4M(v_4-v_3), \quad \varepsilon+\delta_1+\delta_2 < \Psi(v_3)$$

where $M = \max |z(u)|$, $v_3 \leqq u \leqq v_4$. Thus if ε is small enough $M \leqq K|z_3|+Kp_4 < K$ since z_3 is near $\left(\frac{\partial y(v_3)}{\partial a}, \frac{\partial \tau_1}{\partial a}\right)$.

Since $|z(u)| < K$ for $v_B \leqq v \leqq v_3$ we have

$$|z(u)| \leqq K , \quad v_B \leqq u \leqq v_4 . \tag{7.5}$$

By (7.4) and (7.5) we find from (7.1)

$$|z_4-z_3| \leqq K(v_4-v_3)+Kp_4\,, \quad \varepsilon+\delta_1+\delta_2 < \Psi(v_3)\,. \tag{7.6}$$

At P_4 we change again from u to t as the independent variable. We have, if $t = t_4$ at P_4, from $t(u, a) = t_4$

$$\frac{\partial u(t_4)}{\partial a} = -\frac{\partial t(u_4)}{\partial a}\frac{du(t_4)}{dt} \tag{7.7}$$

where $\partial u(t_4)/\partial a$ is $\partial u(t)/\partial a$ at $t = t_4+0$, etc. analogous to (5.1) and (5.2). We also have

$$\begin{aligned} \frac{\partial x(t_4)}{\partial a} &= \frac{\partial x(u_4)}{\partial a} - \frac{dx(t_4)}{dt}\frac{\partial t(u_4)}{\partial a} \\ \frac{\partial}{\partial a}\frac{du(t_4)}{dt} &= -\frac{1}{p_4^2}\frac{\partial p(u_4)}{\partial a} + \frac{1}{p_4^3}\frac{dp(u_4)}{du}\frac{\partial t(u_4)}{\partial a}. \end{aligned} \tag{7.8}$$

We define $t_5 = t_4+\varepsilon \log^2 \dfrac{1}{\varepsilon}$ and denote the point (x_5, u_5, t_5) by P_5. We have from (1.2)

$$\frac{du(t)}{dt} = \frac{du(t_4)}{dt} e^{-\frac{1}{\varepsilon}\int_{t_4}^t g\,dt} - \frac{1}{\varepsilon}\int_{t_4}^t h e^{-\frac{1}{\varepsilon}\int_\sigma^t g\,d\sigma}\,d\sigma\,. \tag{7.9}$$

It follows easily that by integrating

$$\int_{t_4}^t \left(\frac{h}{g}\right) g e^{-\frac{1}{\varepsilon}\int_\sigma^t g\,d\sigma}\,d\sigma$$

by parts we have for small ε

$$\left|\frac{du(t_5)}{dt}+\frac{h_5}{g_5}\right| \leqq \frac{\varepsilon}{p_4}+K(t_5-t_4) < \varepsilon^{\frac{1}{2}} \tag{7.10}$$

where h_5 denotes $h(x, u, t, \varepsilon)$ at P_5 etc.

By (6.0) and Lemma 1 if $\bar{z} = \left(\dfrac{\partial x(t)}{\partial a}, \dfrac{\partial u(t)}{\partial a}\right)$ then for $t_4 \leqq t \leqq t_5$

$$|\bar{z}| \leqq |\bar{z}_4| e^{K\int_{t_4}^t \left|\frac{du}{dt}\right| dt+K(t-t_4)} + K\int_{t_4}^t \left|\frac{d}{d\sigma}\frac{\partial u}{\partial a}\right| e^{K\int_\sigma^t \left|\frac{du}{dt}\right| dt+K(t-\sigma)}\,d\sigma\,. \tag{7.11}$$

From (7.9) we have easily if $g \geqq K_1 > 0$ near C where K_1 is clearly a K,

$$\left|\frac{du}{dt}\right| \leqq \frac{1}{p_4} e^{-K_1(t-t_4)/\varepsilon}+K\,. \tag{7.12}$$

Thus for $t_4 \leqq t \leqq t_5$

(7.13) $$\int_{t_4}^{t}\left|\frac{du}{dt}\right|dt \leqq \frac{K\varepsilon}{p_4}+K(t_5-t_4) \leqq \frac{K\varepsilon}{p_4}.$$

Also from

(7.14) $$\frac{d}{dt}\frac{\partial u}{\partial a}=\frac{d}{dt}\frac{\partial u(t_4)}{\partial a}e^{-\frac{1}{\varepsilon}\int_{t_4}^{t}g dt}-\frac{1}{\varepsilon}\int_{t_4}^{t}\left(\frac{\partial g}{\partial a}\frac{du}{d\sigma}+\frac{\partial h}{\partial a}\right)e^{-\frac{1}{\varepsilon}\int_{\sigma}^{t}g d\sigma}d\sigma ,$$

(7.15) $$\int_{t_4}^{t}\left|\frac{d}{dt}\frac{\partial u}{\partial a}\right|dt \leqq K\varepsilon\left|\frac{d}{dt}\frac{\partial u(t_4)}{\partial a}\right|+K\int_{t_4}^{t}|\bar{z}|\left(\left|\frac{du}{dt}\right|+1\right)dt .$$

From (7.2), (7.3) and (7.5) we have easily that since $\dfrac{\varepsilon}{p_3^2}\dfrac{\partial p(u_3)}{\partial a}$ is bounded by (7.0) as $\varepsilon, \delta_1, \delta_2 \to 0$

$$\left|\frac{\varepsilon}{p_4^2}\frac{\partial p(u_4)}{\partial a}\right| \leqq K .$$

From (7.8) then and (7.5) and (4.3)

$$\varepsilon\left|\frac{\partial}{\partial a}\frac{du(t_4)}{dt}\right| \leqq K+\frac{K}{p_4}<\frac{K}{p_4}.$$

Using the above, (7.13) and (7.15) in (7.11) we have if $\max|\bar{z}| = M$ for $t_4 \leqq t \leqq t_5$ and if ε is small enough,

$$M \leqq K|\bar{z}_4|+\frac{K}{p_4}+KM\left(\frac{\varepsilon}{p_4}+t_5-t_4\right).$$

Since $K\left(\dfrac{\varepsilon}{p_4}+t_5-t_4\right) < \frac{1}{2}$ if ε is small we have

$$|\bar{z}(t)| \leqq K|\bar{z}_4|+\frac{K}{p_4}.$$

Using (7.5), (7.7) and (7.8), we have $|\bar{z}_4| \leqq K/p_4$ and thus

(7.16) $$\left|\frac{\partial u(t)}{\partial a}\right|+\left|\frac{\partial x(t)}{\partial a}\right| \leqq \frac{K}{p_4}, \quad t_4 \leqq t \leqq t_5 .$$

By integrating (7.14),

$$\frac{\partial u(t)}{\partial a}=\frac{\partial u(t_4)}{\partial a}+\frac{d}{dt}\frac{\partial u(t_4)}{\partial a}\int_{t_4}^{t}e^{-\frac{1}{\varepsilon}\int_{t_4}^{\sigma}g d\sigma}d\sigma-J_1$$

where

$$J_1=\frac{1}{\varepsilon}\int_{t_4}^{t}\left(\frac{du}{dt}\frac{\partial g}{\partial a}+\frac{\partial h}{\partial a}\right)d\sigma\int_{\sigma}^{t}e^{-\frac{1}{\varepsilon}\int_{\sigma}^{s}g d\sigma}ds .$$

Using (7.7) and (7.8)

$$(7.17)\quad \frac{\partial u(t)}{\partial a} = -\frac{1}{p_4}\frac{\partial t(u_4)}{\partial a} + \left[-\frac{1}{p_4^2}\frac{\partial p(u_4)}{\partial a} + \frac{1}{p_4^3}\frac{dp(u_4)}{du}\frac{\partial t(u_4)}{\partial a}\right]\int_{t_4}^{t} e^{-\frac{1}{\varepsilon}\int_{t_4}^{\sigma} g dt}\, d\sigma - J_1 .$$

Clearly using (7.16) and (7.13) for $t_4 \leqq t \leqq t_5$ we find

$$(7.18)\qquad |J_1| \leqq \frac{K}{p_4}\int_{t_4}^{t_5}\left(\left|\frac{du}{dt}\right| + 1\right)dt \leqq \frac{K\varepsilon}{p_4^2} < \varepsilon^{\frac{1}{4}}$$

for small ε. Integrating by parts we have

$$\int_{t_4}^{t_5}\frac{1}{g}\left(ge^{-\frac{1}{\varepsilon}\int_{t_4}^{\sigma} g dt}\right)d\sigma = \frac{\varepsilon + J_2}{g_4}$$

where

$$(7.19)\qquad |J_2| < \varepsilon^{\frac{9}{8}} .$$

Thus from (7.17)

$$\frac{\partial u(t_5)}{\partial a} = -\frac{1}{p_4}\frac{\partial t(u_4)}{\partial a} + \frac{\varepsilon}{g_4 p_4^2}\left[-\frac{\partial p(u_4)}{\partial a} + \frac{1}{p_4}\frac{dp(u_4)}{du}\frac{\partial t(u_4)}{\partial a}\right]\left(1 + \frac{1}{\varepsilon}J_2\right) - J_1 .$$

Since $\varepsilon dp/du = p^2 g + p^3 h$,

$$(7.20)\qquad \frac{\partial u(t_5)}{\partial a} = \left[\frac{h_4}{g_4}\frac{\partial t(u_4)}{\partial a} - \frac{\varepsilon}{g_4 p_3^2}\frac{\partial p(u_3)}{\partial a}\right]\left(1 + \frac{1}{\varepsilon}J_2\right) + \frac{1}{\varepsilon}J_2\frac{1}{p_4}\frac{\partial t(u_4)}{\partial a} + J_3 - J_1$$

where

$$J_3 = \frac{1}{g_4}\left(\frac{\varepsilon}{p_3^2}\frac{\partial p(u_3)}{\partial a} - \frac{\varepsilon}{p_4^2}\frac{\partial p(u_4)}{\partial a}\right)\left(1 + \frac{1}{\varepsilon}J_2\right).$$

From (7.3) and (7.2) we have also using (7.0)

$$(7.21)\qquad \left|\frac{\varepsilon}{p_4^2}\frac{\partial p(u_4)}{\partial a} - \frac{\varepsilon}{p_3^2}\frac{\partial p(u_3)}{\partial a}\right| \leqq Kp_4\frac{\varepsilon}{p_3^2}\left|\frac{\partial p(u_3)}{\partial a}\right| + K(v_4 - v_3) .$$

Since v_3 and v_4 can be as near v_C as we wish we see that by taking ε, δ_1, and δ_2 small enough we can make

$$|J_3| < \varepsilon_9$$

where $\varepsilon_9 > 0$ is any prescribed quantity. By (7.6)

$$\left|\frac{\partial t(u_4)}{\partial a} - \frac{\partial t(u_3)}{\partial a}\right| \leqq Kp_4 + K(v_4 - v_3) .$$

Thus (7.20) yields

$$\frac{\partial u(t_5)}{\partial a} = \frac{h_4}{g_4}\frac{\partial t(u_3)}{\partial a} - \frac{\varepsilon}{g_4 p_3^2}\frac{\partial p(u_3)}{\partial a} + \varepsilon_{10}$$

where $\varepsilon_{10} > 0$ can be chosen arbitrarily small. From (7.0) and continuity considerations the above formula yield

$$\left|\frac{\partial u(t_5)}{\partial a} - \frac{h_C}{g_C}\frac{\partial \tau_1}{\partial a} + \frac{1}{g_C}\int_{v_B}^{v_C}\frac{\partial g}{\partial a}dv\right| < \varepsilon_{11} \tag{7.22}$$

where $\frac{\partial g}{\partial a} = \sum \frac{\partial g}{\partial y_i}\frac{\partial y_i}{\partial a} + \frac{\partial g}{\partial s}\frac{\partial \tau_1}{\partial a}$ and where $\varepsilon_{11} > 0$ can be chosen as small as we wish by taking ε, δ_1, and δ_2 small enough. Comparing with (5.9) we have

$$\left|\frac{\partial u(t_5)}{\partial a} - \frac{\partial v(t_C)}{\partial a}\right| < \varepsilon_{11} . \tag{7.23}$$

From (7.14), if $g_C > 2K_1 > 0$, we have using (7.8), (7.12), (7.16) and (7.21)

$$\left|\frac{d}{dt}\frac{\partial u(t_5)}{\partial a}\right| \leq \frac{K}{\varepsilon p_4} e^{-(t_5-t_4)K_1/\varepsilon} + \frac{K}{p_4^2} < \frac{\varepsilon^{\frac{1}{16}}}{\varepsilon} \tag{7.24}$$

for small ε, δ_1, and δ_2.

Finally from (6.0)

$$\frac{\partial}{dt}\frac{\partial x}{\partial a} = \frac{d}{dt}\left(f\frac{\partial u}{\partial a}\right) - \frac{\partial u}{\partial a}\frac{df}{dt} + \frac{\partial \varphi}{\partial a} + \frac{\partial f}{\partial a}\frac{du}{dt} .$$

Integrating we have

$$\frac{\partial x(t_5)}{\partial a} = \frac{\partial x(t_4)}{\partial a} + f_5\frac{\partial u(t_5)}{\partial a} - f_4\frac{\partial u(t_4)}{\partial a} + J$$

where for small ε, δ_1, and δ_2, by (7.16) and (7.13)

$$|J| < \frac{K\varepsilon}{p_4^2} < \varepsilon^{\frac{1}{4}} .$$

Or by (7.7) and (7.8)

$$\frac{\partial x(t_5)}{\partial a} = \frac{\partial x(u_4)}{\partial a} - \frac{dx(t_4)}{dt}\frac{\partial t(u_4)}{\partial a} + f_5\frac{\partial u(t_5)}{\partial a} + f_4\frac{du(t_4)}{dt}\frac{\partial t(u_4)}{\partial a} + J$$

$$= \frac{\partial x(u_4)}{\partial a} + f_5\frac{\partial u(t_5)}{\partial a} - \varphi_4\frac{\partial t(u_4)}{\partial a} + J .$$

Or by (7.6), (7,23) and continuity considerations

$$\left|\frac{\partial x(t_5)}{\partial a} - \frac{\partial x(u_3)}{\partial a} - f_C\frac{\partial u(t_5)}{\partial a} + \varphi_C\frac{\partial t(u_3)}{\partial a}\right| < \varepsilon_{12}$$

where $\varepsilon_{12} > 0$ can be chosen less than any prescribed quantity by taking ε, δ_1, and δ_2 small enough. Or by Theorem 3, part 2 and by (7.23)

$$\left|\frac{\partial x(t_5)}{\partial a}-\frac{\partial y(v_C)}{\partial a}-f_C\frac{\partial v(t_C)}{\partial a}+\varphi_C\frac{\partial \dot{\tau}_1}{\partial a}\right|<\varepsilon_{13} \tag{7.25}$$

where $\varepsilon_{13}>0$ etc. Comparing with (5.10) we have now

$$\left|\frac{\partial x(t_5)}{\partial a}-\frac{\partial y(t_C)}{\partial a}\right|+\left|\frac{\partial u(t_5)}{\partial a}-\frac{\partial v(t_C)}{\partial a}\right| \tag{7.26}$$

as small as we wish. With (7.24), (7.26) and (7.10) we can proceed now as in Part I from t_5 to β since in the range t_C to t_5 or t_5 to t_C the change in the degenerate system is small because $|t_5-t_C|$ is small.

The case where b is used instead of a proceeds with the usual modifications. This completes the proof of Theorem 3.

8. Let us consider the case where the degenerate system has a solution, as defined in § 2, which is periodic of period T. Let $t=\alpha$ be a point where $g \neq 0$. Suppose in the first place we take the case where the functions $f(x, u, t, \varepsilon)$, φ, g and h are periodic in t of period T. Let the initial values for (1.3) be $y_1(\alpha),\ldots, y_n(\alpha)$, $v(\alpha)$. Let us denote the initial values of $x_1(\alpha),\ldots, x_n(\alpha)$, $u(\alpha)$, $\dfrac{du(\alpha)}{dt}$ by $a_1,\ldots, a_{n+1}$, b. Then for the system (1.2) to have a periodic solution for small ε it suffices that, if $\beta=\alpha+T$, the determinant at $t=\beta$,

$$D(a_1,\ldots, a_{n+1}, b, \varepsilon)=\begin{vmatrix} \dfrac{\partial x_1}{\partial a_1}-1 & \dfrac{\partial x_1}{\partial a_2} & \cdots & \dfrac{\partial x_1}{\partial a_{n+1}} & \dfrac{\partial x_1}{\partial b} \\ \dfrac{\partial x_2}{\partial a_1} & \dfrac{\partial x_2}{\partial a_2}-1 & \cdots & \cdot & \dfrac{\partial x_2}{\partial b} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \dfrac{\partial u}{\partial a_1} & \cdot & \cdots & \dfrac{\partial u}{\partial a_{n+1}}-1 & \dfrac{\partial u}{\partial b} \\ \dfrac{\partial}{\partial a_1}\dfrac{du}{dt} & \cdot & \cdots & \cdot & \dfrac{\partial}{\partial b}\dfrac{du}{dt}-1 \end{vmatrix}$$

should be continuous as

$\varepsilon \to +0;\, a_1,\ldots, a_{n+1} \to y_1(\alpha),\ldots, v(\alpha)$; and $b \to -\dfrac{h_A}{g_A}$, and D should not vanish.

This is a consequence of the fact that the existence of a periodic solution of (1.2) is equivalent to the existence of a solution $(a_1,\ldots, a_{n+1}, b)$ of

$$x_i(\beta, a_1, \ldots, a_{n+1}, b, \varepsilon) = a_i,\ i = 1, 2, \ldots, n\,,$$

$$u(\beta, a_1, \ldots, a_{n+1}, b, \varepsilon) = a_{n+1}\,, \quad \frac{du(\beta, a_1, \ldots, a_{n+1}, b, \varepsilon)}{dt} = b\,,$$

where $(a_1, \ldots, a_{n+1}, b)$ are found as functions of $\varepsilon > 0$. By Theorem 3, the last column of D tends to zero except for the last term in the column which tends to -1. Thus D tends to

$$(8.0) \qquad - \begin{vmatrix} \frac{\partial y_1(\beta)}{\partial a_1} - 1 & \frac{\partial y_1(\beta)}{\partial a_2} & \cdots & \frac{\partial y_1(\beta)}{\partial a_{n+1}} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial v(\beta)}{\partial a_{n+1}} & \cdot & \cdots & \frac{\partial v(\beta)}{\partial a_{n+1}} - 1 \end{vmatrix}$$

as ε, δ_1, and $\delta_2 \to 0$. This is the Jacobian associated with a periodic solution of (1.3) and if it is different from zero we see that (1.2) also has a unique nearby periodic solution for small $\varepsilon > 0$. In particular if the periodic solution of (1.3) is stable in the sense that the associated characteristic roots are all less than one in magnitude then the determinant (8.0) does not vanish.

In case the right members of (1.2) do not involve t the period T is no longer a constant for periodic solutions (if any) of the perturbed system. Making the usual modification for this situation the same result relating the existence of periodic solutions for (1.2) to (1.3) holds again.

The case where the last equation of (1.2) is a vector equation can be treated quite readily on the basis of results obtained here by making a change of coordinates. We shall return to this case later.

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