ON CERTAIN THEOREMS IN OPERATIONAL CALCULUS.

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The object of this paper is twofold: firstly to establish certain theorems in Operational Calculus and secondly to obtain the Laplace transforms of several functions.

I.

1. Let us suppose [1]

$$\Phi(p) = p \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

where p is a positive number (or a number whose real part is positive) and the integral on the right converges. We shall then say that $\Phi(p)$ is operationally related to $f(t)$ and symbolically

$$\Phi(p) \doteqdot f(t) \text{ or } f(t) \doteqdot \Phi(p). \tag{2}$$

Many interesting relations involving $\Phi(p)$ and $f(t)$ have been obtained. The following will be required in the sequel.

$$p\,\Phi(p) \doteqdot \frac{d}{dt} f(t), \text{ if } f(0) = 0 \tag{3}$$

$$p \frac{d}{dp}[\Phi(p)] \doteqdot -t \frac{d}{dt} f(t) \tag{4}$$

$$\frac{\Phi(p)}{p} \doteqdot \int_0^t f(t)\, dt \tag{5}$$

$$p \int_0^\infty \frac{\Phi(p)}{p}\, dp \doteqdot \frac{f(t)}{t} \tag{6}$$

$$p \frac{d}{dp}\left[\frac{\Phi(p)}{p}\right] \doteqdot -t f(t). \tag{7}$$

Also Goldstein [2] has proved that if

$$\Phi(p) \doteq f(t), \quad \psi(p) \doteq g(t),$$

then

$$\int_0^\infty \frac{\Phi(t)\, g(t)\, dt}{t} = \int_0^\infty \frac{\psi(t)\, f(t)}{t}\, dt, \tag{8}$$

provided the integrals converge.

It is known that if $h(t)$ is another function which satisfies (1), then

$$f(t) - h(t) = n(t),$$

where $n(t)$ is a null-function, i.e. a function such that

$$\int_0^t n(t)\, dt = 0, \text{ for every } t \geq 0.$$

If $f(t)$ is *a* continuous function which satisfies (1), then it is the only continuous function which satisfies (1). This theorem is due to Lerch [3].

2. Our object is to investigate that if either of the two functions $f(t)$ and $\Phi(t)$ has an assigned property, then will that property or an analogous property be true of the other function?

We know that

$$\frac{p}{(p^2+b^2)^{n+\frac{1}{2}}} \doteq \frac{\sqrt{\pi}}{2^n\, \Gamma(n+\frac{1}{2})} \left(\frac{t}{b}\right)^n J_n(bt). \tag{9}$$

Applying Goldstein's theorem, we get

$$b^2 \int_0^\infty \frac{f(t)\, dt}{(b^2+t^2)^{n+\frac{1}{2}}} = \frac{\sqrt{\pi}\, b}{2^n\, \Gamma(n+\frac{1}{2})} \int_0^\infty \left(\frac{t}{b}\right)^{n-1} \Phi(t)\, J_n(bt)\, dt, \quad \Re(n) > -\tfrac{1}{2}. \tag{10}$$

Les us now put $b^2 = p$ and interpret. Assuming that $\frac{1}{p} \doteq \varkappa$, we get

$$\varkappa^{n-\frac{1}{2}} \int_0^\infty e^{-t^2 \varkappa} f(t)\, dt \doteq \frac{\sqrt{\pi}}{2^n} \frac{1}{p^{\frac{1}{2}n-1}} \int_0^\infty t^{n-1}\, \Phi(t)\, J_n(\sqrt{p}\, t)\, dt, \tag{11}$$

provided the integrals converge.

Again let us divide both sides of (10) by b and put $b = p$. On interpretation, we get

$$\int_0^\infty \left(\frac{\varkappa}{t}\right)^n f(t) J_n(\varkappa t)\,dt \doteqdot \int_0^\infty \left(\frac{t}{p}\right)^{n-1} \Phi(t) J_n(pt)\,dt, \quad R(n) > -\tfrac{1}{2}. \tag{12}$$

This can also be written in the form

$$\varkappa^{n-\frac{1}{2}} \int_0^\infty \sqrt{\varkappa t}\, t^{-n-\frac{1}{2}} f(t) J_n(\varkappa t)\,dt \doteqdot \frac{1}{p^{n-1}} \int_0^\infty t^{n-1} \Phi(t) J_n(pt)\,dt. \tag{13}$$

Suppose $t^{-n-\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order n. Then

$$f(\varkappa)/\varkappa \doteqdot \int_0^\infty \left(\frac{t}{p}\right)^{n-1} \Phi(t) J_n(pt)\,dt. \tag{14}$$

But by (6),

$$p \int_p^\infty \frac{\Phi(p)}{p}\,dp \doteqdot \frac{f(\varkappa)}{\varkappa}.$$

Therefore

$$\int_0^\infty t^{n-1} \Phi(t) J_n(pt)\,dt = p^n \int_p^\infty \frac{\Phi(p)}{p}\,dp, \tag{15}$$

provided the integrals converge.

Dividing both sides by p^n and differentiating with respect to p (assuming that differentiation under the sign of integration is permissible and that $\Phi(t)/t$ is a continuous function of t in $(0, \infty)$), we get on writing $n-1$ for n,

$$\int_0^\infty \sqrt{pt}\, t^{n-\frac{3}{2}} \Phi(t) J_n(pt)\,dt = p^{n-\frac{3}{2}} \Phi(p), \tag{16}$$

showing that $t^{n-\frac{3}{2}} \Phi(t)$ is self-reciprocal in the Hankel transform of order n, when (16) converges.

Thus we have

Theorem I. If $t^{-n-\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order n and $\Phi(t)/t$ is continuous in $(0, \infty)$ then $t^{n-\frac{3}{2}} \Phi(t)$ is self-reciprocal in the Hankel transform of order n.

We can also write (12) in the form

$$\int_0^\infty \left(\frac{\varkappa}{t}\right)^n f(t)\, J_n(\varkappa t)\, dt \doteqdot \frac{1}{p^{n-\frac{1}{2}}} \int_0^\infty \sqrt{pt}\, t^{n-\frac{3}{2}}\, \Phi(t)\, J_n(pt)\, dt. \tag{17}$$

Let $t^{n-\frac{3}{2}}\Phi(t)$ be self-reciprocal in the Hankel transform of order n. The (17) becomes

$$\int_0^\infty \left(\frac{\varkappa}{t}\right)^n f(t)\, J_n(\varkappa t)\, dt \doteqdot \frac{\Phi(p)}{p}. \tag{18}$$

But by (5),

$$\frac{\Phi(p)}{p} \doteqdot \int_0^\varkappa f(t)\, dt.$$

Hence by Lerch's theorem

$$\int_0^\infty \left(\frac{\varkappa}{t}\right)^n f(t)\, J_n(\varkappa t)\, dt = \int_0^\varkappa f(t)\, dt. \tag{19}$$

Differentiating both sides with respect to $\varkappa$ (assuming that differentation under the sign of integration is permissible and $f(t)$ is a continuous function of t), we get on writing $n+1$ for n

$$\int_0^\infty \sqrt{\varkappa t}\, t^{-n-\frac{1}{2}} f(t)\, J_n(\varkappa t)\, dt = \varkappa^{-n-\frac{1}{2}} f(\varkappa), \tag{20}$$

showing that $t^{-n-\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order n. We thus have conversely,

Theorem II. If $t^{n-\frac{3}{2}}\Phi(t)$ is self-reciprocal in the Hankel transform of order n and $f(t)$ is continuous, then $t^{-n-\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order n.

In (12) let us put $n=\frac{1}{2}$. We obtain

$$\int_0^\infty \frac{f(t)}{t} \sin \varkappa t\, dt \doteqdot \int_0^\infty \frac{\Phi(t)}{t} \sin pt\, dt \tag{21}$$

By (4), we get

$$\varkappa \int_0^\infty f(t) \cos \varkappa t\, dt \doteqdot -p \int_0^\infty \Phi(t) \cos pt\, dt, \tag{22}$$

where we again assume that differentiation under the sign of integration is permissible.

If $\Phi(t)$ is self-reciprocal in the cosine transform, we obtain

$$\sqrt{\frac{2}{\pi}}\,\varkappa\int_0^\infty f(t)\cos\varkappa t\,dt \doteqdot -p\,\Phi(p). \tag{23}$$

But by (3),

$$p\,\Phi(p) \doteqdot f'(\varkappa),\ \text{if}\ f(0)=0.$$

Hence

$$\sqrt{\frac{2}{\pi}}\,\varkappa\int_0^\infty f(t)\cos\varkappa t\,dt = -f'(\varkappa).$$

Integrating the left hand side by parts, we have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f'(t)\sin\varkappa t\,dt = f'(\varkappa),\ \text{when}\ f(\infty)=0, \tag{24}$$

showing that $f'(t)$ is self-reciprocal in the sine transform. We therefore have

Theorem III. If $\Phi(t)$ is self-reciprocal in the cosine transform and $f(0)=f(\infty)=0$, then $f'(\varkappa)$ is self-reciprocal in the sine transform. Again integrating the left hand side of (22), we have

$$\int_0^\infty f'(t)\sin\varkappa t\,dt \doteqdot p\int_0^\infty \Phi(t)\cos pt\,dt,$$

provided $f(\infty)=0$.

If $f'(t)$ is self-reciprocal in the sine-transform, we get

$$f'(\varkappa) \doteqdot \sqrt{\frac{2}{\pi}}\,p\int_0^\infty \Phi(t)\cos pt\,dt. \tag{25}$$

But when $f(0)=0$, we have by (3), $f'(\varkappa)\doteqdot p\,\Phi(p)$, so that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(t)\cos pt\,dt = \Phi(p), \tag{26}$$

showing that $\Phi(t)$ is self-reciprocal in the cosine transform. Hence the converse theorem follows, viz.,

Theorem IV. If $f(0)=f(\infty)=0$ and $f'(x)$ is self-reciprocal in the sine transform, then $\Phi(t)$ is self-reciprocal in the cosine transform.

Again in (22) let $f(t)$ be self-reciprocal in the cosine transform. Then

$$x f(x) \doteqdot -\sqrt{\frac{2}{\pi}}\, p \int_0^\infty \Phi(t) \cos pt\, dt.$$

But by (7),

$$x f(x) \doteqdot -p \frac{d}{dp}\left[\frac{\Phi(p)}{p}\right],$$

so that

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \Phi(t) \cos pt\, dt = \frac{d}{dp}\left[\frac{\Phi(p)}{p}\right]. \tag{27}$$

Integrating both sides with respect to p between the limits zero and p and changing the order of integration on the left (if that is permissible), we notice that if $\Phi(p)/p \to 0$ as $p \to 0$,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\Phi(t)}{t} \sin pt\, dt = \frac{\Phi(p)}{p}, \tag{28}$$

showing that $\Phi(t)/t$ is self-reciprocal in the sine transform. Hence we have

Theorem V. If $f(t)$ is self-reciprocal in the cosine transform and $\Phi(t)/t \to 0$ as $t \to 0$, then $\Phi(t)/t$ is self-reciprocal in the sine transform. Conversely, if $\Phi(t)/t$ is self-reciprocal in the sine transform, we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\Phi(t)}{t} \sin pt\, dt = \frac{\Phi(p)}{p} \doteqdot \int_0^x f(t)\, dt, \text{ by (5).}$$

Hence by (4),

$$\sqrt{\frac{2}{\pi}}\, p \int_0^\infty \Phi(t) \cos pt\, dt \doteqdot -x f(x),$$

provided $f(t)$ is continuous and differentiation under the sign of integration is permissible.

But by (22),

$$x \int_0^\infty f(t) \cos xt\, dt \doteqdot -p \int_0^\infty \Phi(t) \cos pt\, dt.$$

Hence

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\varkappa t\,dt = f(\varkappa), \tag{29}$$

showing that $f(t)$ is self-reciprocal in the cosine transform. Thus we have

Theorem VI. If $\varphi(t)/t$ is self-reciprocal in the sine transform and $f(t)$ is continuous, then $f(t)$ is self-reciprocal in the cosine transform.

Theorem IV can also be extended to reciprocal functions.

$$\text{Let } \Phi(p \doteqdot f(\varkappa),\ \psi(p) \doteqdot g(\varkappa)$$

and

$$f(0) = g(0) = f(\infty) = g(\infty) = 0.$$

Then if $\Phi(p)$ is reciprocal to $\psi(p)$; $f'(\varkappa)$ is reciprocal to $g'(\varkappa)$ in the sine transform.

For, by (22)

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\,\varkappa\int_0^\infty f(t)\cos\varkappa t\,dt &\doteqdot -\sqrt{\frac{2}{\pi}}\,p\int_0^\infty \Phi(t)\cos pt\,dt\\ &\doteqdot -p\,\psi(p)\\ &\doteqdot -g'(\varkappa).\end{aligned}$$

Integrating the left hand side and applying Lerch's theorem, we obtain

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f'(t)\sin\varkappa t\,dt = g'(\varkappa), \tag{30}$$

showing that $f'(\varkappa)$ is reciprocal to $g'(\varkappa)$ in the sine transform.

Conversely, let $f'(\varkappa)$ be reciprocal to $g(\varkappa)$ in the sine transform, where $g(\varkappa)$ is continuous in the arbitrary interval $(0, \varkappa)$. Let $G(\varkappa) = \int_0^\infty g(\varkappa)\,d\varkappa$, $\Phi(p) \doteqdot f(\varkappa)$ and $\psi(p) \doteqdot G(\varkappa)$. Then if $f(\infty) = 0$; $\Phi(p)$ is reciprocal to $\psi(p)$ in the cosine transform. We have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f'(t)\sin\varkappa t\,dt = g(\varkappa). \tag{31}$$

On integration, the left hand side becomes

$$-\sqrt{\frac{2}{\pi}}\,\varkappa\int_0^\infty f(t)\cos\varkappa t\,dt,$$

which, by (22) is equal ($\doteqdot$) to

$$\sqrt{\frac{2}{\pi}}\,p\int_0^\infty \Phi(t)\cos p t\,dt.$$

Therefore

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\,p\int_0^\infty \Phi(t)\cos p t\,dt &\doteqdot g(\varkappa)\\ &\doteqdot G'(\varkappa)\\ &\doteqdot p\,\psi(p).\end{aligned}$$

Hence

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(t)\cos p t\,dt=\psi(p),$$

showing that $\Phi(t)$ is reciprocal to $\psi(p)$ in the cosine transform.

3. A Functional Relation.

Let us now consider the relation (10). Putting $b^2=p$ and interpreting, we obtain

$$\frac{2^n}{\sqrt{\pi}}\,\varkappa^{n-\frac{1}{2}}\int_0^\infty e^{-t^2\varkappa}f(t)\,dt\doteqdot\frac{1}{p^{\frac{1}{2}n-\frac{3}{4}}}\int_0^\infty t^{n-\frac{3}{2}}(\sqrt{p}\,t)^{\frac{1}{2}}\,\Phi(t)\,J_n(\sqrt{p}\,t)\,dt,$$

which is our relation (11).

Suppose $t^{n-\frac{3}{2}}\Phi(t)$ is self-reciprocal in the Hankel transform of order n. The right hand side is $\Phi(\sqrt{p})$. But if $\Phi(p)\doteqdot f(t)$, then

$$\Phi(\sqrt{p})\doteqdot\frac{1}{\sqrt{\pi\varkappa}}\int_0^\infty e^{-t^2/4\varkappa}f(t)\,dt,$$

so that

$$2^n\varkappa^n\int_0^\infty e^{-t^2\varkappa}f(t)\,dt=\int_0^\infty e^{-t^2/4\varkappa}f(t)\,dt. \tag{32}$$

If we write

$$F(\varkappa)=\int_0^\infty e^{-t^2\varkappa}f(t)\,dt,$$

the functional relation becomes

$$2^n \varkappa^n F(\varkappa) = F\left(\frac{1}{4\varkappa}\right). \tag{33}$$

4. If $\Phi(p)$ is given by (1), then by Mellin's inversion formula [4],

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\frac{\Phi(\lambda)}{\lambda}\,d\lambda,\ (c>0) \tag{34}$$

The question naturally arises: if $f(t)$ and $\Phi(t)$ have these assigned properties, are there formulae for determining them otherwise if either of the two functions is known?

We know that

$$\frac{\varkappa^n}{(t+\varkappa)^{n+\frac{1}{2}}} \doteqdot 2^{n+\frac{1}{2}}\Gamma(n+1)\sqrt{p}\,e^{\frac{1}{2}pt}\,D_{-2n-1}(\sqrt{2pt}). \tag{35}$$

Applying Goldstein's theorem, we get after slight changes in the variables

$$\frac{1}{2^{n+\frac{1}{2}}\Gamma(n+1)}\int_0^\infty \frac{t^{n-1}\Phi(t)\,dt}{(t+p)^{n+\frac{1}{2}}} = \int_0^\infty t^{-\frac{1}{2}}e^{\frac{1}{2}pt}D_{-2n-1}(\sqrt{2pt})\,f(t)\,dt. \tag{36}$$

Writing t^2 for t and p^2 for p, the above relation becomes

$$\frac{1}{2^{n+\frac{1}{2}}\Gamma(n+1)}\int_0^\infty \frac{t^{2n-1}\Phi(t^2)\,dt}{(p^2+t^2)^{n+\frac{1}{2}}} = \int_0^\infty e^{\frac{1}{2}p^2t^2}D_{-2n-1}(\sqrt{2}pt)\,f(t^2)\,dt. \tag{37}$$

Multiplying both sides by p and interpreting, we have on simplification,

$$\frac{\varkappa^{n-\frac{1}{2}}}{\Gamma(2n+1)}\int_0^\infty \sqrt{\varkappa t}\,t^{n-\frac{3}{2}}\Phi(t^2)J_n(\varkappa t)\,dt$$

$$\doteqdot \sqrt{2}\,p\int_0^\infty e^{\frac{1}{2}p^2t^2}D_{-2n-1}(\sqrt{2}pt)\,f(t^2)\,dt,\ \mathcal{R}(n)>-\tfrac{1}{2}. \tag{38}$$

If $t^{n-\frac{3}{2}}\Phi(t^2)$ is self-reciprocal in the Hankel transform of order n, we get

$$\Phi(\varkappa^2)\,\varkappa^{2n-2} \doteqdot \sqrt{2}\,\Gamma(2n+1)\,p\int_0^\infty e^{\frac{1}{2}p^2t^2}D_{-2n-1}(\sqrt{2}pt)\,f(t^2)\,dt. \tag{39}$$

If $\Phi(t^2)/t$ is self-reciprocal in the sine transform,

$$\Phi(\varkappa^2)/\varkappa \doteqdot \sqrt{2}\, p \int_0^\infty e^{\frac{1}{2}p^2 t^2} D_{-2}(\sqrt{2}\, p\, t)\, f(t^2)\, dt. \tag{40}$$

Let us revert back to relation (10) once more. We can write it in the form

$$\frac{2^n}{\sqrt{\pi}} \Gamma(n+\tfrac{1}{2}) \int_0^\infty \frac{b^{n+\frac{1}{2}} f(t)\, dt}{(t^2+b^2)^{n+\frac{1}{2}}} = \int_0^\infty \sqrt{bt}\, t^{n-\frac{3}{2}} \Phi(t) J_n(bt)\, dt. \tag{41}$$

If $t^{n-\frac{3}{2}}\Phi(t)$ is self-reciprocal in the Haukel Transform of order n, then

$$\Phi(b) = \frac{2^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}} b^2 \int_0^\infty \frac{f(t)\, dt}{(t^2+b^2)^{n+\frac{1}{2}}}. \tag{42}$$

Conversely if $\Phi(b)$ is given by (42), then putting $b=p$ and interpreting, we get after a bit of reduction that $t^{-n+\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order $n-1$, provided $f(t)$ is continuous and $n>0$. If (42) holds and $t^{-n+\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order $n-1$, then $\Phi(p) \doteqdot f(t)$. Again expressing the right hand side of (1) as a double integral and changing the order of integration (if that is permissible) we can prove that if $t^{-n+\frac{1}{2}} f(t)$ is self-reciprocal in the Hankel transform of order $n-1$, then $\Phi(b)$ is always given by (42).

We might also have derived similar relations by considering that [5]

$$f(t^2) \doteqdot \frac{p}{\sqrt{\pi}} \int_0^\infty e^{-p^2 \varkappa^2/4} \Phi\left(\frac{1}{\varkappa^2}\right) d\varkappa. \tag{43}$$

5. A double Integral theorem for $\Phi(t)$.

Let us consider the relation (12) again, Since by (7)

$$p \frac{d}{dp}\left[\frac{\Phi(p)}{p}\right] \doteqdot -\varkappa f(\varkappa)$$

we get on differentiating under the sign of integration (if that is permissible)

$$\int_0^\infty \frac{\varkappa^{n+1}}{t^n} f(t) J_n(\varkappa t)\, dt \doteqdot \int_0^\infty \frac{t^n}{p^{n-1}} \Phi(t) J_{n+1}(pt)\, dt, \quad \Re(n) > -\tfrac{1}{2}. \tag{44}$$

Also we know

$$\frac{\sqrt{\pi}}{2^{n+1}\Gamma(n+\frac{3}{2})}\,\frac{x^{n+1}J_n(cx)}{c^n} \doteq \frac{p^2}{(p^2+c^2)^{n+\frac{3}{2}}},\quad \Re(n) > -1. \tag{45}$$

Making use of Goldstein's Theorem, we obtain

$$\int_0^\infty\int_0^\infty \frac{x^{n+2}f(t)\,J_n(xt)}{t^n\,(x^2+c^2)^{n+\frac{3}{2}}}\,dt\,dx = \frac{\sqrt{\pi}}{2^{n+1}\,\Gamma(n+\frac{3}{2})\,c^n}\times$$

$$\times\int_0^\infty\int_0^\infty x\,t^n\,\Phi(t)\,J_n(cx)\,J_{n+1}(xt)\,dt\,dx. \tag{46}$$

Let $c = \frac{1}{p}$ where we now assume that $\frac{1}{p} \doteq y$. Then on simplification, we have

$$y^{n+1}\int_0^\infty\int_0^\infty \frac{f(t)}{t^n}\,J_n(xt)\,J_n\left(\frac{y}{x}\right)\frac{dt\,dx}{x} \doteq \frac{1}{p^{n+1}}\int_0^\infty\int_0^\infty x\,t^n\,\Phi(t)\,J_n\left(\frac{x}{p}\right)J_{n+1}(xt)\,dt\,dx. \tag{47}$$

Writing $\frac{x}{t}$ for x, we get since x and t are independent variables,

$$y^{n+\frac{1}{2}}\int_0^\infty\int_0^\infty \frac{f(t)}{t^{n+\frac{1}{2}}}\sqrt{yt}\,J_n(x)\,J_n\left(\frac{yt}{x}\right)dt\,\frac{dx}{x}$$

$$\doteq \frac{1}{p^{n+1}}\int_0^\infty\int_0^\infty x\,t^n\,\Phi(t)\,J_n\left(\frac{x}{p}\right)J_{n+1}(xt)\,dt\,dx. \tag{48}$$

Professor Watson [6] has shown that

$$\tilde{\omega}_{\mu,\nu}(xy) = \sqrt{xy}\int_0^\infty J_\nu(t)\,J_\mu\left(\frac{xy}{t}\right)\frac{dt}{t}, \tag{48'}$$

can be taken as the kernel of a *new transform*. Let $f(x)$ be an arbitrary function, and let $g(x)$ be its transform with the Kernel $\tilde{\omega}_{\mu,\nu}(xy)$, so that

$$g(x) = \int_0^\infty \tilde{\omega}_{\mu,\nu}(xy)\,f(y)\,dy.$$

Then assuming that the various changes in the order of integration are permissible, we have

$$\int_0^\infty \tilde{\omega}_{\mu,\nu}(\varkappa y)\, g(y)\, dy = f(\varkappa). \tag{49}$$

When $f(\varkappa) = g(\varkappa)$, we say that $f(\varkappa)$ is self-reciprocal under this new transform. Hence in (48), if $t^{-n-\frac{1}{2}} f(t)$ is *self-reciprocal under this transform*[1], the left hand side is $f(y)$, so that

$$f(y) \doteqdot \frac{1}{p^{n+1}} \int_0^\infty \int_0^\infty \varkappa\, t^n\, \Phi(t)\, J_n\left(\frac{\varkappa}{p}\right) J_{n+1}(\varkappa t)\, dt\, d\varkappa \doteqdot \Phi(p).$$

Therefore

$$\Phi(p) = \frac{1}{p^{n+1}} \int_0^\infty \int_0^\infty \varkappa\, t^n\, \Phi(t)\, J_n\left(\frac{\varkappa}{p}\right) J_{n+1}(\varkappa t)\, dt\, d\varkappa. \tag{50}$$

This can be written in the more symmetrical form, after considerable simplification,

$$\Phi(p) = -p^2 \frac{d}{dp} \frac{1}{p^n} \left[\int_0^\infty \int_0^\infty t^{n-1}\, \Phi(t)\, J_n\left(\frac{\varkappa}{p}\right) J_n(\varkappa t)\, dt\, d\varkappa \right],\quad \mathcal{R}(n) > -\tfrac{1}{2}. \tag{51}$$

provided $\Phi(p)/p$ is continuous. Conversely if (51) holds, then $t^{-n-\frac{1}{2}} f(t)$ is self-reciprocal under this transform.

II.

6. Laplace transforms of certain functions.

Let us us now consider the relation (11). We know that

$$J_\nu\left(\frac{a}{p}\right) \doteqdot J_\nu(\sqrt{2at})\, I_\nu(\sqrt{2at}). \tag{52}$$

Let

$$f(t) = J_\nu(\sqrt{2at})\, I_\nu(\sqrt{2at}) \text{ and } \Phi(t) = J_\nu\left(\frac{a}{t}\right).$$

We thus obtain

$$\frac{2^n \varkappa^{n-\frac{1}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2 \varkappa} J_\nu(\sqrt{2at})\, I_\nu(\sqrt{2at})\, dt \doteqdot \frac{1}{p^{\frac{1}{2}n-1}} \int_0^\infty J_\nu\left(\frac{a}{t}\right) J_n(\sqrt{p}\, t)\, t^{n-1}\, dt. \tag{53}$$

[1] The senior author has been able to construct certain examples giving functions which are self-reciprocal under this *new transform* and also the formal solutions of (49), when $f(\varkappa) = g(\varkappa)$.

But

$$J_\nu(az)\, I_\nu(az) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2} a z)^{2\nu+4m}}{\Gamma(m+1)\,\Gamma(\nu+m+1)\,\Gamma(\nu+2m+1)}. \tag{54}$$

Integrating term by term and applying a result due to Hanumauta Rao [7], we obtain

$$\frac{2^{n-2\nu-1}\, \varkappa^{n-\frac{1}{2}\nu-1}}{\Gamma(\frac{1}{2}\nu+1)\,\Gamma(\nu+1)}\, {}_0F_2\left(\frac{1}{2}\nu+1,\ \nu+1;\ -\frac{a^2}{16\varkappa}\right)$$

$$\doteqdot \left\{\frac{\Gamma(n-\frac{1}{2}\nu)}{2^{2\nu-n+1}\,\Gamma(\frac{1}{2}\nu+1)\,\Gamma(\nu+1)}\, \frac{1}{p^{n-\frac{1}{2}\nu-1}}\, {}_0F_3\left(\frac{1}{2}\nu+1,\ \nu+1,\ \frac{1}{2}\nu-n+1;\ \frac{a^2 p}{16}\right)\right. \tag{54}$$

$$\left. + \frac{\Gamma(\frac{1}{2}\nu-n)\, a^{2n-\nu}\, p}{2^{3n+1}\,\Gamma(n+1)\,\Gamma(\frac{1}{2}\nu+n+1)}\, {}_0F_3\left(n+1,\ n-\frac{1}{2}\nu+1,\ n+\frac{1}{2}\nu+1;\ \frac{a^2 p}{16}\right)\right\};$$

$$\{-\mathfrak{R}(n+\tfrac{3}{2}) < \mathfrak{R}(n) < \mathfrak{R}(\nu+\tfrac{3}{2}) \text{ and } a > 0\}.$$

Again we know that

$$(2t)^{-1} J_0\left(\frac{y^2}{4t}\right) \doteqdot p\, J_0(y\sqrt{\tfrac{1}{2}p})\, K_0(y\sqrt{\tfrac{1}{2}p}). \tag{55}$$

Let

$$f(t) = (2t)^{-1} J_0\left(\frac{y^2}{4t}\right) \text{ and } \Phi(t) = t\, J_0(y\sqrt{\tfrac{1}{2}t})\, K_0(y\sqrt{\tfrac{1}{2}t}).$$

We get (when $n=0$)

$$\frac{1}{2\sqrt{\pi\varkappa}} \int_0^\infty e^{-t^2\varkappa}\, t^{-1} J_0\left(\frac{y^2}{4t}\right) dt \doteqdot p \int_0^\infty J_0(y\sqrt{\tfrac{1}{2}t})\, K_0(y\sqrt{\tfrac{1}{2}t})\, J_0(\sqrt{p}\,t)\, dt. \tag{56}$$

Putting $t = 2z^2/y^2$, the right hand side becomes

$$4p\,y^{-2} \int_0^\infty z\, J_0(z)\, K_0(z)\, J_0\left(\sqrt{p}\,\frac{2z^2}{y^2}\right) dz.$$

By a result due to Mitra [8], the integral can be evaluated and we finally obtain

$$\frac{1}{\sqrt{\pi\varkappa}} \int_0^\infty e^{-t^2\varkappa}\, t^{-1} J_0\left(\frac{y^2}{4t}\right) dt \doteqdot \sqrt{p}\, I_0\left(\frac{y^2}{8\sqrt{p}}\right) K_0\left(\frac{y^2}{8\sqrt{p}}\right). \tag{57}$$

The integral on the left can be evaluated by expressing it as a contour integral.

Again let

$$\Phi(t) = e^{-1/t}\, t^{-1/2};\ f(t) = (\pi)^{-\frac{1}{2}} \sin 2\sqrt{t}.$$

We get

$$(\pi)^{-\frac{1}{2}} \int_{0}^{\infty} e^{-t^2 x} \sin 2\sqrt{t}\, dt \doteq p^{1/2} \int_{0}^{\infty} e^{-1/t}\, t^{-3/2} \sin \sqrt{p}\, t\, dt$$

$$\doteq (\pi)^{\frac{1}{2}} e^{-\sqrt{2}\, p^{\frac{1}{4}}} \sin \sqrt{2}\, p^{\frac{1}{4}}. \qquad (58)$$

The integral on the left is easily obtainable by direct term by term integration.

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References.

[1] Carson, *Electric circuit theory and the operational calculus* (1926), 23.
[2] Goldstein, S., *Proc. Lond. Math. Soc.* **34** (1931), 103.
[3] Lerch, *Acta Mathematica* **27** (1903), 339—52.
[4] Mellin, *Acta Mathematica* **25** (1902), 156—62.
[5] McLachlan and Humbert, *Formulaire pour le calcul symbolique* (1941), 12.
[6] Watson, G. N., *Quart. Journ. Math.* **2**, No. 8 (1931), 298—309.
[7] Rao, C. V. H., *Messenger of Mathematics* **47** (1918), 134—137.
[8] Mitra, S. C., *Proc. Acad. Sciences.* U. P. **4** Pt I (1934), 47—50.