TRANSFORMATIONS OF CERTAIN HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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1. There are several methods for obtaining transformations of hypergeometric functions of three variables. The first and simplest is by writing the triple series defining a given hypergeometric function as an infinite sum of the hypergeometric functions of two variables; the known transformation theory can then be applied to each term to obtain new transformations.

The second method consists in transforming the system of partial differential equations satisfied by these hypergeometric functions. This method is rather tedious in practice and not very useful for discovering new transformations.

The third method is obtained by transformation of integrals representing these functions. The object of this paper is to apply the third method to obtain some new transformations of such functions. The first two methods have been illustrated by me [4]. The success of the present method, as is obvious, lies in the method of substitution in the integral representations known for our functions and as such it becomes less useful in the cases where the integrals are such that substitutions are not very elegant.

2. Following the notation of [4] the hypergeometric functions of three variables are defined as

$$(2.1) \quad F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n+p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p,$$

$$(2.2) \quad F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \sum \frac{(\alpha_1, m+n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p,$$

(2.3) $F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$=\sum \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p,$$

(2.4) $F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z)$

$$=\sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p,$$

(2.5) $F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$=\sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p,$$

(2.6) $F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$=\sum \frac{(\alpha_1, m)(\alpha_2, n)(\alpha_3, p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p,$$

(2.7) $F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$=\sum \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+n)(\beta_2, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p,$$

(2.8) $F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$=\sum \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p,$$

(2.9) $F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z)$

$$=\sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} x^m y^n z^p,$$

and

(2.10) $F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z)$

$$=\sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} x^m y^n z^p,$$

where

$$(\alpha, m)=\alpha(\alpha+1)\ldots(\alpha+m-1); \qquad (\alpha, 0)=1.$$

The summation in the above triple series extends over all positive integral values of m, n and p from zero to infinity.

The integrals deduced by me for F_E, F_F, F_P and F_R are not capable of simple transformations and hence there does not appear to be any point of interest in deducing transformations for these here. The other six functions can be represented by the following integrals [4]:

$$(2.11)\quad \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\gamma_1-\beta_1)\Gamma(\gamma_2-\beta_2-\beta_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}F_G$$

$$=\iiint u^{\beta_1-1}v^{\beta_2-1}w^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-v-w)^{\gamma_2-\beta_2-\beta_3-1}(1-ux-vy-wz)^{-\alpha_1}\,du\,dv\,dw,$$

$$\operatorname{Re}(\gamma_1)>\operatorname{Re}(\beta_1)>0;\quad \operatorname{Re}(\gamma_2)>\operatorname{Re}(\beta_2+\beta_3)>0,$$

where $\operatorname{Re}(\beta_2)>0$, $\operatorname{Re}(\beta_3)>0$ and the contour of integration is $0\le u\le 1$, $v\ge 0$, $w\ge 0$, $v+w\le 1$, $\varrho+\varrho'+\varrho''<1$ $(|x|\le\varrho,\ |y|\le\varrho',\ |z|\le\varrho'')$,

$$(2.12)\quad \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_2-\beta_2)\Gamma(\gamma_3-\beta_1)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}F_K$$

$$=\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}v^{\beta_1-1}w^{\beta_2-1}(1-u)^{\gamma_1-\alpha_1-1}(1-v)^{\gamma_2-\beta_2-1}(1-w)^{\gamma_3-\beta_1-1}\times$$

$$\times(1-ux)^{\alpha_2-\beta_1}(1-ux-vy-wz+uvxy)^{-\alpha_1}\,du\,dv\,dw,$$

$$\operatorname{Re}(\gamma_1)>\operatorname{Re}(\alpha_1)>0,\quad \operatorname{Re}(\gamma_3)>\operatorname{Re}(\beta_1)>0 \text{ and } \operatorname{Re}(\gamma_2)>\operatorname{Re}(\beta_2)>0.$$

$$(2.13)\quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_2-\alpha_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}F_M$$

$$=\int_0^1\int_0^1 u^{\alpha_1-1}v^{\alpha_2-1}(1-u)^{\gamma_1-\alpha_1-1}(1-v)^{\gamma_2-\alpha_2-1}(1-vy)^{-\beta_2}(1-ux-vz)^{-\beta_1}\,du\,dv$$

$$\operatorname{Re}(\gamma_1)>\operatorname{Re}(\alpha_1)>0,\quad \operatorname{Re}(\gamma_2)>\operatorname{Re}(\alpha_2)>0,\quad \varrho+\varrho''<1\quad (|x|\le\varrho,\ |z|\le\varrho''),$$

$$(2.14)\quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_2-\alpha_2-\alpha_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}F_N$$

$$=\iiint u^{\alpha_1-1}v^{\alpha_2-1}w^{\alpha_3-1}(1-u)^{\gamma_1-\alpha_1-1}(1-v-w)^{\gamma_2-\alpha_2-\alpha_3-1}\times$$

$$\times(1-vy)^{-\beta_2}(1-ux-wz)^{-\beta_1}\,du\,dv\,dw.$$

$$\operatorname{Re}(\gamma_1)>\operatorname{Re}(\alpha_1)>0,\quad \operatorname{Re}(\gamma_2)>\operatorname{Re}(\alpha_2+\alpha_3)>0$$

where $\operatorname{Re}(\alpha_2)>0$, $\operatorname{Re}(\alpha_3)>0$ and the integral is taken over the contour $0\le u\le 1$, $v\ge 0$, $w\ge 0$, $v+w\le 1$, $\varrho+\varrho''<1$ $(|x|\le\varrho,\ |z|\le\varrho'')$,

$$(2.15)\quad \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\gamma_1-\beta_1-\beta_2-\beta_3)}{\Gamma(\gamma_1)}F_S$$

$$=\iiint u^{\beta_1-1}v^{\beta_2-1}w^{\beta_3-1}(1-u-v-w)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}\times$$

$$\times(1-ux)^{-\alpha_1}(1-vy-wz)^{-\alpha_2}\,du\,dv\,dw.$$

$\operatorname{Re}(\gamma_1)>\operatorname{Re}(\beta_1+\beta_2+\beta_3)>0$ where $\operatorname{Re}(\beta_1)>0$, $\operatorname{Re}(\beta_2)>0$, $\operatorname{Re}(\beta_3)>0$ and the integral is taken over the region $u\ge 0$, $v\ge 0$, $w\ge 0$, $u+v+w\le 1$, $\varrho'+\varrho''<1$ $(|y|\le\varrho',\ |z|\le\varrho'')$.

(2.16) $$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_1-\alpha_1-\alpha_2)}{\Gamma(\gamma_1)}F_S$$
$$=\iint u^{\alpha_1-1}v^{\alpha_2-1}(1-u-v)^{\gamma_1-\alpha_1-\alpha_2-1}(1-ux)^{-\beta_1}(1-vy)^{-\beta_2}(1-vz)^{-\beta_3}\,du\,dv\text{-}$$

$\mathrm{Re}(\gamma_1)>\mathrm{Re}(\alpha_1+\alpha_2)>0$ where $\mathrm{Re}(\alpha_1)>0$, $\mathrm{Re}(\alpha_2)>0$ and the double integral is taken over the region $u\geq 0$, $v\geq 0$, $u+v\leq 1$, and,

(2.17) $$\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1-\beta_1-\beta_2)}{\Gamma(\gamma_1)}F_T$$
$$=\iint u^{\beta_2-1}v^{\beta_1-1}(1-u-v)^{\gamma_1-\beta_1-\beta_2-1}(1-vx)^{-\alpha_1}(1-uy-vz)^{-\alpha_2}\,du\,dv.$$

$\mathrm{Re}(\gamma_1)>\mathrm{Re}(\beta_1+\beta_2)>0$ where $\mathrm{Re}(\beta_1)>0$, $\mathrm{Re}(\beta_2)>0$ and the integral is taken over the region $u\geq 0$, $v\geq 0$, $u+v\leq 1$, $\varrho'+\varrho''<1$ $(|y|\leq\varrho', |z|\leq\varrho'')$.

Transformations of F_G-function

3. Consider the integral of (2.11) for the F_G-function, namely

(3.1) $$\iiint u^{\beta_1-1}v^{\beta_2-1}w^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-v-w)^{\gamma_2-\beta_2-\beta_3-1}(1-ux-vy-wz)^{-\alpha_1}\,du\,dv\,dw.$$

Putting $v=s(1-t)$, $w=st$, in (3.1) it becomes

(3.2) $$\int_0^1\int_0^1\int^1 u^{\beta_1-1}s^{\beta_2+\beta_3-1}t^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-s)^{\gamma_2-\beta_2-\beta_3-1}\times$$
$$\times(1-t)^{\beta_2-1}(1-ux-sy+sty-stz)^{-\alpha_1}\,du\,ds\,dt.$$

Now, $(1-ux-sy+sty-stz)^{-\alpha_1}$ can be expanded in the form

(3.3) $$\sum_{m=0}^{\infty}\frac{(\alpha_1,m)}{(1,m)}(z-y)^m s^m t^m(1-ux-sy)^{-\alpha_1-m} \quad\text{if}\quad \left|\frac{z-y}{1-x-y}\right|<1,$$

and

(3.4) $$\sum_{m=0}^{\infty}\frac{(\alpha_1,m)}{(1,m)}s^m y^m(1-ux)^{-\alpha_1-m}\left\{1-t\left(1-\frac{z}{y}\right)\right\}^m \quad\text{if}\quad \left|\frac{zy}{1-x}\right|+\left|\frac{z}{1-x}\right|<1,$$
$$0\leq s\leq 1,\ 0\leq t\leq 1 \text{ and } 0\leq u\leq 1.$$

Using (3.3) in (3.2) which is justified for $\left|\frac{z-y}{1-x-y}\right|<1$ we get after changing the order of integration and summation

$$\sum_{m=0}^{\infty}\frac{(\alpha_1,m)}{(1,m)}(z-y)^m\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}s^{\beta_2+\beta_3+m-1}t^{\beta_3+m-1}\times$$
$$\times(1-u)^{\gamma_1-\beta_1-1}(1-s)^{\gamma_2-\beta_2-\beta_3-1}(1-t)^{\beta_2-1}(1-ux-sy)^{-\alpha_1-m}\,du\,ds\,dt.$$

Evaluating the t-integral and using the integral representation

$$\frac{\Gamma(\beta)\Gamma(\beta_1)\Gamma(\gamma-\beta)\Gamma(\gamma_1-\beta_1)}{\Gamma(\gamma)\Gamma(\gamma_1)} F_2(\alpha, \beta, \beta_1; \gamma, \gamma_1; x, y) \tag{3.5}$$

$$= \int_0^1\int_0^1 u^{\beta-1} v^{\beta_1-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma_1-\beta_1-1}(1-ux-vy)^{-\alpha}\,du\,dv.$$

$$\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0,\ \operatorname{Re}(\gamma_1) > \operatorname{Re}(\beta_1) > 0$$

we obtain the transformation

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) \tag{3.6}$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_3, m)}{(1, m)(\gamma_2, m)} (z-y)^m F_2(\alpha_1+m, \beta_1, \beta_2+\beta_3+m; \gamma_1, \gamma_2+m; x, y).$$

Putting $y=z$ in (3.6) we get

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, y) = F_2(\alpha_1; \beta_1, \beta_2+\beta_3; \gamma_1, \gamma_2; x, y).$$

Now, using (3.4) in (3.2) and changing the order of integration and summation which is valid for

$$\left|\frac{2y}{1-x}\right| + \left|\frac{z}{1-x}\right| < 1,$$

we get after some simplification that

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)\Gamma(\beta_2+\beta_3+m)\Gamma(\gamma_2-\beta_2-\beta_3)}{\Gamma(\gamma_2+m)} y^m \times$$

$$\times \int_0^1\int_0^1 u^{\beta_1-1} t^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-t)^{\beta_2-1}(1-ux)^{-\alpha_1-m}\left\{1-t\left(1-\frac{z}{y}\right)\right\}^m du\,dt$$

valid if $\operatorname{Re}(\gamma_2) > \operatorname{Re}(\beta_2+\beta_3) > 0$.

In the above series using the known relation

$$\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta; \gamma; x) = \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha}\,du. \tag{3.7}$$

$$(\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0)$$

we obtain the transformation

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) \tag{3.8}$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_2+\beta_3, m)}{(1, m)(\gamma_2, m)} y^m {}_2F_1(\alpha_1+m, \beta_1; \gamma_1; x)\, {}_2F_1\left(-m, \beta_3, \beta_2+\beta_3; 1-\frac{z}{y}\right).$$

$\alpha_1=0$ leads to the known result, giving the expansion of F_1 ([1], 8, p. 34). Using the well-known transformations

$$(3.9)\quad {}_2F_1(\alpha,\beta;\gamma;x)=(1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{-x}{1-x}\right)$$
$$=(1-x)^{-\beta}\,{}_2F_1\left(\gamma-\alpha,\beta;\gamma;\frac{-x}{1-x}\right)$$
$$=(1-x)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;x)$$

in (3.8) we can get eight more transformations of the F_G-function.

Next let us use the substitution

$$n=1-s,\quad w=st$$

in (3.2). This transforms the integral (2.11) to

$$\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}s^{\gamma_2-\beta_2-1}t^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-s)^{\beta_2-1}(1-t)^{\gamma_2-\beta_2-\beta_3-1}\times$$
$$\times(1-y-ux+sy-stz)^{-\alpha_1}\,du\,ds\,dt.$$

But

$$(1-y-ux+sy-stz)^{-\alpha_1}=\sum_{m=0}^{\infty}\frac{(\alpha_1,m)}{(1,m)}s^m(-y)^m(1-y-ux)^{-\alpha_1-m}\left(1-\frac{z}{y}t\right)^m,$$

and since $0\le s\le 1$, $0\le t\le 1$, the series is absolutely convergent if

$$\left|\frac{y}{1-x-y}\right|+\left|\frac{z}{1-x-y}\right|<1.$$

Using this expansion in the above integral and changing the order of integration and summation, which is easily justified by absolute convergence, we get

$$(1-y)^{-\alpha_1}\sum_{m=0}^{\infty}\frac{(\alpha_1,m)\,\Gamma(\gamma_2-\beta_2+m)\,\Gamma(\beta_2)}{(1,m)\,\Gamma(\gamma_2+m)}\left(\frac{-y}{1-y}\right)^m\times$$
$$\times\int_0^1\int_0^1 u^{\beta_1-1}t^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-t)^{\gamma_2-\beta_2-\beta_3-1}\left(1-u\frac{x}{1-y}\right)^{-\alpha_1-m}\left(1-\frac{z}{y}t\right)^m du\,dt.$$

Applying (3.7), we finally get the transformation

$$(3.10)\quad F_G(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$
$$=(1-y)^{-\alpha_1}\sum_{m=0}^{\infty}\frac{(\alpha_1,m)(\gamma_2-\beta_2,m)}{(1,m)(\gamma_2,m)}\left(\frac{y}{y-1}\right)^m\times$$
$$\times{}_2F_1\left(\alpha_1+m,\beta_1;\gamma_1;\frac{x}{1-y}\right){}_2F_1\left(-m,\beta_3;\gamma_2-\beta_2;\frac{z}{y}\right).$$

In case we use the transformations (3.9) in (3.10) we can get eight more transformations of the F_G-function.

Transformations of F_K and F_M

4. Consider now the integral (2.12) for F_K, namely

$$\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1} v^{\beta_2-1} w^{\beta_1-1} (1-u)^{\gamma_1-\alpha_1-1} (1-v)^{\gamma_2-\beta_2-1} (1-w)^{\gamma_3-\beta_1-1}\times$$
$$\times (1-ux)^{\alpha_2-\beta_1} (1-ux-vy-wz+uvxy)^{-\alpha_2}\, du\, dv\, dw.$$

Making the substitutions

$$\text{(i)}\quad u=1-u_1,\quad v=v_1,\quad w=w_1$$
$$\text{(ii)}\quad u=1-u_1,\quad v=1-v_1,\quad w=w_1$$

we get after a simple transformation the two relations

$$(4.1)\qquad F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$
$$=(1-x)^{-\beta_1} F_K\left(\gamma_1-\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; \frac{-x}{1-x}, y, z\right),$$

$$(4.2)\qquad =(1-x)^{-\beta_1}(1-y)^{-\alpha_2}\times$$
$$\times F_K\left(\gamma_1-\alpha_1, \alpha_2, \alpha_2, \beta_1, \gamma_2-\beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; \frac{-x}{1-x}, \frac{-y}{1-y}, \frac{z}{(1-x)(1-y)}\right).$$

These two results have also been obtained by me otherwise ([4], 5.4 and 5.6). Next, let us consider the integral (2.13) for F_M, namely

$$\int_0^1\int_0^1 u^{\alpha_1-1} v^{\alpha_2-1} (1-u)^{\gamma_1-\alpha_1-1} (1-v)^{\gamma_2-\alpha_2-1} (1-vy)^{-\beta_2} (1-ux-vz)^{-\beta_1}\, du\, dv.$$

Making the substitutions

$$\text{(i)}\quad u=1-u_1,\quad v=v_1$$
$$\text{(ii)}\quad u=1-u_1,\quad v=1-v_1$$
$$\text{(iii)}\quad u=u_1,\quad v=1-v_1$$

we can easily deduce the following three transformations:

$$(4.3)\qquad F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$
$$=(1-x)^{-\beta_1} F_M\left(\gamma_1-\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; \frac{-x}{1-x}, y, \frac{z}{1-x}\right),$$

(4.4) $$=(1-y)^{-\beta_2}(1-x-z)^{-\beta_1}\times$$

$$\times F_M\left(\gamma_1-\alpha_1,\ \gamma_2-\alpha_2,\ \gamma_2-\alpha_2,\ \beta_1,\ \beta_2,\ \beta_1;\ \gamma_1,\ \gamma_2,\ \gamma_2;\ \frac{-x}{1-x-z},\ -\frac{y}{1-y},\ -\frac{z}{1-x-z}\right),$$

(4.5) $$=(1-y)^{-\beta_2}(1-z)^{-\beta_1}\times$$

$$\times F_M\left(\alpha_1,\ \gamma_2-\alpha_2,\ \gamma_2-\alpha_2,\ \beta_1,\ \beta_2,\ \beta_1;\ \gamma_1,\ \gamma_2,\ \gamma_2;\ \frac{x}{1-z},\ -\frac{y}{1-y},\ -\frac{z}{1-z}\right).$$

Transformations of F_N, F_S and F_T

5. In this section we shall again make the following two substitutions

(5.1) $$\begin{cases} \text{(i)} & p=s(1-t),\ q=st \\ \text{(ii)} & p=1-s \qquad q=st. \end{cases}$$

The substitution 5.1 (i) reduces an expression of the type

$$(1-px-qy)^{-\lambda}$$

to

(5.2) $$\sum_{m=0}^{\infty}\frac{(\lambda,\ m)}{(1,\ m)}s^m t^m (y-x)^m (1-sx)^{-\lambda-m}$$

valid if $\left|\frac{y-x}{1-x}\right|<1$, and also to

(5.3.) $$\sum_{m=0}^{\infty}\frac{(\lambda,\ m)}{(1,\ m)}s^m x^m \left[1-t\left(1-\frac{y}{x}\right)\right]^m$$

valid if $|x|+|y|<1$.

The substitution 5.1 (ii) however changes an expression of the type

$$(1-px-qy)^{-\lambda}$$

to

(5.4) $$\sum_{m=0}^{\infty}\frac{(\lambda,\ m)}{(1,\ m)}s^m t^m y^m (1-x-sx)^{-\lambda-m}$$

valid if $2|x|+|y|<1$.

In particular, if μ is a positive integer

$$(1-px-qy)^{\mu}$$

(5.5) $$=(-)^{\mu}\sum_{m+m=0}^{m+n=\mu}\frac{(-\mu,\ m+n)}{(1,\ m)(1,\ n)}(1-px)^m(1-qy)^m.$$

Thus using 5.1 (i) in the integral (2.14) for F_N it is transformed into the integral

$$(5.6) \quad \int_0^1\int_0^1\int_0^1 u^{\alpha_1-1} s^{\alpha_2+\alpha_3-1} t^{\alpha_3-1} (1-u)^{\gamma_1-\alpha_1-1} (1-s)^{\gamma_2-\alpha_2-\alpha_3-1} \times$$
$$\times (1-t)^{\alpha_2-1} (1-sy+sty)^{-\beta_2} (1-ux-stz)^{-\beta_1} du\, ds\, dt.$$

But

$$(1-sy+sty)^{-\beta_2} = \sum_{m=0}^{\infty} \frac{(\beta_2, m)}{(1, m)} s^m y^m (1-t) \text{ for } |y|<1.$$

Using this expansion if $|y|<1$ (5.6) becomes after changing the order of integration and summation

$$\sum_{m=0}^{\infty} \frac{(\alpha_2, m)}{(1, m)} y^m \int_0^1\int_0^1\int_0^1 u^{\alpha_1-1} s^{\alpha_2+\alpha_3+m-1} t^{\alpha_3-1} (1-u)^{\gamma_1-\alpha_1-1} \times$$
$$\times (1-s)^{\gamma_2-\alpha_2-\alpha_3-1} (1-t)^{\alpha_2+m-1} (1-ux-stz)^{-\beta_1} du\, ds\, dt.$$

Applying (3.5) this gives

$$\frac{\Gamma(\alpha_2)\,\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)} F_N = \sum_{m=0}^{\infty} \frac{(\beta_2, m)\,(\alpha_2+\alpha_3, m)}{(1, m)\,(\gamma_2, m)} y^m \times$$
$$\times \int_0^1 t^{\alpha_3-1} (1-t)^{\alpha_2+m-1} F_2(\beta_1; \alpha_1, \alpha_2+\alpha_3+m; \gamma_1, \gamma_2+m; x, tz)\, dt.$$

Transforming the F_2 on the right with the help of the known transformation

$$(5.7) \quad F_2(\alpha, \beta, \beta_1; \gamma, \gamma_1; x, y) = (1-x)^{-\alpha} F_2\left(\alpha, \gamma-\beta, \beta_1; \gamma, \gamma_1; \frac{-x}{1-x}, \frac{y}{1-x}\right).$$

we get

$$(5.8) \quad F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$
$$= (1-x)^{-\beta_1} F_N\left(\gamma_1-\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; \frac{-x}{1-x}, y, \frac{z}{1-x}\right).$$

Next, the substitution 5.1 (ii) transforms the integral (5.6) to the form

$$\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1} s^{\gamma_2-\alpha_2-1} t^{\alpha_3-1} (1-u)^{\gamma_1-\alpha_1-1} (1-s)^{\alpha_2-1} \times$$
$$\times (1-t)^{\gamma_2-\alpha_2-\alpha_3-1} (1-y+sy)^{-\beta_2} (1-ux-stz)^{-\beta_1} du\, ds\, dt.$$

Again, applying (3.5), we obtain

$$\frac{\Gamma(\gamma_2-\alpha_2)\,\Gamma(\alpha_2)}{\Gamma(\gamma_2)} F_N = (1-y)^{-\beta_2} \int_0^1 s^{\gamma_2-\alpha_2-1} (1-s)^{\alpha_2-1} \left(1+\frac{sy}{1-y}\right)^{-\beta_2} \times$$
$$\times F_2(\beta_1; \alpha_1, \alpha_3; \gamma_1, \gamma_2-\alpha_2; x, sz)\, ds.$$

Using the three known transformations of F_2 ([1], p. 32) on the right and then expanding the new F_2 and integrating term by term we obtain the three transformations of F_N, one of which is

$$(5.9)\quad F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = (1-x)^{-\beta_1}(1-y)^{-\beta_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta_1, m+n)(\gamma_1-\alpha_1, m)(\alpha_3, n)}{(1, m)(1, n)(\gamma_1, m)(\gamma_2, n)}\times \times\left(\frac{-x}{1-x}\right)^m\left(\frac{z}{1-x}\right)^n {}_2F_1\left(\beta_2, \gamma_2-\alpha_2+n; \gamma_2+n; -\frac{y}{1-y}\right).$$

In particular, putting $\gamma_2=\alpha_2+\alpha_3$ in (5.9) we get the interesting transformation

$$(5.10)\quad F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \alpha_2+\alpha_3, \alpha_2+\alpha_3; x, y, z) = (1-x)^{-\beta_1}(1-y)^{-\beta_2}\times \times F_M\left(\gamma_1-\alpha_1, \alpha_3, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \alpha_2+\alpha_3, \alpha_2+\alpha_3; \frac{-x}{1-x}, \frac{y}{1-y}, \frac{z}{1-x}\right).$$

A similar transformation between F_N and F_M is obtained by putting $\gamma_1=\alpha_2+\alpha_3$ in the other two expansions.

Using (5.5) in the integral (2.14) for F_N when $\beta_1=-p$ (a negative integer), we get after changing the order of integration and summation

$$F_N(\alpha_1, \alpha_2, \alpha_3, -p, \beta_2, -p; \gamma_1, \gamma_2, \gamma_2; x, y, z) = (-)^p\sum_{m+n=0}^{m+n=p}\frac{(-p, m+n)}{(1, m)(1, n)}\int\int\int u^{\alpha_1-1}v^{\alpha_2-1}w^{\alpha_3-1}(1-u)^{\gamma_1-\alpha_1-1}\times \times(1-v-w)^{\gamma_2-\alpha_2-\alpha_3-1}(1-vy)^{-\beta_2}(1-ux)^m(1-wy)^n\,du\,dv\,dw.$$

Replacing the inner integrals by the corresponding functions by means of the well-known formula we get the transformation

$$(5.11)\quad = (-)^p\sum_{m+n=0}^{m+n=p}\frac{(-p, m+n)}{(1, m)(1, n)}\,{}_2F_1(-m, \alpha_1; \gamma_1; x)\,F_3(\beta_2, -n; \alpha_2, \alpha_3; \gamma_2; y, z).$$

$\beta_2=0$ leads to a known expansion of the F_2-function in terms of series of ordinary hypergeometric function ([1], Result 15, p. 36).

Next, consider the integral (2.15) for F_S, namely

$$\iiint u^{\beta_1-1}v^{\beta_2-1}w^{\beta_3-1}(1-u-v-w)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}(1-ux)^{-\alpha_1}(1-vy-wz)^{-\alpha_2}\,du\,dv\,dw.$$

Using (5.2) to expand $(1-vy-wz)^{-\alpha_2}$ and integrating term by term which is valid for $\left|\frac{z-y}{1-y}\right|<1$, we get

$$\sum_{m=0}^{\infty} \frac{(\alpha_2, m)}{(1, m)} (z-y)^m \int\int\int u^{\beta_1-1} s^{\beta_2+\beta_3+m-1} t^{\beta_3+m-1} \times$$
$$\times (1-u-s)^{\gamma_1-\beta_1-\beta_2-\beta_3-1} (1-t)^{\beta_2-1} (1-ux)^{-\alpha_1} (1-sy)^{-\alpha_2-m}\, du\, ds\, dt.$$

Applying the known integral for F_3

$$(5.12)\quad \frac{\Gamma(\alpha)\Gamma(\alpha_1)\Gamma(\gamma-\alpha-\alpha_1)}{\Gamma(\gamma)} F_3(\alpha, \alpha_1, \beta, \beta_1; \gamma; x, y)$$
$$= \iint u^{\alpha-1} v^{\alpha_1-1} (1-u-v)^{\gamma-\alpha-\alpha_1-1} (1-ux)^{-\beta} (1-vy)^{-\beta}\, du\, dv.$$
$$\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha+\alpha_1) > 0,\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\alpha_1) > 0$$

and the region of integration is $u \geq 0$, $v \geq 0$, $u+v \leq 1$, we obtain the transformation

$$(5.13)\quad F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z)$$
$$= \sum_{m=0}^{\infty} \frac{(\alpha_2, m)(\beta_3, m)}{(1, m)(\gamma_1, m)} (z-y)^m F_3(\alpha_1, \alpha_2+m; \beta_1, \beta_2+\beta_3+m; \gamma_1+m; x, y).$$

Putting $y=z$ in (5.13), we get

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, y) = F_3(\alpha_1, \alpha_2, \beta_1, \beta_2+\beta_3; \gamma_1; x, y).$$

Again, using (5.3) if $|y|+|z|<1$, the integral (2.15) for F_S, after term by term integration gives

$$\sum_{m=0}^{\infty} \frac{(\alpha_2, m)}{(1, m)} y^m \int\int\int u^{\beta_1-1} s^{\beta_2+\beta_3+m-1} t^{\beta_3-1} \times$$
$$\times (1-u-s)^{\gamma_1-\beta_1-\beta_2-\beta_3-1} (1-t)^{\beta_2-1} (1-ux)^{-\alpha_1} \left\{1-t\left(1-\frac{z}{y}\right)\right\}^m du\, ds\, dt.$$

If we put $s=(1-u)p$ this equals to

$$\sum_{m=0}^{\infty} \frac{(\alpha_2, m)}{(1, m)} y^m \int_0^1\int_0^1\int_0^1 u^{\beta_1-1} t^{\beta_3-1} p^{\beta_2+\beta_3+m-1} \times$$
$$\times (1-u)^{\gamma_1-\beta_1+m-1} (1-t)^{\beta_2-1} (1-p)^{\gamma_1-\beta_1-\beta_2-\beta_3-1} (1-ux)^{-\alpha_1} \left\{1-t\left(1-\frac{z}{y}\right)\right\}^m du\, dt\, dp.$$

Evaluating the p-integral and using (3.7) we obtain the transformation

$$(5.14)\quad F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1\, \gamma_1; x, y, z)$$
$$= \sum_{m=0}^{\infty} \frac{(\alpha_2, m)(\beta_2+\beta_3, m)}{(1, m)(\gamma_1, m)} y^m\, {}_2F_1(\alpha_1, \beta_1; \gamma_1+m; x)\, {}_2F_1\left(-m, \beta_3; \beta_2+\beta_3; 1-\frac{z}{y}\right).$$

Either α_1 or $\beta_1 = 0$ leads to a known result giving an expression for F_1 ([1], p. 34).

Further using (3.9) in (5.14) to transform the ${}_2F_1$'s we get eight more transformations. Two interesting transformations are

$$F_S = z^{\beta_2} y^{-\beta_2} F_N\left(\alpha_2,\ \alpha_1,\ \beta_2;\ \beta_2+\beta_3,\ \beta_1,\ \beta_2+\beta_3;\ \gamma_1,\ \gamma_1,\ \beta_2+\beta_3;\ z,\ x,\ 1-\frac{z}{y}\right)$$

$$= y^{\beta_3} z^{-\beta_3} F_N\left(\alpha_2,\ \alpha_1,\ \beta_3;\ \beta_2+\beta_3,\ \beta_1,\ \beta_2+\beta_3;\ \gamma_1,\ \gamma_1,\ \beta_2+\beta_3;\ y,\ x,\ 1-\frac{y}{z}\right).$$

Further, using the substitution 5.1 (ii) in the second integral (2.16) for F_S, namely,

$$\iint u^{\alpha_1-1} v^{\alpha_2-1} (1-u-v)^{\gamma_1-\alpha_1-\alpha_2-1} (1-ux)^{-\beta_1} (1-vy)^{-\beta_2} (1-vz)^{-\beta_3}\, du\, dv,$$

it transforms into

$$\int_0^1\int_0^1 s^{\gamma_1-\alpha_1-1} t^{\alpha_2-1} (1-s)^{\alpha_1-1} (1-t)^{\gamma_1-\alpha_1-\alpha_2-1} (1-x+sx)^{-\beta_1} (1-sty)^{-\beta_2} (1-stz)^{-\beta_3}\, ds\, dt.$$

Replacing the t-integral by means of the known integral

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1(\alpha,\ \beta,\ \beta_1;\ \gamma;\ x,\ y) = \int_0^1 u^{\alpha_1-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} (1-uy)^{-\beta_1}\, du$$

$$\operatorname{Re}(\gamma) > \operatorname{Re}(\alpha) > 0,$$

we obtain

$$\frac{\Gamma(\alpha_1)\,\Gamma(\gamma_1-\alpha_1)}{\Gamma(\gamma_1)} F_S = \int_0^1 s^{\gamma_1-\alpha_1-1} (1-s)^{\alpha_1-1} (1-x+sx)^{-\beta_1} F_1(\alpha_2,\ \beta_2,\ \beta_3;\ \gamma_1-\alpha_1;\ sy,\ sz)\, ds.$$

Transforming the F_1 on the right by the formula

$$F_1(\alpha,\ \beta,\ \beta_1;\ \gamma;\ x,\ y) = (1-x)^{-\beta} (1-y)^{-\beta_1} F_1\left(\gamma-\alpha,\ \beta,\ \beta_1;\ \gamma\colon \frac{-x}{1-x},\ \frac{-y}{1-y}\right)$$

we obtain after term by term integration and simplification the transformation

$$(5.15)\quad F_S = (1-x)^{-\beta_1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma_1-\alpha_1-\alpha_2,\ m+n)\,(\beta_2,\ m)\,(\beta_3,\ m)}{(1,\ m)\,(1,\ n)\,(\gamma_1,\ m+n)} \times$$

$$\times(-y)^m (-z)^n F_D\left(\gamma_1-\alpha_1+m+n;\ \beta_1,\ \beta_2+m,\ \beta_3+n;\ \gamma_1+m+n;\ \frac{-x}{1-x},\ y,\ z\right)$$

where F_D is Lauricella's hypergeometric function of the fourth type ([1], p. 114).

If $\gamma_1 = \alpha_1 + \alpha_2$, we get the interesting relation

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1+\alpha_2; x, y, z)$$
$$= (1-x)^{-\beta_1} F_D\left(\alpha_2, \beta_1, \beta_2, \beta_3; \alpha_1+\alpha_2; \frac{-x}{1-x}, y, z\right)$$

Next, using (5.5) in (2.16) it becomes after some simplification

$$(-)^p \sum_{m+n=0}^{m+n=p} \frac{(-p, m+n)}{(1, m)(1, n)} \int\int\int u^{\beta_1-1} v^{\beta_2-1} w^{\beta_3-1} \times$$
$$\times (1-u-v-w)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}(1-ux)^{-\alpha_1}(1-vy)^m(1-wz)^n\, du\, dv\, dw.$$

Using a known integral for the Lauricella hypergeometric function F_B we obtain the transformation

$$F_S = (-)^p \sum_{m+n=0}^{m+n=p} \frac{(-p, m+n)}{(1, m)(1, n)} F_B(\beta_1, \beta_2, \beta_3; \alpha_1, -m, -n; \gamma_1; x, y, z).$$

Either $\alpha_1 = 0$ or $\beta_1 = 0$ will lead to the known result giving a relation between F_1 and F_3 ([1], 14, p. 34).

Finally, coming to the transformation of F_T-function we use the expansion (5.2) in the following integral (2.17) for F_T

$$\int\int u^{\beta_2-1} v^{\beta_1-1}(1-u-v)^{\gamma_1-\beta_1-\beta_2-1}(1-vx)^{-\alpha_1}(1-uy-vz)^{-\alpha_2}\, du\, dv.$$

For $\left|\frac{z-y}{1-y}\right| < 1$ this becomes on expansion and term by term integration

$$\sum_{m=0}^{\infty} \frac{(\alpha_2, m)}{(1, m)}(z-y)^m \int_0^1\int_0^1 s^{\beta_1+\beta_2+m-1} t^{\beta_1+m-1} \times$$
$$\times (1-s)^{\gamma_1-\beta_1-\beta_2-1}(1-t)^{\beta_2-1}(1-stx)^{-\alpha_1}(1-sy)^{-\alpha_2-m}\, ds\, dt.$$

This can now be written as

$$\frac{\Gamma(\gamma_1-\beta_1-\beta_2)\Gamma(\beta_1+\beta_2)}{\Gamma(\gamma_1)} F_T = \sum_{m=0}^{\infty} \frac{(\alpha_2, m)(\beta_1, m)}{(1, m)(\beta_1+\beta_2, m)}(z-y)^m \times$$
$$\times \int_0^1 s^{\beta_1+\beta_2+m-1}(1-s)^{\gamma_1-\beta_1-\beta_2-1}(1-sy)^{-\alpha_2-m}\, {}_2F_1(\alpha_1, \beta_1+m; \beta_1+\beta_2+m; sx)\, ds.$$

Applying (3.9), we get the transformation

$$(5.16)\quad F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n)(\beta_2, m)(\beta_1, n)}{(1, m)(1, n)(\gamma_1, m+n)}(-x)^m(z-y)^n \times$$
$$\times F_1(\beta_1+\beta_2+m+n; \alpha_1+m, \alpha_2+n; \gamma_1+m+n; x, y).$$

As a particular case if $\gamma_1 = \beta_1 + \beta_2$ in (5.16)

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \beta_1+\beta_2, \beta_1+\beta_2, \beta_1+\beta_2; x, y, z)$$
$$= (1-x)^{-\alpha_1}(1-y)^{-\alpha_2} F_3\left(\alpha_1, \alpha_2, \beta_2, \beta_1; \beta_1+\beta_2; \frac{-x}{1-x}, \frac{z-y}{1-x}\right).$$

Again, applying the substitution 5.1 (ii) in (2.17) it becomes

$$(5.17) \quad \int_0^1\int_0^1 s^{\gamma_1-\beta_2-1} t^{\beta_1-1}(1-s)^{\beta_2-1}(1-t)^{\gamma_1-\beta_1-\beta_2-1}(1-stx)^{-\alpha_1}(1-y+sy-stz)^{-\alpha_2}\,ds\,dt.$$

If $2|y|+|z|<1$, we can use the expansion (5.4) and term by term integration gives

$$\sum_{m=0}^{\infty}\frac{(\alpha_2, m)}{(1, m)} z^m \int_0^1\int_0^1 s^{\gamma_1-\beta_2+m-1} t^{\beta_1+m-1}(1-s)^{\beta_2-1}\times$$
$$\times(1-t)^{\gamma_1-\beta_1-\beta_2-1}(1-stx)^{-\alpha_1}(1-y+sy)^{-\alpha_2-m}\,ds\,dt.$$

Thus, we get

$$(5.18) \quad \frac{\Gamma(\beta_2)\Gamma(\gamma_1-\beta_2)}{\Gamma(\gamma_1)} F_T = (1-y)^{-\alpha_2}\sum_{m=0}^{\infty}\frac{(\alpha_2, m)(\beta_1, m)}{(1, m)(\gamma_1-\beta_2, m)}\left(\frac{z}{1-y}\right)\times$$
$$\times\int_0^1 s^{\gamma_1-\beta_2+m-1}(1-s)^{\beta_2-1}\left(1+\frac{sy}{1-y}\right)^{-\alpha_2-m} {}_2F_1(\alpha_1, \beta_1+m; \gamma_1-\beta_2+m; sx)\,ds.$$

Using the three transformations of (3.9) then expanding the new ${}_2F_1$, in each case, we get the three transformations

$$(5.19) \quad F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z)$$
$$= (1-y)^{-\alpha_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1, m)(\alpha_2, n)(\gamma_1-\beta_1-\beta_2, m)(\beta_1, n)}{(1, m)(1, n)(\gamma_1, m+n)}\times$$
$$\times(-x)^m\left(\frac{z}{1-y}\right)^n F_1\left(\gamma_1-\beta_2+m+n; \alpha_1+m, \alpha_2+n; \gamma_1+m+n; x, \frac{-y}{1-y}\right)$$

$$(5.20) \quad = (1-y)^{-\alpha_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1-\alpha_1-\beta_2+n, m)(\alpha_2, n)(\beta_1, m+n)}{(1, m)(1, n)(\gamma_1, m+n)}(-x)^m\left(\frac{z}{1-y}\right)^n\times$$
$$\times F_1\left(\gamma_1-\beta_2+m+n; \beta_1+m+n, \alpha_2+n; \gamma_1+m+n; x, \frac{-y}{1-y}\right)$$

$$(5.21) \quad = (1-y)^{-\alpha_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_2, n)(\gamma_1-\alpha_1-\beta_2+n, m)(\beta_1, n)(\gamma_1-\beta_1-\beta_2, m)}{(1, m)(1, n)(\gamma_1, m+n)}\times$$
$$\times x^m\left(\frac{z}{1-y}\right)^n F_1\left(\gamma_1-\beta_2+m+n; \alpha_1+\beta_1+\beta_2-\gamma_1, \alpha_2+n; \gamma_1+m+n; x, \frac{-y}{1-y}\right)$$

For $\gamma_1 = \beta_1 + \beta_2$ (5.19–21) we get

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \beta_1+\beta_2, \beta_1+\beta_2, \beta_1+\beta_2; x, y, z)$$
$$= (1-y)^{-\alpha_2} F_1\left(\beta_1; \alpha_1, \alpha_2; \beta_1+\beta_2; x, \frac{z-y}{1-y}\right).$$

Convergence conditions

6. In order that the formal proofs of our expansions deduced in previous sections be justified we must prove the conditions of absolute convergence for the said expansions. We take all the parameters in the hypergeometric functions to be real and positive; the variables x, y and z have been replaced by their absolute values $|x|$, $|y|$ and $|z|$ in the cases where they are not positive, i.e., when the series run in positive and negative terms.

We need then to know the bounds for the hypergeometric functions with positive variables when their positive parameters diverge to infinity in certain ways. We establish these bounds by first proving the following simple inequalities which have been given in the forms of lemmas.

Lemmas 1

(1) $(1-|x|)\,{}_2F_1(\alpha_1+1, \beta_1; \gamma_1; |x|) < {}_2F_1(\alpha_1, \beta_1; \gamma_1; |x|)$

if $\gamma_1 > \beta_1$ and $\beta_1 + \alpha_1 > \gamma_1$.

(2) ${}_2F_1(\alpha_1, \beta_1; \gamma_1+1; x) < {}_2F_1(\alpha_1, \beta_1; \gamma_1; x)$.

(3) $(1-x-y)\,F_2(\alpha_1+1; \beta_1, \beta_2+1; \gamma_1, \gamma_2+1; x, y) < \dfrac{\gamma_2}{\alpha_1} F_2(\alpha_1; \beta_1, \beta_2; \gamma_1, \gamma_2; x, y)$

if $\gamma_1 > \beta_1$, $\gamma_2 > \alpha_1$.

(4) $(1-y)\,F_3(\alpha_1, \alpha_2+1, \beta_1, \beta_2+1; \gamma_1+1; x, y) < \dfrac{\gamma_1}{\alpha_2} F_3(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1; x, y)$

if $\gamma_1 > \alpha_2$.

(5) $(1-|y|)\,F_D(\alpha_1+1; \beta_1, \beta_2+1, \beta_3; \gamma_1+1; |x|, |y|, |z|) <$

$$< \frac{\gamma_1}{\alpha_1} F_D(\alpha_1; \beta_1, \beta_2, \beta_3; \gamma_1; |x|, |y|, |z|)$$

if $\gamma_1 > \alpha_1$.

Corollaries

Repeating the above lemmas m times we get the following inequalities:

(1) $(1-|x|)^m {}_2F_1(\alpha_1+m, \beta_1; \gamma_1; |x|) < {}_2F_1(\alpha_1, \beta_1; \gamma_1; |x|)$

if $\gamma_1 > \beta_1$.

(2) ${}_2F_1(\alpha_1, \beta_1; \gamma_1+m; x) < {}_2F_1(\alpha_1, \beta_1; \gamma_1; x)$.

(3) $$(1-x-y)^m F_2(\alpha_1+m; \beta_1, \beta_2+m; \gamma_1, \gamma_2+m; x, y) < \frac{(\gamma_2, m)}{(\alpha_1, m)} F_2(\alpha_1; \beta_1, \beta_2; \gamma_1, \gamma_2; x, y)$$

if $\gamma_2 > \alpha_1$ and $\gamma_1 > \beta_1$.

(4) $$(1-y)^m F_3(\alpha_1, \alpha_2+m, \beta_1, \beta_2+m; \gamma_1+m; x, y) < \frac{(\gamma_1, m)}{(\alpha_2, m)} F_3(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1; x, y)$$

if $\gamma_1 > \gamma_2$.

(5) Repeating the process first m times with respect to x and then n times with respect to y, we get

$$(1-|y|)^m (1-|z|)^n F_D(\alpha_1+m+n; \beta_1, \beta_2+m, \beta_3+n; \gamma_1+m+n; |x|, |y|, |z|) < \frac{(\gamma_1, m+n)}{(\alpha_1, m+n)} F_D(\alpha_1; \beta_1, \beta_2\ \beta_3; \gamma_1; |x|, |y|, |z|)$$

if $\gamma_1 > \alpha_1$.

Lemma 2

(1) $$(1-|x|)^{m+n} (1-|y|)^n F_1(\alpha_1+m+n; \beta_1+m+n, \beta_2+n; \gamma_1+m+n; |x|, |y|) < \frac{(\gamma_1, m+n)}{(\alpha_1, m+n)} F_1(\alpha_1; \beta_1, \beta_2; \gamma_1; |x|, |y|)$$

if $\gamma_1 > \alpha_1$.

(ii) $$(1-|y|)^n F_1(\alpha_1+m+n; \beta_1, \beta_2+n; \gamma_1+m+n; |x|, |y|) < \frac{(\gamma_1, m+n)}{(\alpha_1, m+n)} F_1(\alpha_1; \beta_1, \beta_2, \gamma_1; |x|, |y|).$$

Proofs of the Lemmas 1

In each case we compare the ratios of corresponding coefficients on the two sides. More precisely, we denote by R_m (or $R_{m,n}$ or $R_{m,n,p}$) the ratio of the coefficients of x^m (or $x^m y^n$ or $x^m y^n z^p$) on the left to the corresponding coefficient on the right and show that $R_m < 1$ (or $R_{m,n} < 1$ or $R_{m,n,p} < 1$). It may be noted that the factor $(1-x-y)$ where it occurs on the left is positive by virtue of conditions necessary for the convergence of the series F_2 while $1-y$ for convergence of F_3 and so on.

$$(1) \quad R_m = \frac{\alpha_1 + m}{\alpha_1} - \frac{m(\gamma_1 + m - 1)}{\alpha_1(\beta_1 + m - 1)} = \frac{\alpha_1(\beta_2 - 1) + m(\beta_1 + \alpha_1 - \gamma_1)}{\alpha_1(\beta_1 + m - 1)} < 1$$

if $\gamma_1 > \beta_1$, $\beta_1 + \alpha_1 > \gamma_1$.

$$(2) \quad R_m = \frac{\alpha_1}{\alpha_1 + m} < 1.$$

$$(3) \quad R_{m,n} = \frac{(\alpha_1 + m + n)(\beta_2 + n)}{\beta_2(\gamma_2 + n)} - \frac{m(\beta_2 + n)(\gamma_1 + m - 1)}{\beta_2(\beta_1 + m - 1)(\gamma_2 + n)} - \frac{n}{\beta_2}$$

$$< \frac{1}{\beta_2}\left[\frac{(\alpha_1 + m + n)(\beta_2 + n)}{\gamma_2 + n} - \frac{m(\beta_2 + n)}{\gamma_2 + n} - n\right]$$

if $\beta_1 < \gamma_1$.

$$= \frac{\alpha_1 \beta_2 + n(\alpha_1 + \beta_2 - \gamma_2)}{\beta_2(\gamma_2 + n)} < 1$$

if $\gamma_2 > \alpha_1$ and $\gamma_1 > \beta_1$ and $\gamma_2 > \alpha_1 + \beta_2$.

$$(4) \quad R_{m,n} = \left[\frac{(\alpha_2 + n)(\beta_2 + n)}{\beta_2(\gamma_1 + m + n)} - \frac{n}{\beta_2}\right] \leq \frac{(\alpha_2 + n)(\beta_2 + n)}{\beta_2(\gamma_1 + n)} - \frac{n}{\beta_2},$$

for every positive integral m,

$$= \frac{\alpha_2 \beta_2 + n(\alpha_2 + \beta_2 - \gamma_1)}{\beta_2 \gamma_1 + \beta_2 n} < 1$$

if $\gamma_1 > \alpha_2$ and $\gamma_1 > \alpha_2 + \beta_2$.

$$(5) \quad R_{m,n,p} = \frac{(\alpha_1 + m + n + p)(\beta_2 + n)}{\beta_2(\gamma_1 + m + n + p)} - \frac{n}{\beta_2} = \frac{\alpha_1 \beta_2 + (m + p)\beta_2 + n(\alpha_1 + \beta_2 - \gamma_1)}{\beta_2(\gamma_1 + m + n + p)} < 1$$

if $\gamma_1 > \alpha_1$, $\gamma_1 < \alpha_1 + \beta_2$.

Proof of Lemma 2

We know that [1]

$$\frac{\Gamma(\alpha_1 + m + n)\,\Gamma(\gamma_1 - \alpha_1)}{\Gamma(\gamma_1 + m + n)} F_1(\alpha_1 + m + n; \beta_1 + m + n, \beta_2 + n; \gamma_1 + m + n; |x|, |y|)$$

$$= \int_0^1 u^{\alpha_1 + m + n - 1}(1 - u)^{\gamma_1 - \alpha_1 - 1}(1 - u|x|)^{-\beta_1 - m - n}(1 - u|y|)^{-\beta_2 - n}\,du$$

$$\operatorname{Re}(\gamma_1) > \operatorname{Re}(\alpha_1) > 0$$

$$= \int_0^1 u^{\alpha_1 - 1}(1 - u)^{\gamma_1 - \alpha_1 - 1}(1 - u|x|)^{-\beta_1}(1 - u|y|)^{-\beta_2}\left[\frac{u^{m+n}}{(1 - u|x|)^{m+n}(1 - u|y|)^n}\right] du.$$

For m and n large, $0 \leq u \leq 1$,

$$\frac{u^{m+n}}{(1-u|x|)^{m+n}(1-u|y|)^{n}} \leq \frac{1}{(1-|x|)^{m+n}(1-|y|)^{n}}.$$

Thus

$$\frac{\Gamma(\alpha_1+m+n)\,\Gamma(\gamma_1-\alpha_1)}{\Gamma(\gamma_1+m+n)} F_1(\alpha_1+m+n;\ \beta_1+m+n,\ \beta_2+n;\ \gamma_1+m+n;\ |x|,\ |y|)$$
$$\leq (1-|x|)^{-m-n}(1-|y|)^{-n}\int_0^1 u^{\alpha_1-1}(1-u)^{\gamma_1-\alpha_1-1}(1-u|x|)^{-\beta_1}(1-u|y|)^{-\beta_2}\,du.$$

Hence replacing the integral by its corresponding function F_1 we get the inequality (i).

Similarly, to prove (ii) we have

$$\frac{\Gamma(\alpha_1+m+n)\,\Gamma(\gamma_1-\alpha_1)}{\Gamma(\gamma_1+m+n)} F_1(\alpha_1+m+n;\ \beta_1,\ \beta_2+n;\ \gamma_1+m+n;\ |x|,\ |y|)$$
$$=\int_0^1 u^{\alpha_1+m+n-1}(1-u)^{\gamma_1-\alpha_1-1}(1-u|x|)^{-\beta_1}(1-u|y|)^{-\beta_2-n}\,du$$
$$\operatorname{Re}(\gamma_1) > \operatorname{Re}(\alpha_1) > 0.$$

For m and n large, $0 \leq u \leq 1$,

$$\frac{1}{(1-u|y|)^{n}} \leq \frac{1}{(1-|y|)^{n}}.$$

Hence as before we prove that (ii) of Lemma 2.

It may be noted that five corollaries of Lemma 1 can also be obtained by the help of integrals.

We shall also make use of the following two inequalities given by Burchnall and Chaundy ([2], page 264).

(i) $\quad {}_2F_1(\alpha+m,\ \beta;\ \gamma+m;\ x) \leq \dfrac{(\gamma,\ m)}{(\alpha,\ m)} F_1(\alpha,\ \beta;\ \gamma;\ x)$

where $\alpha = \min(\alpha,\ \gamma)$.

(ii) $\quad (1-x)^m(1-y)^n F_1(\alpha+m+n;\ \beta_1+m,\ \beta_2+n;\ \gamma_1+m+n;\ x,\ y)$
$$< \frac{(\gamma_1,\ m+n)}{(\alpha_1,\ m+n)} F_1(\alpha;\ \beta_1,\ \beta_2;\ \gamma_1;\ x,\ y)$$

if $\gamma_1 > \alpha_1$.

Using these asymptotic forms in our expansions we get the following regions of convergence:

(1) $\left.\begin{array}{l}|z-y|<|1-x-y| \\ |x|+|y|<1\end{array}\right\}$ in (3.6)

(2) $|x|<1, \quad \left|\frac{y}{1-x}\right|<1, \quad \left|\frac{z}{1-x}\right|<1$ in (3.8)

(3) $|y|<1, \quad \left|\frac{x}{1-y}\right|<1 \quad \left|\frac{y-z}{1-x-y}\right|<1$ in (3.10)

(4) $|x|<1, \quad |y|<1, \quad \left|\frac{y}{1-y}\right|<1, \quad \left|\frac{z}{1-x}\right|+\left|\frac{x}{1-x}\right|<1$ in (5.9)

(5) $|x|<1, \quad |y|<1, \quad \left|\frac{z-y}{1-y}\right|<1$ in (5.13)

(6) $|y|<1, \quad |y|<1, \quad |z|<1$ in (5.14)

(7) $|x|<1, \quad \left|\frac{x}{1-x}\right|<1, \quad |y|<1, \quad \left|\frac{y}{1-y}\right|<1, \quad |z|<1, \quad \left|\frac{z}{1-z}\right|<1$ in (5.15)

(8) $|x|<1, \quad \left|\frac{x}{1-x}\right|<1, \quad |y|<1, \quad \left|\frac{z-y}{1-y}\right|<1$ in (5.16)

(9) $|x|<1, \quad |y|<1, \quad \left|\frac{x}{1-x}\right|<1, \quad \left|\frac{z}{1-y}\right|+\left|\frac{y}{1-y}\right|<1$ in (5.19)

(10)[1] $|x|<1, \quad |y|<1, \quad \left|\frac{y}{1-y}\right|+\left|\frac{z}{1-x}\right|<1$ in (5.20) and (5.21).

It may be recalled that the regions of convergence for the hypergeometric functions of three variables are as below:

$$F_G: \quad r+s=1$$
$$r+t=1$$

$$F_K: \quad t=(1-r)(1-s)$$

$$F_M: \quad r+t=1$$
$$s=1$$

$$F_N: \quad s(1-r)+t(1-s)=0$$

$$F_S: \quad \frac{1}{r}+\frac{1}{s}=\frac{1}{r}+\frac{1}{t}=1$$

and

$$F_T: \quad t=r-rs+s,$$

where $|x|<r$, $|y|<s$ and $|z|<t$.

[1] [3]. Use result (15), p. 146.

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References

[1]. P. Appell et J. Kampé de Feriet, *Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite*. Paris, Gauthier-Villars et C^ie 1926, 30–8.

[2]. J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions, Quart. J. of Maths. (Oxford), 11 (1940), 249–70.

[3]. A. Erdélyi, Hypergeometric functions of two variables. *Acta Math.* 83 (1950), 131-64.

[4]. Shanti Saran, Hypergeometric functions of three variables (*Ganita*) 5 (1954).