ON THE DIFFERENTIAL EQUATIONS OF HILL IN THE THEORY OF THE MOTION OF THE MOON (II)

BY

J. F. STEFFENSEN

in Copenhagen

1. In a former paper with the same title[1] (quoted below as "I") polar coordinates r and l were employed to represent the orbit, and power series in $x \equiv \cos 2l$ were used for satisfying the differential equations. In the present paper the time t will be employed as independent variable, and the expansions will be in powers of t or some simple function of t.

We put in I (4)

$$\cos 2l = x, \qquad \sin 2l = y, \qquad \varepsilon = \xi + \tfrac{1}{2} \tag{1}$$

so that

$$\frac{dx}{dl} = -2y, \qquad \frac{dy}{dl} = 2x, \qquad x^2 + y^2 = 1 \tag{2}$$

and find, taking I (3) into account, the following system of equations[2]

$$\frac{d\varrho}{dt} = \omega^2 - \varrho^2 - \xi + \frac{3}{2}x \tag{3}$$

$$\frac{d\omega}{dt} = -2\varrho\omega - \frac{3}{2}y \tag{4}$$

$$\frac{d\xi}{dt} = -3\varrho\xi - \frac{3}{2}\varrho \tag{5}$$

$$\frac{dx}{dt} = -2y\omega + 2y \tag{6}$$

$$\frac{dy}{dt} = 2x\omega - 2x. \tag{7}$$

[1] *Acta mathematica*, 93 (1955), 169–177.

[2] The possibility of reducing to this form was, in principle, already indicated in my thesis *Analytiske Studier med Anvendelser paa Taltheorien*, Copenhagen 1912, p. 146–147.

Jacobi's integral I (5) becomes

$$C = (2\xi - \varrho^2 - \omega^2 + 2\omega + \tfrac{3}{2}x + \tfrac{3}{2})(\xi + \tfrac{1}{2})^{-\frac{2}{3}}. \tag{8}$$

The five differential equations (3)–(7) are for sufficiently small $|t|$ satisfied by power series of the forms

$$\varrho = \sum_{\nu=0}^{\infty} \alpha_\nu t^\nu \tag{9}$$

$$\omega = \sum_{\nu=0}^{\infty} \beta_\nu t^\nu \tag{10}$$

$$\xi = \sum_{\nu=0}^{\infty} \gamma_\nu t^\nu \tag{11}$$

$$x = \sum_{\nu=0}^{\infty} \delta_\nu t^\nu \tag{12}$$

$$y = \sum_{\nu=0}^{\infty} \varkappa_\nu t^\nu. \tag{13}$$

Inserting these series and demanding that the coefficients of t^n shall vanish, we obtain the following recurrence formulas for the determination of the coefficients

$$(n+1)\,\alpha_{n+1} = \sum_{\nu=0}^{n} \beta_\nu \beta_{n-\nu} - \sum_{\nu=0}^{n} \alpha_\nu \alpha_{n-\nu} - \gamma_n + \frac{3}{2}\delta_n. \tag{14}$$

$$(n+1)\,\beta_{n+1} = -2\sum_{\nu=0}^{n} \alpha_\nu \beta_{n-\nu} - \frac{3}{2}\varkappa_n. \tag{15}$$

$$(n+1)\,\gamma_{n+1} = -3\sum_{\nu=0}^{n} \alpha_\nu \gamma_{n-\nu} - \frac{3}{2}\alpha_n. \tag{16}$$

$$(n+1)\,\delta_{n+1} = -2\sum_{\nu=0}^{n} \varkappa_\nu \beta_{n-\nu} + 2\varkappa_n. \tag{17}$$

$$(n+1)\,\varkappa_{n+1} = 2\sum_{\nu=0}^{n} \beta_\nu \delta_{n-\nu} - 2\delta_n. \tag{18}$$

From $x^2 + y^2 = 1$ we obtain furthermore

$$\delta_0^2 + \varkappa_0^2 = 1 \tag{19}$$

and

$$\sum_{\nu=0}^{n} (\delta_\nu \delta_{n-\nu} + \varkappa_\nu \varkappa_{n-\nu}) = 0 \qquad (n > 0). \tag{20}$$

The values of (9)–(13) for $t = 0$, that is α_0, β_0, γ_0, δ_0, $\varkappa_0$, may with the reservation resulting from (19) be employed as constants of integration.

2. We will now assume that $t=0$ corresponds to $l=0$ which leads to certain simplifications. In that case we have

$$\varkappa_0=0, \qquad \delta_0=1, \qquad \alpha_0=0, \tag{21}$$

the latter because, by I (6), $\varrho=\zeta \sin 2l$. Thus, only the two constants of integration β_0 and γ_0 are left for free disposition.

By (21) we obtain at once from (15), (16) and (17)

$$\beta_1=\gamma_1=\delta_1=0, \tag{22}$$

and (21) and (22) together with the recurrence formulas (14)–(18) then show that for all ν

$$\alpha_{2\nu}=\varkappa_{2\nu}=0; \qquad \beta_{2\nu+1}=\gamma_{2\nu+1}=\delta_{2\nu+1}=0. \tag{23}$$

We therefore obtain from (14)–(18)

$$(2n+1)\,\alpha_{2n+1}=\sum_{\nu=0}^{n}\beta_{2\nu}\beta_{2n-2\nu}-\sum_{\nu=1}^{n}\alpha_{2\nu-1}\,\alpha_{2n-2\nu+1}-\gamma_{2n}+\frac{3}{2}\delta_{2n}. \tag{24}$$

$$2n\,\beta_{2n}=-2\sum_{\nu=1}^{n}\alpha_{2\nu-1}\beta_{2n-2\nu}-\frac{3}{2}\varkappa_{2n-1}. \tag{25}$$

$$2n\,\gamma_{2n}=-3\sum_{\nu=1}^{n}\alpha_{2\nu-1}\gamma_{2n-2\nu}-\frac{3}{2}\alpha_{2n-1}. \tag{26}$$

$$2n\,\delta_{2n}=-2\sum_{\nu=1}^{n}\varkappa_{2\nu-1}\beta_{2n-2\nu}+2\varkappa_{2n-1}. \tag{27}$$

$$(2n+1)\,\varkappa_{2n+1}=2\sum_{\nu=0}^{n}\delta_{2\nu}\beta_{2n-2\nu}-2\delta_{2n}. \tag{28}$$

For l we find by $\dfrac{dl}{dt}=\omega-1$ and (10) the expansion

$$l=(\beta_0-1)\,t+\sum_{\nu=1}^{\infty}\frac{\beta_{2\nu}}{2\nu+1}t^{2\nu+1}. \tag{29}$$

3. For use in connection with a numerical example we state below the first few recurrence formulas for the coefficients. These are, taking $\delta_0=1$ into account:

$$\left.\begin{aligned}\alpha_1&=\beta_0^2-\gamma_0+\tfrac{3}{2}\\ \varkappa_1&=2(\beta_0-1)\end{aligned}\right\}. \tag{30}$$

$$\left.\begin{aligned}\beta_2&=-(\alpha_1\beta_0+\tfrac{3}{4}\varkappa_1)\\ \gamma_2&=-\tfrac{3}{2}\alpha_1(\gamma_0+\tfrac{1}{2})\\ \delta_2&=-\varkappa_1(\beta_0-1)\end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned}\alpha_3 &= \tfrac{1}{3}(2\beta_0\beta_2 - \alpha_1^2 - \gamma_2) + \tfrac{1}{2}\delta_2 \\ \varkappa_3 &= \tfrac{2}{3}[\delta_2(\beta_0 - 1) + \beta_2].\end{aligned}\right\}. \tag{32}$$

$$\left.\begin{aligned}\beta_4 &= -\tfrac{1}{2}(\alpha_1\beta_2 + \alpha_3\beta_0 + \tfrac{3}{4}\varkappa_3) \\ \gamma_4 &= -\tfrac{3}{4}[\alpha_1\gamma_2 + \alpha_3(\gamma_0 + \tfrac{1}{2})] \\ \delta_4 &= -\tfrac{1}{2}[\varkappa_3(\beta_0 - 1) + \beta_2\varkappa_1]\end{aligned}\right\}. \tag{33}$$

$$\left.\begin{aligned}\alpha_5 &= \tfrac{1}{5}(2\beta_0\beta_4 + \beta_2^2 - 2\alpha_1\alpha_3 - \gamma_4 + \tfrac{3}{2}\delta_4) \\ \varkappa_5 &= \tfrac{2}{5}[\delta_4(\beta_0 - 1) + \beta_2\delta_2 + \beta_4]\end{aligned}\right\}. \tag{34}$$

$$\left.\begin{aligned}\beta_6 &= -\tfrac{1}{3}(\alpha_1\beta_4 + \alpha_3\beta_2 + \alpha_5\beta_0) - \tfrac{1}{4}\varkappa_5 \\ \gamma_6 &= -\tfrac{1}{2}[\alpha_1\gamma_4 + \alpha_3\gamma_2 + \alpha_5(\gamma_0 + \tfrac{1}{2})] \\ \delta_6 &= -\tfrac{1}{3}[(\beta_0 - 1)\varkappa_5 + \beta_2\varkappa_3 + \beta_4\varkappa_1]\end{aligned}\right\}. \tag{35}$$

In order to compare with the earlier numerical example we observe that, since $t=0$ corresponds to $l=0$, β_0 is the value of $\omega = 1 + \eta$ for $x=1$, and γ_0 the value of $\xi = \varepsilon - \frac{1}{2}$ for $x=1$. In this way we find as constants of integration, besides $\delta_0 = 1$,

$$\beta_0 = 13{\cdot}62217, \qquad \gamma_0 = 182{\cdot}5881.$$

The results are as follows:

ν	α_ν	$\varkappa_\nu$
1	4·475416	25·24434
3	− 481·8885	− 2734·537
5	18005·94	104202·57

ν	β_ν	γ_ν	δ_ν
0	13·62217	182·5881	1·
2	− 79·89813	− 1229·0931	− 318·63835
4	4486·424	70296·56	18266·383
6	− 127337·5	− 2101782·8	− 549001·3

It is true that these coefficients increase rapidly, but for the orbit under consideration small values of t correspond to relatively large values of l, as appears from the expansion[1]

[1] J. F. Steffensen, Les orbites périodiques dans le problème de Hill. *Académie royale de Danemark, Bulletin* n° 3 (1909), 335.

$$\left.\begin{aligned} t = {} & \cdot 08085761\ l \\ & - \cdot 00082585 \sin 2l \\ & + \cdot 00000501 \sin 4l \\ & - \cdot 00000004 \sin 6l \end{aligned}\right\} \tag{36}$$

From (29) we obtain

$$\left.\begin{aligned} l = {} & \quad 12 \cdot 62217\, t \\ & - 26 \cdot 63271\, t^3 \\ & + 897 \cdot 2848\, t^5 \\ & - 18191 \cdot 07\, t^7 \end{aligned}\right\} \tag{37}$$

For $t = \cdot 01$ we find $l = \cdot 12619516$. If we insert this value in (36), we find to eight figures $t = \cdot 01000000$. For $t = \cdot 02$ we find $l = \cdot 25223319$ and, inserting this in (36), $t = \cdot 02000001$.

I have further calculated Jacobi's constant C by (8) for various values of t and found the following results

$$t = 0, \qquad C = 6 \cdot 5085378$$

$$t = \cdot 01, \qquad C = 6 \cdot 5085379$$

$$t = \cdot 02, \qquad C = 6 \cdot 5085378$$

where Bauschinger and Peters' table of logarithms to eight decimals has been used for calculating $(\xi + \tfrac{1}{2})^{-\frac{3}{2}}$.

4. In order to examine the convergence theoretically we begin by writing (24)–(28) in the following form where we have isolated the constants of integration β_0 and γ_0.

$$(2n+1)\,\alpha_{2n+1} = 2\beta_0\beta_{2n} + \sum_{\nu=1}^{n-1} \beta_{2\nu}\beta_{2n-2\nu} - \sum_{\nu=1}^{n} \alpha_{2\nu-1}\alpha_{2n-2\nu+1} - \gamma_{2n} + \frac{3}{2}\delta_{2n}. \tag{38}$$

$$-n\beta_{2n} = \beta_0\alpha_{2n-1} + \sum_{\nu=1}^{n-1} \alpha_{2\nu-1}\beta_{2n-2\nu} + \frac{3}{4}\varkappa_{2n-1}. \tag{39}$$

$$-\frac{2n}{3}\gamma_{2n} = \gamma_0\alpha_{2n-1} + \sum_{\nu=1}^{n-1} \alpha_{2\nu-1}\gamma_{2n-2\nu} + \frac{1}{2}\alpha_{2n-1}. \tag{40}$$

$$-n\delta_{2n} = \beta_0\varkappa_{2n-1} + \sum_{\nu=1}^{n-1} \varkappa_{2\nu-1}\beta_{2n-2\nu} - \varkappa_{2n-1}. \tag{41}$$

$$(n+\tfrac{1}{2})\,\varkappa_{2n+1} = \beta_0\delta_{2n} + \sum_{\nu=1}^{n-1} \beta_{2\nu}\delta_{2n-2\nu} + \beta_{2n} - \delta_{2n}. \tag{42}$$

We now put

$$k_\nu = \frac{\lambda^\nu}{\nu} \qquad (\lambda > 0,\ \nu \geq 1) \tag{43}$$

and will assume that for $1 \leq \nu \leq 2n$ we have proved that

$$\left.\begin{matrix} |\alpha_\nu| \leq \alpha\, k_\nu, \quad |\beta_\nu| \leq \beta\, k_\nu, \quad |\gamma_\nu| \leq \gamma\, k_\nu, \\ |\delta_\nu| \leq \delta\, k_\nu, \quad |\varkappa_\nu| \leq \varkappa\, k_\nu \end{matrix}\right\}, \tag{44}$$

whereafter we find conditions which are sufficient to ensure that (44) is always valid. The argumentation proceeds on the same lines as the corresponding one in paper I.

From (43) we obtain

$$k_\nu\, k_{m-\nu} = \frac{\lambda^m}{m}\left(\frac{1}{\nu} + \frac{1}{m-\nu}\right) \tag{45}$$

and if we write for abbreviation

$$s_n' = \sum_{\nu=1}^{n} \frac{1}{2\nu - 1} = s_{2n-1} - \tfrac{1}{2}\, s_{n-1} \tag{46}$$

where, as in the earlier paper, $s_n = \sum_{\nu=1}^{n} \frac{1}{\nu}$, we find

$$\sum_{\nu=1}^{n-1} k_{2\nu}\, k_{2n-2\nu} = \frac{\lambda^{2n}}{2n}\, s_{n-1}, \tag{47}$$

$$\sum_{\nu=0}^{n-1} k_{2\nu+1}\, k_{2n-2\nu-1} = \frac{\lambda^{2n}}{n}\, s_n', \tag{48}$$

$$\sum_{\nu=1}^{n-1} k_{2\nu-1}\, k_{2n-2\nu} = \frac{\lambda^{2n-1}}{2n-1}\, s_{2n-2}. \tag{49}$$

We first obtain from (38) by (43) and (47)–(49)

$$(2n+1)\,|\alpha_{2n+1}| \leq (2\,|\beta_0|\,\beta + \gamma + \tfrac{3}{2}\delta)\,\frac{\lambda^{2n}}{2n} + \beta^2\,\frac{\lambda^{2n}}{2n}\, s_{n-1} + \alpha^2\,\frac{\lambda^{2n}}{n}\, s_n'$$

and require that the right-hand side of this inequality shall not exceed

$$(2n+1)\,\alpha\, k_{2n+1} = \alpha\,\lambda^{2n+1}.$$

The condition may therefore, after dividing by λ^{2n}, be written

$$(2\,|\beta_0|\,\beta + \gamma + \tfrac{3}{2}\delta)\,\frac{1}{2n} + \frac{\beta^2}{2}\,\frac{s_{n-1}}{n} + \alpha^2\,\frac{s_n'}{n} \leq \alpha\,\lambda. \tag{50}$$

The four other conditions, obtained from (39)–(42) in the same way, are

$$\left(2\alpha\,|\beta_0|+\frac{3}{2}\varkappa\right)\frac{1}{2n-1}+2\alpha\beta\frac{s_{2n-2}}{2n-1}\le\beta\lambda, \tag{51}$$

$$\left(|\gamma_0|+\frac{1}{2}\right)\frac{1}{2n-1}+\gamma\,\frac{s_{2n-2}}{2n-1}\le\frac{\gamma\lambda}{3\alpha}, \tag{52}$$

$$(|\beta_0|+1)\frac{1}{2n-1}+\beta\frac{s_{2n-2}}{2n-1}\le\frac{\delta\lambda}{2\varkappa}, \tag{53}$$

$$(\delta\,|\beta_0|+\beta+\delta)\frac{1}{n}+\beta\delta\frac{s_{n-1}}{n}\le\varkappa\lambda. \tag{54}$$

It follows from the relations

$$s_n=s_{n-1}+\frac{1}{n},\quad s'_{n+1}=s'_n+\frac{1}{2n+1},$$

$$s_{2n}=s_{2n-2}+\frac{1}{2n-1}+\frac{1}{2n}$$

that none of the quantities s_{n-1}/n, s'_n/n and $s_{2n-2}/(2n-1)$ increases with n for $n>1$. Thus, for instance, the inequality

$$\frac{s_{2n-2}}{2n-1}>\frac{s_{2n}}{2n+1}$$

may be written in the form

$$2\,s_{2n-2}>2-\frac{1}{2n}$$

and so on.

We now assume that (44) has been proved for $1\le\nu\le6$. We may then obtain sufficient conditions for the validity of (44) for all ν by putting $n=3$ in (50)–(54). We find first

$$\frac{s_2}{3}=\frac{1}{2},\qquad\frac{s'_3}{3}=\frac{23}{45},\qquad\frac{s_4}{5}=\frac{5}{12}$$

and thereafter

$$\frac{1}{6}\left(2\,|\beta_0|\,\beta+\gamma+\frac{3}{2}\delta\right)+\frac{\beta^2}{4}+\frac{23}{45}\alpha^2\le\alpha\lambda, \tag{55}$$

$$\frac{1}{5}\left(2\alpha\,|\beta_0|+\frac{3}{2}\varkappa\right)+\frac{5}{6}\alpha\beta\le\beta\lambda, \tag{56}$$

$$\frac{1}{5}\left(|\gamma_0|+\frac{1}{2}\right)+\frac{5}{12}\gamma\leq\frac{\gamma\lambda}{3\alpha}, \tag{57}$$

$$\frac{1}{5}(|\beta_0|+1)+\frac{5}{12}\beta\leq\frac{\delta\lambda}{2\varkappa}, \tag{58}$$

$$\frac{1}{3}(\delta|\beta_0|+\beta+\delta)+\frac{\beta\delta}{2}\leq\varkappa\lambda. \tag{59}$$

In our numerical example (44) is satisfied for $1\leq\nu\leq 6$ if we put, for instance,

$$\lambda=20,\qquad \alpha=\beta=1,\qquad \gamma=7,\qquad \delta=\varkappa=2, \tag{60}$$

and since (55)–(59) are also satisfied, the expansions (9)–(13) are at least convergent for $|t|<1/20$.

5. We proceed to show that the system (3)–(7) can be satisfied by expansions of the form

$$\varrho=\cos(t+\theta)\cdot\sum_{\nu=0}^{\infty}d_\nu\sin^\nu(t+\theta) \tag{61}$$

$$\omega=\sum_{\nu=0}^{\infty}e_\nu\sin^\nu(t+\theta) \tag{62}$$

$$\xi=\sum_{\nu=0}^{\infty}f_\nu\sin^\nu(t+\theta) \tag{63}$$

$$x=\sum_{\nu=0}^{\infty}g_\nu\sin^\nu(t+\theta) \tag{64}$$

$$y=\cos(t+\theta)\cdot\sum_{\nu=0}^{\infty}h_\nu\sin^\nu(t+\theta) \tag{65}$$

where θ is an arbitrary constant, and $|\sin(t+\theta)|$ is assumed to be sufficiently small.

Differentiating (61), we obtain after a simple rearrangement of the terms, making use of $\cos^2(t+\theta)=1-\sin^2(t+\theta)$ and interpreting d_{-1} and h_{-1} as 0,

$$\frac{d\varrho}{dt}=\sum_{\nu=0}^{\infty}[(\nu+1)\,d_{\nu+1}-\nu\,d_{\nu-1}]\sin^\nu(t+\theta) \tag{66}$$

and hence, by a simple exchange of letters, from (65)

$$\frac{dy}{dt}=\sum_{\nu=0}^{\infty}[(\nu+1)\,h_{\nu+1}-\nu\,h_{\nu-1}]\sin^\nu(t+\theta). \tag{67}$$

Furthermore, we find by (62)–(64)

$$\frac{d\omega}{dt} = \cos(t+\theta) \cdot \sum_{\nu=0}^{\infty} (\nu+1)\, e_{\nu+1} \sin^{\nu}(t+\theta) \tag{68}$$

$$\frac{d\xi}{dt} = \cos(t+\theta) \cdot \sum_{\nu=0}^{\infty} (\nu+1)\, f_{\nu+1} \sin^{\nu}(t+\theta) \tag{69}$$

$$\frac{dx}{dt} = \cos(t+\theta) \cdot \sum_{\nu=0}^{\infty} (\nu+1)\, g_{\nu+1} \sin^{\nu}(t+\theta). \tag{70}$$

If we insert all these expansions in (3)–(7), the factor $\cos(t+\theta)$ disappears, and only power series in $\sin(t+\theta)$ remain. We now demand that the coefficients of $\sin^n(t+\theta)$ shall vanish, and find in this way the recurrence formulas

$$\left.\begin{aligned}(n+1)\, d_{n+1} = n\, d_{n-1} + \sum_{\nu=0}^{n} e_\nu\, e_{n-\nu} - \sum_{\nu=0}^{n} d_\nu\, d_{n-\nu} + \\ + \sum_{\nu=0}^{n-2} d_\nu\, d_{n-\nu-2} - f_n + \tfrac{3}{2} g_n\end{aligned}\right\}. \tag{71}$$

$$(n+1)\, e_{n+1} = -2 \sum_{\nu=0}^{n} d_\nu\, e_{n-\nu} - \tfrac{3}{2} h_n. \tag{72}$$

$$(n+1)\, f_{n+1} = -3 \sum_{\nu=0}^{n} d_\nu\, f_{n-\nu} - \tfrac{3}{2} d_n. \tag{73}$$

$$(n+1)\, g_{n+1} = -2 \sum_{\nu=0}^{n} h_\nu\, e_{n-\nu} + 2\, h_n. \tag{74}$$

$$(n+1)\, h_{n+1} = n\, h_{n-1} + 2 \sum_{\nu=0}^{n} g_\nu\, e_{n-\nu} - 2\, g_n. \tag{75}$$

6. We now assume that $\sin(t+\theta) = 0$ corresponds to $l = 0$, so that, corresponding to (21),

$$h_0 = 0, \qquad g_0 = 1, \qquad d_0 = 0. \tag{76}$$

As arbitrary constants of integration besides θ there are, then, only left e_0 and f_0.

We first obtain from (72)–(74) $e_1 = f_1 = g_1 = 0$, and (71)–(75) show thereafter that for all ν

$$d_{2\nu} = h_{2\nu} = 0; \qquad e_{2\nu+1} = f_{2\nu+1} = g_{2\nu+1} = 0. \tag{77}$$

The recurrence formulas may therefore be written as follows:

$$\left.\begin{aligned}(2n+1)\, d_{2n+1} = 2n\, d_{2n-1} + \sum_{\nu=0}^{n} e_{2\nu}\, e_{2n-2\nu} - \sum_{\nu=1}^{n} d_{2\nu-1}\, d_{2n-2\nu+1} + \\ + \sum_{\nu=1}^{n-1} d_{2\nu-1}\, d_{2n-2\nu-1} - f_{2n} + \tfrac{3}{2} g_{2n}\end{aligned}\right\}. \tag{78}$$

$$2n\,e_{2n} = -2\sum_{\nu=1}^{n} d_{2\nu-1}\,e_{2n-2\nu} - \tfrac{3}{2}h_{2n-1}. \tag{79}$$

$$2n\,f_{2n} = -3\sum_{\nu=1}^{n} d_{2\nu-1}\,f_{2n-2\nu} - \tfrac{3}{2}d_{2n-1}. \tag{80}$$

$$2n\,g_{2n} = -2\sum_{\nu=1}^{n} h_{2\nu-1}\,e_{2n-2\nu} + 2\,h_{2n-1}. \tag{81}$$

$$(2n+1)\,h_{2n+1} = 2n\,h_{2n-1} + 2\sum_{\nu=0}^{n} g_{2\nu}\,e_{2n-2\nu} - 2\,g_{2n}. \tag{82}$$

7. The first few of these recurrence formulas are

$$\left.\begin{aligned} d_1 &= e_0^2 - f_0 + \tfrac{3}{2} \\ h_1 &= 2\,(e_0 - 1) \end{aligned}\right\}. \tag{83}$$

$$\left.\begin{aligned} e_2 &= -(d_1\,e_0 + \tfrac{3}{4}h_1) \\ f_2 &= -\tfrac{3}{2}d_1\,(f_0 + \tfrac{1}{2}) \\ g_2 &= -h_1\,(e_0 - 1) \end{aligned}\right\}. \tag{84}$$

$$\left.\begin{aligned} d_3 &= \tfrac{1}{3}\,(2\,d_1 + 2\,e_0\,e_2 - d_1^2 - f_2) + \tfrac{1}{2}g_2 \\ h_3 &= \tfrac{2}{3}\,[h_1 + g_2\,(e_0 - 1) + e_2] \end{aligned}\right\}. \tag{85}$$

$$\left.\begin{aligned} e_4 &= -\tfrac{1}{2}\,(d_1\,e_2 + d_3\,e_0 + \tfrac{3}{4}h_3) \\ f_4 &= -\tfrac{3}{4}\,[d_1\,f_2 + d_3\,(f_0 + \tfrac{1}{2})] \\ g_4 &= -\tfrac{1}{2}\,[h_1\,e_2 + h_3\,(e_0 - 1)] \end{aligned}\right\}. \tag{86}$$

$$\left.\begin{aligned} d_5 &= \tfrac{1}{5}\,(4\,d_3 + 2\,e_0\,e_4 + e_2^2 - 2\,d_1\,d_3 + d_1^2 - f_4 + \tfrac{5}{2}g_4) \\ h_5 &= \tfrac{2}{5}\,[2\,h_3 + e_4 + g_2\,e_2 + g_4\,(e_0 - 1)] \end{aligned}\right\}. \tag{87}$$

$$\left.\begin{aligned} e_6 &= -\tfrac{1}{3}\,(d_1\,e_4 + d_3\,e_2 + d_5\,e_0) - \tfrac{1}{4}h_5 \\ f_6 &= -\tfrac{1}{2}\,[d_1\,f_4 + d_3\,f_2 + d_5\,(f_0 + \tfrac{1}{2})] \\ g_6 &= -\tfrac{1}{3}\,[h_1\,e_4 + h_3\,e_2 + h_5\,(e_0 - 1)] \end{aligned}\right\}. \tag{88}$$

Now a comparison of (62) and (10) shows for $t = \theta = 0$ that $e_0 = \beta_0$; and (63) and (11) show in the same way that $f_0 = \gamma_0$. Using the same initial values as above, we therefore have

$$e_0 = 13{\cdot}62217, \qquad f_0 = 182{\cdot}5881$$

and obtain the following results:

ν	d_ν	h_ν
1	4·475416	25·24434
3	—478·9049	—2717·707
5	17526·43	101481·48

ν	e_ν	f_ν	g_ν
0	13·62217	182·5881	1·
2	—79·89813	—1229·0931	—318·63835
4	4459·791	69886·87	18160·168
6	—124360·7	—2055136·1	—536880·2

A few of the first of these coefficients are identical with the corresponding ones in the expansions in powers of t; the remaining ones do not differ much from them.

The calculation of Jacobi's constant C for various values of $\sin(t+\theta)$ produces the following results:

$$\sin(t+\theta)=0, \qquad C=6{\cdot}5085378$$

$$\sin(t+\theta)={\cdot}01, \qquad C=6{\cdot}5085378$$

$$\sin(t+\theta)={\cdot}02, \qquad C=6{\cdot}5085381.$$

8. We now write (78)–(82) in the following form, corresponding to (38)–(42):

$$\left.\begin{aligned}(2n+1)\,d_{2n+1} &= 2\,e_0\,e_{2n}+\sum_{\nu=1}^{n-1} e_{2\nu}\,e_{2n-2\nu}-\sum_{\nu=0}^{n-1} d_{2\nu+1}\,d_{2n-2\nu-1}+\\ &+\sum_{\nu=1}^{n-1} d_{2\nu-1}\,d_{2n-2\nu-1}+2n\,d_{2n-1}-f_{2n}+\tfrac{3}{2}g_{2n}\end{aligned}\right\}. \tag{89}$$

$$-n\,e_{2n}=e_0\,d_{2n-1}+\sum_{\nu=1}^{n-1} d_{2\nu-1}\,e_{2n-2\nu}+\tfrac{3}{4}h_{2n-1}. \tag{90}$$

$$-\frac{2n}{3}f_{2n}=f_0\,d_{2n-1}+\sum_{\nu=1}^{n-1} d_{2\nu-1}\,f_{2n-2\nu}+\tfrac{1}{2}d_{2n-1}. \tag{91}$$

$$-n\,g_{2n}=e_0\,h_{2n-1}+\sum_{\nu=1}^{n-1} h_{2\nu-1}\,e_{2n-2\nu}-h_{2n-1}. \tag{92}$$

$$(n+\tfrac{1}{2})\,h_{2n+1}=n\,h_{2n-1}+e_0\,g_{2n}+\sum_{\nu=1}^{n-1} g_{2\nu}\,e_{2n-2\nu}+e_{2n}-g_{2n}. \tag{93}$$

We assume next that for $1 \le \nu \le 2n$ we have proved that

$$\left.\begin{array}{c} |d_\nu| \le D k_\nu, \quad |e_\nu| \le E k_\nu, \quad |f_\nu| \le F k_\nu, \\ |g_\nu| \le G k_\nu, \quad |h_\nu| \le H k_\nu \end{array}\right\}, \tag{94}$$

k_ν being defined by (43).

We then obtain from (89)–(93)

$$\left.\begin{array}{l} (2n+1)\,|d_{2n+1}| \le 2\,|e_0|\,E k_{2n} + E^2 \sum\limits_{\nu=1}^{n-1} k_{2\nu} k_{2n-2\nu} + \\ \qquad + D^2 \sum\limits_{\nu=0}^{n-1} k_{2\nu+1} k_{2n-2\nu-1} + D^2 \sum\limits_{\nu=1}^{n-1} k_{2\nu-1} k_{2n-2\nu-1} + \\ \qquad + 2nD k_{2n-1} + F k_{2n} + \frac{3}{2} G k_{2n} \end{array}\right\} \tag{95}$$

$$n\,|e_{2n}| \le |e_0|\,D k_{2n-1} + DE \sum_{\nu=1}^{n-1} k_{2\nu-1} k_{2n-2\nu} + \tfrac{3}{4} H k_{2n-1}. \tag{96}$$

$$\frac{2n}{3}\,|f_{2n}| \le |f_0|\,D k_{2n-1} + DF \sum_{\nu=1}^{n-1} k_{2\nu-1} k_{2n-2\nu} + \tfrac{1}{2} D k_{2n-1}. \tag{97}$$

$$n\,|g_{2n}| \le |e_0|\,H k_{2n-1} + HE \sum_{\nu=1}^{n-1} k_{2\nu-1} k_{2n-2\nu} + H k_{2n-1}. \tag{98}$$

$$(n+\tfrac{1}{2})\,|h_{2n+1}| \le |e_0|\,G k_{2n} + GE \sum_{\nu=1}^{n-1} k_{2\nu} k_{2n-2\nu} + E k_{2n} + G k_{2n} + nH k_{2n-1}. \tag{99}$$

By (47)–(49) and the relation

$$\sum_{\nu=1}^{n-1} k_{2\nu-1} k_{2n-2\nu-1} = \frac{\lambda^{2n-2}}{n-1} s'_{n-1}, \tag{100}$$

resulting from (48), writing $n-1$ for n, (95) may be written

$$\left.\begin{array}{l} (2n+1)\,|d_{2n+1}| \le (2\,|e_0|\,E + F + \frac{3}{2} G)\,\dfrac{\lambda^{2n}}{2n} + E^2 \dfrac{\lambda^{2n}}{2n} s_{n-1} + \\ \qquad + D^2 \dfrac{\lambda^{2n}}{n} s'_n + D^2 \dfrac{\lambda^{2n-2}}{n-1} s'_{n-1} + 2nD \dfrac{\lambda^{2n-1}}{2n-1} \end{array}\right\}. \tag{101}$$

We require that the right-hand side of this shall not exceed $(2n+1) D k_{2n+1} = D \lambda^{2n+1}$, so that, after dividing by λ^{2n-2},

$$\left.\begin{array}{l} (2\,|e_0|\,E + F + \frac{3}{2} G)\,\dfrac{\lambda^2}{2n} + \dfrac{E^2 \lambda^2}{2} \dfrac{s_{n-1}}{n} + D^2 \lambda^2 \dfrac{s'_n}{n} + D^2 \dfrac{s'_{n-1}}{n-1} + \\ \qquad + D\lambda \left(1 + \dfrac{1}{2n-1}\right) \le D \lambda^3 \end{array}\right\}. \tag{102}$$

In precisely the same way we obtain from (96)–(99) the sufficient conditions

$$(2\,|e_0|\,D+\tfrac{3}{2}H)\frac{1}{2n-1}+2DE\,\frac{s_{2n-2}}{2n-1}\leq E\lambda, \tag{103}$$

$$(|f_0|+\tfrac{1}{2})\,\frac{1}{2n-1}+F\,\frac{s_{2n-2}}{2n-1}\leq\frac{F\lambda}{3D}, \tag{104}$$

$$(|e_0|+1)\frac{1}{2n-1}+E\,\frac{s_{2n-2}}{2n-1}\leq\frac{G\lambda}{2H}, \tag{105}$$

$$(|e_0|\,G+E+G)\frac{\lambda}{n}+EG\lambda\frac{s_{n-1}}{n}+H\left(1+\frac{1}{2n-1}\right)\leq H\lambda^2. \tag{106}$$

None of the quantities depending on n increases for $n>1$. If therefore (94) has been proved, for instance, for $1\leq\nu\leq 6$, we may find sufficient conditions for the validity of (94) for all ν by putting $n=3$ in (102)–(106). We thus find

$$\frac{\lambda^2}{6}\left(2\,|e_0|\,E+F+\frac{3}{2}G+\frac{3}{2}E^2+\frac{46}{15}D^2\right)+\frac{2}{3}D^2+\frac{6}{5}D\lambda\leq D\lambda^3, \tag{107}$$

$$\frac{1}{5}\left(2\,|e_0|\,D+\frac{3}{2}H\right)+\frac{5}{6}DE\leq E\lambda, \tag{108}$$

$$\frac{1}{5}\left(|f_0|+\frac{1}{2}\right)+\frac{5}{12}F\leq\frac{F\lambda}{3D}, \tag{109}$$

$$\frac{1}{5}(|e_0|+1)+\frac{5}{12}E\leq\frac{G\lambda}{2H}, \tag{110}$$

$$\frac{\lambda}{3}\left(|e_0|\,G+E+G+\frac{3}{2}EG\right)+\frac{6}{5}H\leq H\lambda^2. \tag{111}$$

In our numerical example (94) is satisfied for $1\leq\nu\leq 6$ if we put, for instance,

$$\lambda=20,\qquad D=E=1,\qquad F=7,\qquad G=H=2, \tag{112}$$

and since (107)–(111) are, then, also satisfied, the expansions (61)–(65) are at least convergent for $|\sin(t+\theta)|<1/20$.