# THE AUTOMORPHISMS AND THE ENDOMORPHISMS OF THE GROUP ALGEBRA OF THE UNIT CIRCLE

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**1.** Professor Boas recently suggested the following problem in a letter:
*For what integer-valued sequences* $\{t(n)\}$, $n=0, \pm 1, \pm 2, \ldots$, *is it true that*

$$\sum_{-\infty}^{\infty} a\,(t(n))\, e^{in\theta}$$

*is a Fourier series whenever*

$$\sum_{-\infty}^{\infty} a\,(n)\, e^{in\theta}$$

*is a Fourier series?*

Theorem I of the present paper contains the solution of this problem, and leads to a complete description of all automorphisms and endomorphisms of the group algebra of the unit circle, i.e., the algebra whose members are the Lebesgue integrable complex-valued functions on the unit circle, with convolution as multiplication. The algebra of all bounded complex Borel measures on the circle is also discussed from this standpoint.

Boas' question was prompted by the following theorem recently obtained by Leibenson [7] and Kahane [6] (the latter removed the differentiability conditions imposed on $w$ by the former):

*The only real functions $w$ which have the property that $f(e^{iw(\theta)})$ has an absolutely convergent Fourier series whenever the Fourier series of $f(e^{i\theta})$ converges absolutely, are of the form $w(\theta)=n\,\theta+\alpha$, where $n$ is an integer and $\alpha$ a real number.*

The basic "reason" for the contrast between the simplicity of this result and the rather complicated Theorem I of the present paper seems to be the fact that the circle group is connected, whereas the additive group of the integers is discrete. This point is further illustrated by the following result of Beurling and Helson:

*If $G$ is a locally compact abelian group with connected dual group and if $T$ is an automorphism of the group algebra $L(G)$, then $T$ is given by the formula*

$$(T f)(x) = k \cdot y(x) \cdot f(\gamma(x)) \qquad (x \in G,\ f \in L(G)), \tag{1.1}$$

*where $y$ is a continuous character of $G$, $\gamma$ is a topological automorphism of $G$, and $k$ is a positive number which compensates for the change in Haar measure caused by $\gamma$.*

This theorem is not explicitly stated by Beurling and Helson, but is an easy consequence of the second theorem of [1] and the first two paragraphs of [2]. Since the present paper is primarily devoted to the group algebra of the circle group $C$, we omit the details.

Thus $L(G)$ admits only the trivial automorphisms (1.1) if the dual of $G$ is connected, whereas $L(C)$ admits the much larger variety of automorphisms described toward the end of this paper.

**2.** This paragraph is devoted to a quick review of some of the principal facts concerning Fourier and Fourier-Stieltjes series, measures, and convolutions, which will be needed later. By the circle group $C$ we mean the set of all complex numbers of absolute value 1, with multiplication as group operation. The additive group of all integers will be denoted by $J$. Instead of writing $e^{i\theta}$, etc., for the elements of $C$, we shall use the letters $x$, $y$, $z$; instead of Lebesgue measure on $C$ we shall use the Haar measure $m(E)$ (which is nothing but the Lebesgue measure of $E$, divided by $2\pi$); this simplifies the formalism. $L(C)$ is the set of all complex functions on $C$ which are integrable with respect to Haar measure; with convolution defined by

$$(f * g)(x) = \int_C f(x y^{-1})\, g(y)\, dm(y) \tag{2.1}$$

and norm

$$\|f\| = \int_C |f(x)|\, dm(x), \tag{2.2}$$

$L(C)$ is a commutative Banach algebra. The Fourier coefficients (or simply the *coefficients*) of a function $f \in L(C)$ are given by

$$a(n) = \int_C x^{-n} f(x)\, dm(x) \qquad (n \in J); \tag{2.3}$$

the statement

$$f(z) \sim \sum_{n \in J} a(n) z^n \tag{2.4}$$

is equivalent to (2.3). With these definitions the coefficients of $f * g$ are the products of the coefficients of $f$ and $g$.

By a *measure* we mean a countably additive bounded complex-valued set function defined for all Borel subsets of $C$; $M(C)$ is the set of all measures. With the norm $\|\mu\|$ of $\mu$ defined as the total variation of $\mu$ on $C$ and convolution of two measures $\mu$ and $\lambda$ given by

$$(\mu * \lambda)(E) = \int_C \mu(E y^{-1})\, d\lambda(y) \tag{2.5}$$

(where $E y^{-1}$ is the set of all elements $x y^{-1}$ with $x \in E$), $M(C)$ is a commutative Banach algebra; for the details of this, see for instance [11]. The measure which is concentrated at the point $x = 1$ and which assigns the value 1 to that point is the unit element of $M(C)$. The coefficients of a measure $\mu$ are the numbers

$$a(n) = \int_C x^{-n}\, d\mu(x) \qquad (n \in J); \tag{2.6}$$

the Fourier-Stieltjes series of $\mu$ is

$$d\mu(z) \sim \sum_{n \in J} a(n) z^n, \tag{2.7}$$

with $a(n)$ given by (2.6). Again, convolution of measures corresponds to multiplication of coefficients.

For every $\mu \in M(C)$ there is a unique decomposition

$$\mu = \mu_1 + \mu_2 + \mu_3 \tag{2.8}$$

where $\mu_1$ is discrete (i.e., $\mu_1$ is concentrated on an at most countable set of points), $\mu_2$ is absolutely continuous with respect to Haar measure, and $\mu_3$ is singular (i.e., continuous, but concentrated on a set of Haar measure zero). From (2.5) it follows immediately that if one factor of a convolution is continuous or absolutely continuous, then the same is true of the convolution; the convolution of two discrete measures is discrete.

With every $f \in L(C)$ there is associated a measure

$$\mu_f(E) = \int_E f(x)\, dm(x). \tag{2.9}$$

This formula furnishes an isometric isomorphism of $L(C)$ onto the set of all absolutely continuous measures and the identification of $f$ with $\mu_f$ allows us to consider $L(C)$ as a closed ideal of $M(C)$.

3. Besides the well-known facts outlined in the preceding section, the following will be used several times in our investigation ([3]; see also [4]):

HELSON'S THEOREM. *If $\mu \in M(C)$ and if the coefficients of $\mu$ have only a finite number of distinct values, then $\mu = \sigma + \tau$, where $\sigma$ is a discrete measure whose coefficients form a periodic sequence, and $\tau$ has only finitely many coefficients different from zero.*

Conversely, it is easy to see that every periodic sequence of complex numbers, with period $p$, is the sequence of Fourier-Stieltjes coefficients of a discrete measure, concentrated at the $p$th roots of unity.

A set $S \subset J$ is said to be periodic if, for some $p > 0$, $n \in S$ if and only if $n + p \in S$. A set $N \subset J$ will be called a $P$-set if $N$ can be made periodic by adding or deleting a finite number of elements.

With this terminology, Helsons's theorem furnishes the following characterization of the idempotent measures (i.e., those measures which satisfy the equation $\mu * \mu = \mu$), since their coefficients are all 0 or 1:

*A trigonometric series $\sum_{n \in J} a(n) z^n$ is the Fourier-Stieltjes series of an idempotent measure if and only if there is a $P$-set $N$ such that*

$$a(n) = \begin{cases} 1 & \textit{if } n \in N, \\ 0 & \textit{otherwise.} \end{cases}$$

On the other hand, this characterization of the idempotent measures immediately implies Helson's theorem. The extension of this result to arbitrary compact abelian groups would be a major step toward a complete description of the automorphisms and endomorphisms of their group algebras.

4. In order not to interrupt the main argument, we insert here a measure-theoretic lemma which depends on the theory of analytic sets [9].

LEMMA. *Let $f$ be a complex-valued Borel measurable function on the topological product $C \times C$, and suppose that for each $x \in C$ there is an at most countable set $A_x \subset C$ such that $f(x, y) = 0$ if $y \notin A_x$. If*

$$g(x) = \sum_{y \in A_x} f(x, y) \qquad (x \in C),$$

*the series converging absolutely for each $x$, then $g$ is Lebesgue measurable.*

To prove this, assume first that the only two values of $f$ are 0 and 1, let $E$ be the set of all points $(x, y)$ at which $f(x, y) = 1$, and, for every positive integer $k$, let $E_k$ be the set of all $x \in C$ at which $g(x) \geq k$. Since $g(x)$ is, in this case, nothing but the number of points in $A_x$, the Lebesgue measurability of $g$ will be established if it is proved that every $E_k$ is Lebesgue measurable.

To this end, fix $k$ and let $\{V_n\}$, $n = 1, 2, 3, \ldots$, be a countable base for $C \times C$. Let $W_n$ be the set of all $x \in C$ such that $(x, y) \in E \cap V_n$ for some $y \in C$. Then $W_n$ is Lebesgue measurable (since projections of Borel sets are analytic sets ([9], p. 144) and the latter are Lebesgue measurable ([9], p. 152)).

For any choice of positive integers $n_1, \ldots, n_k$ such that the open sets $V_{n_i}$ $(i = 1, \ldots, k)$ are pairwise disjoint, put

$$Q(n_1, \ldots, n_k) = W_{n_i} \cap \ldots \cap W_{n_k}.$$

It is easy to see that

$$E_k = \bigcup Q(n_1, \ldots, n_k),$$

the union being taken over all $k$-tuples subject to the above condition. Since this a countable union, $E_k$ is Lebesgue measurable.

Thus the lemma is true if $f$ is the characteristic function of a Borel set, hence if $f$ is a simple function (i.e., one with a finite set of values), then if $f$ is real and non-negative (since $f$ is then the pointwise limit of an increasing sequence of simple functions), and the general case follows by noting that

$$f = f_1 - f_2 + i f_3 - i f_4 \text{ with } f_n \geq 0 \ (n = 1, 2, 3, 4).$$

REMARK. The proof shows that the hypothesis of the lemma can be weakened and the conclusion strengthened by replacing Borel and Lebesgue measurability, respectively, by measurability with respect to the analytic sets. However, the statement of the lemma becomes false if "Borel" is replaced by "Lebesgue" in the hypothesis or if "Lebesgue" is replaced by "Borel" in the conclusion.

**5. THE PRINCIPAL THEOREM.** Let $N$ be a subset of $J$ and let $t$ be a mapping of $N$ into $J$. We say that $t$ carries $L(C)$ into $L(C)$ if the series

$$\sum_{n \in N} a(t(n)) z^n \tag{5.1}$$

is a Fourier series[1] whenever the series

$$\sum_{n \in J} a(n) z^n \tag{5.2}$$

is a Fourier series.

THEOREM I. *Let $t$ be a mapping of $N$ into $J$, with $N \subset J$. Then $t$ carries $L(C)$ into $L(C)$ if and only if the following conditions are satisfied:*

A: *$N$ is a $P$-set.*

B: *There is a mapping $s$ of $J$ into $J$ and a positive integer $q$ such that*

B 1: *$t(n) = s(n)$ for all $n \in N$, except possibly on a finite subset of $N$;*

B 2: *for every $n \in J$, $s(n+q) + s(n-q) = 2\,s(n)$;*

B 3: *for every $n \in J$, $s(n+q) \neq s(n)$.*

The special case $N = J$ answer Boas' question.

**6.** PROOF THAT THE CONDITIONS ARE NECESSARY. This proof is rather long and will be broken up into several steps. We assume now that $t$ carries $L(C)$ into $L(C)$.

STEP 1. *Extend $t$ to a mapping of $J$ into $J$ by defining $t(n) = n$ if $n \notin N$. Let*

$$\psi(n) = \begin{cases} 1 & \textit{if } n \in N \\ 0 & \textit{otherwise.} \end{cases}$$

*For every $x \in C$ there is then a measure $\nu_x$ such that*

$$d\nu_x(z) \sim \sum_{n \in J} \psi(n)\, x^{t(n)} z^n. \tag{6.1}$$

*These measures are bounded in norm.*

For every $f \in L(C)$ with Fourier series (5.2) there is a function $Tf \in L(C)$ with Fourier series (5.1). The coefficients of $Tf$ are accordingly given by

$$\int_C x^{-n} (Tf)(x)\, dm(x) = \psi(n)\, a(t(n)) = \psi(n) \int_C x^{-t(n)} f(x)\, dm(x).$$

Putting

$$Q_n(z, x) = \sum_{k=-n}^{n} \left[1 - \frac{|k|}{n+1}\right] \psi(k)\, x^{t(k)} z^k,$$

the Cesàro means of the Fourier series of $Tf$ are therefore

[1] This means, of course, that there is a function in $L(C)$ whose $n$th Fourier coefficient is 0 if $n \notin N$ and is $a(t(n))$ if $n \in N$.

$$\sigma_n(Tf;\, z) = \int_C f(x)\, Q_n(z,\, x^{-1})\, dm(x) \qquad (n = 0,\, 1,\, 2,\, \ldots).$$

Suppose the functions

$$g_n(x) = \int_C |Q_n(z,\, x^{-1})|\, dm(z) \qquad (n = 0,\, 1,\, 2, \ldots;\, x \in C)$$

are not uniformly bounded. Then there exist sets $E_n \subset C$ such that the functions

$$h_n(x) = \int_{E_n} Q_n(z,\, x^{-1})\, dm(z)$$

are not uniformly bounded, and by the Banach-Steinhaus theorem there is a function $f \in L(C)$ for which

$$\int_{E_n} \sigma_n(Tf;\, z)\, dm(z) = \int_C f(x)\, h_n(x)\, dm(x)$$

is unbounded as $n \to \infty$. This implies that for this particular $f$

$$\int_C |\sigma_n(Tf;\, z)|\, dm(z)$$

is unbounded, contradicting the fact that $Tf \in L(C)$ [13, p. 84]. Consequently there is a constant $K$ such that

$$\int_C |Q_n(z,\, x)|\, dm(z) < K \qquad (n = 0,\, 1,\, 2,\, \ldots;\, x \in C).$$

Observing that $Q_n(z, x)$ is the $n$th Cesàro mean of the series (6.1), the assertions of Step 1 follow ([13], p. 79), with $\|\nu_x\| \le K$ for all $x \in C$.

STEP 2. *The set $N$ satisfies condition $A$ of the theorem.*

Taking $x = 1$ in (6.1), we see that $\{\psi(n)\}$ is a sequence of Fourier-Stieltjes coefficients of an idempotent measure. The definition of $\psi(n)$ now shows that $N$ is a $P$-set.

STEP 3. *If $t$ is extended as in Step 1, then the extended mapping also carries $L(C)$ into $L(C)$, and for every $x \in C$ there is a measure $\mu_x$ such that*

$$d\mu_x(z) \sim \sum_{n \in J} x^{t(n)} z^n. \tag{6.2}$$

*These measures are bounded in norm.*

Taking $x = 1$ in (6.1), we see that

$$\sum_{n \in J} [1 - \psi(n)]\, z^n$$

is a Fourier-Stieltjes series. Hence, if (5.2) is a Fourier series, so is

$$\sum_{n \in J} [1 - \psi(n)]\, a(n)\, z^n = \sum_{n \notin N} a(t(n))\, z^n;$$

adding this to (5.1), we conclude that

$$\sum_{n \in J} a(t(n))\, z^n$$

is a Fourier series. We can now apply Step 1 with $N = J$, and Step 3 follows.

STEP 4. *The measures $\mu_x$ of Step 3 satisfy the equation*

$$\mu_x * \mu_y = \mu_{xy} \qquad (x \in C,\ y \in C). \tag{6.3}$$

*If $\lambda_x$ is the discrete part of $\mu_x$, then we also have*

$$\lambda_x * \lambda_y = \lambda_{xy} \qquad (x \in C,\ y \in C). \tag{6.4}$$

Since convolution of measures corresponds to multiplication of coefficients, (6.3) is an immediate consequence of (6.2); and (6.4) follows from (6.3) by equating the discrete parts on both sides of (6.3).

STEP 5. *There is a mapping $s$ of $J$ into $J$ such that*

$$d\lambda_x(z) \sim \sum_{n \in J} x^{s(n)} z^n. \tag{6.5}$$

Let the coefficients of $\lambda_x$ be denoted by $c_n(x)$; we compute $c_n(x)$ in terms of the coefficients of $\mu_x$: for $0 < r < 1$, let

$$u_x(r, z) = \sum_{n \in J} x^{t(n)} z^n r^{|n|} = \int_C \frac{1 - r^2}{1 - 2r \operatorname{Re}(zy^{-1}) + r^2}\, d\mu_x(y),$$

where $\operatorname{Re}(z)$ denotes the real part of $z$, and put

$$f(x, z) = \lim_{r \to 1} \frac{1 - r}{2}\, u_x(r, z) \qquad (x \in C,\ z \in C).$$

From the Poisson integral representation of $u_x$ it follows easily that $f(x, z)$ is equal to the mass which $\mu_x$ assigns to the set consisting of the single point $z$, so that

$$\sum_{z \in C} |f(x, z)| < \infty \qquad (x \in C),$$

and

$$c_n(x) = \int_C z^{-n}\, d\lambda_x(z) = \sum_{z \in C} z^{-n} f(x, z).$$

Since $f(x, z)$ is the pointwise limit of a sequence of continuous functions on $C \times C$, the lemma of section 4 is applicable and shows that the functions $c_n$ are Lebesgue measurable. From (6.4) we infer that

$$c_n(x)\, c_n(y) = c_n(xy) \qquad (x \in C,\ y \in C). \tag{6.6}$$

Now it is well known ([10], p. 479) that the only Lebesgue measurable solutions of

$$g(\theta)\, g(\phi) = g(\theta + \phi) \qquad (\theta,\ \phi \text{ real})$$

which are not identically zero are of the form

$$g(\theta) = e^{(\alpha + i\beta)\theta} \qquad (\alpha,\ \beta \text{ real}).$$

If $g$ is to have period $2\pi$, then $\alpha = 0$, $\beta \in J$.

Since $\lambda_1 = \mu_1$ (the unit of $M(C)$), we see that $c_n(1) = 1$, and we conclude that for each $n \in J$ there is an $s(n) \in J$ such that $c_n(x) = x^{s(n)}$. This completes Step 5.

STEP 6. *The mapping $s$ of Step 5 satisfies condition* B 2.

Since $\lambda_x$ is a discrete measure, the integral

$$x^{s(n)} = \int_C z^{-n}\, d\lambda_x(z)$$

reduces, for each fixed $x \in C$, to a series of characters of $J$ which converges absolutely and uniformly on $J$. Hence $x^{s(n)}$ is, for each $x$, an almost periodic function on $J$ ([10], p. 448). Every infinite set of translates of an almost periodic function contains a uniformly convergent subsequence; all we need here is that for each $x \in C$ there is a positive integer $k_x$ such that

$$\left| x^{s(n)} - x^{s(n+k_x)} \right| < 1 \qquad (n \in J).$$

It follows that there exists a positive integer $k$ and a set $E \subset C$ with $m(E) > 0$, such that

$$\left| x^{s(n)} - x^{s(n+k)} \right| < 1 \qquad (n \in J,\ x \in E).$$

Putting $b(n) = s(n+k) - s(n)$, this becomes

$$\left| 1 - x^{b(n)} \right| < 1 \qquad (n \in J,\ x \in E). \tag{6.7}$$

If $\{b(n)\}$ were unbounded, then the sequence $\{x^{b(n)}\}$ would be dense on $C$, for almost every $x$ ([12], p. 344); by (6.7) this is false for every $x \in E$; hence $\{b(n)\}$ is bounded.

Since $\{x^{s(n+k)}\}$ and $\{x^{-s(n)}\}$ are sequences of Fourier-Stieltjes coefficients of discrete measures, so is their product $\{x^{b(n)}\}$. Helson's theorem now tells us that $\{x^{b(n)}\}$ is a periodic sequence, for every $x \in C$. Considering an $x \in C$ which is not a root of unity, we conclude that $\{b(n)\}$ is periodic; i.e., $b(n+p) = b(n)$ for some positive integer $p$ and every $n \in J$.

The definition of $b(n)$ shows that

$$s(n+kp)-s(n)=\sum_{j=0}^{p-1}[s(n+(j+1)k)-s(n+jk)]$$
$$=\sum_{j=0}^{p-1} b(n+jk),$$

and similarly that

$$s(n)-s(n-kp)=\sum_{j=0}^{p-1} b(n-kp+jk).$$

Since $p$ is a period of $\{b(n)\}$, these last two expressions are equal. Putting $q=kp$ we thus obtain condition B 2:

$$s(n+q)-s(n)=s(n)-s(n-q).$$

STEP 7. *For every $x \in C$, put*

$$\tau_x=(\lambda_x-\mu_x) * \lambda_{x^{-1}}.$$

*Since $\lambda_x-\mu_x$ is a continuous measure, so is $\tau_x$. Putting $r(n)=t(n)-s(n)$,* (6.2) *and* (6.5) *imply*

$$d\tau_x(z) \sim \sum_{n \in J}[1-x^{r(n)}]z^n. \tag{6.8}$$

This is obvious.

STEP 8. *The sequence $\{r(n)\}$ has at most a finite number of terms different from zero; this implies that condition* B 1 *holds.*

For every continuous complex function $g$ on $C$ the integral

$$\int_C g(z)\,d\tau_x(z)$$

is a continuous function of $x$; if $g$ is a trigonometric polynomial, this follows from the fact that the coefficients of $\tau_x$ are continuous functions of $x$, and the general case follows if we approximate $g$ uniformly by trigonometric polynomials and note that $\|\tau_x\|$ is bounded.

For any open set $E \subset C$, the total variation $|\tau_x|(E)$ of $\tau_x$ on $E$ is given by

$$|\tau_x|(E)=\sup\left|\int_C g(z)\,d\tau_x(z)\right|,$$

where the supremum is taken over all continuous functions $g$ which vanish outside $E$ and are bounded by 1 in absolute value. Being the supremum of a collection of continuous functions, $|\tau_x|(E)$ is a lower semi-continuous function of $x$. Taking $E=C$, we see that the same is true of $\|\tau_x\|$.

For any closed set $F \subset C$ with complement $E$ we have

$$|\tau_x|(F) = \|\tau_x\| - |\tau_x|(E),$$

so that $|\tau_x|(F)$ is the difference of two lower semi-continuous functions of $x$ *and is therefore continuous on a dense set of type* $G_\delta$ ([5], p. 310).

Let $R$ be the set of all $x \in C$ which are roots of unity. If $y \in R$, (6.8) shows that $\tau_y$ has only finitely many distinct coefficients, and Helson's theorem, together with the continuity of $\tau_y$, implies that $\tau_y$ is absolutely continuous. From (6.8) it follows also that

$$\tau_y \times \tau_z = \tau_y + \tau_z - \tau_{yz} \qquad (y \in C,\, z \in C),$$

so that $\tau_{yz}$ and $\tau_z$ have the same singular part if $y \in R$.

Suppose now that $\tau_z$ fails to be absolutely continuous for some $z \in C$. Then there is a closed set $F$ with $m(F) = 0$, such that

$$|\tau_{yz}|(F) = |\tau_z|(F) > 0$$

for every $y \in R$. But $|\tau_y|(F) = 0$ for every $y \in R$. Since $R$ is dense in $C$, this means that $|\tau_x|(F)$ is discontinuous at every $x \in C$, a contradiction.

We conclude that $\tau_x$ is absolutely continuous, for every $x \in C$, so that

$$\lim_{|n| \to \infty} [1 - x^{r(n)}] = 0. \tag{6.9}$$

If $\{r(n)\}$ were unbounded, (6.9) would be false for almost all $x$ [12]. Thus $\{r(n)\}$ is bounded. Taking $x \notin R$, (6.9) now shows that $r(n) = 0$ except possibly for a finite set of values of $n$.

STEP 9. *The mapping $s$ satisfies condition* B 3.

The fact that $s$ satisfies B 2 (proved in Step 6) means that $s$ is linear on each residue class modulo $q$. Suppose $s(n+q) = s(n)$ for some $n$; then $s$ is constant on some residue class $H$. If $H \cap N$ is finite, then $s(n) = n$ for all $n \in H$, by B 1 and the way in which $t$ was extended. Hence $H \cap N$ is infinite, so that for some $n_0 \in N$ we have $t(n) = n_0$ for infinitely many $n \in N$. If $a(n_0) \neq 0$ in (5.2), the coefficients of (5.1) do not tend to 0 as $|n| \to \infty$. This contradiction shows that B 3 holds.

The proof of the necessity of the conditions A and B is now complete.

7. PROOF THAT THE CONDITIONS ARE SUFFICIENT. Suppose the conditions A and B hold, and extend $t$ to $J$ by defining $t(n) = s(n)$ if $n \notin N$. The transformation of (5.2) into (5.1) can be considered as the product of three transformations:

$$\sum_{n \in J} a(n) z^n \to \sum_{n \in J} a(s(n)) z^n, \tag{7.1}$$

$$\sum_{n \in J} a(s(n)) z^n \to \sum_{n \in J} a(t(n)) z^n, \tag{7.2}$$

$$\sum_{n \in J} a(t(n)) z^n \to \sum_{n \in N} a(t(n)) z^n. \tag{7.3}$$

Since $N$ is a $P$-set, (7.3) is nothing but convolution with an idempotent measure; since B 1 holds, (7.2) changes only a finite set of coefficients; hence $t$ transforms $L(C)$ into $L(C)$ if (7.1) transforms $L(C)$ into $L(C)$.

Condition B 2 means that $s$ is linear on each residue class modulo $q$; that is to say, there exist integers $b_1, \ldots, b_q$ and $c_1, \ldots, c_q$ such that

$$s(kq+j) = k c_j + b_j \qquad (j = 1, \ldots, q).$$

By B 3, $c_j \neq 0$.

The second series in (7.1) is therefore the sum of the $q$ series

$$\sum_{k \in J} a(k c_j + b_j) z^{kq+j} \qquad (j = 1, \ldots, q), \tag{7.4}$$

and it is sufficient to show that each of these is a Fourier series if the first series in (7.1) is the Fourier series of a function $f \in L(C)$.

To do this, fix $j$, put $b_j = b$, $c_j = c$, and define

$$f_1(z) = f(z) z^{-b} \sim \sum_{n \in J} a(n) z^{n-b}. \tag{7.5}$$

Denoting the $|c|$ distinct $c$th roots of $z$ by $\alpha_r(z)$, $r = 1, \ldots, |c|$, observe that

$$\sum_{r=1}^{|c|} \alpha_r(z)^{q(n-b)} = \begin{cases} |c| z^h & \text{if } q(n-b) = hc \text{ for some } h \in J, \\ 0 & \text{otherwise,} \end{cases}$$

and that the function

$$f_2(z) = \frac{1}{|c|} \sum_{r=1}^{|c|} f_1(\alpha_r(z)^q)$$

belongs to $L(C)$; the Fourier series of $f_2$ is consequently

$$\sum_{h \in H} a(q^{-1} hc + b) z^h,$$

where $H$ is the set of all $h \in J$ such that $q$ divides $hc$.

Let $\nu$ be the idempotent discrete measure defined by

$$d\nu(z) \sim \sum_{k \in J} z^{qk}.$$

Then if $f_3 = f_2 * \nu$, we have

$$f_3(z) \sim \sum_{k \in J} a(kc+b) z^{qk}.$$

Hence (7.4) is a Fourier series.

This completes the proof of Theorem I.

**8.** THEOREM II. *Let $t$ be a mapping of $N$ into $J$, with $N \subset J$. The following three statements are equivalent:*

(i) *$t$ carries $L(C)$ into $M(C)$.*

(ii) *Conditions* A, B 1, B 2 *hold.*

(iii) *$t$ carries $M(C)$ into $M(C)$.*

The statement (i) means, of course, that (5.1) is a Fourier-Stieltjes series whenever (5.2) is a Fourier series; (iii) has the obvious analogous meaning.

Note that Step 1 depended on the fact that the Cesàro means of the Fourier series of $Tf$ are bounded in the norm of $L(C)$; but this is equally true for Fourier-Stieltjes series ([13], p. 79). Hence if (i) holds, the assertion of Step 1 is still true, and the remainder of the proof of Theorem I, up to and including Step 8, is unaffected. Thus (i) implies (ii).

A very slight change in the argument of Section 7 proves that (ii) implies (iii), and it is trivial that (iii) implies (i).

Theorem II is proved.

**9.** ENDOMORPHISMS AND HOMOMORPHISMS. If $T$ is a mapping of $L(C)$ into $L(C)$ such that

$$\begin{cases} T(f+g) = Tf + Tg \\ T(\alpha f) = \alpha Tf \\ T(f * g) = Tf * Tg \end{cases} \tag{9.1}$$

for any $f$, $g \in L(C)$ and any complex number $\alpha$, then $T$ is an *endomorphism* of $L(C)$. If, for every $f \in L(C)$, we have $Tf \in M(C)$ and (9.1) holds, then $T$ is a *homomorphism* of $L(C)$ into $M(C)$. An endomorphism of $M(C)$ is a mapping of $M(C)$ into $M(C)$ which satisfies (9.1).

We mention in passing that these definitions are purely algebraic and that no continuity assumptions are made. However, if the argument of Section 7 (slightly modified if $M(C)$ is involved) is examined in detail, it will be seen that all mappings $T$ considered in this paper are actually continuous with respect to the norm topology. This could also be deduced from the general theory of semi-simple commutative Banach algebras ([8], pp. 76–77).

If the conditions of Theorem I hold, then the operator $T$ which associates with a function $f$ whose Fourier series is (5.2) the function $Tf$ whose Fourier series is (5.1) is evidently an endomorphism of $L(C)$. We next prove that there are no other endomorphisms of $L(C)$:

THEOREM III. *Every endomorphism $T$ of $L(C)$ is of the form*

$$\sum_{n \in J} a(n) z^n \rightarrow \sum_{n \in N} a(t(n)) z^n, \tag{9.2}$$

*where $N$ and $t$ satisfy the conditions* A *and* B *of Theorem* I.

To prove this, denote the coefficients of $Tf$ by $c_n(f)$. For each $n \in J$, $c_n$ is a homomorphism of $L(C)$ into the complex field. Let $N$ be the set of all $n$ for which $c_n$ is not the zero homomorphism. For each $n \in N$, $c_n(f)$ is then a Fourier coefficient of $f$ [[8], p. 136), so that there is an integer $t(n)$ for which

$$c_n(f) = \int_C x^{-t(n)} f(x)\, dm(x) \qquad (f \in L(C),\ n \in N). \tag{9.3}$$

That is to say that $T$ transforms the function $f$ whose Fourier series is (5.2) into the function $Tf$ whose Fourier series is (5.1), and Theorem III follows from Theorem I.

THEOREM IV. *If $T$ is a homomorphism of $L(C)$ into $M(C)$, then $T$ is of the form* (9.2), *where $N$ and $t$ satisfy the conditions* A, B 1, *and* B 2 *of Theorem* I.

*Every homomorphism of $L(C)$ into $M(C)$ can be extended to an endomorphism of $M(C)$. This extension is unique if and only if $N = J$.*

The proof of the first assertion is precisely the same as the proof of Theorem III, except that we replace the reference to Theorem I by a reference to Theorem II. Thus $T$ is given by (9.2), with A, B 1, B 2 holding, and Theorem II shows that (9.2) also induces an endomorphism of $M(C)$; this furnishes one extension of $T$ to $M(C)$.

To prove the last part of Theorem IV we let $T$ be an endomorphism of $M(C)$ and investigate the extent to which $T$ is determined by its action on $L(C)$. The restriction of $T$ to $L(C)$ is a homomorphism of $L(C)$ into $M(C)$, and is therefore of the form (9.2), with $N$ and $t$ subject to A, B 1, B 2.

For any $\mu \in M(C)$, denote the coefficients of $T\mu$ by $c_n(\mu)$. Then for every $n \in J$, $c_n$ is a homomorphism of $M(C)$ into the complex field.

Suppose now that $n \in N$. The restriction of $c_n$ to $L(C)$ is given by (9.3). Thus, if $\mu \in M(C)$ and $f \in L(C)$, we have

$$c_n (f * \mu) = c_n (f) \, c_n (\mu);$$

since $f * \mu$ is absolutely continuous, (9.3) applies, and

$$c_n (f * \mu) = \int_C x^{-t(n)} \, d (f * \mu) (x) = c_n (f) \cdot \int_C x^{-t(n)} \, d \mu (x).$$

Choosing an $f \in L(C)$ for which $c_n (f) \neq 0$, we conclude that

$$c_n (\mu) = \int_C x^{-t(n)} \, d \mu (x) \qquad (\mu \in M(C), \, n \in N).$$

In particular, there is only one extension of $T$ from $L(C)$ to $M(C)$ if $N = J$.

Let us now assume that $N$ is a proper subset of $J$. Since $L(C)$ is a closed ideal of the normed ring $M(C)$, $L(C)$ is contained in at least one maximal ideal of $M(C)$ ([8], p. 58), so that there exists a homomorphism of $M(C)$ *onto* the complex field which maps $L(C)$ into 0. If $A$ is any finite subset of the complement of $N$, the mapping

$$\sum_{n \in J} a(n) z^n \to \sum_{n \in N} a(t(n)) z^n + \sum_{n \in A} h_n(\mu) z^n, \tag{9.4}$$

where the $h_n$ are homomorphisms of the type just described, is an endomorphism of $M(C)$.

This completes the proof of Theorem IV.

A natural question to raise at this point concerns the restrictions one has to impose on the homomorphisms $h_n$ if the set $A$ of (9.4) is infinite; the answer would provide us with a complete description of the endomorphisms of $M(C)$. The homomorphisms of measure algebras on groups are discussed in [11].

**10.** AUTOMORPHISMS. An automorphism is an endomorphism which is one-to-one and onto.

THEOREM V. *If $t$ is a one-to-one mapping of $J$ onto $J$ which satisfies condition* B *of Theorem* I *(with $N = J$), then the mapping $T$ given by*

$$\sum_{n \in J} a(n) z^n \to \sum_{n \in J} a(t(n)) z^n \tag{10.1}$$

*is an automorphism of $L(C)$ (and of $M(C)$); every automorphism of $L(C)$ (and of $M(C)$) is obtained in this manner.*

Our previous results show that (10.1) is an endomorphism of $L(C)$ and of $M(C)$; since $t$ is one-to-one and onto, the inverse mapping of $t$ also satisfies B; it follows that $T$ is an automorphism.

The second part of the theorem can now be proved in several ways, of which the following is perhaps the simplest. The continuous characters of $C$ are the only idempotents in $L(C)$ and in $M(C)$ which are not sums of two non-zero idempotents. Since automorphisms preserve all algebraic properties and since $J$ is the dual group of $C$, every automorphism must be induced by a one-to-one mapping of $J$ onto $J$ and must therefore be of the form (10.1). The theorem follows.

COROLLARY. *Suppose $T$ is an automorphism of $M(C)$.*

(i) *If $\mu$ is absolutely continuous, so is $T\mu$.*

(ii) *If $\mu$ is a continuous measure, so is $T\mu$.*

(iii) *If $\mu$ has no singular component, then $T\mu$ has no singular component.*

Assertion (i) is part of the statement of Theorem V. A slight change in the argument of Section 7 shows that (7.1) carries continuous measures into continuous measures and discrete measures into discrete measures (it is again sufficient to consider the series (7.4)). The transformation (7.2) adds at most an absolutely continuous component. This proves (ii) and (iii).

The corollary shows that certain structural properties of measures are invariant under every automorphism of $M(C)$. Theorem V exhibits quite a large variety of such automorphisms. The trivial ones (i.e., those which are of the form (1.1)) are given by $t(n)=c+n$ and by $t(n)=c-n$, where $c$ is constant. To mention just one non-trivial case we consider the example

$$t(3k)=2k;\ t(3k+1)=4k+1;\ t(3k+2)=4k+3 \qquad (k\in J).$$

**11.** We conclude with a remark which concerns our method of proof and restrict ourselves for simplicity to the problem of finding the automorphisms.

It is known [2] that every automorphism $T$ of $L(C)$ can be extended to an automorphism of $M(C)$. The extended mapping carries idempotents into idempotents, so that the mapping $t$ of $J$ onto $J$ which induces $T$ carries $P$-sets into $P$-sets. It is tempting to try to deduce from this alone that $t$ satisfies condition B of Theorem I. The following example shows that this is impossible.

Let $E$ be the set consisting of the integers $n!$ and $-n!$, and denote the elements of $E$ by $k_i$ $(i\in J)$, in such a way that $k_{i-1}<k_i$. Define $t(n)=n$ if $n\in E$, and $t(k_i)=k_{i-1}$. Then $t$ is a one-to-one mapping of $J$ onto $J$ which evidently does not satisfy condition B. We shall show that the image of every $P$-set under $t$ (as well as under $t^{-1}$) is again a $P$-set.

To this end, let $H$ be a residue class (mod $p$) for some $p > 0$. If $H$ consists of all integers divisible by $p$ then all but a finite number of elements of $E$ are contained in $H$. If $H$ is any other residue class, then $H \cap E$ is finite. Consequently $t(H)$ differs from $H$ by at most a finite number of elements, and is therefore a $P$-set. Now, any periodic set with period $p$ is the union of a finite number of residue classes (mod $p$). This makes it evident that $t(S)$ is a $P$-set for every $P$-set $S$.

The assertion concerning $t^{-1}$ is obtained in the same manner, and implies that every $P$-set is the image, under $t$, of some $P$-set. Thus $t$ induces a one-to-one mapping of the set of all idempotent measures onto itself which cannot be extended to an automorphism of $M(C)$.

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