# NOTE ON A LEMMA OF FINN AND GILBARG

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In the preceding paper [1] it was shown by Finn and Gilbarg that if $u$ satisfies the elliptic differential equation

$$(a^{ij} u_{,i})_{,j} = 0 \tag{1}$$

in an $n$-dimensional domain $D$ containing the exterior $S_{r_0}$ of a sphere of radius $r_0$ and if the coefficients $a^{ij}$ and the solution $u$ behave suitably at infinity, then

$$A(r) = \int_{S_r} a^{ij} u_{,i} u_{,j} \, dV \tag{2}$$

exists for $r > r_0$ and

$$A(r) \leqslant A(r_0) \left(\frac{r_0}{r}\right)^{\lambda}, \tag{3}$$

where $\lambda = \min [(n-2), 2\sqrt{n-1}]$. In the case of Laplace's equation, it is easily seen that (3) holds with $\lambda$ replaced by $n-2$, which is greater than $\lambda$ for large $n$. Thus, (3) is not sharp.

We shall replace (3) by a sharp asymptotic estimate under the assumption that $a^{ij}$ approaches the unit matrix at infinity. As is pointed out in [1], if $a^{ij}$ approaches any positive definite matrix, one can make this limit the unit matrix by means of a coordinate transformation.

We define the two functions

$$a_0(r) = \inf_{\substack{\xi_1, \ldots, \xi_n \\ |x| \geq r}} \frac{\sum_{i,j} a^{ij}(x) \xi_i \xi_j}{\sum_{i=1}^{n} \xi_i^2} \tag{4}$$

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and

$$a_1(r) = \frac{1}{r^2} \sup_{|x|=r} \sum_{i,j} a^{ij} x^i x^j, \tag{5}$$

where $|x|$ is the length of the vector $x$.

We assume that $a^{ij}$ approaches the unit matrix in the sense that $a_0$ and $a_1$ approach 1 as $r \to \infty$.

Let $D_R(r)$ be the Dirichlet integral of $u$ over the annular region between spheres of radii $r$ and $R > r$. Clearly $D_R(r)$ increases with $R$, and by (4)

$$D(r) = \lim_{r \to \infty} D_R(r) \leqslant A(r)/a_0(r). \tag{6}$$

We now write

$$D_R(r) = \int_{S(1)} \left[ \int_r^R \left(\frac{\partial u}{\partial \varrho}\right)^2 \varrho^{n-1} d\varrho \right] d\Omega + \int_r^R \varrho^{n-3} \left[ \int_{S(1)} |\operatorname{grad}_\Omega u|^2 d\Omega \right] d\varrho, \tag{7}$$

where $S(1)$ is the surface of the unit sphere, $d\Omega$ its surface element, and $\operatorname{grad}_\Omega u$ the projection of the gradient of $u$ on the unit sphere.

$$|\operatorname{grad}_\Omega u|^2 = r^2 \left[ |\operatorname{grad} u|^2 - \left(\frac{\partial u}{\partial r}\right)^2 \right]. \tag{8}$$

By Schwarz's and Wirtinger's inequalities

$$\left.\begin{aligned} D_R(r) &\geqslant \int_{S(1)} \left[ \frac{\left( \int_r^R \frac{\partial u}{\partial \varrho} d\varrho \right)^2}{\int_r^R \frac{d\varrho}{\varrho^{n-1}}} \right] d\Omega + (n-1) \int_r^R \left[ \int_{S(1)} u^2 d\Omega - \frac{1}{\omega_n} \left( \int_{S(1)} u\, d\Omega \right)^2 \right] \varrho^{n-3} d\varrho \\ &\geqslant (n-2) r^{n-2} \int_{S(1)} [u(R) - u(r)]^2 d\Omega + (n-1) \int_r^R \left[ \int_{S(1)} u^2 d\Omega - \frac{1}{\omega_n} \left( \int_{S(1)} u\, d\Omega \right)^2 \right] \varrho^{n-3} d\varrho, \end{aligned}\right\} \tag{9}$$

where $\omega_n$ is the area of $S(1)$. Since both integrals on the right are non-negative, each is separately bounded by $D_R(r)$ and hence uniformly in $R$ by $D(r)$. By considering the first integral for $r$ sufficiently large and $R > r$, we see that $u(r)$ considered as a function on the unit sphere converges in the mean square sense as $r \to \infty$. This implies that the limit of $\int_{S(1)} u(r)\, d\Omega$ exists. By adding a suitable constant to $u$, we may make this limit zero. We suppose this to have been done.

The second integral on the right of (9) must converge, and hence its integrand must approach zero. Thus we find

$$\lim_{R\to\infty} \int_{S(1)} u^2(R)\, d\Omega = 0. \tag{10}$$

Neglecting the second term in (9) and letting $R\to\infty$ shows that

$$D(r) \geqslant (n-2)\, r^{n-2} \int_{S(1)} u^2(r)\, d\Omega = \frac{(n-2)}{r} \oint_{S(r)} u^2\, dS, \tag{11}$$

where $S(r)$ is the surface of the sphere of radius $r$.

It is easily seen from Schwarz's inequality together with (10) and the Dirichlet integrability of $u$ that

$$\lim_{R\to\infty} \oint_{S(R)} u\, a^{ij} u_{,i} \nu_j\, dS = 0. \tag{12}$$

Hence, using Schwarz's inequality

$$[A(r)]^2 = \left[\int_{S(r)} u\, a^{ij} u_{,i} \nu_j\, dS\right]^2 \leqslant a_1(r) \oint_{S(r)} a^{ij} u_{,i} u_{,j}\, dS \oint_{S(r)} u^2\, dS = -a_1(r) \frac{\partial A(r)}{\partial r} \oint_{S(r)} u^2\, dS.$$

Using (11) and (6) gives

$$[A(r)]^2 \leqslant \frac{-r\, a_1(r)}{(n-2)\, a_0(r)} \frac{\partial A}{\partial r} A(r). \tag{13}$$

This means that for $r > r_0$

$$A(r) \leqslant A(r_0) \exp\left\{-(n-2)\int_{r_0}^{r} \frac{a_0(\varrho)}{\varrho\, a_1(\varrho)}\, d\varrho\right\}. \tag{14}$$

Weakening this inequality, we find

$$A(r) \leqslant A(r_0) \left(\frac{r_0}{r}\right)^{(n-2) \inf\limits_{\varrho > r_0} \left(\frac{a_0(\varrho)}{a_1(\varrho)}\right)}. \tag{15}$$

If we assume that $a_0(\varrho)/a_1(\varrho) \to 1$ as $\varrho\to\infty$, the exponent in (15) is arbitrarily close to $n-2$ for sufficiently large $r_0$.

In general, (14) can also be written in the form

$$A(r) \leqslant A(r_0) \left(\frac{r_0}{r}\right)^{n-2} \exp\left\{(n-2)\int_{r_0}^{r} (1 - a_0/a_1)\, d\varrho/\varrho\right\} \tag{16}$$

which clearly displays the effect of the deviation of $a_0/a_1$ from one.

## Reference

[1]. R. Finn & D. Gilbarg, Three-dimensional subsonic flows, and asymptotic estimates for elliptic partial differential equations. *Acta Math.*, 98 (1957), 265—296.