SOLUTION OF A MIXED PROBLEM FOR A HYPERBOLIC DIFFERENTIAL EQUATION BY RIEMANN'S METHOD

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1. Introduction

This paper deals with a mixed initial and boundary value problem for a linear, hyperbolic partial differential equation of order n and with two independent variables. The values of the unknown function and its first $n-1$ normal derivatives are specified on an initial curve, and, in addition, the values of an appropriate number of normal derivatives are given on a boundary curve which intersects the initial curve. A solution of the differential equation which assumes the given initial and boundary values will be found by an extension of Riemann's well known solution of the initial value problem for a second order hyperbolic equation. The problem considered in the present paper is a special case of a mixed problem for which another method of solution has been given by Campbell and Robinson [2].

Hadamard [6, 7] adapted Riemann's method to deal with mixed problems for the second order equation. More recently, Bureau [1] and Durand [4] have treated mixed problems for second order equations by the same method. Rellich [9] has generalized Riemann's method to solve the initial value problem for linear, hyperbolic equations of order greater than two. In the present paper, a mixed problem for an equation of order greater than two is solved by an extension of the methods of Rellich and Hadamard. The complete existence proof will not be given here, but a method of obtaining the associated Riemann function will be outlined. A more complete proof is given in the author's thesis [3].

2. Preliminary results

The general linear partial differential equation of order n may be written

$$L[u] \equiv \sum_{k=0}^{n} \sum_{j=0}^{k} {}_kC_j A_{kj}(x, y) \frac{\partial^k u}{\partial x^{k-j} \partial y^j} = a_0(x, y), \tag{1}$$

where ${}_kC_j$ is the binominal coefficient, $k!/j!(k-j)!$.

We assume that equation (1) is hyperbolic in the region under consideration. Hence, if ξ and η are arbitrary parameters,

$$\sum_{k=0}^{n} {}_nC_k A_{nk} \xi^{n-k} \eta^k = \prod_{j=1}^{n} (p^j \xi + q^j \eta), \tag{2}$$

where p^j and q^j are real and

$$p^i q^j - p^j q^i \neq 0 \quad (i \neq j). \tag{3}$$

We assume further that (1) is normalized so that

$$(p^j)^2 + (q^j)^2 = 1 \quad (j = 1, 2, \ldots, n). \tag{4}$$

With this normalization, p^j and q^j are the direction cosines of the characteristic curves of (1).

Define the quantity Z^σ by

$$Z^\sigma = \sum_{k=0}^{n-2} {}_{n-2}C_k \frac{\partial^{n-2} w^\sigma}{\partial x^{n-2-k} \partial y^k} [p^\sigma q^\sigma A_{nk} + (q^\sigma)^2 A_{n,k+1} - (p^\sigma)^2 A_{n,k+1} - p^\sigma q^\sigma A_{n,k+2}] \quad (\sigma = 1, 2, \ldots, n), \tag{5}$$

where $w^\sigma(x, y)$ is a function which vanishes, together with all its derivatives of order $n-3$ and less, on the characteristic curve with direction cosines p^σ and q^σ. Rellich [9] has proved that the following formulas hold on this characteristic curve:

$$\sum_{k=0}^{n-2} {}_{n-2}C_k \frac{\partial^{n-2} w^\sigma}{\partial x^{n-2-k} \partial y^k} \left[q^\sigma A_{nk} \frac{\partial u}{\partial x} - p^\sigma A_{n,k+1} \frac{\partial u}{\partial x} + q^\sigma A_{n,k+1} \frac{\partial u}{\partial y} - p^\sigma A_{n,k+2} \frac{\partial u}{\partial y} \right] = \frac{\partial u}{\partial s_\sigma} Z^\sigma \quad (\sigma = 1, 2, \ldots, n), \tag{6}$$

$$\sum_{k=0}^{n-1} {}_{n-1}C_k \frac{\partial^{n-1} w^\sigma}{\partial x^{n-1-k} \partial y^k} (q^\sigma A_{nk} - p^\sigma A_{n,k+1})$$
$$= -(n-1) \sum_{k=0}^{n-2} {}_{n-2}C_k \frac{\partial^{n-2} w^\sigma}{\partial x^{n-2-k} \partial y^k} \frac{\partial}{\partial s_\sigma} [p^\sigma q^\sigma A_{nk} - (p^\sigma)^2 A_{n,k+1} + $$
$$+ (q^\sigma)^2 A_{n,k+1} - p^\sigma q^\sigma A_{n,k+2}] + (n-1) \frac{\partial Z^\sigma}{\partial s_\sigma} \quad (\sigma = 1, 2, \ldots, n), \tag{7}$$

and $$\frac{\partial^{n-2} w^\sigma}{\partial x^{n-2-k}\partial y^k} = (-1)^{n-1-k} n (p^\sigma)^k (q^\sigma)^{n-2-k} \frac{Z^\sigma}{D_\sigma} \quad (\sigma = 1, 2, \dots, n). \tag{8}$$

In (6) and (7) the derivative with respect to arc length on the characteristic curve, $p^\sigma \partial/\partial x + q^\sigma \partial/\partial y$, is denoted by $\partial/\partial s_\sigma$. In (8), D_σ is defined by

$$D_\sigma = \prod_{\substack{k=1\\ k\neq\sigma}}^{n} (p^\sigma q^k - p^k q^\sigma) \quad (\sigma = 1, 2, \dots, n). \tag{9}$$

Finally, Green's theorem for the plane shows that

$$\iint_G (v L[u] - u M[v])\, dx\, dy$$
$$= \int_\Gamma \Bigg\{ \sum_{m=1}^{n} \sum_{k=0}^{m-1} \sum_{i=0}^{k} \sum_{j=0}^{m-k-1} (-1)^{m-k} {}_kC_i \, {}_{m-k-1}C_j \frac{\partial^k u}{\partial x^{k-i}\partial y^i} \cdot$$
$$\cdot \left[\frac{\partial^{m-k-1}(A_{m,i+j+1} v)}{\partial x^{m-k-j-1}\partial y^j} dx - \frac{\partial^{m-k-1}(A_{m,i+j} v)}{\partial x^{m-k-j-1}\partial y^j} dy \right] \Bigg\}, \tag{10}$$

where $$M[v] = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^k {}_kC_j \frac{\partial^k (A_{kj} v)}{\partial x^{k-j}\partial y^j}. \tag{11}$$

In (10), G is a closed region in the xy-plane with the boundary Γ. The integral on Γ is to be taken in the counter-clockwise sense.

3. Application of Green's theorem to the mixed problem

In the mixed problem considered in this paper the initial curve, I, is the segment $0 \leqslant y \leqslant a$ of the y-axis, and the boundary curve, B, is the segment $0 \leqslant x \leqslant c$ of the x-axis. We assume that no characteristic curve is parallel to the y-axis and that no characteristic curve is tangent to the x-axis in the region under consideration. That is, we assume that $p^\sigma(x, y) \neq 0$ and $q^\sigma(x, 0) \neq 0$ for $\sigma = 1, 2, \dots, n$.

Let $q^\sigma(x, 0)/p^\sigma(x, 0)$ be negative for $\sigma = 1, 2, \dots, n-K$ and positive for $\sigma = n-K+1, \dots, n$. Thus, there are K characteristics of positive slope and $n-K$ characteristics of negative slope at each point of B. Let $0 < K < n$.

On I, the values of u, $\partial u/\partial x, \dots, \partial^{n-1} u/\partial x^{n-1}$ are given as functions of y. These functions are to be sufficiently differentiable with respect to y so that all the derivatives of u of order n or less are continuous functions of y on I. On B, the values of K of the quantities u, $\partial u/\partial y, \dots, \partial^{n-1} u/\partial y^{n-1}$ are given functions of x. These boundary values must also be differentiable often enough so that the derivatives

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of order n of u which can be formed from them are continuous functions of x. Moreover, we assume that the behavior of the initial values and the boundary values in the neighborhood of the origin is such that the derivatives of u of order n and less are continuous in the neighborhood of the origin. The reason for giving K boundary values is discussed elsewhere by Campbell and Robinson [2]. It is known from their results that a function u which satisfies the differential equation (1) and the initial and boundary conditions can be found in a region of the first quadrant of the xy-plane provided that the coefficients of (1) are sufficiently regular.

We assume that all the derivatives of the coefficients of equation (1) which appear in Green's formula, equation (10), possess continuous first derivatives with respect to x and y in the region under consideration.

Let R be a closed region in the first quadrant of the xy-plane with the following properties: If the point (x, y) is in R, it shall be possible to draw the n characteristic curves through (x, y) in the direction of decreasing x until they all intersect either I or B. The characteristics so drawn must remain in R. If the point is on B or I some or all of these characteristics may have zero length. Such a region R will be called a region of determinacy of $I+B$.

The application of Rellich's method to the mixed problem is complicated by the fact that not all of the derivatives of u which appear in Green's formula are known on the boundary. Thus the function v must be made to satisfy certain subsidiary conditions on the boundary in order to make the unknown terms disappear.

Let u be a solution of the mixed problem in a region, R, of determinacy of $I+B$. We wish to obtain an explicit representation of u at the point $P(x_0, y_0)$ in R

We draw the n characteristic curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ back from P to meet I or B at $P_1, P_2, \ldots, P_n$ and we suppose, for convenience, that the characteristics are numbered in order of increasing slope. At most K of the points P_i fall on B. From each of those points P_i which do fall on B we draw the $n-K$ characteristic curves which lead back to I. Let the characteristic curve leading from P_i on B to I and with direction cosines p^j, q^j be denoted by Γ^i_j. Let Γ^i_j meet I at P^i_j (Fig. 1). These curves break up the region bounded by Γ_1, Γ_n, I and B into a finite number o regions. Let $v(x, y)$ by a solution of the adjoint equation,

$$M[v]=0, \tag{12}$$

in each of these regions, with continuous derivatives of order n. On Γ_1 and Γ_n, $v(x, y)$ and its derivatives of order less than $n-2$ are to vanish. In the interior of the region bounded by Γ_1, Γ_n, B and I, v and its derivatives of order less than $n-2$

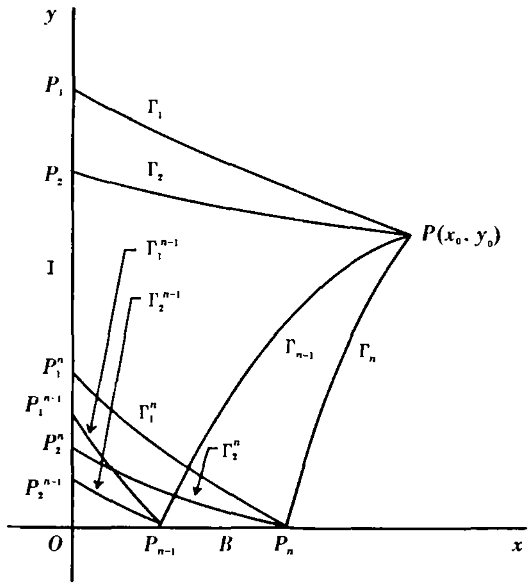

Fig. 1.

are to be continuous everywhere, but the derivatives of order $n-2$ and greater will have discontinuities across each of the characteristic curves Γ_i and Γ_j^i.

We now apply Green's formula (10) to each of the regions formed by the various characteristics and add the results. The left hand side contributes the integral J_s, where

$$J_s = \iint_{PP_1 OP_n P} v a_0 \, dx\, dy. \tag{13}$$

This is a known quantity when v is known. The various integrals over I contribute J_I, where

$$J_I = \int_O^{P_1} \sum_{m=1}^{n} \sum_{k=0}^{m-1} \sum_{i=0}^{k} \sum_{j=0}^{m-k-1} (-1)^{m-k} {}_kC_i \, {}_{m-k-1}C_j \frac{\partial^k u}{\partial x^{k-i} \partial y^i} \frac{\partial^{m-k-1} (A_{m,i+j} v)}{\partial x^{m-k-j-1} \partial y^j} \, dy. \tag{14}$$

J_I is also known when v is known. The integrals on B contribue J_B, where

$$J_B = \int_O^{P_n} \sum_{m=1}^{n} \sum_{k=0}^{m-1} \sum_{i=0}^{k} \sum_{j=0}^{m-k-1} (-1)^{m-k} {}_kC_i \, {}_{m-k-1}C_j \frac{\partial^k u}{\partial x^{k-i} \partial y^i} \frac{\partial^{m-k-1} (A_{m,i+j+1} v)}{\partial x^{m-k-j-1} \partial y^j} \, dx. \tag{15}$$

Let C_μ represent a segment of one of the characteristic curves drawn from P, P_n, P_{n-1}, etc. Let (x_1, y_1) be a point on C_μ. Then we define $w^\mu(x, y)$ by

$$w^{\mu}(x_1, y_1) = v(x_1, y_1 - 0) - v(x_1, y_1 + 0), \tag{16}$$

where $v(x_1, y_1 - 0)$ and $v(x_1, y_1 + 0)$ denote limits of v as (x_1, y_1) is approached from below and above respectively. On Γ_1 and Γ_n, w^1 and w^n are defined by $w^1 = v$ and $w^n = -v$ respectively. Partial derivatives of w^{μ} will denote the corresponding differences of derivatives of the function v on the two sides of C_{μ}. According to the assumptions made about v, w^{μ} and all its derivatives of order $n-3$ or less vanish on C_{μ}.

Then the integral on C_{μ} which results from the application of Green's theorem is given by

$$J_{\mu} = \int_{C_{\mu}} \Bigg\{ (-1)^{n-2} \sum_{k=0}^{n-2} {}_{n-2}C_k \left[q^{\mu} \frac{\partial u}{\partial x} \frac{\partial^{n-2}(A_{nk} w^{\mu})}{\partial x^{n-2-k} \partial y^k} - p^{\mu} \frac{\partial u}{\partial x} \frac{\partial^{n-2}(A_{n,k+1} w^{\mu})}{\partial x^{n-2-k} \partial y^k} + \right.$$
$$\left. + q^{\mu} \frac{\partial u}{\partial y} \frac{\partial^{n-2}(A_{n,k+1} w^{\mu})}{\partial x^{n-2-k} \partial y^k} - p^{\mu} \frac{\partial u}{\partial y} \frac{\partial^{n-2}(A_{n,k+2} w^{\mu})}{\partial x^{n-2-k} \partial y^k} \right] +$$
$$+ u \sum_{m=n-1}^{n} \sum_{k=0}^{m-1} (-1)^{m-1} {}_{m-1}C_k \left[q^{\mu} \frac{\partial^{m-1}(A_{mk} w^{\mu})}{\partial x^{m-1-k} \partial y^k} - p^{\mu} \frac{\partial^{m-1}(A_{m,k+1} w^{\mu})}{\partial x^{m-1-k} \partial y^k} \right] \Bigg\} ds. \tag{17}$$

The remainder of the terms disappears because w^{μ} and its derivatives of order $n-3$ and less vanish on C_{μ}. The direction of increasing s on C_{μ} has been chosen so that $dx/ds = p^{\mu}$ and $dy/ds = q^{\mu}$, where p^{μ} and q^{μ} are the direction cosines of C_{μ}.

Thus, the application of Green's formula yields the result

$$J_s = J_I + J_B + \Sigma J_{\mu}, \tag{18}$$

where J_s and J_I are known quantities.

The integrals J_{μ} can be evaluated in terms of the values of u at the end-points of C_{μ} provided that v satisfies certain conditions on C_{μ}. Let P'_{μ} and P''_{μ} be the end-points of C_{μ}, where P'_{μ} is the point with the larger abscissa. Then Rellich [9] has shown that

$$J_{\mu} = (-1)^n [Z^{\mu}(P''_{\mu}) u(P''_{\mu}) - Z^{\mu}(P'_{\mu}) u(P'_{\mu})], \tag{19}$$

provided that
$$n \frac{\partial Z^{\mu}}{\partial s_{\mu}} = \sum_{k=0}^{n-2} {}_{n-2}C_k E_k^{\mu} \frac{\partial^{n-2} w^{\mu}}{\partial x^{n-2-k} \partial y^k}, \tag{20}$$

on C_{μ}, where

$$E_k^{\mu} = (n-1) \frac{\partial}{\partial s_{\mu}} [p^{\mu} q^{\mu} A_{nk} - (p^{\mu})^2 A_{n,k+1} + (q^{\mu})^2 A_{n,k+1} - p^{\mu} q^{\mu} A_{n,k+2}] -$$
$$- (n-1) \left[q^{\mu} \left(\frac{\partial A_{nk}}{\partial x} + \frac{\partial A_{n,k+1}}{\partial y} \right) - p^{\mu} \left(\frac{\partial A_{n,k+1}}{\partial x} + \frac{\partial A_{n,k+2}}{\partial y} \right) \right] +$$
$$+ q^{\mu} A_{n-1,k} - p^{\mu} A_{n-1,k+1}. \tag{21}$$

Equation (19) results from the use of (6) and (7) and an integration by parts. Z^{μ} is defined by (5). Equation (8) shows that (20) is an ordinary linear first-order differential equation for Z^{μ} on C_{μ}.

4. Intersections of characteristics in the interior of the region

Consider a point P_0 which is in the interior of the region bounded by I, B, Γ_1 and Γ_n and which is the intersection of N of the characteristic curves which were drawn through the region. It will now be shown that there is no contribution to ΣJ_{μ} from this point. In fact, it will be shown that the functions Z^{μ} associated with the characteristics which cross at the intersection are continuous across the intersection.

We demonstrate first that $N < n$. In view of inequality (3) it is clear that two different characteristic curves cannot intersect twice. Thus the N characteristic curves must originate at N different points from among the point P and the points P_n, P_{n-1}, etc. on the boundary. There are, at most, K such points on the boundary. Now $K < n$ by hypothesis. If $K < n-1$ then there are at most $n-1$ points from which characteristics could be drawn to meet at P_0 and in this case $N < n$. If $K = n-1$ there is only one family of characteristic curves with negative slopes on B. Since two members of this family drawn from different points can never meet, N can only have the value 2 in this case. Since we are concerned with the case $n > 2$ we again have $N < n$. Thus, in all cases, $2 \leqslant N < n$.

Let the segments C'_{μ_i} and C''_{μ_i} $(i = 1, 2, \ldots, N)$ meet at P_0, where C'_{μ_i} and C''_{μ_i} are two segments of the same characteristic curve with direction cosines p^{μ_i} and q^{μ_i}. Let $Z^{\mu_i\prime}$, $Z^{\mu_i\prime\prime}$, $w^{\mu_i\prime}$ and $w^{\mu_i\prime\prime}$ be the corresponding functions on C'_{μ_i} and C''_{μ_i}. The situation is illustrated in Figure 2.

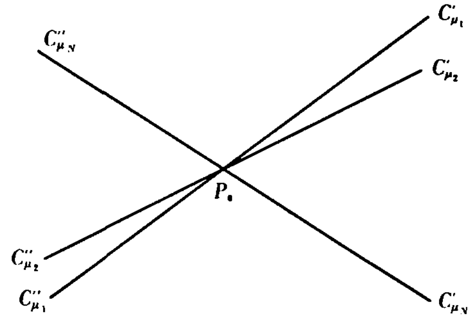

Fig. 2.

Since $w^{\mu_i\prime}$ and $w^{\mu_i\prime\prime}$ are the differences of the function v across the characteristics C'_{μ_i} and C''_{μ_i} it is easily seen that, at P_0,

$$\sum_{i=1}^{N}\left(\frac{\partial^{n-2} w^{\mu_i\prime}}{\partial x^k \partial y^{n-2-k}} - \frac{\partial^{n-2} w^{\mu_i\prime\prime}}{\partial x^k \partial y^{n-2-k}}\right) = 0 \quad (k=0, 1, \ldots, n-2). \tag{22}$$

Since the derivatives of w^μ of order $n-3$ vanish on C_μ it follows that

$$p^\mu \frac{\partial^{n-2} w^\mu}{\partial x^{k+1} \partial y^{n-3-k}} + q^\mu \frac{\partial^{n-2} w^\mu}{\partial x^k \partial y^{n-2-k}} = 0 \quad (k=0, 1, \ldots, n-3), \tag{23}$$

and hence that

$$\frac{\partial^{n-2} w^\mu}{\partial x^k \partial y^{n-2-k}} = \left(-\frac{q^\mu}{p^\mu}\right)^k \frac{\partial^{n-2} w^\mu}{\partial y^{n-2}} \quad (k=0, 1, \ldots, n-2). \tag{24}$$

If we substitute (24) into (22) we obtain, at P_0,

$$\sum_{i=1}^{N}\left(\frac{q^{\mu_i}}{p^{\mu_i}}\right)^k \left[\frac{\partial^{n-2} w^{\mu_i\prime}}{\partial y^{n-2}} - \frac{\partial^{n-2} w^{\mu_i\prime\prime}}{\partial y^{n-2}}\right] = 0 \quad (k=0, 1, \ldots, n-2). \tag{25}$$

Equations (25) are a system of $n-1$ linear homogeneous equations for the N variables $\partial^{n-2} w^{\mu_i\prime}/\partial y^{n-2} - \partial^{n-2} w^{\mu_i\prime\prime}/\partial y^{n-2}$, where $N \leqslant n-1$. It is easily shown that when inequality (3) holds these equations have only the solution

$$\frac{\partial^{n-2} w^{\mu_i\prime}}{\partial y^{n-2}} = \frac{\partial^{n-2} w^{\mu_i\prime\prime}}{\partial y^{n-2}}. \tag{26}$$

It then follows from (8) that $\qquad Z^{\mu_i\prime}(P_0) = Z^{\mu_i\prime\prime}(P_0). \qquad (27)$

Thus Z^μ has no discontinuity at the intersection P_0.

Finally, from (19) and (27), the contribution to ΣJ_μ from the point P_0 is zero. Thus, there is no contribution to ΣJ_μ from points of intersection of characteristics in the interior of the region bounded by I, B, Γ_1, and Γ_n.

There will, however, be a contribution to ΣJ_μ from the points P_n, P_{n-1}, etc. on B. If u is given on B, this contribution is a known function and is of no further concern. However, when u is not given on B, these contributions must be considered further.

5. Conditions on the boundary

Before considering the contributions to ΣJ_μ on B, we consider the boundary integral J_B. On the boundary, K of the quantities u, $\partial u/\partial y$, $\ldots$, $\partial^{n-1} u/\partial y^{n-1}$ are known. The general plan in the treatment of J_B is to integrate by parts on B until all the differentiations of u with respect to x are removed. Then the coefficients of the unknown derivatives of u with respect to y are set equal to zero. This gives some further conditions on v. Finally, the terms at the points P_n, P_{n-1}, etc., which arise from the integration by parts, are combined with the terms from ΣJ_μ on the boundary. Then the coefficient of u at these points is also set equal to zero if u is not given on B. This gives the last of the conditions to be applied to v.

Let

$$I=\int_{P'}^{P''}\sum_{m=1}^{n}\sum_{k=0}^{m-1}\sum_{i=0}^{k}\sum_{j=0}^{m-k-1}(-1)^{m-k}{}_kC_i\,{}_{m-k-1}C_j\frac{\partial^k u}{\partial x^{k-i}\partial y^i}\frac{\partial^{m-k-1}(A_{m,\,i+j+1}v)}{\partial x^{m-k-j-1}\partial y^j}dx, \tag{28}$$

where P' and P'' are two points on B such that v and its derivatives are continuous on $P'P''$. When I is integrated by parts and the order of summation is altered the result is

$$I=\int_{P'}^{P''}\sum_{m=1}^{n}\sum_{i=0}^{m-1}(-1)^{m-i}\frac{\partial^i u}{\partial y^i}\sum_{j=0}^{m-i-1}{}_mC_{i+j+1}\frac{\partial^{m-i-1}(A_{m,\,i+j+1}v)}{\partial x^{m-i-j-1}\partial y^j}dx+$$
$$+\left[\sum_{m=1}^{n}\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-2}\sum_{k=i+1}^{m-j-1}\sum_{l=0}^{k-i-1}(-1)^{m-k+l}{}_kC_i\,{}_{m-k-1}C_j\frac{\partial^{k-l-1}u}{\partial x^{k-i-l-1}\partial y^i}\frac{\partial^{m-k+l-1}(A_{m,\,i+j+1}v)}{\partial x^{m-k-j+l-1}\partial y^j}\right]_{P'}^{P''}. \tag{29}$$

The identity

$$\sum_{k=i}^{m-j-1}({}_kC_i)({}_{m-k-1}C_j)={}_mC_{i+j+1}\quad(0\leqslant i+j\leqslant m-1) \tag{30}$$

has been used in deriving (29). A proof of this identity is outlined by Feller [5].

Now the integration on B from O to P_n may be broken up into a sum of integrals on a finite number of segments on which v and its derivatives of order up to n are continuous. Thus, if we integrate by parts on each segment, we obtain

$$J_B=\int_{O}^{P_n}\left\{\sum_{i=0}^{n-1}(-1)^{n-i}\frac{\partial^i u}{\partial y^i}\left[\sum_{j=0}^{n-i-1}{}_nC_{i+j+1}\frac{\partial^{n-i-1}(A_{n,\,i+j+1}v)}{\partial x^{n-i-j-1}\partial y^j}+R_i\right]\right\}dx+S, \tag{31}$$

where R_i is a linear combination of derivatives of v of order less than $n-i-1$ and S depends on the values of u and v at the points O, P_n, P_{n-1}, etc. on B.

Now K of the values $\partial^i u/\partial y^i$ $(i=0, 1, \dots, n-1)$ are known functions of x on B. We require that the coefficients of the other $n-K$ derivatives $\partial^i u/\partial y^i$ vanish on B. That is, if $\partial^k u/\partial y^k$ is not given on B, we require that

$$\sum_{j=0}^{n-k-1} {}_nC_{j+k+1} \frac{\partial^{n-k-1}(A_{n,j+k+1}v)}{\partial x^{n-j-k-1}\partial y^j} + R_k = 0, \tag{32}$$

where R_k is a linear combination of derivatives of v of order less than $n-k-1$.

6. Intersections of characteristics with the boundary

Finally, we must consider the remaining terms at the points P_n, P_{n-1}, etc. on B. These terms result from the integrations by parts on the boundary B and on the characteristics which intersect at the points P_n, P_{n-1}, etc. We observe, from (29), that integration by parts introduces no unknown functions at O because u and all its derivatives of order up to $n-1$ are known on the initial segment I.

Let α characteristic curves drawn from P meet the boundary, B. Then the points at which these curves intersect B are $P_{n-\alpha+1}$, $P_{n-\alpha+2}$, ..., P_n where $\alpha \leqslant K$ (see Fig. 1). From each of these points P_m $(m=n-\alpha+1, \ldots, n)$ the $n-K$ characteristics Γ_j^m $(j=1, 2, \ldots, n-K)$ with direction cosines p^j and q^j are drawn.

Let us now fix our attention on the point P_m on B. Since it has been shown that the function Z^μ, associated with the segment C_μ of a characteristic curve, is continuous across an intersection with another characteristic curve in the interior of the region, we may denote the functions Z^μ by Z^i and Z_j^m which are associated with Γ_i $(i=1, 2, \ldots, n)$ and Γ_j^m $(m=n-\alpha+1, \ldots, n;\ j=1, 2, \ldots, n-K)$ respectively. Then P_m is at the intersection of Γ_m, the characteristic drawn from P, and the characteristics Γ_j^m $(j=1, 2, \ldots, n-K)$, the characteristics drawn from P_m to I (see Fig. 3).

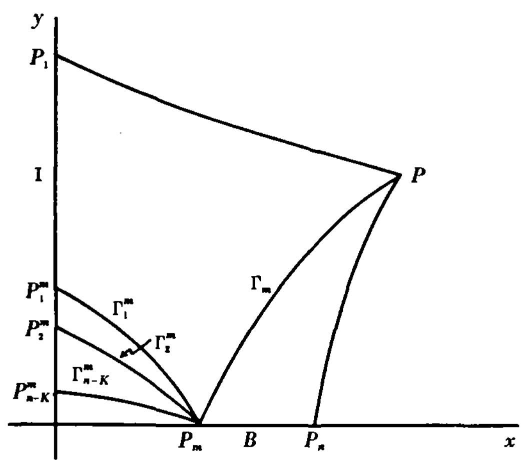

Fig. 3.

Then, from equation (19), the contribution at P_m due to integration by parts along the characteristics is

$$(-1)^n u(P_m)\left[Z^m(P_m)-\sum_{j=1}^{n-K} Z_j^m(P_m)\right]. \tag{33}$$

From (29), the contribution due to integration by parts on B is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-i-2}\sum_{k=i+1}^{n-j-1}\sum_{l=0}^{k-i-1}(-1)^{n-k+l}({}_kC_i)({}_{n-k-1}C_j)\frac{\partial^{k-l-1}u(P_m)}{\partial x^{k-i-l-1}\partial y^i}T_{ijkl}, \tag{34}$$

where

$$T_{ijkl}=\left[\frac{\partial^{n-k+l-1}(A_{n,i+j+1}v)}{\partial x^{n-k-j+l-1}\partial y^j}+U_{ijkl}\right]_{P_m+0}^{P_m-0}. \tag{35}$$

In (35), U_{ijkl} denotes a linear combination of derivatives of v of order less than $n-k+l-1$, and P_m-0 and P_m+0 denote limits as P_m is approached from the left and from the right respectively.

It follows from (16) that

$$v(P_m-0)-v(P_m+0)=-w^m(P_m)+\sum_{j=1}^{n-K} w_j^m(P_m). \tag{36}$$

A similar relation holds for the derivatives of v. But the derivatives of w^m and w_j^m of order less than $n-2$ were assumed to vanish on Γ_m and Γ_j^m. Hence, at P_m, only the derivatives of order $n-2$ in T_{ijkl} in (35) play any part. Thus U_{ijkl} plays no part and only those derivatives for which $n-k+l-1=n-2$ need be considered.

Hence, we may simplify the expression (34) considerably. Since $l=k-1$, we must have $i=0$. Thus the terms at P_m due to integration by parts on the boundary, which do not drop out, are given by

$$\sum_{j=0}^{n-2}(-1)^{n+1}\,{}_{n-1}C_{j+1}\,u(P_m)\,A_{n,j+1}(P_m)\left[\frac{\partial^{n-2}v}{\partial x^{n-2-j}\partial y^j}\right]_{P_m+0}^{P_m-0}. \tag{37}$$

The identity

$$\sum_{k=1}^{n-j-1}{}_{n-k-1}C_j={}_{n-1}C_{j+1}\quad (j=0,1,\ldots,n-2) \tag{38}$$

has been used in deriving (37). This identity is the special case of equation (30) with $i=0$.

If P_m is the point P_n, which is the point at which the integration on B ends, the same result is obtained. In this case we may consider that v is identically zero outside the region to which Green's theorem was applied. Then all the statements made apply to this special case.

From (36) it is seen that

$$\left[\frac{\partial^{n-2} v}{\partial x^{n-2-j}\partial y^j}\right]_{P_m+0}^{P_m-0} = \sum_{i=1}^{n-K} \frac{\partial^{n-2} w_i^m (P_m)}{\partial x^{n-2-j}\partial y^j} - \frac{\partial^{n-2} w^m (P_m)}{\partial x^{n-2-j}\partial y^j}. \tag{39}$$

From (8), we have that

$$\left[\frac{\partial^{n-2} v}{\partial x^{n-2-j}\partial y^j}\right]_{P_m+0}^{P_m-0} = \sum_{i=1}^{n-K} (-1)^{n-j-1}\, n\,(p^i)^j (q^i)^{n-j-2} \frac{Z_i^m}{D_i} + \\ + (-1)^{n-j}\, n\,(p^m)^j (q^m)^{n-j-2} \frac{Z^m}{D_m} \quad (j=0,\,1,\,\dots,\,n-2). \tag{40}$$

Then from (33), (37), and (40) it follows that the contribution at P_m, $F_m(P_m)\,u(P_m)$ say, which is due to integration by parts on the characteristics and the boundary, is

$$F_m(P_m)\,u(P_m) = u(P_m)\Biggl\{\Biggl[(-1)^n + \sum_{j=0}^{n-2} (-1)^{j+1}\,{}_{n-1}C_{j+1}\,A_{n,\,j+1} \frac{n\,(p^m)^j (q^m)^{n-j-2}}{D_m}\Biggr] Z^m - \\ - \sum_{i=1}^{n-K} \Biggl[(-1)^n + \sum_{j=0}^{n-2} (-1)^{j+1}\,{}_{n-1}C_{j+1}\,A_{n,\,j+1} \frac{n\,(p^i)^j (q^i)^{n-j-2}}{D_i}\Biggr] Z_i^m\Biggr\}. \tag{41}$$

If u is given on the boundary then $F_m(P_m)\,u(P_m)$ is a known quantity when v is known. If u is not given on the boundary we require that $F_m(P_m)$ vanish at P_m.

Finally, we make $Z^1, Z^2, \dots, Z^n$ satisfy

$$\sum_{r=1}^{n} Z^r(P) = (-1)^n. \tag{42}$$

This completes the set of conditions which v must satisfy.

7. Explicit representation of the solution

Let $\varepsilon_k = 1$ if $\partial^k u/\partial y^k$ is given on B and let $\varepsilon_k = 0$ if $\partial^k u/\partial y^k$ is not given on B. Further, let

$$J'_B = \int_0^{P_n} \sum_{m=1}^{n} \sum_{i=0}^{m-1} (-1)^{m-i} \varepsilon_i \frac{\partial^i u}{\partial y^i} \sum_{j=0}^{m-i-1} {}_mC_{i+j+1} \frac{\partial^{m-i-1}(A_{m,\,i+j+1}\,v)}{\partial x^{m-i-j-1}\partial y^j}\,dx - \\ - \Biggl[\sum_{m=1}^{n} \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-2} \sum_{k=i+1}^{m-j-1} \sum_{l=0}^{k-i-1} (-1)^{m-k+l}\,{}_kC_i\,{}_{m-k-1}C_j \times \\ \times \frac{\partial^{k-l-1} u}{\partial x^{k-l-i-1}\partial y^i} \frac{\partial^{m-k+l-1}(A_{m,\,i+j+1}\,v)}{\partial x^{m-k-j+l-1}\partial y^j}\Biggr]_{x=y=0}. \tag{43}$$

It will be seen from (15), (28) and (29) that J'_B is what remains of J_B after the integrations by parts are performed and the values at the points $P_{n-\alpha+1}, \ldots, P_n$ are removed. J'_B contains only known functions.

We can now express $u(P)$ explicitly in terms of known functions. Because of (42) we have, from (18),

$$u(P) = -J_s + J_I + J'_B + \sum_{m=1}^{n-\alpha} (-1)^n Z^m(P_m)\, u(P_m) + \varepsilon_0 \sum_{m=n-\alpha+1}^{n} F_m(P_m)\, u(P_m) + \\ + \sum_{j=1}^{n-K} \sum_{m=n-\alpha+1}^{n} (-1)^n Z_j^m(P_j^m)\, u(P_j^m). \quad (44)$$

As before, $\varepsilon_0 = 1$ if u is given on B and $\varepsilon_0 = 0$ if u is not given on B. The points $P_1, P_2, \ldots, P_{n-\alpha}$ are the intersections with I of characteristic curves drawn from P. The points $P_{n-\alpha+1}, \ldots, P_n$ are the intersections with B of characteristic curves drawn from P. The points P_j^m are the intersections with I of characteristic curves drawn from P_m $(m = n-\alpha+1, \ldots, n)$. J_s, J_I, J'_B and $F_m(P_m)$ are given by (13), (14), (43) and (41) respectively.

8. Discontinuities of v on the characteristics

The Riemann function, $v(x, y)$ may be found by a modification of the method used by Campbell and Robinson [2] to solve the mixed problem. To see this, we must examine the conditions which $v(x, y)$ must satisfy on the characteristic curves and on the boundary.

The first step is to show that the jumps of the derivatives of v of order $n-2$ are known across all the characteristics Γ_i and Γ_j^i. This means that we must show that all the functions Z^i and Z_j^i, are known on the corresponding characteristics. According to (8), all the derivatives of w^i and w_j^i of order $n-2$ are multiples of Z^i and Z_j^i, and the derivatives of w^i and w_j^i are just the jumps of the derivatives of v across the corresponding characteristics. Equation (20) can be written

$$n \frac{\partial Z^\mu}{\partial s_\mu} + K^\mu Z^\mu = 0. \quad (45)$$

where K^μ is a function of the coefficients of equation (1). Hence each function Z^i and Z_j^i is a solution of an ordinary, first-order, linear differential equation. In order to specify Z^i and Z_j^i completely, the values of these functions must be known at the points P and P_i $(i = n-\alpha+1, \ldots, n)$.

At P, Rellich [9] has shown that

$$Z^i(P) = (-1)^n/n \quad (i = 1, 2, \dots, n). \tag{46}$$

Equation (46) is a consequence of (42) and the definition of the functions w^i. The functions $Z^i\,(i = 1, 2, \dots, n)$ are completely specified by (45) and (46).

The functions Z^i_j, which correspond to characteristics leading from points on the boundary to the initial segment, also satisfy the differential equations (45). It will now be shown that their values of the points P_i on the boundary are specified by the conditions imposed on v.

It is convenient to define new quantities $A_r^{(i)}$ by the equation

$$\prod_{\substack{j=1\\ j\neq i}}^{n} (p^j \xi + q^j \eta) = \sum_{r=0}^{n-1} A_r^{(i)} \xi^{n-r-1} \eta^r \quad (i = 1, 2, \dots, n). \tag{47}$$

We must now derive some identities connecting the functions $A_r^{(i)}$, and the coefficients, A_{nr}, of the differential equation. It is clear from (2) that

$$\sum_{r=0}^{n} {}_nC_r A_{nr} \xi^{n-r} \eta^r = (p^i \xi + q^i \eta) \sum_{r=0}^{n-1} A_r^{(i)} \xi^{n-r-1} \eta^r \quad (i = 1, 2, \dots, n). \tag{48}$$

From (48) we deduce that

$$p^i A_0^{(i)} = A_{no}, \tag{49}$$

$$p^i A_r^{(i)} + q^i A_{r-1}^{(i)} = {}_nC_r A_{nr} \quad (r = 1, 2, \dots, n-1), \tag{50}$$

and

$$q^i A_{n-1}^{(i)} = A_{nn}. \tag{51}$$

Also, if we put $\xi = -q^i$ and $\eta = p^i$ in (47), we have, from the definition of D_i in equation (9),

$$D_i = \sum_{r=0}^{n-1} A_r^{(i)} (p^i)^r (-q^i)^{n-r-1} \quad (i = 1, 2, \dots, n). \tag{52}$$

It is easily shown from these equations that

$$\sum_{j=0}^{n-k-1} {}_nC_{j+k+1} A_{n,\,j+k+1} (-1)^{n-j-1} (p^i)^j (q^i)^{n-2-j} = (-q^i)^{n-1} A_k^{(i)}$$
$$(i = 1, 2, \dots, n;\ k = 0, 1, \dots, n-1), \tag{53}$$

and that

$$D_i + \sum_{j=0}^{n-2} (-1)^{n-j-1}\, n\, {}_{n-1}C_{j+1} A_{n,\,j+1} (p^i)^j (q^i)^{n-2-j} = n\,(-q^i)^{n-1} A_0^{(i)} \quad (i = 1, 2, \dots, n). \tag{54}$$

Equation (53) may be verified by using (50) and (51) to substitute for ${}_nC_{j+k+1} A_{n,\,j+k+1}$. Equation (54) results from the use of the identity

$$n\,({}_{n-1}C_{j+1}) = (n-j-1)\,({}_nC_{j+1}) \tag{55}$$

and the use of (50) and (52) to substitute for $A_{n,j+1}$ and D_i. Equations (53) and (54) are the identities connecting $A_k^{(i)}$ and A_{nk} which it will be necessary to use.

Let us consider first the case where u is given on the boundary. Then, at all points of continuity of v and its derivatives on the boundary, we have $n-K$ conditions of the form

$$\sum_{j=0}^{n-k-1} {}_nC_{j+k+1} \frac{\partial^{n-k-1}(A_{n,j+k+1}v)}{\partial x^{n-j-k-1}\partial y^j} + R_k = 0, \tag{32}$$

where k is an integer between 1 and $n-1$ inclusive, and R_k is a linear combination of derivatives of v of order less than $n-k-1$. Next, differentiate this expression $k-1$ times with respect to x to obtain

$$\sum_{j=0}^{n-k-1} {}_nC_{j+k+1} A_{n,j+k+1} \frac{\partial^{n-2} v}{\partial x^{n-j-2}\partial y^j} + R_k' = 0, \tag{56}$$

where R_k' contains only derivatives of v of order less than $n-2$.

We now concentrate our attention on the point P_m of Figure 3. Since the derivatives of v of order less than $n-2$ are continuous everywhere in the region under consideration, we have, from (56),

$$\left[\sum_{j=0}^{n-k-1} {}_nC_{j+k+1} A_{n,j+k+1} \frac{\partial^{n-2} v}{\partial x^{n-j-2}\partial y^j}\right]_{P_m+0}^{P_m-0} = 0. \tag{57}$$

From (40) it then follows that

$$\sum_{i=1}^{n-K} \sum_{j=0}^{n-k-1} {}_nC_{j+k+1} A_{n,j+k+1} (-1)^{n-j-1} (p^i)^j (q^i)^{n-2-j} Z_i^m / D_i$$
$$= \sum_{j=0}^{n-k-1} {}_nC_{j+k+1} A_{n,j+k+1} (-1)^{n-j-1} (p^m)^j (q^m)^{n-j-2} Z^m / D_m, \tag{58}$$

or, because of (53),

$$\sum_{i=1}^{n-K} (q^i)^{n-1} \frac{A_k^{(i)} Z_i^m}{D_i} = (q^m)^{n-1} A_k^{(m)} \frac{Z^m}{D_m}. \tag{59}$$

Since k takes on $n-K$ values of the integers from 1 to $n-1$, and since Z^m is a known function, (59) is a system of $n-K$ linear equations for the $n-K$ quantities $Z_i^m(P_m)$ $(i=1, 2, \dots, n-K)$. It will be shown presently that the determinant of this system does not vanish.

Before showing that (59) possesses a solution we will consider the case when u is not given on the boundary. In this instance one of equations (32) contains derivatives of order $n-1$. This is the case $k=0$ which was excluded earlier. Such a boundary condition is not suitable for determining the discontinuities in the derivatives of order $n-2$ at P_m. Thus, there are only $n-K-1$ equations in the system (59) and we need one more to specify the functions Z_i^m uniquely. This is provided by (41). It was assumed that, if u is not given on the boundary, then the coefficient of u in (41) vanishes. This yields the equation

$$\sum_{i=1}^{n-K} \frac{(q^i)^{n-1}}{D_i} A_0^{(i)} Z_i^m = (q^m)^{n-1} \frac{A_0^{(m)}}{D_m} Z^m. \tag{60}$$

Equation (54) was used to simplify (41) and hence (60).

Equation (60) is just what would result if k were allowed to equal zero in (59). Thus, whether or not u is given on the boundary, we have the $n-K$ equations

$$\sum_{i=1}^{n-K} (q^i)^{n-1} \frac{A_k^{(i)} Z_i^m (P_m)}{D_i} = (q^m)^{n-1} \frac{A_k^{(m)} Z^m (P_m)}{D_m}, \tag{59}$$

where k takes on $n-K$ values from 0 to $n-1$. If we regard these as equations for the $n-K$ quantities $(q^i)^{n-1} Z_i^m/D_i$ the determinant of the equation is

$$\Delta_1 = |A_{k_i}^{(j)}| \quad (i, j = 1, 2, \ldots, n-K), \tag{61}$$

where $k_1, k_2, \ldots, k_{n-K}$ are the $n-K$ values which k assumes. This determinant may also be written as

$$\Delta_1 = \pm \begin{vmatrix} A_0^{(1)} & A_0^{(2)} \ldots A_0^{(n-K)} & 0 \ldots 0 \\ A_1^{(1)} & A_1^{(2)} \ldots A_1^{(n-K)} & 0 \ldots 0 \\ & & 0 \\ & & 1 \\ \cdot & \cdot \quad\quad \cdot & 0 \ldots 0 \\ & & 1 \quad \cdot \\ & & 0 \quad \cdot \\ & & \cdot \quad \cdot \\ A_{n-1}^{(1)} & A_{n-1}^{(2)} \ldots A_{n-1}^{(n-K)} & 0 \ldots 0 \end{vmatrix}, \tag{62}$$

where each of the last K columns contains $n-1$ zeros and a one. There will be a one in the rth row of one of these columns if the functions $A_r^{(1)}, A_r^{(2)}, \ldots, A_r^{(n-K)}$ do not appear in Δ_1 (i.e. if $\partial^r u/\partial y^r$ is given on the boundary).

In order to demonstrate that the determinant Δ_1 does not vanish, we pre-multiply it by the non-vanishing determinant Δ_2, whose element in the rth row and jth column is $(p^r)^j\ (-q^r)^{n-j-1}$, where $r=1, 2, \ldots, n$ and $j=0, 1, \ldots, n-1$. Now it follows from equation (47) that

$$\sum_{j=0}^{n-1} (p^r)^j (-q^r)^{n-j-1} A_j^{(i)} = \delta_{ri} D_i \quad (i, r=1, 2, \ldots, n), \tag{63}$$

where δ_{ri} is the Kronecker delta. Therefore

$$\Delta_1 \Delta_2 = \pm D_1 D_2 \ldots D_{n-K} \,|(p^r)^j (-q^r)^{n-j-1}| \quad (r=n-K+1, n-K+2, \ldots, n;\ j=a_1, a_2, \ldots, a_K), \tag{64}$$

where $a_1, a_2, \ldots, a_K$ are K integers between 0 and $n-1$. The exponents a_i are the orders of the derivatives of u which are given on the boundary. Let

$$\gamma_r = q^r/p^r \quad (r=1, 2, \ldots, n). \tag{65}$$

Then γ_r is the slope of the characteristic curve with direction cosines p^r and q^r. Thus

$$\Delta_1 \Delta_2 = \pm D_1 D_2 \ldots D_{n-K} (p^{n-K+1} p^{n-K+2} \ldots p^n)^{n-1} \,|(\gamma_r)^{n-j-1}| \quad (r=n-K+1, \ldots, n;\ j=a_1, a_2, \ldots, a_K). \tag{66}$$

Now γ_r is, by hypothesis, positive on the boundary for $r=n-K+1, \ldots, n$ and the exponents $n-j-1$ are K different integers chosen from among $0, 1, \ldots, n-1$. Under these conditions, it can be shown that the determinant $|(\gamma_r)^{n-j-1}|$ does not vanish. One proof of this has been given by Campbell and Robinson [2] in connection with their solution of the mixed problem. The non-vanishing character of the determinant $|(\gamma_r)^{n-j-1}|$ can also be deduced from a theorem due to Rosenbloom [8, Theorem 4] on symmetric polynomials.

From the definition of D_r in (9) and from inequality (3) it will be seen that D_r does not vanish for $r=1, 2, \ldots, n$. Moreover, it was assumed earlier that $p^j(x, y) \neq 0$ for $j=1, 2, \ldots, n$. Therefore Δ_1 does not vanish and hence (59) has an unique solution $(q^i)^{n-1} Z_i^m/D_i$ $(i=1, 2, \ldots, n-K)$. Since q^i does not vanish on the boundary, $Z_i^m(P_m)$ is determined by (59).

Since this result holds for all points P_m and since the functions Z_i^m satisfy the differential equation (45), all the functions $Z_i^m(x, y)$ are known. Therefore, the jumps in the derivatives of order $n-2$ of v are known across every characteristic in the region.

9. Determination of the Riemann function

In this section we outline a method by which $v(x, y)$ may be determined. A more complete existence proof may be found in the author's thesis [3]. The procedure is based on Robinson's [10] solution of the initial value problem and on the solution by Campbell and Robinson [2] of the mixed problem. The reader is referred to these papers for many details which are omitted here.

The function $v(x, y)$ must satisfy the adjoint equation, (12), in the interior of each region formed by the characteristics which have been drawn. On the characteristics Γ_1 and Γ_n, which form part of the boundary of the region R, v and its first $n-2$ derivatives are known. Across the characteristics in the interior of R, the jumps of the derivatives of order $n-2$ of v are known. The lower order derivatives are continuous across these characteristics. On the boundary, B, v must satisfy the $n-K$ conditions (32).

We define $f_m(x, y)$ by

$$f_m(x, y) = \sum_{k=0}^{n-1} A_k^{(m)}(x, y) \frac{\partial^{n-1} v}{\partial x^{n-1-k} \partial y^k} + \sum_{k=0}^{n-2} \sum_{l=0}^{k} b_{kl}^{(m)}(x, y) \frac{\partial^k v}{\partial x^{k-l} \partial y^l} \quad (m = 1, 2, \ldots, n). \tag{67}$$

In (67), $b_{kl}^{(m)}(x, y)$ is an undetermined function of x and y, while $A_k^{(m)}$ is defined by (47). Then, if $v(x, y)$ satisfies the adjoint equation (12) and the functions $b_{kl}^{(m)}(x, y)$ are chosen properly, the functions $f_m(x, y)$ satisfy the linear system of first-order equations

$$p^m \frac{\partial f_m}{\partial x} + q^m \frac{\partial f_m}{\partial y} = \sum_{k=1}^{n} b_{mk} f_k \quad (m = 1, 2, \ldots, n). \tag{68}$$

In (68), b_{mk} is a known function of the coefficients of the differential equation and of the functions $b_{kl}^{(m)}$. The condition which $b_{kl}^{(m)}$ must satisfy in order that (68) hold is that it shall satisfy the first-order equations

$$p^m \frac{\partial b_{kl}^{(m)}}{\partial x} + q^m \frac{\partial b_{kl}^{(m)}}{\partial y} = G_{kl}^{(m)}. \tag{69}$$

The function $G_{kl}^{(m)}$ is a quadratic function of the coefficients $b_{kl}^{(m)}$. Explicit expressions for $G_{kl}^{(m)}$ and b_{mk} are given by Robinson [10].

The system (68) is a hyperbolic system with the same characteristic curves as (1) and (12). The quantity

$$p^m \frac{\partial f_m}{\partial x} + q^m \frac{\partial f_m}{\partial y}$$

is the derivative of f_m with respect to arc length along a characteristic curve with direction cosines p^m, q^m.

Now, from (47) it can be seen that

$$\sum_{k=0}^{n-1} A_k^{(m)} \frac{\partial^{n-1} v}{\partial x^{n-1-k} \partial y^k} = \left(p^1 \frac{\partial}{\partial x} + q^1 \frac{\partial}{\partial y}\right) \left[\prod_{\substack{j=2 \\ j \neq m}}^{n} \left(p^j \frac{\partial}{\partial x} + q^j \frac{\partial}{\partial y}\right) v\right] + L_m \quad (m = 2, 3, \ldots, n). \tag{70}$$

where L_m is a linear combination of derivatives of v of order $n-2$ and less. Moreover, the expression in the square brackets on the right hand side of (70) is itself a linear combination of derivatives of v of order $n-2$ and less. Since, on Γ_1, v and its derivatives of order $n-2$ and less are known functions which are differentiable with respect to arc length, the right hand side of (70) is known on Γ_1. It follows therefore, from (67), that $f_m(x, y)$ is determined on Γ_1 for $m = 2, 3, \ldots, n$. Similarly, f_m can be calculated on Γ_n for $m = 1, 2, \ldots, n-1$. It also follows, by much the same reasoning, that we can calculate the jump of $f_1, f_2, \ldots, f_{m-1}, f_{m+1}, \ldots, f_n$ across a characteristic Γ_m or Γ_m^j with direction cosines p^m, q^m. It is not necessary to know f_1 on Γ_1, because in determining f_1 we integrate the equation

$$p^1 \frac{\partial f_1}{\partial x} + q^1 \frac{\partial f_1}{\partial y} = \sum_{k=1}^{n} b_{1k} f_k$$

along characteristics with direction cosines p^1, q^1. Since none of these curves can intersect Γ_1, knowledge of f_1 on Γ_1 is unnecessary. Similar remarks apply to f_n on Γ_n and f_m on Γ_m or Γ_m^j.

Finally, $f_1, f_2, \ldots, f_{n-K}$ must be determined on B in terms of $f_{n-K+1}, \ldots, f_n$. This is done with the aid of the boundary conditions (32) and a suitable choice of the boundary conditions for $b_{kl}^{(m)}$. The method is exactly the same as that used in [2]. It is not difficult to show that the determinant D of that paper [2, equation (7)] does not vanish for the boundary conditions (32).

The method of determining $f_1, f_2, \ldots, f_n$ is then as follows: From a point P' in the interior of R, the n characteristics are drawn in the direction of increasing x to meet Γ_1, Γ_n or B. Each of the equations (68) is then integrated with respect to arc length along the corresponding characteristic. This yields the Volterra-type system of integral equations

$$f_m(P') = f_{m0}(P'_m) + \sum_\alpha J_{m\alpha} + \int_{P'}^{P'_m} \sum_{k=1}^{n} b_{mk} f_k \, ds \quad (m = 1, 2, \ldots, n). \tag{71}$$

The points P'_m are the intersections of the characteristics with Γ_1, Γ_n or B. The values $f_{m0}(P'_m)$ are the values of f_m which have been calculated on Γ_1, Γ_n or B. If P'_m is on B, $f_{m0}(P'_m)$ may be given in terms of other functions $f_{n-K+1}(P'_m), \ldots, f_n(P'_m)$ for which similar integral equations may be written. The functions $J_{m\alpha}$ are the jumps of f_m as it crosses the characteristics Γ_r and Γ_r^s. These jumps are known. The integral equations (71) can be solved by the method of successive approximation under conditions which are similar to those usually required for Picard's method to be applied.

The functions $b_{kl}^{(m)}(x, y)$ satisfy a similar set of integral equations and may be found in the same way. In the case of these functions there is no discontinuity across the interior characteristics and their values on Γ_1 and Γ_n may be chosen more or less arbitrarily.

The above reasoning follows the discussion of the mixed problem [2] quite closely. The main differences are that Γ_1 and Γ_n have replaced the initial segment of that discussion and that discontinuities across characteristics have been introduced.

Finally, once $f_m(x, y)$ is known, equation (67) is a hyperbolic equation of order $n-1$ for v. This equation can be treated in the same way. Because all the derivatives and jumps in derivatives of order $n-2$ or less are given, the functions corresponding to f_m can be determined. Thus, the order of the equation is successively reduced until we have a first-order equation for v which can be solved. Once v is known, equation (44) gives an explicit representation of the solution to the mixed problem which was originally posed.

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References

[1]. F. Bureau, Les séries de fonctions fondamentales et les problèmes aux limites pour les équations aux dérivées partielles linéaires hyperboliques. *Acta Math.*, 89 (1953), 1–43.

[2]. L. L. Campbell & A. Robinson, Mixed problems for hyperbolic partial differential equations. *Proc. London Math. Soc.* (3) 5 (1955), 129–147.

[3]. L. L. Campbell, Mixed boundary value problems for hyperbolic differential equations. Thesis, University of Toronto, 1955.

[4]. E. Durand, Recherche des solutions de l'équation des ondes planes. *Annales de la Faculté des Sciences de l'Université de Toulouse* (4) 17 (1953), 229–264.

[5]. W. Feller, *An Introduction to Probability Theory and Its Applications,* Vol. I, p. 48. John Wiley and Sons, Inc., New York, 1950.

[6]. J. Hadamard, Sur un problème mixte aux dérivées partielles. *Bull. Soc. Math. France* 31 (1903), 208–214.

[7]. ——, Résolution d'un problème aux limites pour les équations linéaires du type hyperbolique. *Bull. Soc. Math. France,* 32 (1904), 242–268.

[8]. L. Kuipers & B. Meulenbeld, Symmetric polynomials with non-negative coefficients. *Proc. Amer. Math. Soc.*, 6 (1955), 88–93.

[9]. F. Rellich, Verallgemeinerung der Riemannschen Integrationsmethode auf Differentialgleichungen n-ter Ordnung in zwei Veränderlichen. *Math. Ann.*, 103 (1930), 249–278.

[10]. A. Robinson, On the integration of hyperbolic differential equations. *J. London Math. Soc.*, 25 (1950), 209–217.