ON A THEOREM OF DAVENPORT AND HEILBRONN

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1. Let $\lambda_1, \ldots, \lambda_5$ be any 5 real numbers, not all of the same sign and none of them 0. It was proved by Davenport and Heilbronn [7] that, for any $\varepsilon > 0$, the inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < \varepsilon \tag{1}$$

is soluble in integers $x_1, \ldots, x_5$, not all 0. The mention of ε here is in reality superfluous, for if the solubility of

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < 1 \tag{2}$$

is proved for all $\lambda_1, \ldots, \lambda_5$, then the solubility of (1) for any $\varepsilon > 0$ follows, on applying (2) with λ_j replaced by $\varepsilon^{-1}\lambda_j$.

Our object in the present paper is to make this result more precise by giving an estimate for a solution of (2) in terms of $\lambda_1, \ldots, \lambda_5$. It would not be easy to do this with much precision by following the original line of argument, which depended on considering the continued fraction development of one of the ratios λ_i/λ_j.

The result we shall prove is as follows.

THEOREM. *For $\delta > 0$ there exists C_δ with the following property. For any real $\lambda_1, \ldots, \lambda_5$, not all of the same sign and all of absolute value* 1 *at least, there exist integers $x_1, \ldots, x_5$ which satisfy both* (2) *and*

$$0 < |\lambda_1| x_1^2 + \cdots + |\lambda_5| x_5^2 < C_\delta |\lambda_1 \ldots \lambda_5|^{1+\delta}. \tag{3}$$

In another paper [1] we have applied this result to general indefinite quadratic forms. We have proved that any real indefinite quadratic form in 21 or more variables assumes values arbitrarily near to 0, provided that when the form is expressed as a sum of squares of real linear forms with positive and negative signs, there are either not more than 4 positive signs or not more than 4 negative signs.

An important part in the proof of the present theorem is played by a result of Cassels [3] which gives an estimate for the magnitude of a solution of a homogeneous quadratic equation with integral coefficients. We use Cassels's result in the modified form which we have published recently [2]; this modified form is essentially the same as the special case of our present theorem when $\lambda_1, \dots, \lambda_5$ are integers, but without the parameter δ. It is remarkable that the result obtained here when $\lambda_1, \dots, \lambda_5$ are real is almost as good as that known for the special case when $\lambda_1, \dots, \lambda_5$ are integers.

It may be of interest to remark that the use of Cassels's result is not, in principle, essential. The special case of the theorem in which $\lambda_1, \dots, \lambda_5$ are integers could be proved analytically, and we could then use this in place of Cassels's result in the present work.

As a corollary to the theorem, we note that if $\lambda_1, \dots, \lambda_5$ are fixed, and ε is an arbitrarily small positive number, there exist solutions of (1) with $x_1, \dots, x_5$ all $O(\varepsilon^{-2-\delta})$ for any fixed $\delta > 0$.

2. Throughout the paper $\lambda_1, \dots, \lambda_5$ are real, not all of the same sign, and all of absolute value 1 at least. Let

$$\Lambda = \max |\lambda_j|. \tag{4}$$

Let P be a positive number with the property that the inequality (2) has no solution in integers $x_1, \dots, x_5$ satisfying

$$0 < |\lambda_1| x_1^2 + \dots + |\lambda_5| x_5^2 \leqslant 500 P^2. \tag{5}$$

We shall ultimately deduce a contradiction when P is taken to be of the form $C_\delta |\lambda_1 \dots \lambda_5|^{(1+\delta)/2}$, with a suitable (large) C_δ, and this will prove the theorem. But for the time being (until Lemma 6) we make only the weaker supposition that $P\Lambda^{-\frac{1}{2}}$ is large (i.e. greater than a suitable absolute constant). This supposition is plainly essential from the nature of (5).

We define exponential sums $S_1(\alpha), \dots, S_5(\alpha)$ by

$$S_j(\alpha) = \sum_{P < |\lambda_j|^{\frac{1}{2}} x_j < 10P} e(\alpha \lambda_j x_j^2), \tag{6}$$

where $e(\theta) = e^{2\pi i \theta}$.

LEMMA 1. *There exists, for any positive integer n, a real function $K(\alpha)$ of the positive variable α, satisfying*

$$|K(\alpha)| < C(n) \min(1, \alpha^{-n-1}), \tag{7}$$

with the following property. Let

$$\psi(\theta) = \Re \int_0^\infty e(\theta\alpha) K(\alpha)\, d\alpha. \tag{8}$$

Then

$$0 \leqslant \psi(\theta) \leqslant 1 \quad \textit{for all real } \theta, \tag{9}$$

$$\psi(\theta) = 0 \quad \textit{for } |\theta| \geqslant 1, \tag{10}$$

$$\psi(\theta) = 1 \quad \textit{for } |\theta| \leqslant \tfrac{1}{3}. \tag{11}$$

For a proof, see Lemma 1 of a paper by Davenport [4].

COROLLARY. *We have*

$$\Re \int_0^\infty S_1(\alpha) \ldots S_5(\alpha) K(\alpha)\, d\alpha = 0. \tag{12}$$

Proof. By (8) and (6), the left hand side is

$$\sum_{x_1} \ldots \sum_{x_5} \psi(\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2),$$

where the summations are over the intervals occurring in the sums (6). By (10) and the hypothesis that (2) has no solutions satisfying (5), the sum is 0.

3. We define $I(\alpha)$ by

$$I(\alpha) = \int_P^{10P} e(\alpha\xi^2)\, d\xi. \tag{13}$$

LEMMA 2. *For*

$$0 < \alpha < \frac{1}{40} P^{-1} |\lambda_j|^{-\frac{1}{2}} \tag{14}$$

we have

$$S_j(\alpha) = |\lambda_j|^{-\frac{1}{2}} I(\pm\alpha) + O(1), \tag{15}$$

where the $\pm$ *sign is the sign of* λ_j.

Proof. This is a special case of van der Corput's lemma ([8], Chapter I, Lemma 13). If $f(x) = \alpha|\lambda_j|x^2$, then for $P|\lambda_j|^{-\frac{1}{2}} < x < 10P|\lambda_j|^{-\frac{1}{2}}$ we have

$$f''(x) > 0, \qquad 0 < f'(x) < \tfrac{1}{2},$$

by (14). Hence

$$S_j(\alpha) = \int_{P|\lambda_j|^{-\frac{1}{2}}}^{10P|\lambda_j|^{-\frac{1}{2}}} e(\alpha \lambda_j \xi^2)\, d\xi + O(1),$$

and (15) follows on changing the variable of integration.

LEMMA 3. *For $\alpha > 0$ we have* [1]

$$|I(\pm\alpha)| \ll \min(P, P^{-1}\alpha^{-1}). \tag{16}$$

Proof. The estimate P is obvious from the definition (13). The second estimate follows from the alternative representation

$$I(\alpha) = \frac{1}{2} \int_{P^2}^{100P^2} \eta^{-\frac{1}{2}} e(\alpha\eta)\, d\eta$$

and the second mean value theorem.

LEMMA 4. *We have*

$$\Re \int_0^{\frac{1}{40}P^{-1}\Lambda^{-\frac{1}{2}}} S_1(\alpha) \dots S_5(\alpha) K(\alpha)\, d\alpha = M + R, \tag{17}$$

where
$$M > \frac{1}{6} P^3 |\lambda_1 \dots \lambda_5|^{-\frac{1}{2}}, \qquad |R| \ll P^2 \Lambda^{\frac{1}{2}} |\lambda_1 \dots \lambda_5|^{-\frac{1}{2}}. \tag{18}$$

Proof. In the interval of integration, the condition (14) of Lemma 2 is satisfied for $j = 1, \dots, 5$, and so (15) holds for $j = 1, \dots, 5$. By (16),

$$|\lambda_j|^{-\frac{1}{2}} |I(\pm\alpha)| \ll |\lambda_j|^{-\frac{1}{2}} \min(P, P^{-1}\alpha^{-1}),$$

and the right hand side is $\gg 1$ for all α in the range of integration. Hence (15) gives

$$\left| \prod_{j=1}^{5} S_j(\alpha) - |\lambda_1 \dots \lambda_5|^{-\frac{1}{2}} \prod_{j=1}^{5} I(\pm\alpha) \right| \ll \left(\sum |\lambda_2 \dots \lambda_5|^{-\frac{1}{2}}\right) \min(P^4, P^{-4}\alpha^{-4}),$$

where the summation is over all selections of four suffixes from 1, ..., 5. Obviously

$$\sum |\lambda_2 \dots \lambda_5|^{-\frac{1}{2}} \ll \Lambda^{\frac{1}{2}} |\lambda_1 \dots \lambda_5|^{-\frac{1}{2}},$$

[1] The notation $\ll$ indicates an inequality with an unspecified constant factor. In general, these constants are absolute until Lemma 7, after which they may depend on δ. There is an obvious exception when an unspecified constant occurs in a hypothesis, as in Lemma 9.

by (4). Hence

$$\int_0^{\frac{1}{40}P^{-1}\Lambda^{-\frac{1}{2}}} S_1(\alpha)\,\dots\,S_5(\alpha)\,K(\alpha)\,d\alpha = |\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}} \int_0^{\frac{1}{40}P^{-1}\Lambda^{-\frac{1}{2}}} I(\pm\alpha)\,\dots\,I(\pm\alpha)\,K(\alpha)\,d\alpha$$

$$+\,O\left\{\Lambda^{\frac{1}{2}}\,|\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}}\int_0^{\infty}\min\,(P^4,\,P^{-4}\alpha^{-4})\,d\alpha\right\},$$

and the last error term is $\quad O(\Lambda^{\frac{1}{2}}\,|\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}}\,P^2)$

and can be absorbed in R by (18). Thus it suffices to consider

$$\Re\,|\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}} \int_0^{\frac{1}{40}P^{-1}\Lambda^{-\frac{1}{2}}} I(\pm\alpha)\,\dots\,I(\pm\alpha)\,K(\alpha)\,d\alpha,$$

where the signs are those of $\lambda_1, \dots, \lambda_5$.

The error introduced by extending the integral to ∞ is

$$\ll |\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}} \int_{\frac{1}{40}P^{-1}\Lambda^{-\frac{1}{2}}}^{\infty} P^{-5}\alpha^{-5}\,d\alpha$$

$$\ll |\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}}\,P^{-5}P^4\,\Lambda^2$$

$$\ll P^2\,\Lambda^{\frac{1}{2}}\,|\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}},$$

since $P > \Lambda^{\frac{1}{2}}$.

It remains to give a lower bound for

$$M = \Re\,|\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}} \int_0^{\infty} I(\pm\alpha)\,\dots\,I(\pm\alpha)\,K(\alpha)\,d\alpha. \tag{19}$$

By (13) and (8), we have

$$M = |\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}} \int_P^{10P} \dots \int_P^{10P} \psi(\pm\xi_1^2 \pm \dots \pm \xi_5^2)\,d\xi_1\,\dots\,d\xi_5$$

$$= 2^{-5}\,|\lambda_1\,\dots\,\lambda_5|^{-\frac{1}{2}} \int_{P^2}^{100P^2} \dots \int_{P^2}^{100P^2} \psi(\pm\eta_1 \pm \dots \pm\eta_5)\,(\eta_1\,\dots\,\eta_5)^{-\frac{1}{2}}\,d\eta_1\,\dots\,d\eta_5.$$

We can suppose without loss of generality that the sign attached to η_1 is $+$ and that attached to η_2 is $-$. The region defined by

$$P^2 < \eta_3 < 4P^2, \quad P^2 < \eta_4 < 4P^2, \quad P^2 < \eta_5 < 4P^2, \quad 14P^2 < \eta_2 < 87P^2,$$

$$|\eta_1 - \eta_2 \pm \eta_3 \pm \eta_4 \pm \eta_5| < \frac{1}{3}$$

is contained in the region of integration. Hence, by (9) and (11),

$$M > 2^{-5}|\lambda_1 \ldots \lambda_5|^{-\frac{1}{2}} \frac{2}{3}(100P^2)^{-\frac{1}{2}} \int_{14P^2}^{87P^2} \eta_2^{-\frac{1}{2}} d\eta_2 \left\{ \int_{P^2}^{4P^2} \eta^{-\frac{1}{2}} d\eta \right\}^3 > \frac{1}{6} P^3 |\lambda_1 \ldots \lambda_5|^{-\frac{1}{2}}.$$

This completes the proof.

4. Lemma 5. *We have*

$$\int_0^{|\lambda_j|^{-1}} |S_j(\alpha)|^4 \, d\alpha \ll |\lambda_j|^{-2} P^2 \log P. \tag{20}$$

Proof. Putting $\alpha = |\lambda_j|^{-1}\theta$, we see that the integral on the left is

$$|\lambda_j|^{-1} \int_0^1 |\sum_x e(\theta x^2)|^4 \, d\theta,$$

where the summation is over

$$P < |\lambda_j|^{\frac{1}{2}} x < 10P. \tag{21}$$

Now

$$\int_0^1 |\sum_x e(\theta x^2)|^4 \, d\theta$$

is the number of solutions of $x^2 + y^2 = z^2 + w^2$, where x, y, z, w all range over the interval (21). This number does not exceed

$$\sum_{n<N} r^2(n), \tag{22}$$

where $r(n)$ is the number of representations of n as a sum of two integral squares, and

$$N = 200P^2|\lambda_j|^{-1}.$$

It is well known that the sum (22) is $\ll N \log N$, and the estimate (20) follows from this.

Let

$$\lambda = \min |\lambda_j|, \tag{23}$$

$$\Pi = \prod_{j=1}^{5} |\lambda_j|. \tag{24}$$

LEMMA 6. *Provided* $P > \Pi^{\frac{1}{2}}$, *we have*

$$\int_{\frac{1}{40}P^{-1}\Lambda^{-\frac{1}{2}}}^{\frac{1}{40}P^{-1}\lambda^{-\frac{1}{2}}} |S_1(\alpha) \dots S_5(\alpha)|\, d\alpha \ll \Pi^{-\frac{1}{2}} P^2 (\log P)\Lambda^{\frac{1}{2}}. \tag{25}$$

Proof. We suppose, during this proof only, that $|\lambda_1| \leqslant |\lambda_2| \leqslant \dots \leqslant |\lambda_5|$, so that $\Lambda = |\lambda_5|$ and $\lambda = |\lambda_1|$. We split up the interval of integration into the 4 intervals

$$I_k: \frac{1}{40}P^{-1}|\lambda_k|^{-\frac{1}{2}} < \alpha < \frac{1}{40}P^{-1}|\lambda_{k-1}|^{-\frac{1}{2}}, \tag{26}$$

where $k = 2, \dots, 5$. In (26), the condition (14) of Lemma 2 is satisfied provided $j \leqslant k-1$. Thus, for α in I_k, (15) and (16) give

$$|S_j(\alpha)| \ll |\lambda_j|^{-\frac{1}{2}} P^{-1}\alpha^{-1} + 1 \ll |\lambda_j|^{-\frac{1}{2}} P^{-1}\alpha^{-1} \tag{27}$$

for $j \leqslant k-1$. For $j \geqslant k$ we use merely the trivial estimate $|S_j(\alpha)| \ll P|\lambda_j|^{-\frac{1}{2}}$. Hence, in I_k,

$$|S_1(\alpha) \dots S_5(\alpha)| \ll \Pi^{-\frac{1}{2}} (P^{-1}\alpha^{-1})^{k-1} P^{5-(k-1)}.$$

Thus, provided $k \geqslant 3$,

$$\int_{I_k} |S_1(\alpha) \dots S_5(\alpha)|\, d\alpha \ll \Pi^{-\frac{1}{2}} P^{5-2(k-1)} (P^{-1}|\lambda_k|^{-\frac{1}{2}})^{-k+2}$$
$$= \Pi^{-\frac{1}{2}} P^3 (P^{-1}|\lambda_k|^{\frac{1}{2}})^{k-2} \ll \Pi^{-\frac{1}{2}} P^3 P^{-1} \Lambda^{\frac{1}{2}}.$$

There remains the case $k=2$, corresponding to the interval I_2. Here (27) is still valid for $j=1$, and implies

$$|S_1(\alpha)| \ll |\lambda_1|^{-\frac{1}{2}} |\lambda_2|^{\frac{1}{2}}. \tag{28}$$

For the remaining factors we use Lemma 5. For $j = 2, 3, 4$ we have

$$|\lambda_j|^{-1} \geqslant |\lambda_j \lambda_5|^{-\frac{1}{2}} \geqslant \Pi^{-\frac{1}{2}} > P^{-1} > \frac{1}{40} P^{-1} |\lambda_1|^{-\frac{1}{2}}.$$

Hence I_2 is contained in the interval $0 < \alpha < |\lambda_j|^{-1}$ of Lemma 5, and

$$\int_{I_2} |S_j(\alpha)|^4\, d\alpha \ll |\lambda_j|^{-2} P^2 \log P. \tag{29}$$

For $j = 5$ this argument fails, but as $S_5(\alpha)$ is periodic with period $|\lambda_5|^{-1}$, we can say that

$$\int_{I_2} |S_5(\alpha)|^4 \, d\alpha \ll |\lambda_5|^{-2} P^2 (\log P) \{1 + P^{-1} |\lambda_1|^{-\frac{1}{2}} |\lambda_5|\}. \tag{30}$$

The estimate (29) for $j=2, 3, 4$ and the estimate (30) imply, by Hölder's inequality, that

$$\int_{I_2} |S_2(\alpha) \ldots S_5(\alpha)| \, d\alpha \ll |\lambda_2 \ldots \lambda_5|^{-\frac{1}{2}} P^2 (\log P) \{1 + P^{-1} |\lambda_1|^{-\frac{1}{2}} |\lambda_5|\}^{\frac{1}{4}}.$$

In view of (28), we obtain

$$\begin{aligned}\int_{I_2} |S_1(\alpha) \ldots S_5(\alpha)| \, d\alpha &\ll \prod^{-\frac{1}{2}} P^2 (\log P) |\lambda_2|^{\frac{1}{2}} \{1 + P^{-1} |\lambda_1|^{-\frac{1}{2}} |\lambda_5|\}^{\frac{1}{4}} \\ &\ll \prod^{-\frac{1}{2}} P^2 (\log P) \Lambda^{\frac{1}{2}} + \prod^{-\frac{1}{2}} P^{7/4} (\log P) |\lambda_2|^{\frac{1}{2}} |\lambda_1|^{-\frac{1}{8}} |\lambda_5|^{\frac{1}{4}}.\end{aligned}$$

Now
$$|\lambda_2|^{\frac{1}{2}} |\lambda_1|^{-\frac{1}{8}} |\lambda_5|^{\frac{1}{4}} \leqslant \prod^{\frac{1}{8}} |\lambda_2|^{\frac{3}{8}} |\lambda_1|^{-\frac{1}{4}} |\lambda_5|^{\frac{1}{8}} \leqslant \prod^{\frac{1}{8}} \Lambda^{\frac{1}{2}} < P^{\frac{1}{4}} \Lambda^{\frac{1}{2}}.$$

This gives the estimate (25), and the proof of Lemma 6 is complete.

5. LEMMA 7. *For $\delta > 0$ there exists $C_\delta > 0$ such that if $P > C_\delta \prod^{\delta + \frac{1}{2}}$ then*

$$\int_{\frac{1}{40} P^{-1} \lambda^{-\frac{1}{2}}}^{P^\delta} |S_1(\alpha) \ldots S_5(\alpha)| \, d\alpha > \frac{1}{7} P^3 \prod^{-\frac{1}{2}}. \tag{31}$$

Proof. By the Corollary to Lemma 1 and by Lemmas 4 and 6, we have

$$M + R + R' + R'' + R''' = 0,$$

where
$$|R| + |R'| \ll \prod^{-\frac{1}{2}} P^2 (\log P) \Lambda^{\frac{1}{2}}$$

and
$$R'' = \int_{\frac{1}{40} P^{-1} \lambda^{-\frac{1}{2}}}^{P^\delta} S_1(\alpha) \ldots S_5(\alpha) K(\alpha) \, d\alpha,$$

$$R''' = \int_{P^\delta}^{\infty} S_1(\alpha) \ldots S_5(\alpha) K(\alpha) \, d\alpha.$$

By (7) and the trivial estimate $|S_j(\alpha)| \ll P |\lambda_j|^{-\frac{1}{2}}$ we have

$$|R'''| \ll P^5 \prod^{-\frac{1}{2}} C(n) P^{-n\delta}.$$

Choosing $n = [5\delta^{-1}] + 1$, we get

$$|R'''| < C_1(\delta) \prod^{-\frac{1}{2}}.$$

Hence $\quad |M+R''| \ll \prod^{-\frac{1}{2}} P^2 (\log P)\Lambda^{\frac{1}{2}} + C_1(\delta)\prod^{-\frac{1}{2}} \ll P^2 \log P + C_1(\delta)\prod^{-\frac{1}{2}}.$

In view of (18), the desired result, namely

$$|R''| > \frac{1}{7} P^3 \prod^{-\frac{1}{2}},$$

will hold provided that both

$$\frac{P}{\log P} > C_2 \prod^{\frac{1}{2}} \quad \text{and} \quad P^3 > C_2(\delta),$$

for a suitable absolute constant C_2 and a suitable positive $C_2(\delta)$. Both these conditions are satisfied if $P > C_\delta \prod^{\delta+\frac{1}{2}}$, with a suitable C_δ.

From now onwards we shall be concerned solely with values of α in the interval

$$\mathfrak{J}: \frac{1}{40} P^{-1} \lambda^{-\frac{1}{2}} < \alpha < P^\delta. \tag{32}$$

6. For any integers a, q with $q > 0$ and $(a, q) = 1$ we define

$$S_{a,q} = \sum_{x=1}^{q} e(a x^2/q), \tag{33}$$

and for any integer ν we write

$$S_{a,q,\nu} = \sum_{x=1}^{q} e((a x^2 + \nu x)/q). \tag{34}$$

LEMMA 8. *We have, for* $(a, q) = 1$,

$$|S_{a,q}| \ll q^{\frac{1}{2}}, \quad |S_{a,q,\nu}| \ll q^{\frac{1}{2}}. \tag{35}$$

Proof. The first result is well known ([8], Chapter 2, Lemma 6), and the second follows from it; for instance, if q is odd then

$$a x^2 + \nu x \equiv a(x + \nu b)^2 - a b^2 \nu^2$$

for $2ab \equiv 1 \pmod q$, so we have $\quad |S_{a,q,\nu}| = |S_{a,q}|.$

The following approximation to $S(\alpha)$ is well known in principle. (1)

LEMMA 9. *Suppose that* $A > 1$ *and that* α *is a real number satisfying*

$$\alpha = \frac{a}{q} + \beta, \tag{36}$$

(1) For the corresponding results for higher powers instead of squares, see [5] and [6].

where $$(a, q) = 1, \quad 0 < q \leqslant A, \quad 40\,|\beta| < q^{-1} A^{-1}. \tag{37}$$

Then $$\sum_{A < x < 10A} e(\alpha x^2) = q^{-1} S_{a,q} \int_A^{10A} e(\beta \xi^2)\, d\xi + O(q^{\frac{1}{2}} \log 2q). \tag{38}$$

Proof. It will be convenient to suppose that neither A nor $10A$ is near an integer; this supposition is permissible since the sum and the integral in (38) vary only by an amount $O(1)$ if α is varied by an amount $O(1)$. Strictly speaking, such a variation may disturb the condition $A > 1$ or the condition (37), but these conditions are not used with any precision in the proof.

We first dissect the sum according to residue classes modulo q:

$$\sum_{A<x<10A} e(\alpha x^2) = \sum_{z=1}^{q} e(az^2/q) \sum_{\substack{A<x<10A \\ x\equiv z \pmod q}} e(\beta x^2). \tag{39}$$

The inner sum is $$\sum_{(A-z)/q<y<(10A-z)/q} e(\beta(qy+z)^2),$$

and by Poisson's summation formula this is

$$\int_{(A-z)/q}^{(10A-z)/q} e(\beta(q\eta+z)^2)\, d\eta + \sum_{\nu}{}' \int_{(A-z)/q}^{(10A-z)/q} e(\beta(q\eta+z)^2 + \nu\eta)\, d\eta,$$

where $\sum'$ is over $\nu \neq 0$ with the terms ν, $-\nu$ taken together. Putting $q\eta + z = \xi$, and substituting in (39), we obtain

$$\sum_{A<x<10A} e(\alpha x^2) = q^{-1} S_{a,q} \int_A^{10A} e(\beta\xi^2)\, d\xi + E,$$

where $$E = q^{-1} \sum_{\nu}{}' S_{a,q,-\nu} \int_A^{10A} e(\beta\xi^2 + \nu\xi/q)\, d\xi. \tag{40}$$

Suppose for simplicity that $\beta > 0$. We have

$$\beta\xi^2 + \nu\xi/q = \beta\left(\xi + \frac{\nu}{2q\beta}\right)^2 - \frac{\nu^2}{4\beta q^2},$$

and we note that $$\frac{|\nu|}{2q\beta} > 20A$$

by (37). This ensures that $\xi + \nu/(2q\beta)$ does not vanish even when ν is negative, so putting $\{\xi + \nu/(2q\beta)\} = \beta^{-1}\zeta$, the integral in (40) becomes

$$e\left(-\frac{\nu^2}{4\beta q^2}\right)\frac{1}{2}\beta^{-\frac{1}{2}}\int_{\zeta_1}^{\zeta_2}(\operatorname{sgn}\nu)\,\zeta^{-\frac{1}{2}}\,e(\zeta)\,d\zeta,$$

where
$$\zeta_1=\beta\left(A+\frac{\nu}{2q\beta}\right)^2,\quad \zeta_2=\beta\left(10A+\frac{\nu}{2q\beta}\right)^2.$$

Since $\zeta_1>0$ and $\zeta_2>0$, integration by parts gives

$$\int_{\zeta_1}^{\zeta_2}\zeta^{-\frac{1}{2}}\,e(\zeta)\,d\zeta=\frac{1}{2\pi i}\{\zeta_2^{-\frac{1}{2}}\,e(\zeta_2)-\zeta_1^{-\frac{1}{2}}\,e(\zeta_1)\}+O(\zeta_1^{-\frac{3}{2}})+O(\zeta_2^{-\frac{3}{2}})$$

$$=\frac{2q\beta^{\frac{1}{2}}}{2\pi i}(\operatorname{sgn}\nu)\left\{\frac{1}{\nu+20Aq\beta}e(100A^2\beta+10A\nu/q)\right.$$
$$\left.-\frac{1}{\nu+2Aq\beta}e(A^2\beta+A\nu/q)\right\}e(\nu^2/(4\beta q^2))+O(q^3\beta^{\frac{3}{2}}|\nu|^{-3}).$$

Substituting for the integral in (40), we obtain

$$E=\frac{1}{2\pi i}\sum_{\nu}{}'\,S_{a,q,-\nu}\,\frac{1}{\nu+20Aq\beta}\,e(100A^2\beta+10A\nu/q)$$
$$-\frac{1}{2\pi i}\sum_{\nu}{}'\,S_{a,q,-\nu}\,\frac{1}{\nu+2Aq\beta}\,e(A^2\beta+A\nu/q)+O\Big(\sum_{\nu}{}'\,|S_{a,q,-\nu}|\,q^2\beta\,|\nu|^{-3}\Big).$$

By (35) and (37), the last error term is

$$O(q^{2+\frac{1}{2}}\beta)=O(q^{\frac{1}{2}}).$$

It remains to consider the two sums over ν, and as they are essentially the same it will suffice to treat the second.

We have
$$\sum_{|\nu|\leqslant q^2}{}'\,|S_{a,q,-\nu}|\,|\nu+2Aq\beta|^{-1}\ll q^{\frac{1}{2}}\sum_{|\nu|\leqslant q^2}{}'\,|\nu|^{-1}\ll q^{\frac{1}{2}}\log 2q.$$

The sum over $|\nu|>q^2$ can be written as

$$\sum_{z=1}^{q}e(az^2/q)\sum_{|\nu|>q^2}{}'\,\frac{1}{\nu+2Aq\beta}\,e(\nu(A-z)/q), \tag{41}$$

apart from a factor of absolute value 1. The inner sum here, by Abel's lemma, has absolute value

$$\ll q^{-2}\,\|(A-z)/q\|^{-1},$$

where $\|\theta\|$ denotes the difference between θ and the nearest integer, taken positively. As we supposed that A was not near to an integer, we have

$$\|(A-z)/q\| > q^{-1}.$$

Hence the double sum in (41) is $O(1)$, and we have

$$|E| \ll q^{\frac{1}{2}} \log 2q.$$

This completes the proof of (38).

COROLLARY. *Suppose that*

$$\lambda_j \alpha = \frac{a_j}{q_j} + \beta_j, \tag{42}$$

where $$(a_j, q_j) = 1, \quad 0 < q_j \ll P|\lambda_j|^{-\frac{1}{2}}, \quad 40|\beta_j| < q_j^{-1} P^{-1} |\lambda_j|^{\frac{1}{2}}. \tag{43}$$

Then $$|S_j(\alpha)| \ll q_j^{-\frac{1}{2}} (\log P) \min (P|\lambda_j|^{-\frac{1}{2}},\ P^{-1}|\lambda_j|^{\frac{1}{2}}|\beta_j|^{-1}). \tag{44}$$

Proof. The hypotheses of Lemma 9 are satisfied when $A = P|\lambda_j|^{-\frac{1}{2}}$, $a = a_j$, $q = q_j$ and α is replaced by $\lambda_j\alpha$. The sum on the left of (38) then becomes $S_j(\alpha)$. Hence [1]

$$S_j(\alpha) = q_j^{-1} S_{a_j, q_j} \int_{P|\lambda_j|^{-\frac{1}{2}}}^{10P|\lambda_j|^{-\frac{1}{2}}} e(\beta_j \xi^2)\, d\xi + O(q_j^{\frac{1}{2}} \log 2q_j).$$

Using (35) and estimating the integral as in the proof of Lemma 3, we obtain

$$|S_j(\alpha)| \ll q_j^{-\frac{1}{2}} \min (P|\lambda_j|^{-\frac{1}{2}},\ P^{-1}|\lambda_j|^{\frac{1}{2}}|\beta_j|^{-1}) + q_j^{\frac{1}{2}} \log 2q_j.$$

Since $$q_j^{\frac{1}{2}} \ll q_j^{-\frac{1}{2}} P|\lambda_j|^{-\frac{1}{2}} \quad \text{and} \quad q_j^{\frac{1}{2}} \ll q_j^{-\frac{1}{2}} P^{-1}|\lambda_j|^{\frac{1}{2}}|\beta_j|^{-1},$$

by (43), the result follows.

7. For any α in the interval (32), and for each $j = 1, \ldots, 5$, there exist integers a_j, q_j such that

$$(a_j, q_j) = 1, \quad 0 < q_j \leqslant 40P|\lambda_j|^{-\frac{1}{2}} \tag{45}$$

and $$\lambda_j \alpha = \frac{a_j}{q_j} + \beta_j, \tag{46}$$

where $$|\beta_j| < q_j^{-1}(40P|\lambda_j|^{-\frac{1}{2}})^{-1}. \tag{47}$$

Thus (42) and (43) are satisfied, and consequently (44) is valid.

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[1] It is of interest to note that if $q_j = 1$ and $a_j = 0$, so that $S_{a_j, q_j} = 1$, this approximation reduces to that of Lemma 2.

It is important to note that none of $a_1, \ldots, a_5$ is 0, for $a_j = 0$ would imply

$$|\lambda_j \alpha| = |\beta_j| < \frac{1}{40} P^{-1} |\lambda_j|^{\frac{1}{2}},$$

contrary to (32).

Let $\mathfrak{I}$ denote the subset of the interval $\mathfrak{J}$, defined in (32), consisting of those α for which

$$|S_j(\alpha)| > P^{1-2\delta} \prod{}^{-\frac{1}{4}} |\lambda_j|^{-\frac{1}{4}} \qquad (j = 1, \ldots, 5). \tag{48}$$

LEMMA 10. *We have*

$$\int_{\mathfrak{J}-\mathfrak{I}} |S_1(\alpha) \ldots S_5(\alpha)| \, d\alpha \ll P^{3-\delta/2} \prod{}^{-\frac{1}{2}}. \tag{49}$$

Proof. In $\mathfrak{J} - \mathfrak{I}$, one of the inequalities (48) is false, say that for $j = 5$. Thus

$$|S_5(\alpha)| \leqslant P^{1-2\delta} \prod{}^{-\frac{1}{4}} |\lambda_5|^{-\frac{1}{4}}. \tag{50}$$

By Lemma 5 and the periodicity of $S_j(\alpha)$, with period $|\lambda_j|^{-1}$, we have

$$\int_0^{P^\delta} |S_j(\alpha)|^4 \, d\alpha \ll |\lambda_j|^{-1} P^{2+\delta} \log P.$$

It follows from the cases $j = 1, 2, 3, 4$ of this, and Hölder's inequality, that

$$\int_0^{P^\delta} |S_1(\alpha) \ldots S_4(\alpha)| \, d\alpha \ll |\lambda_1 \ldots \lambda_4|^{-\frac{1}{4}} P^{2+\delta} \log P = \prod{}^{-\frac{1}{4}} |\lambda_5|^{\frac{1}{4}} P^{2+\delta} \log P.$$

From this and (50), it follows that

$$\int_{\mathfrak{J}-\mathfrak{I}} |S_1(\alpha) \ldots S_5(\alpha)| \, d\alpha \ll P^{3-\delta} (\log P) \prod{}^{-\frac{1}{2}},$$

whence (49).

8. It follows from Lemmas 7 and 10 that

$$\int_{\mathfrak{I}} |S_1(\alpha) \ldots S_5(\alpha)| \, d\alpha > \frac{1}{8} P^3 \prod{}^{-\frac{1}{2}}, \tag{51}$$

provided that the constant C_δ of Lemma 7 is taken sufficiently large. The remainder of the paper will be concerned with deducing from (51) a contradiction to the basic hypothesis made at the beginning of § 2.

It will be convenient to consider the parts of $\mathfrak{J}$ in which $|S_1(\alpha)|, \ldots, |S_5(\alpha)|$, $q_1, \ldots, q_5$ are all of particular orders of magnitude. Let $T_1, \ldots, T_5, U_1, \ldots, U_5$ be positive numbers, and let $\mathcal{G} = \mathcal{G}(T_1, \ldots, U_5)$ denote the set of those α in $\mathfrak{J}$ for which

$$\frac{1}{2} T_j P < |\lambda_j^{\frac{1}{2}} S_j(\alpha)| \leqslant T_j P \qquad (j = 1, \ldots, 5), \tag{52}$$

$$\frac{1}{2} U_j < |\lambda_j|^{\frac{1}{2}} q_j \leqslant U_j \qquad (j = 1, \ldots, 5). \tag{53}$$

By (48) and the trivial upper bound for $|S_j(\alpha)|$, we can suppose that

$$P^{-2\delta} \prod{}^{-\frac{1}{2}} < T_j < 20 |\lambda_j|^{-\frac{1}{2}}. \tag{54}$$

By (44),
$$T_j \ll U_j^{-\frac{1}{2}} (\log P) \min (1, P^{-2} |\lambda_j| \, |\beta_j|^{-1}),$$
whence
$$U_j \ll (\log P)^2 T_j^{-2} \tag{55}$$
and
$$|\lambda_j|^{-1} |\beta_j| \ll P^{-2} (\log P) T_j^{-1} U_j^{-\frac{1}{2}}. \tag{56}$$

We have also $U_j \geqslant |\lambda_j|^{\frac{1}{2}} q_j \geqslant 1$ by (53).

LEMMA 11. *There exist $T_1, \ldots, T_5$, $U_1, \ldots, U_5$ such that the measure of the set $\mathcal{G}$, say $m(\mathcal{G})$, satisfies*

$$m(\mathcal{G}) \gg P^{-2} (\log P)^{-10} \prod{}^{-\frac{1}{2}} (T_1 \ldots T_5)^{-1}. \tag{57}$$

Proof. Since the numbers T_j and U_j are bounded above and below by fixed powers of P, it is clear from the nature of (52) and (53) that the number of choices for $T_1, \ldots, T_5$, $U_1, \ldots, U_5$ that need to be made to cover all α in $\mathfrak{J}$ is $\ll (\log P)^{10}$. Hence, by (51), there is some choice such that

$$\int_{\mathcal{G}} |S_1(\alpha) \ldots S_5(\alpha)| \, d\alpha \gg P^3 \prod{}^{-\frac{1}{2}} (\log P)^{-10}.$$

For any α in $\mathcal{G}$, we have

$$|S_1(\alpha) \ldots S_5(\alpha)| \leqslant P^5 T_1 \ldots T_5 \prod{}^{-\frac{1}{2}}$$

by (52). Hence (57).

9. From now onwards we shall be concerned only with a particular set $T_1, \ldots, T_5$, $U_1, \ldots, U_5$ for which (57) holds. We shall suppose, as we may without loss of generality, that

$$T_1 \geqslant T_j \qquad (j = 2, \ldots, 5). \tag{58}$$

LEMMA 12. *For each j $(1 \leqslant j \leqslant 5)$, the number N_j of distinct integer pairs a_j, q_j which arise from all α in $\mathcal{G}$ satisfies*

$$N_j \gg (\log P)^{-11} \prod^{-\frac{1}{2}} (T_1 \ldots T_5)^{-1} T_j U_j^{\frac{1}{2}}. \tag{59}$$

Proof. The fact that α gives rise to a_j, q_j implies, by (46) and (56), that

$$\left| \lambda_j \alpha - \frac{a_j}{q_j} \right| \ll |\lambda_j| P^{-2} (\log P) T_j^{-1} U_j^{-\frac{1}{2}}.$$

This limits α to an interval of length

$$\ll P^{-2} (\log P) T_j^{-1} U_j^{-\frac{1}{2}}.$$

By (57), the number of such intervals must be

$$\gg P^{-2} (\log P)^{-10} \prod^{-\frac{1}{2}} (T_1 \ldots T_5)^{-1} P^2 (\log P)^{-1} T_j U_j^{\frac{1}{2}},$$

and (59) follows.

LEMMA 13. *For $j = 2, 3, 4, 5$, the integers a_1, q_1, a_j, q_j corresponding to any α in $\mathcal{G}$ satisfy*

$$0 < |a_1| q_j \ll |\lambda_1|^{\frac{1}{2}} |\lambda_j|^{-\frac{1}{2}} P^{\delta} U_1 U_j, \tag{60}$$

$$|a_1 q_j \lambda_j / \lambda_1 - a_j q_1| \ll |\lambda_1|^{-\frac{1}{2}} |\lambda_j|^{\frac{1}{2}} (U_1 U_j)^{\frac{1}{2}} (T_1 T_j)^{-1} P^{-2} (\log P)^2. \tag{61}$$

Proof. As remarked in § 7, we have $a_j \neq 0$ $(j = 1, \ldots, 5)$. Also $|\beta_j|$ is small compared with q_j^{-1} by (47). Hence, by (46),

$$|\lambda_j| \alpha q_j \ll |a_j| \ll |\lambda_j| \alpha q_j.$$

Thus, in particular,

$$|a_1| q_j \ll |\lambda_1| \alpha q_1 q_j \ll |\lambda_1|^{\frac{1}{2}} |\lambda_j|^{-\frac{1}{2}} U_1 U_j P^{\delta},$$

by (53) and the fact that $\alpha < P^{\delta}$. This proves (60).

Next, by (46),

$$a_1 q_j \lambda_j / \lambda_1 - a_j q_1 = a_1 q_1 (a_j + q_j \beta_j)(a_1 + q_1 \beta_1)^{-1} - a_j q_1 = q_1 (a_1 q_j \beta_j - a_j q_1 \beta_1)(a_1 + q_1 \beta_1)^{-1}.$$

This has absolute value

$$\begin{aligned} &\ll |\lambda_1|^{-1} \alpha^{-1} (|a_1| q_j |\beta_j| + |a_j| q_1 |\beta_1|) \\ &\ll |\lambda_1|^{-1} \alpha^{-1} (|\lambda_1| \alpha q_1 q_j |\beta_j| + |\lambda_j| \alpha q_j q_1 |\beta_1|) \\ &= q_1 q_j (|\beta_j| + |\lambda_j \lambda_1^{-1} \beta_1|). \end{aligned}$$

Using (56) and (53), we obtain the estimate

$$\lambda_j|^{\frac14}\,|\lambda_1|^{-\frac14}\,U_1\,U_j\,(T_j^{-1}\,U_j^{-\frac12}+T_1^{-1}\,U_1^{-\frac12})\,P^{-2}\,(\log P)\ll|\lambda_j|^{\frac14}\,|\lambda_1|^{-\frac14}\,U_1^{\frac12}\,U_j^{\frac12}\,T_1^{-1}\,T_j^{-1}\,P^{-2}\,(\log P)^2,$$

the last step by (55). The lemma is proved.

LEMMA 14. *Suppose θ is real, and suppose there exist N distinct integer pairs x, y satisfying*

$$|\theta x-y|<\zeta, \tag{62}$$

$$0<|x|<X, \tag{63}$$

where $\zeta>0$ and $X>0$. Then either

$$N<24\,\zeta\,X \tag{64}$$

or all integer pairs x, y satisfying (62) *and* (63) *have the same ratio y/x.*

Proof. We can suppose that $X>1$, since otherwise $N=0$ by (63), and (64) is trivially satisfied. We can suppose also that $\zeta<\frac12$; for if $\zeta\geqslant\frac12$ we have

$$N\leqslant(2X+1)(2\zeta+1)\leqslant 12\,X\,\zeta.$$

There exist integers p, q such that

$$(p,q)=1,\quad 0<q<2X,\quad |q\theta-p|\leqslant(2X)^{-1}.$$

If x and y satisfy (62) and (63), then

$$|xp-yq|\leqslant|x(p-q\theta)|+|q(x\theta-y)|<X(2X)^{-1}+q\zeta=\tfrac12+q\zeta.$$

If $q\zeta\leqslant\frac12$ we obtain $xp-yq=0$, which gives the second alternative of the enunciation. If $q\zeta>\frac12$, the number of possible residue classes for x (mod q) is less than $2(\frac12+q\zeta)+$ $+1<6q\zeta$, and consequently the number of possibilities for x is less than

$$6q\zeta\,(2Xq^{-1}+1)<12\,\zeta\,X+12\,\zeta\,X.$$

Since x determines y with at most one possibility by (62), we obtain (64).

LEMMA 15. *Suppose that*

$$P^{2-9\delta}>C_\delta\prod, \tag{65}$$

for a suitable C_δ. Then, for each $j=2,\ldots,5$ and for any α in $\mathcal{G}$, we have

$$\frac{a_1 q_j}{a_j q_1}=\frac{A_j}{B_j}, \tag{66}$$

where A_j, B_j are relatively prime integers which are independent of α, and $B_j>0$, $A_j\neq 0$.

Proof. By Lemma 13, the integers $x = a_1 q_j$ and $y = a_j q_1$ corresponding to any α in $\mathcal{G}$ satisfy

$$|x\lambda_j/\lambda_1 - y| < \zeta, \tag{67}$$

$$0 < |x| < X, \tag{68}$$

where
$$\zeta \ll |\lambda_j|^{\frac{1}{2}} |\lambda_1|^{-\frac{1}{2}} (U_1 U_j)^{\frac{1}{2}} (T_1 T_j)^{-1} P^{-2} (\log P)^2, \tag{69}$$

$$X \ll |\lambda_j|^{-\frac{1}{2}} |\lambda_1|^{\frac{1}{2}} P^{\delta} U_1 U_j. \tag{70}$$

The values of x and y determine those of a_1, q_1, a_j, q_j with $\ll P^{\varepsilon}$ possibilities, for any fixed $\varepsilon > 0$ (note that $a_j \neq 0$, as remarked earlier). Hence the number N of distinct integer pairs x, y that arise from all α in $\mathcal{G}$ satisfies

$$N \gg P^{-\varepsilon} N_1, \tag{71}$$

where N_1 has the significance of Lemma 12.

By Lemma 14 there are two possibilities: either all y/x have the same value, independent of α, which gives the desired conclusion for the particular j under consideration, or (64) holds. In the latter case we have

$$P^{-\varepsilon} N_1 \ll P^{-2+\delta} (\log P)^2 (U_1 U_j)^{\frac{3}{2}} (T_1 T_j)^{-1},$$

by (69), (70), (71). Using (59), we obtain

$$(\log P)^{-11} \prod{}^{-\frac{1}{2}} (T_1 \dots T_5)^{-1} T_1 U_1^{\frac{1}{2}} \ll P^{-2+\delta+\varepsilon} (\log P)^2 (U_1 U_j)^{\frac{3}{2}} (T_1 T_j)^{-1},$$

that is,
$$P^{2-2\delta} T_1^2 T_j \ll \prod{}^{\frac{1}{2}} T_1 \dots T_5 U_1 U_j^{\frac{3}{2}}.$$

By (55), this gives
$$T_1^{\frac{1}{2}} T_j^{\frac{1}{4}} \ll P^{-2+3\delta} \prod{}^{\frac{1}{2}} T_1 \dots T_5.$$

Cancelling T_j from both sides and using (58), we obtain

$$T_1^{\frac{1}{2}} T_j^{\frac{3}{4}} \ll P^{-2+3\delta} \prod{}^{\frac{1}{2}} T_1^{4}.$$

Since $T_j \gg P^{-2\delta} \prod^{-\frac{1}{2}}$ by (54), this implies

$$P^{-6\delta} \prod{}^{-\frac{3}{2}} \ll P^{-2+3\delta} \prod{}^{\frac{1}{2}},$$

which contradicts (65) if C_δ is suitably chosen. Hence the result.

LEMMA 16. *Suppose* (65) *holds. Then the integers* $a_1, q_1, \dots, a_5, q_5$, *corresponding to any* α *in* $\mathcal{G}$, *are of the form*

$$a_j = a a_j', \quad q_j = q q_j' \qquad (j = 1, \dots, 5), \tag{72}$$

where $q>0$, $a>0$, *and* $$a'_j \mid H, \quad q'_j \mid H, \tag{73}$$

where H *is independent of* α *and*

$$0<H \ll P^{17}. \tag{74}$$

Proof. Since $(a_1, q_1)=(a_j, q_j)=(A_j, B_j)=1$, the equation (66), which can be written as

$$\frac{a_j}{q_j}=\frac{a_1 B_j}{q_1 A_j},$$

implies that $$a_j=\frac{a_1 B_j}{(a_1, A_j)(q_1, B_j)}, \quad q_j=\frac{q_1 A_j}{(a_1, A_j)(q_1, B_j)}.$$

Define a and q by

$$a=\frac{|a_1|}{(a_1, A_2 A_3 A_4 A_5)}, \quad q=\frac{q_1}{(q_1, B_2 B_3 B_4 B_5)}.$$

Then a and q are integers, and a_1/a, q_1/q are integers, say a'_1 and q'_1. Also a'_1 and q'_1 are divisors of $A_2 A_3 A_4 A_5$ and $B_2 B_3 B_4 B_5$ respectively.

Further, for $j=2, \ldots, 5$, we have

$$\frac{a_j}{a}=\pm\frac{B_j}{(B_j, q_1)} \cdot \frac{(a_1, A_2 A_3 A_4 A_5)}{(a_1, A_j)},$$

and the expression on the right is an integer, say a'_j, which divides $B_j A_2 A_3 A_4 A_5$. Similarly $q_j/q=q'_j$ is an integer which divides $A_j B_2 B_3 B_4 B_5$.

Taking $$H=|A_2 A_3 A_4 A_5| B_2 B_3 B_4 B_5,$$

so that H is independent of α, it remains only to prove (74). By (66) and (60), we have

$$|A_j| \ll |\lambda_1|^{\frac{1}{2}} |\lambda_j|^{-\frac{1}{2}} P^{\delta} U_1 U_j,$$

and similarly $$B_j \ll |\lambda_1|^{-\frac{1}{2}} |\lambda_j|^{\frac{1}{2}} P^{\delta} U_1 U_j.$$

Using (55) and (54), we obtain

$$|A_j| B_j \ll P^{2\delta} U_1^2 U_j^2 \ll P^{3\delta} T_1^{-4} T_j^{-4} \ll P^{19\delta} \prod{}^2.$$

Hence, by (65), $$H \ll P^{76\delta} \prod{}^8 \ll P^{17}.$$

This proves the lemma.

10. Lemma 17. *For any non-zero integers* $f_1, \ldots, f_5$, *not all of the same sign, there exist integers* $y_1, \ldots, y_5$ *such that*

$$f_1 y_1^2+\cdots+f_5 y_5^2=0 \tag{75}$$

and $$0<|f_1|y_1^2+\cdots+|f_5|y_5^2\ll|f_1f_2\ldots f_5|. \tag{76}$$

Proof. This is contained in our modified form of Cassels's result referred to in § 1.

LEMMA 18. *Suppose* (65) *holds. Then the integers a, q which correspond to any α in $\mathcal{G}$, in the manner of Lemma* 16, *satisfy*

$$a^4q^6\ll P^{-2+5\delta}(U_1\ldots U_5)^2\max_{1\leqslant k\leqslant 5}T_k^{-1}U_k^{-\frac{1}{2}}. \tag{77}$$

Proof. By (46) and (72) we have

$$\alpha(\lambda_1x_1^2+\cdots+\lambda_5x_5^2)=\frac{a}{q}\left(\frac{a_1'}{q_1'}x_1^2+\cdots+\frac{a_5'}{q_5'}x_5^2\right)+(\beta_1x_1^2+\cdots+\beta_5x_5^2)$$

for any α in $\mathcal{G}$ and any $x_1,\ldots,x_5$. Putting $x_j=q_j'y_j$ for $j=1,\ldots,5$, we obtain

$$\alpha(\lambda_1x_1^2+\cdots+\lambda_5x_5^2)=\frac{a}{q}(a_1'q_1'y_1^2+\cdots+a_5'q_5'y_5^2)+(\beta_1q_1'^2y_1^2+\cdots+\beta_5q_5'^2y_5^2).$$

The signs of $a_1'q_1',\ldots,a_5'q_5'$ are the same as the signs of $a_1/q_1,\ldots,a_5/q_5$, and these are the same as the signs of $\lambda_1,\ldots,\lambda_5$. Hence $a_1'q_1',\ldots,a_5'q_5'$ are non-zero integers, not all of the same sign. It follows from Lemma 17 that there exist integers $y_1,\ldots,y_5$ such that

$$a_1'q_1'y_1^2+\cdots+a_5'q_5'y_5^2=0$$

and $$0<|a_1'q_1'|y_1^2+\cdots+|a_5'q_5'|y_5^2\ll|a_1'\ldots a_5'|q_1'\ldots q_5'. \tag{78}$$

For the corresponding integers $x_1,\ldots,x_5$, we have

$$|\lambda_1x_1^2+\cdots+\lambda_5x_5^2|\leqslant\alpha^{-1}(|\beta_1|q_1'^2y_1^2+\cdots+|\beta_5|q_5'^2y_5^2)$$

and $$0<|\lambda_1|x_1^2+\cdots+|\lambda_5|x_5^2\leqslant 2\alpha^{-1}aq^{-1}(|a_1'q_1'|y_1^2+\cdots+|a_5'q_5'|y_5^2).$$

By the basic hypothesis made at the beginning of § 2, the conditions (2) and (5) cannot both be satisfied. Hence *either*

$$|\beta_1|q_1'^2y_1^2+\cdots+|\beta_5|q_5'^2y_5^2>\alpha \tag{79}$$

or $$|a_1'q_1'|y_1^2+\cdots+|a_5'q_5'|y_5^2>250\,\alpha a^{-1}qP^2. \tag{80}$$

We examine the second alternative first. By (78) it implies that

$$|a_1'\ldots a_5'|q_1'\ldots q_5'\gg\alpha a^{-1}qP^2,$$

whence $$a^{-5} q^{-5} |a_1 \dots a_5| q_1 \dots q_5 \gg \alpha a^{-1} q P^2.$$

Since $|a_j| \ll |\lambda_j| \alpha q_j$, this gives

$$|\lambda_1 \dots \lambda_5| q_1^2 \dots q_5^2 \gg \alpha^{-4} a^4 q^6 P^2,$$

whence $$(U_1 \dots U_5)^2 \gg \alpha^{-4} a^4 q^6 P^2, \tag{81}$$

by (53).

We now examine the first alternative, namely (79). By (56),

$$|\beta_k| q_k'^2 \ll P^{-2} (\log P) T_k^{-1} U_k^{-\frac{1}{2}} |\lambda_k| q_k'^2,$$

and $$|\lambda_k| q_k'^2 \ll \alpha^{-1} \frac{|a_k|}{q_k} q_k'^2 = \alpha^{-1} a q^{-1} |a_k'| q_k'.$$

Hence (79) and (78) imply

$$\alpha \ll P^{-2} (\log P) \alpha^{-1} a q^{-1} (\max_k T_k^{-1} U_k^{-\frac{1}{2}}) |a_1' \dots a_5'| q_1' \dots q_5'.$$

Simplifying this as before, we obtain

$$(U_1 \dots U_5)^2 \max_k T_k^{-1} U_k^{-\frac{1}{2}} \gg \alpha^{-3} a^4 q^6 P^2 (\log P)^{-1}. \tag{82}$$

Since $$T_k^{-1} U_k^{-\frac{1}{2}} \gg (\log P)^{-1}$$

by (55), and since $\alpha < P^{\delta}$, both the alternatives (81) and (82) imply (77), and this proves the lemma.

11. *Completion of the proof of the theorem.* Assuming (65) to hold, Lemma 18 gives the estimate (77) for the integers a, q corresponding to any α in $\mathcal{G}$. The expression on the right of (77) must, of course, be $\gg 1$, otherwise the set $\mathcal{G}$ would be empty. Since the number of solutions of $a^4 q^6 < Z$ in positive integers a, q is $\ll Z^{\frac{1}{4}}$ for $Z \gg 1$, it follows that the number N of distinct pairs a, q which can arise from all α in $\mathcal{G}$ satisfies

$$N^4 \ll P^{-2+5\delta} (U_1 \dots U_5)^2 \max_k T_k^{-1} U_k^{-\frac{1}{2}}. \tag{83}$$

By (73) and (74), the number of distinct possibilities for $a_1', \dots, a_5'$, $q_1', \dots, q_5'$ is $\ll P^{\varepsilon}$ for any fixed $\varepsilon > 0$. Hence the number N_j of distinct possibilities for a_j, q_j for any j from 1 to 5 satisfies

$$N_j^4 \ll P^{\delta} N^4. \tag{84}$$

A lower bound for N_j was obtained in (59). Combining this with (83) and (84), we deduce that

$$(\log P)^{-44} \prod^{-1} (T_1 \dots T_5)^{-4} T_j^4 U_j^2 \ll P^{-2+6\delta} (U_1 \dots U_5)^2 \max_k T_k^{-1} U_k^{-\frac{1}{2}}$$

for $j = 1, \dots, 5$.

Let k be a suffix for which the maximum of $T_k^{-1} U_k^{-\frac{1}{2}}$ is attained. Take j to be any suffix other than k. The last inequality implies that

$$T_j^4 U_j^2 T_k U_k^{\frac{1}{2}} (T_1 \dots T_5)^{-4} (U_1 \dots U_5)^{-2} \ll P^{-2+7\delta} \prod.$$

It is convenient to put $$V_i = T_i U_i^{\frac{1}{2}} (\log P)^{-1}$$

for $i = 1, \dots, 5$, so that $$V_i \ll 1 \tag{85}$$

by (55). We now have $$V_j^4 V_k (V_1 \dots V_5)^{-4} \ll P^{-2+8\delta} \prod.$$

Since $j \neq k$, it follows from (85) that

$$1 \ll P^{-2+8\delta} \prod.$$

This contradicts (65) if C_δ is chosen sufficiently large, and this contradiction completes the proof of the theorem.

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