LEBESGUE AREA OF MAPS FROM HAUSDORFF SPACES

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By a surface is meant a pair (f, X) where X is a compact m-dimensional Hausdorff space and f is a map of X into a Euclidean space of dimension n. The purpose of this paper is to define a Lebesgue type area, $L_m^*(f)$, for such surfaces and to show that it has two desirable properties, (2.6) and (3.3). In this setting, "approximately elementary" surfaces, defined in terms of the nerves of open covers of X, form natural substitutes for elementary ones. Indeed, with this simple substitution, the usual definition of Lebesgue area—in which limits are taken only once—becomes a reasonable definition of area (called L_m^p below). However, this functional is not well understood at present and the alternative, L_m^*, is defined using two limiting stages.

In section 1, L_m^p and L_m^* are defined; the inequalities $L_m^p \leqslant L_m^*$, $L_m^* \leqslant L_m$, (L_m is the usual Lebesgue m-dimensional area, see, e.g. [2]) follow easily from the definitions. In section 2 the case in which X is a compact 2-dimensional manifold (with or without boundary) is considered and it is shown that $L_m^* = L_m$ for such surfaces. This result depends essentially on a countability lemma (Cesari–Radó) and when it fails—as it does for $m \geqslant 3$, even for cells—L_m^* and L_m are different. In section 3 an inequality, essentially $L_n^*(f) \leqslant \int_{E_n} M_f(p)\,dp$, is proved for X a compact Hausdorff space and f a light map. This is the so-called "flat case", i.e., $m = n$; M_f is the "crude" multiplicity function. An example is given (2.9) in which the inequalities $L_2^*(f) < L_2(f)$ and $L_2(f) > \int_{E_2} M_f(p)\,dp$ occur though X is finitely triangulable and $f: X \to E_2$ is light.

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1. Definitions

1.0. If n and m are positive integers, Y is an m-dimensional (geometric) complex, and $f: Y \to E_n$ is simplicial, relative to Y, then the *elementary m-area*, e_m, of f is defined by: $e_m(f) = \sum_{\sigma \in Y} a_m(f(\sigma))$, i.e., the sum extends over all m-dimensional simplexes σ of Y, and a_m denotes (Lebesgue) m-dimensional measure. If X is a Hausdorff space, X is said to be: (i) *compact*, if and only if each open cover of X has a finite refinement which covers X and (ii) *of dimension* $\leqslant m$, if and only if each open cover of X has a refinement $\mathfrak{V}$ which covers X, such that no point of X lies in more than $m+1$ elements of $\mathfrak{V}$. Such a cover $\mathfrak{V}$ will be said to be an *m-dimensional* cover of X. If α is an open coper of X, then X_α will denote a realized nerve of α, (called $P(\alpha)$ in [6]). Barycentric α-maps are as in [6]; canonical maps are as in [3]; simplexes are always "open-simplexes".

If K is a complex and v is a vertex of K, then $\mathrm{St}(v)$ will denote the open star about v, relative to K, i.e., the union of all simplexes σ of K such that v is a vertex of σ. $\mathrm{St}(K)$ will denote $\{\mathrm{St}(v): v \text{ is a vertex of } K\}$. Note that if K is n-dimensional, then $\mathrm{St}(K)$ is an n-dimensional open cover of $|K|$ and that K and $|K|_{\mathrm{St}(K)}$ are isomorphic complexes.

1.1. If X is a Hausdorff space and Y is a triangulated space, a triple (α, g, h) will be said to be an *m-canonical map triple of X into Y*, if and only if:

a) α is a finite, m-dimensional open cover of X;

b) $g: X \to |X_\alpha|$ is a canonical map;

and c) $h: X_\alpha \to Y$ is simplicial (relative to a subdivision of Y).

1.2. Suppose X is a compact, m-dimensional Hausdorff space and (α, g, h) is an m-canonical map triple of X into E^n. Then $|X_\alpha|$ is metrizable; let ϱ be a metric for $|X_\alpha|$. Define the **-elementary* m-area, $e_m^*(\alpha, g, h)$ to be the least number k such that for each positive number ε and each open cover $\mathfrak{U}$ of X, there exists an m-canonical map triple (β, g', h') of X into X_α such that $\varrho(h' g', g) < \varepsilon$, β refines $\mathfrak{U}$ and $e_m(h h') < k + \varepsilon$.

1.3. Suppose X is a compact Hausdorff space and $f: X \to E^n$. Define the (Lebesgue) m-dimensional area $L_m^*(f)$ [respectively, $L_m^p(f)$], to be the least number k such that for each positive number ε and each finite open cover $\mathfrak{U}$ of X, there exists an m-canonical map triple (α, g, h) of X into E^n such that $\varrho(hg, f) < \varepsilon$, α refines $\mathfrak{U}$ and $e_m^*(\alpha, g, h) < k + \varepsilon$, [respectively, $e_m(h) < k + \varepsilon$]. If X is a metric space, the notion of "refinement" for open covers of X can be replaced by familiar considerations of the *mesh* and *Lebesgue number* of open covers of X, defined as usual.

1.4. *If X is a compact Hausdorff space of dimension $\leqslant m$ and $f: X \to E_n$, then $L_m^p(f) \leqslant L_m^*$.*

Proof. Suppose L_m^* is finite, $\varepsilon > 0$, and $\mathcal{U}$ is an open cover of X. Then there exists an m-canonical map triple $(\alpha, g, h): X \to E_n$ such that α refines $\mathcal{U}$, $\varrho(f, hg) < \frac{1}{2}\varepsilon$, and $e_m^*(\alpha, g, h) < L_m^*(f) + \frac{1}{2}\varepsilon$. Let $\delta > 0$ be such that if $x, x' \varepsilon |X_\alpha|$ and $\varrho(x, x') < \delta$, then $\varrho[h(x), h(x')] < \frac{1}{2}\varepsilon$. By the definition of e_m^*, there exists an m-canonical map triple $(\beta, g', h'): X \to |X_\alpha|$ such that β refines $\mathcal{U}$, $\varrho(g, h'g') < \delta$, and $e_m(hh') < e_m^*(\alpha, g, h) + \frac{1}{2}\varepsilon < L_m^*(f) + \varepsilon$. Therefore, $\varrho(f, hh'g') < \varepsilon$ so that $L_m^p(f) \leqslant L_m^*(f)$.

1.5. *If $T = (t, K)$ is a finite m-dimensional triangulation of a compact space X and $f': X \to E_n$ is simplicial relative to T, then there exists an m-canonical map triple (α, g, h) of X into E_n such that:*

1) $e_m^*(\alpha, g, h) \leqslant e_m(f')$;
2) $f' = hg$;
3) *mesh* $\alpha \leqslant 2$ *mesh* T.

Proof. Let α be the collection of all open stars about vertices of T. Then K is a realization of the nerve of α and hence we may set $X_\alpha = K$. Furthermore, $g = t^{-1}: X \to |X_\alpha|$ is canonical. Let $h = f't$ and suppose $\varepsilon > 0$. Then there is a subdivision $T' = (t', K')$ of K such that $T'' = (tt', K')$ has mesh less than $\frac{1}{2}\varepsilon$. Let β be the collection of all open stars about vertices of T''; again we may set $X_\beta = K'$. Then $(\beta, t'^{-1}t^{-1}, t')$: $|X_\beta| \to |X_\alpha|$ is m-canonical and $e_m(ht') = e_m(f')$, so that $e_m^*(\alpha, g, h) \leqslant e_m(f')$. Parts (2) and (3) follow from the definition of α, g, and h.

If X is a triangulable space and $f': X \to E_n$ in semi-linear, then there are triangulations of X of arbitrarily small mesh, relative to which f' is simplicial. Therefore we have the following corollary to (1.5):

1.6. *If X is finitely triangulable,* dim $X \leqslant m$, *and $f: X \to E_n$, then $L_m^*(f) \leqslant L_m(f)$.*

2. The Case: X = A Compact Manifold

Definitions. **2.0.** If m is a positive integer, an m-dimensional space will be said to be *Euclidean* if and only if it is homeomorphic to a subset of E_n; *planar*, if it is Euclidean and of dimension 2. If Y is a subset of X, Y^0 will denote the interior of Y.

2.1. Suppose X is a compact m-manifold, with or without boundary,

$$(\alpha, g, h): X \to E_n$$

is m-canonical, and that $g^{-1}(\sigma)$ is Euclidean, for all m-simplexes σ of X_α. Define:

$$e_m^D(\alpha, g, h) = \sum_{\sigma \in X_\alpha} D(g, \sigma) \cdot a_m[h(\sigma)],$$

where $D(g, \sigma)$ is as defined by Federer [4].

2.3. Let X, Y, f and σ be as in (2.5), below. Then a point $p \in \sigma$ is said to be a branch point of f, (relative to σ) if and only if the number $S(f, p)$ of essential components of $f^{-1}(p)$ is less than $D(f, \sigma)$. (See Cesari [1] and Federer [4]). Similarly, let K, T, and f be as in (2.5.0), below. A point $p \in T^0$ is said to be on O-branch point of f (relative to K) if and only if $S(f, p)$ is less than $|O(f, K, T^0)|$, where $O(f, k, T^0)$ is the topological index (ordinarily defined, [7], relative to a point, e.g., $O(f, K, p)$ for $p \in T$. However, the notation $O(K, f, T^0)$ is not ambiguous, as $O(f, K, p)$, considered as a function of p, is constant throughout T^0).

2.4 (Federer). *If X is a compact, triangulated, m-dimensional manifold, with or without boundary, $(\alpha, g, h): X \to E_n$ is canonical, and each element of α is Euclidean, then there exist subdivisions X_1 of X and $X_{\alpha 1}$ of X_α, and a simplicial map $g': X_1 \to X_\alpha$ which approximates g relative to X_α* [3; II 7.1], *such that $e_m(hg') = e_m^D(\alpha, g, h)$.*

Proof. There exist subdivisions, X_1 of X and $X_{\alpha 1}$ of X_α and a simplificial map $g': X_1 \to X_{\alpha 1}$ such that for each m-simplex $\Delta \in X_\alpha$, $g'^{-1}(y)$ has $D(g, \Delta)$ elements, for almost all (Lebesgue m-dimensional measure) points y of Δ, and such that g' approximates g relative to X_α. (See Federer, [4], p. 6.13. Federer is concerned only with the case $|X_\alpha| = E_m$, though his proof applies here.) Then hg' is simplificial. Suppose σ is an m-simplex in E_n such that for some simplex Δ in X_α, $\sigma = h(\Delta)$. Then for almost all $y \in \sigma$, y is the image of $\sum D(g, \Delta)$ points of X, where the sum extends over all m-simplexes Δ of X such that $h(\Delta) = \sigma$. Therefore

$$e_m(hg') = \sum_{\Delta \in X_\alpha} D(g, \Delta) \cdot a_m[h(\Delta)],$$

and by definition, this is $e_m^D(\alpha, g, h)$.

(2.5.0) (Cesari). *Suppose K is a finitely connected closed Jordan region with mutually exclusive boundary curves $J_1, \ldots, J_m$, $J = \bigcup_{i=1}^{m} J_i$, that T is a solid triangle with boundary B, $f: K \to T$, and $f(J) \subset B$. Then the subset A of T^0 of all O-branch points of f is finite.*

Proof. We first show that A is locally finite, by induction on m. For $m = 1$, this is proved by Cesari [1; Theorem B]. Suppose that for all positive integers less than m, A is locally finite. Let Q_0 be a point of T^0 and assume as case 1 that $f^{-1}(Q_0)$ does not separate any two of the sets $J_1, \dots, J_m$. Then there exist closed non-overlapping discs, $K_1, \dots, K_n$, such that $f^{-1}(Q_0) \subset \bigcup_{i=1}^{n} K_i^0$. Then $\sum_{i=1}^{n} O(f, K_i, Q_0) = O(f, K, Q_0)$ and, by Cesari's theorem, there exists a neighborhood N_i of Q_0 such that if $p \in N_i - Q_0$, then $f^{-1}(p)$ has at least $|O(f, K_i, Q_0)|$ essential components in K_i^0, $i = 1, \dots, n$. Therefore $A \cap (\bigcap_{i=1}^{n} N_i)$ contains at most the point Q_0 so that A is locally finite.

Case 2: $f^{-1}(Q_0)$ separates some two of the sets, $J_1, \dots, J_m$. Let $K_1, \dots, K_n$ denote those components of $K - f^{-1}(Q_0)$ which contain one of the sets $J_1, \dots, J_m$. For $i = 1, \dots, n$, let $\hat{K}_i$ denote the decomposition space of $\bar{K}_i$, in which all points of $\bar{K}_i \cap f^{-1}(Q_0)$ are identified, let $g_i : \bar{K}_1 \to \hat{K}_i$ be the decomposition map and $h_i = f g_i^{-1}$. Then $\hat{K}_i$ is a finitely connected closed Jordan region, whose boundary curves consist of certain of the sets $J_1, \dots, J_m$, say $\{J_{n_{ij}}\}_{j=1}^{m_i}$, $i = 1, \dots, n$. Note that $n_{ij} \neq n_{i'j'}$, unless $(i, j) = (i', j')$, and that $g_i | J_{n_{ij}}$ is a homeomorphism, $i = 1, \dots, n$, $j = 1, \dots, m_i$, so that $\sum_{i=1}^{n} (h_i, K_i, T^0) = O(f, K, T^0)$. Furthermore, $m_i < m$, $i = 1, \dots, n$, so that by the induction hypothesis, there exists a neighborhood N_i of Q_0, such that the set A_i of all branch points of h_i, intersects N_i in at most the point Q_0. As before, $A \cap (\bigcap_{i=1}^{n} N_i)$ contains at most the point Q_0, so that A is locally finite.

We complete the proof by showing that this weaker conclusion implies the stronger. Enlarge K by adding annular regions, one for each boundary curve, J_i $i = 1, \dots, m$, obtaining K' with boundary J'; similarly, add an annular region to T obtaining T' with boundary B'. Extend f to $f' : K' \to T'$, so that $J' \to B'$, and $K' - K \to T' - T$, by homotopies. Then $O(f', K', T'^0) = O(f, K, T^0)$ so that $A' \supset A$. The fact that A' is locally finite implies $A = A' \cap T^0 \subset A' \cap T$ is finite.

2.5. LEMMA. (Cesari–Radó). *Suppose X is a compact 2-manifold, with or without boundary, Y is a 2-complex, $f : X \to |Y|$, σ is a 2-simplex of Y, U is an open subset of X, U is planar, and that $f^{-1}(\sigma) \subset U$. Then the subset A of σ of all branch points of f has no limit point in σ.*

Proof. Assume, on the contrary, that A has a limit point $y \in \sigma$. Let $\mathfrak{V}$ denote the collection of all components of $f^{-1}(\sigma)$. Then $D(f, \sigma) = \sum_{V \in \mathfrak{V}} D(f, \sigma, V)$, and for only finitely many elements of $\mathfrak{V}$, say $V_1, \dots, V_k$, is this index different from zero. Let

A_i denote the set of all branch points of $f\,|\,\overline{V}_i$, $i=1, \ldots, k$. Then $\bigcup_{i=1}^{k} A_i \supset A$, so that there is an integer n such that y is a limit point of A_n; let $g = f\,|\,\overline{V}_n$.

Let r be a 2-simplex such that $y \in r \subset \bar{r} \subset \sigma$. There exists a continuum M such that $g^{-1}(r) \subset M \subset V_n$. Let r' be a simplex such that $g(M) \subset r' \subset \bar{r}' \subset \sigma$. From the hypothesis concerning U it follows that there exists a finitely connected Jordan region R, with boundary J, such that $g(\overline{V}_n - R^0) \cap \bar{r}' = 0$. Let B denote the boundary of σ, let $s : \bar{\sigma} \to \bar{\sigma}$ be a map such that $s\,|\,\bar{r}' \cup B$ is the identity and $s(g(\overline{V}_n - R^0)) \subset B$, and let $h = sg$. It follows, (see [8; VI.1.4]), that $|O(h, R, \sigma)| = D(h, \sigma)$. As

$$g^{-1}(r) \subset M \subset g^{-1}(r') = h^{-1}(r') \subset h^{-1}(\sigma),$$

some component, say V', of $h^{-1}(\sigma)$ contains $g^{-1}(r) = h^{-1}(r)$. From this last it follows that $h(\overline{V}_n - V') \cap r = 0$, so that $D(h, \sigma) = D(h, \sigma, V')$.

In the diagram

$$\begin{tikzcd}
H^2(\overline{V}_n, \overline{V}_n - V_n) & H^2(\overline{V}_n, \overline{V}_n - R^0) \arrow[l, "i_1^*"'] \arrow[r, "i_2"] & H^2(R, J) & \\
H^2(\bar{\sigma}, B) \arrow[u, "g_1^*"] \arrow[ur, "g_2^*"'] & H^2(\bar{\sigma}, B) \arrow[l, "s^*"'] \arrow[u, "h_1^*"] \arrow[ur, "h_2^*"'] \arrow[dr, "h_6^*"'] & H^2(\bar{\sigma}, \bar{\sigma} - r) \arrow[l, "i_5^*"'] \arrow[u, "h_3^*"] \arrow[d, "h_5^*"] & H^2(R, R - V') \arrow[l, "h_4^*"'] \arrow[ul, "i_3^*"'] \arrow[dl, "i_4^*"] \\
 & & H^2(\overline{V}', \overline{V}' - V') &
\end{tikzcd}$$

g_j^* is induced by g, $j = 1, 2$; h_j^* by h, $j = 1, \ldots, 6$; i_j^* by inclusions, $j = 1, \ldots, 5$; and s^* by s. Then s^* is the identity and i_j^* is an isomorphism onto, $j = 1, \ldots, 5$ (see [3; X, 5]).

These groups are all infinite cyclic, [3; XI, 6.8], and commutativity holds throughout. As the homomorphisms g_1^* and h_6^* are "connected" by isomorphisms, the indices they define are equal, i.e., $D(g, \sigma, V_n) = D(h, \sigma, V')$. Therefore $D(f, \sigma, V_n) = |O(h, R, \sigma)|$. Thus, as f and h agree on $R \cap h^{-1}(\bar{r})$, it follows that the set A_n' of all O-branch points of $h\,|\,R$ satisfies $A_n' \cap r = A_n \cap r$. But (2.5.0) applies so that A_n' has no limit point in σ, contradicting the fact that y is a limit point of A_n.

2.6. LEMMA. *Let X, Y, f, σ, A, and B be as in* (2.5). *Suppose, in addition that U and V are open subsets of $\bar{\sigma}$ such that $A \cup B \subset V \subset \overline{V} \subset U$ and let $X_0 = f^{-1}(\bar{\sigma} - U)$. Then there exist positive numbers ε, δ, such that if $g : X_0 \to \bar{\sigma}$ is a map with $\varrho(g, f\,|\,X_0) < \varepsilon$,* *then for each* $y \in \bar{\sigma} - U$, $g^{-1}(y)$ *has at least* $D(f, \sigma)$ *essential components, no two of which are within* δ *of each other.*

Proof. Let $m = D(f, \sigma)$, $f_0 = f \mid X_0$, and let C denote the space of all maps $g: X_0 \to \bar{\sigma}$, metrized as usual. For each $(g, y) \in C \times \bar{\sigma}$, let $S(g, y)$ denote the number of essential components of $g^{-1}(y)$. Then, as S is lower-semi-continuous and $\geqslant m$ on $\{f_0\} \times (\bar{\sigma} - U)$, it follows from the compactness of $\bar{\sigma} - U$ that there is a neighborhood N of f_0 in C such that if $(g, y) \in N \times (\bar{\sigma} - U)$, then $S(g, y) \geqslant m$.

For each $(g, y) \in N \times (\bar{\sigma} - U)$, let $d_m(g, y)$ be the greatest lower bound of the set of all numbers δ such that: (a) $g^{-1}(y)$ does not have m essential components, no two of which are within δ of each other. Then $d_m > 0$ on $N \times (\bar{\sigma} - U)$. Furthermore, d_m is lower-semi-continuous, for assume the contrary. Then there exist a sequence $\{g_i\}_{i=0}^{\infty}$ of maps of X_0 into $\bar{\sigma}$, a sequence $\{y_i\}_{i=0}^{\infty}$ of points of $\bar{\sigma} - U$, and a positive number δ, such that (i) $g_i \to g_0$ and $y_i \to y_0$ as $i \to \infty$, (ii) (g_i, y_i) satisfies (a) for δ, $i = 1, 2, \ldots$, and (iii) (g_0, y_0) does not satisfy (a) for δ. Let $L_1, \ldots, L_m$ be essential components of $g_0^{-1}(y_0)$ such that $\varrho(L_i, L_j) > \delta$, for $i \neq j$, $i, j = 1, \ldots, m$, and let N_i be a neighborhood of L_i such that $\varrho(N_i, N_j) > \delta$, for $i \neq j$, $i, j = 1, \ldots, m$. It follows from the definition of "essential component", that for sufficiently large i, $g_i^{-1}(y_i)$ has an essential component $L_{ij} \subset N_{ij}$, for each $j = 1, \ldots, m$, contradicting (ii).

As above, there exist a neighborhood N' of f_0 and a positive number δ, such that for all $(g, y) \in N' \times (\bar{\sigma} - U)$, $d_m(g, y) > \delta$, so that $g^{-1}(y)$ contains m essential components, no two of which are within δ of each other.

2.7. LEMMA. *If* X *is a 2-manifold, with or without boundary, and* $(\alpha, g, h): X \to E_n$ *is 2-canonical, then* $e_2^D(\alpha, g, h) \leqslant e_2^*(\alpha, g, h)$.

Proof. Suppose $\varepsilon > 0$ and that $e_2^*(\alpha, g, h)$ is finite. For each 2-simplex σ of X_α, let $M_\sigma = D(g, \sigma)$, $R_\sigma = g^{-1}(\sigma)$, $B_\sigma =$ the boundary of σ, and A_σ be the subset of σ of all branch points of the map g, relative to σ. Let N be the number of 2-simplexes in X. By (2.5) there exist neighborhoods U_σ, V_σ of $A_\sigma \cup B_\sigma$ such that $\bar{V}_\sigma \subset U_\sigma$ and $a_2[h(\bar{U}_\sigma)] < \varepsilon/3(M_\sigma + 1)(N + 1)$. Let ε_σ, δ_σ be as guaranteed by (2.6).

By the definition of e_2^* there exists a 2-canonical triple $(\beta, g', h'): X \to X_\alpha$ such that:

a) $\varrho(h' g', g) < \varepsilon_\sigma$, for all 2-simplexes $\sigma \in X_\alpha$;

b) mesh $\beta < \delta_\sigma$, for all two simplexes $\sigma \in X_\alpha$;

and c) $e_2(h h') < e_2^*(\alpha, g, h) + \varepsilon/4$.

Then $h': X_\beta \to X_\alpha$ is simplicial relative to a subdivision $X_{\alpha 1}$ of X_α. For each 2-simplex σ of X_α, let σ_1 be the subdivision of σ induced by $X_{\alpha 1}$. Then

$$e_2(hh') = \sum_{\Delta \in X_\beta} a_2[hh'(\Delta)] = \sum_{\Delta \in X_{\alpha 1}} M_\Delta \cdot a_2[h(\Delta)] = \sum_{\sigma \in X_\alpha} \sum_{\Delta \in \sigma_1} M_\Delta \cdot a_2[h(\Delta)],$$

where M_Δ is the number of 2-simplexes of X_β which map onto Δ under h'.

Suppose σ is a 2-simplex of X_α. Let $\sigma_1' = \{\Delta \in \sigma_1 : \Delta - U \neq 0\}$. Then

$$\sum_{\Delta \in \sigma_1 - \sigma_1'} a_2[h(\Delta)] \leqslant a_2[h(U_\sigma)] \leqslant \varepsilon/(M_\sigma + 1)(N+1).$$

Furthermore, for each 2-simplex $\Delta \in \sigma_1'$, $M_\Delta \geqslant M_\sigma$, for suppose on the contrary that $M_\Delta < M_\sigma$, for some 2-simplex $\Delta \in \sigma_1'$. Let $y \in \Delta - U_\sigma$, $z = M_\sigma$, and let $s_1, \ldots, s_z$ be those 2-simplexes of X_β which map onto Δ under h'. As no 1-simplex of X_β maps onto y under h', it follows that $\{s_i\}_{i=1}^z$ covers $h'^{-1}(y)$; therefore $\{g^{-1}(s_i)\}_{i=1}^z$ covers $g'^{-1} h'^{-1}(y)$. But for each $i = 1, \ldots, z$, $g'^{-1}(s_i)$ lies in some element of β and hence has diameter $< \delta_\sigma$. This contradicts the fact that $g'^{-1} h'^{-1}(y)$ has M_σ essential components, no two of which are within δ_σ of each other.

Therefore

$$e_2(hh') \geqslant \sum_{\sigma \in X_\alpha} \sum_{\Delta \in \sigma_1'} M_\Delta a_2[h(\Delta)] \geqslant \sum_{\sigma \in X_\alpha} M_\sigma \sum_{\Delta \in \sigma_1'} a_2[h(\Delta)]$$

$$\geqslant \sum_{\sigma \in X_\alpha} M_\sigma(a_2[h(\sigma)] - a_2[h(U_\sigma)]) \geqslant \sum_{\sigma \in X_\alpha} M_\sigma \cdot a_2[h(\sigma)] - \varepsilon = e_2^D(\alpha, g, h) - \varepsilon,$$

and the proof is complete.

2.8. THEOREM. *If X is a compact 2-manifold, with or without boundary and $f: X \to E_n$, then $L_2^*(f) = L_2(f)$.*

Proof. Let a triangulation of X be specified. By (1.6), $L_2^*(f) \leqslant L_2(f)$. To show the other inequality, for $i = 1, 2, \ldots$, let $(\alpha_i, g_i, h_i): X \to E_n$ be a 2-canonical map triple such that $h_i g_i$ converges uniformly to f, *mesh* $\alpha_i \to 0$, and $e_2^*(\alpha_i, g_i, h_i) \to L_2^*(f)$, as $i \to \infty$. For each positive integer i, lemma (2.4) implies the existence of a semi-linear map $g_i': X \to X_{\alpha_i}$ which approximates g_i relative to X_{α_i}, such that $e_2(hg_i') = e_2^D(\alpha_i, g_i, h_i)$. But for sufficiently large i, $e_2^D(\alpha_i, g_i, h_i) \leqslant e_2^*(\alpha_i, g_i, h_i)$, by lemma 2.7. As hg_i' converges uniformly to f, we have

$$L_2(f) \leqslant \liminf_i e_2(hg_i') = \liminf_i e_2^D(\alpha_i, g_i, h_i) \leqslant \liminf_i e_2^*(\alpha_i, g_i, h_i) = L_2^*(f).$$

2.9. EXAMPLE. *There exists a triangulable space X and a map f of X onto the unit square Q such that $L_2^*(f) < L_2(f)$.*

Proof. Let X be the union of two (solid) squares, Q_1 and Q_2 whose intersection is a diagonal of each. There exists an isometry $h: Q_2 \to Q_1$ such that $h(x) = x$ for all

$x \in Q_1 \cap Q_2$. There exists a homeomorphism $f_1 : Q_1 \to Q$ such that $f_1(Q_1 \cap Q_2)$ is a "heavy arc", i.e., an arc having positive 2-dimensional Lebesgue measure, say d. Define $f : X \to Q$ by $f(x) = f_1(x)$, for $x \in Q_1$ and $f(x) = f_1 h(x)$, for $x \in Q_2$.

By over-additivity and symmetry, we have $L_2(f) \geqslant L_2(f \mid Q_1) + L_2(f \mid Q_2) = 2 \cdot L_2(f) = 2$; the last equality follows from the fact that f_1 is a homeomorphism. That $L_2^*(f) \leqslant 2 - d$ follows from theorem 3.3, below.

3. An inequality for flat mappings

3.0. In this section we are interested in a compact Hausdorff space X and a light map $f : X \to E_n$. (It follows that X is of dimension $\leqslant n$.) For each point $p \in E_n$, let $M_f(p)$ be the number (possibly infinite) of points of $f^{-1}(p)$. Then M_f is the so-called "crude multiplicity" function; it need not be measurable, indeed.

(3.0.1). *Suppose Y is a topological space and $M : Y \to \{0, 1, \dots, \infty\}$ is a function. Then, in order that there exist a compact Hausdorff space X and a light map $f : X \to Y$ such that $M = M_f$, it is necessary and sufficient that the support of M be a compact Hausdorff space.*

The necessity is just the fact that the continuous image of a compact Hausdorff space is again a compact Hausdorff space. To prove the other half, assume the support of M is Y', a compact Hausdorff space. Let $X = \{(y, j) : j$ is a positive integer $\leqslant M(y)\}$, and define $f : X \to Y$, by $f(y, j) = y$, all $(y, j) \in X$. A basis $\mathcal{N}$ of neighborhoods for X is defined as follows: a subset N of X is in $\mathcal{N}$ if and only if either (1) N is a single point (y, j), where $y \in Y$ and $1 < j \leqslant M(y)$ or (2) $N = f^{-1}(U) - F$, where U is an open set in Y and F is a finite set of points $(y, j) \in X$ such that $1 < j \leqslant M(y)$. The axioms of Hausdorff can easily be verified.

Let $X' = \{(y, 1) : y \in Y'\}$. Then $f \mid X'$ is open and one-to-one and therefore a homeomorphism. Hence X' is compact. Therefore if $\mathcal{U}$ is a sub-collection of $\mathcal{N}$ which covers X, some finite subcollection $\mathcal{V}$, of $\mathcal{U}$, covers X', and hence all but finitely many points of X, so that X is compact. The projection f is clearly continuous and $M_f = M$.

Thus, in the main result of this section we are forced to use the lower Riemann integral of M_f, $\underline{\int} M_f(p)\,dp$. However, if X satisfies the second axiom of countability—i.e. is metric—then M_f is measurable, as may be seen as follows: there exists a countable collection $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite open covers of X such that if $\mathcal{U}$ is an open cover of X, then some $\mathcal{U}_i$ refines $\mathcal{U}$. For $p \in E_n$, let $M_i(p)$ be the least integer $j \geqslant 0$ such that some subcollection of j elements of $\mathcal{U}_i$ covers $f^{-1}(p)$. Then

M_i is upper-semi-continuous, finite valued, and $M_i(p) \leqslant M_f(p)$ for all $p \in E_n$. Finally, $\sup_i M_i = M_f$, so that M_f is measurable.

(3.1). Lemma. *Suppose X is a Hausdorff space, Y is a finite n-complex $f: X \to |Y|$, and that for each vertex v of Y, there exist non-empty open sets $\alpha_{v1}, \ldots, \alpha_{v, m(v)}$ such that:*

1. $\bigcup_{i=1}^{m(v)} \alpha_{vi} = f^{-1}(\mathrm{St}\,(v))$; *and*
2. $\bar{\alpha}_{vi} \cap \bar{\alpha}_{vj} = 0$, *unless* $i = j$.

Let $\alpha = \{\alpha_{vi} : v$ is a vertex of Y and $i = 1, \ldots, m(v)\}$, and let v_i be the vertex of X_α corresponding to α_{vi}. Then the correspondence $v_i \to v$ determines a simplicial map $h: X_\alpha \to Y$ and there exists a (unique) barycentric α-map g completing the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & {|Y|} \\
& {|X_\alpha|} \arrow[ur, "h"'] &
\end{tikzcd}$$

Proof. First, if v and w are vertices of Y such that w_j and w_k are adjacent in X_α then $\alpha_{vj} \cap \alpha_{wk} \neq 0$ so that $\mathrm{St}\,(v) \cap \mathrm{St}\,(w) \supset f(\alpha_{vj} \cap \alpha_{wk}) \neq 0$, and hence v and w are adjacent. Note also that $v \neq w$ so that h collapses no simplex of X_α.

To define g, suppose $x \in X$ and let $\alpha_{v_0 j_0}, \ldots, \alpha_{v_r j_r}$ be the only elements of α which contain x. Let σ_x be the simplex of X_α with vertices $v_{0 j_0}, \ldots, v_{r j_r}$. (Here, and below, $v_{i j_i}$ is used instead of $(v_i)_{j_i}$.) Let $g(x) = \sigma_x \cap h^{-1} f(x)$; this is a single point as $h \mid \sigma_x$ is a homeomorphism.

To show that g is continuous suppose $x \in X$ and that $\alpha_{v_0 j_0}, \ldots, \alpha_{v_r j_r}$ are the only elements of α which contain x. Let $\alpha_{v_{r+1} j_{r+1}}, \ldots, \alpha_{v_s j_s}$ be the only other elements of α which intersect $\bigcap_{i=0}^{r} \alpha_{v_i j_i}$. Let σ_0 be the simplex with vertices $v_{0 j_0}, \ldots, v_{r j_r}$ and let $\sigma_0, \ldots, \sigma_k$ be the only simplices of X_α which have σ_0 as a face. Note that $\sigma_0, \ldots, \sigma_k$ are characterized as having each of $v_{0 j_0}, \ldots, v_{r j_r}$ as a vertex and all remaining vertices among $v_{r+1\, j_{r+1}}, \ldots, v_{s j_s}$. Let $\sigma_i' = h(\sigma_i)$, $i = 0, \ldots, k$.

Suppose N is a neighborhood of $g(x)$ in X_α. Then N contains an open subset of $\bar{\sigma}_i$ and hence $h(N)$ contains an open subset of $\bar{\sigma}_i'$, $i = 0, \ldots, k$. Hence there exists a neighborhood R of $h(g(x))$ such that $R \cap \bar{\sigma}_i' \subset h(N) \cap \bar{\sigma}_i'$. Let N' be a neighborhood of x such that $N' \subset \bigcap_{i=0}^{r} \alpha_{v_i j_i}$ and $f(N') \subset R$. Then $g(N') \subset \bigcup_{i=0}^{s} \sigma_i$ and $g(N') \subset h^{-1} f(N') \subset h^{-1}(R)$, so that $g(N') \subset (\bigcup_{i=1}^{s} \sigma_i) \cap h^{-1} h(N) \subset N$.

Then if $x \in \alpha_{vj}$, $g(x)$ lies in a simplex having v_j as a vertex and conversely, for all vertices v of Y and $j = 1, \dots, m(v)$, so that g is a barycentric α-map.

Furthermore, if g' is such a barycentric α-*map*, $g'(x) \in \sigma_x \cap h^{-1} f(x)$ (notation as above) so that $g' = g$.

(3.2). LEMMA. *Suppose both the diagrams*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & {|Y|} \\ & {|X_\alpha|} \arrow[ur, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g'"'] & & {|Y'|} \\ & {|X_\beta|} \arrow[ur, "h'"'] & \end{tikzcd}$$

are as in the previous lemma and that

a) Y' *is a subdivision of* Y; *and*

b) *if* v *is a vertex of* Y *and* v' *is a vertex of* Y', *then no one of the sets* $\beta_{v'1}, \dots, \beta_{v'm'(v')}$ *intersects two of the sets* $\alpha_{v1}, \dots, \alpha_{vm(v)}$. *Then there exists a map* $\Phi : X_\beta \to X_\alpha$, *simplicial relative to a subdivision of* X_α, *such that commutativity holds throughout the diagram:*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "g"] \arrow[ddr, "g'"'] & & {|Y|} \\ & {|X_\alpha|} \arrow[ur, "h"'] & \\ & {|X_\beta|} \arrow[u, "\phi"] \arrow[uur, "h'"'] & \end{tikzcd}$$

Proof. Suppose v_t' is a vertex of X_β, that is v' is a vertex of Y' and t one of the integers $1, \dots, m'(v')$; let $x \in \beta_{v't}$. Let $v_0, \dots, v_r$ be the only vertices v of Y

such that $v' \in \mathrm{St}(v)$. Then there exist unique integers $j_0, \dots, j_r$ such that $x \in \alpha_{v_i j_i}$, for $i = 0, \dots, r$. As $\beta_{v't} \subset f^{-1}\mathrm{St}'(v')$ (here and below, $\mathrm{St}'(v')$ denotes the open star about v', relative to Y'), and $\mathrm{St}'(v') \subset \mathrm{St}(v_i)$, it follows that $\beta_{v't} \subset \bigcup_{j=1}^{m(v_i)} \alpha_{v_i j}$, $i = 0, \dots, r$. Then from part (b), it follows that $\beta_{v't} \subset \alpha_{v_i j_i}$, $i = 0, \dots, r$. Let $\sigma_{v_t'}$ be the simplex of X_α with vertices v_{ij_i}, $i = 0, \dots, r$. As $h \,|\, \sigma_{v't}$ is a homeomorphism, $h^{-1}h'(v_t') \cap \sigma_{v_t'}$ is a single point, v_t'', so that the correspondence $v_t' \to v_t''$ is single valued.

We must show that if v_t' and w_u' are adjacent vertices of X_β, then v_t'' and v_u'' lie together in a single closed simplex of X_α. To this end, let the notation of the previous paragraph hold for v_t' and let $w_0, \dots, w_s$, $k_0, \dots, k_s$ and $\sigma_{w_u'}$ be the corresponding elements for w_u'. Then $\beta_{w'u} \subset \alpha_{w_i k_i}$, $i = 0, \dots, s$. As above the non-empty set $\beta_{v't} \cap \beta_{w'u} \subset (\bigcap_{i=0}^{r} \alpha_{v_i j_i}) \cap (\bigcap_{i=0}^{s} \alpha_{w_i k_i})$ and thus it follows that there is a simplex σ in X_α which has both $\sigma_{v_t'}$ and $\sigma_{w_u'}$ as faces. Then v_t'' and w_u'' lie together in $\bar{\sigma}$. Therefore the map $\Phi : X_\beta \to X_\alpha$ is determined and is simplicial relative to a subdivision of X_α.

We conclude by proving the commutativity of the diagram. As $h\Phi$ and h' are simplicial and agree on the vertices of X_β, it follows that $h\Phi = h'$. To show that $\Phi g' = g$, suppose $x \in X$. As $h\Phi g'(x) = h' g'(x) = f(x) = h g(x)$ and as $h \,|\, \bar{\sigma}$ is a homeomorphism for each simplex σ of X_α, it will suffice to show that $\Phi g'(x)$ and $g(x)$ lie together in a closed simplex of X_α.

Let ω be the simplex of X_β containing $g'(x)$, $v_{0j_0}', \dots, v_{kj_k}'$ its vertices. Let σ_i be the simplex of X_α used above in defining v_{ij_i}', let $v_{i0}, \dots, v_{ir_i}$ be its vertices, and α_{ie} be the element of α corresponding to v_{ie}, $e = 0, \dots, r_i$, $i = 0, \dots, k$. (Note that the symbols "v_{ie}" and "α_{ie}" constitute a change from the notation of the previous paragraphs.) As $x \in \beta_{v_i' j_i}$, and $\beta_{v_i' j_i} \subset \alpha_{ie}$, $e = 0, \dots, r_i$, $i = 0, \dots, k$, it follows that $x \in \bigcap_{\substack{i=0,\dots,k\\ e=0,\dots,r_i}} \alpha_{ie}$ so that there is a simplex σ of X_α with vertices $\{v_{ie}\}_{\substack{i=0,\dots,k\\ e=0,\dots,r_i}}$. Then $\Phi(v_{ij_i}') \in \sigma$, $i = 0, \dots, k$, so that $\Phi g'(x) \in \Phi(\omega) \subset \bar{\sigma}$. But as g is a barycentric α-map, $g(x)$ is in a simplex σ' which has σ as a face so that both $g(x)$ and $\Phi g'(x)$ lie in $\bar{\sigma}'$.

(3.3). THEOREM. *If X is a compact Hausdorff space and $f : X \to E_n$ is light, then $L_n^*(f) \leqslant \int M_f(p)\,dp$.*

Proof. Suppose $\mathcal{U}$ is an open cover of X. If $p \in f(X)$, there exists a neighborhood N_p of p, such that $f^{-1}(p)$ is the union of mutually exclusive sets, $A_{p1}, \dots, A_{pk}$, such that A_{pi} lies in some element of $\mathcal{U}$, $i = 1, \dots, k$. Then by the covering theorem, there exists a $\delta > 0$, such that if B is a subset of E_n of diameter less than δ, then $f^{-1}(B)$ is the union of mutually exclusive sets $A_1, \dots, A_l$, such that A_i lies in an element of $\mathfrak{U}$, $i = 1, \dots, l$. Let T be a triangulation of E_n such that St (T) has mesh $< \delta$. For each vertex v of T, let $\alpha_{v,1}, \dots, \alpha_{v,m(v)}$ be open sets such that

1. $\bigcup_{i=1}^{m(v)} \alpha_{vi} = f^{-1}(\mathrm{St}\,(v))$;
2. $\bar{\alpha}_{vi} \cap \bar{\alpha}_{vj} = 0$, for $i \neq j$, $i, j = 1, \dots, m(v)$; and
3. α_{vi} lies in some element of $\mathfrak{U}$, $i = 1, \dots, m(v)$.

Let $\alpha = \{\alpha_{vi} : v$ is a vertex of T and $i = 1, \dots, m(v)\}$. Lemma (3.1) applies, yielding the commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & E_n \\ & {|X_\alpha|} \arrow[ur, "h"'] & \end{tikzcd}$$

Then $(\alpha, g, h) : X \to E_n$ is n-canonical, α refines $\mathfrak{U}$, and $\varrho(hg, f) = 0$; it will therefore suffice to show that $e_n^*(\alpha, g, h) \leqq \int M_f(p)\, dp$.

To this end, suppose $\varepsilon > 0$ and $\mathfrak{V}$ is an open cover of X. As X is normal, there exists an open cover $\mathfrak{V}'$ of X such that if v is a vertex of T, then no element of $\mathfrak{V}'$ intersects two of the elements $\bar{\alpha}_{v1}, \dots, \bar{\alpha}_{v,m(v)}$. Let $\mathfrak{W}$ be a finite open cover of X which refines both $\mathfrak{V}$ and $\mathfrak{V}'$. As before, there exist arbitrarily fine subdivisions T' of T, such that for each vertex v' of T', there are open sets $\beta_{v',1}, \dots, \beta_{v',m'(v')}$ such that:

1. $f^{-1}(\mathrm{St}'\,(v')) = \bigcup_{i=1}^{m'(v')} \beta_{v',i}$;
2. $\beta_{v',i} \cap \beta_{v',j} = 0$, unless $i = j$; and
3. $\{\beta_{v',i}\}_{i=1}^{m'(v')}$ refines $\mathfrak{W}$.

By (3) and the definition of $\mathfrak{V}'$ we obtain

3′. If v is a vertex of T, then no one of the sets $\beta_{v',1}, \dots, \beta_{v',m'(v')}$ intersects two of the sets $\bar{\alpha}_{v,1}, \dots, \bar{\alpha}_{v,m(v)}$.

Let $\beta = \{\beta_{v',i} : v'$ is a vertex of T' and $i = 1, \dots, m'(v')\}$. Then Lemma 3.2 applies, yielding the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, "g'"'] & & E_n \\
& {|X_\alpha|} \arrow[ur, "h"] & \\
& {|X_\beta|} \arrow[u, "\Phi"] \arrow[uur, "h'"'] &
\end{tikzcd}$$

in which g' is a barycentric β-map, h' is simplicial relative to T', and Φ is simplicial relative to a subdivision $X_{\alpha 1}$ of X_α. Now as $h\,|\,\bar{\sigma}$ is a homeomorphism for each simplex σ of X_α, the mesh of $X_{\alpha 1}$ is directly related to that of T'; we suppose that T' was chosen to be so fine that the mesh of $X_{\alpha 1}$ is less than ε.

For each n-simplex σ' of T', let $p_{\sigma'}$ be a point of σ' at which the minimum of $M_f\,|\,\sigma'$ is attained. Let L denote the collection of all n-simplexes Δ of X_β such that $h'(\Delta) \cap \{p_{\sigma'} : \sigma' \text{ is an } n\text{-simplex of } T'\} = 0$, and let X_γ denote the complex $X_\beta - L$. Then for any n-simplex σ' of T', it follows from the fact that $f = h' g'$ that there are at most $M_f(p_{\sigma'})$ simplexes of X_γ which map onto σ'.

The notation X_γ anticipates the following definition of an open cover γ having X_v as its realized nerve: as $g'(X)$ covers no simplex of L, there exists a retraction s of $g'(X)$ into $|X_\gamma|$. Let $g'' = s g'$ and $v = \{g''^{-1}(\mathrm{St}\,(v)) : v \text{ is a vertex of } X_v\}$. This suffices; furthermore, $g'' : X \to |X_\gamma|$ is a barycentric γ-map and γ refines β and hence $\mathfrak{V}$. The triple $(\gamma, g'', \Phi\,|\,X_\gamma|)$ is n-canonical and, as the mesh of $X_{\alpha 1}$ is less than ε, $\varrho(\Phi g'', g) = \varrho(\Phi g'', \Phi g') = \varrho(\Phi s, \Phi\,|\,g'(X)) < \varepsilon$. Lastly, $e_n(h\Phi\,|\,|X_\gamma|) = e_n(h'\,|\,|X_\gamma|) = \sum_{\sigma' \in T'} M_f(p_{\sigma'}) \cdot a_n(\sigma')$, a lower Riemann sum of M_f. Thus, as ε and $\mathfrak{V}$ are arbitrary, $e_n^*(\alpha, g, h) \leqslant \underline{\int} M_f(p)\,dp$ and the proof is complete.

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