ON JORDAN ARCS AND LIPSCHITZ CLASSES OF FUNCTIONS DEFINED ON THEM

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Introduction

Let the equations

$$x = f(t), \quad y = g(t) \qquad (0 \leqslant t \leqslant 1),$$

define a continuous arc in the plane E_2 and let us assume that the derivative of $g(t)$ with respect to $f(t)$ vanishes everywhere. According to Lebesgue ([4], p. 296) this means that

$$\lim_{h \to 0} \frac{g(t+h) - g(t)}{f(t+h) - f(t)} = 0 \quad (0 \leqslant t \leqslant 1),$$

where we ignore as $h \to 0$ those values of h which produce simultaneously vanishing increments Δf and Δg and where the above limit relation is assumed to hold, by definition, in the interior of any common interval of constancy for f and g. Lebesgue showed that $g(t)$ is necessarily constant provided that we assume $f(t)$ to be of bounded variation. R. Caccioppoli [2] and J. Petrovski [6] showed that $g(t)$ is constant even without the last additional assumption concerning $f(t)$.

H. Whitney [8] showed that the situation is different for skew arcs: Whitney constructs in the complex x-plane a Jordan arc

$$J : x = f(t) \quad (0 \leqslant t \leqslant 1), \tag{1}$$

(1) The main results of the present paper were announced in the note: *Sur les arcs ascendants à pente partout nulle et des problèmes qui s'y rattachent, C. R. Acad. Paris,* 249 (1959), 1079–1080. Subsequently M. G. Glaeser kindly brought to our attention the references [3] and [8] which helped us to shorten and improve our paper.

and also a real-valued, non-decreasing, non-constant continuous function $g(t)$ in $[0, 1]$ such that

$$\lim_{h\to 0} \frac{g(t+h)-g(t)}{|f(t+h)-f(t)|}=0 \quad \text{for all } t \text{ in } [0, 1]. \tag{2}$$

It is clear that the point $(f(t), g(t))$ describes a Jordan arc, in the 3-dimensional space, which is rising while having, in view of (2), everywhere vanishing slopes with respect to the complex x-plane which is thought of as horizontal. For a particularly simple example of such a skew arc (whose projection J is the arc of H. von Koch) see G. Glaeser ([3], 57–58).

Our first result is

THEOREM 1. *There exists in the complex x-plane a Jordan arc J, having the following properties: Let v and v' be distinct points of J and let $J(v, v')$ be the subarc of J having v, v' as end points while $m_2 J(v, v')$ denotes its 2-dimensional Lebesgue measure. To every positive ε there corresponds a constant C_ε such that for all subarcs*

$$0 < m_2 J(v, v') < C_\varepsilon |v-v'|^{2-\varepsilon}. \tag{3}$$

An arc enjoying these properties will be constructed in § 1 below. Before we discuss the significance of Theorem 1 let us first show how it furnishes one more example of an arc of the kind first constructed by Whitney. To obtain it we erect at each point v, of J, an ordinate $y = G(v) = m_2 J(0, v)$. This is a continuous point-function on J which increases strictly in view of the first inequality (3): If $J(0, v)$ is a proper subarc of $J(0, v')$ then $G(v') - G(v) = m_2 J(v, v') > 0$. By (3)

$$\frac{G(v')-G(v)}{|v'-v|} = \frac{m_2 J(v, v')}{|v-v'|} < C_\varepsilon |v-v'|^{1-\varepsilon}.$$

If we select $\varepsilon < 1$ and let $v' \to v$ we see that the skew arc described by $(v, G(v))$, $v \in J$, has everywhere a vanishing slope.

Observe that the ε appearing in Theorem 1 is required to be positive. This is not an accident because of

THEOREM 2. *Let J be a plane Jordan arc such that $m_2 J > 0$. Then*

$$\overline{\lim_{v'\to v}} \frac{m_2 J(v, v')}{|v-v'|^2} = +\infty \tag{4}$$

holds at almost all points v, of J, in the sense of the m_2-measure.

This result allows an application to the notion of lower quadratic length of arcs. We use the following

DEFINITION 1. *Let the complex-valued function* $x = f(t)$, $(0 \leqslant t \leqslant 1)$, *describe a continuous arc* B *in the plane. If* $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$, *we define the lower quadratic length of* B *by*

$$L^{(2)} B = \varliminf \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2, \tag{5}$$

where the limes inferior is taken as $\max |t_i - t_{i-1}| \to 0$.

It was shown by A. Ville [7] that $L^{(2)} B = 0$ provided that $m_2 B = 0$. It now turns out that the additional condition may be ignored since we have the following

THEOREM 3. *The lower quadratic length of any plane continuous arc vanishes.*

Using Theorem 2 we first prove Theorem 3 for the case of a Jordan arc (Section 2.2). A lemma to the effect that any continuous arc may be reduced to a Jordan arc by removing appropriate loops easily allows to complete a general proof of Theorem 3 (Section 2.3).

In contrast to Theorem 2 we have a different situation for Jordan arcs of finite α-dimensional Hausdorff measure; we state this as

THEOREM 4. *Let* $1 < \alpha < 2$. *There are plane Jordan arcs of finite and positive* Λ^{α}*-measure such that*

$$\frac{\Lambda^{\alpha} J(v, v')}{|v - v'|^{\alpha}} < K \tag{6}$$

for all subarcs $J(v, v')$.

However, a weaker analogue of Theorem 2 still holds which shows that the exponent α of $|v - v'|^{\alpha}$ in (6) can not be increased. Indeed, we have

THEOREM 5. *If* J *is a plane Jordan arc such that* $0 < \Lambda^{\alpha} J < \infty$, $1 < \alpha < 2$ *then*

$$\varlimsup_{v' \to v} \frac{\Lambda^{\alpha} J(v, v')}{|v - v'|^{\alpha}} \geqslant 1 \tag{7}$$

at almost all points v *in the sense of the* Λ^{α}*-measure.*

We turn now to a discussion of Lipschitz classes of point-functions $G(v)$ defined on an arc J. We shall use the following

DEFINITION 2. *Let* $\phi(x)$ *be defined for* $x > 0$ *and be positive, continuous, nondecreasing and such that* $\phi(+0) = 0$. *Let* J *be a Jordan arc and* $G(v)$ *be defined on* J.

We write $G(v) \in \mathrm{Lip}_J\, \phi(x)$ *provided that there is a finite-valued positive function* $A(v)$ *such that*

$$|G(v) - G(v')| < A(v)\, \phi(|v - v'|), \quad (v,\, v' \in J,\ v \neq v'), \tag{8}$$

and we say that $G(v)$ *is of Lipschitz class* $\phi(x)$ *along* J. *If* $A(v)$ *is bounded we write*

$$G(v) \in \mathrm{U\ Lip}_J\, \phi(x)$$

and say that $G(v)$ *is uniformly of Lipschitz class* $\phi(x)$ *along* J.

It is well known that if J is the segment $[0, 1]$ and $\phi(x) = o(x)$, as $x \to 0$, then constants are the only elements of the class $\mathrm{Lip}_J\, \phi(x)$. The situation is different for plane arcs J: For the arc J of Theorem 1 and the function $G(v) = m_2 J(0, v)$ we see from (3) that

$$G(v) \in \mathrm{U\ Lip}_J\, x^{2-\varepsilon} \quad (\varepsilon > 0),$$

while $G(v)$ is certainly not constant.

What about the class $\mathrm{U\ Lip}_J\, x^2$ obtained by letting here ε become zero? The answer becomes obvious if we apply our Theorem 3. Indeed, let $G(v)$ satisfy the inequality

$$|G(v) - G(v')| < A\, |v - v'|^2 \quad (v,\, v' \in J;\ A \text{ const.}).$$

If J is traced out by $x = f(t)$, $0 \leqslant t \leqslant 1$, and if α and β are the endpoints of J then

$$|G(\beta) - G(\alpha)| \leqslant \sum |G(f(t_i)) - G(f(t_{i-1}))| < A \sum_1^n |f(t_i) - f(t_{i-1})|^2.$$

However, we know that the last-written sum will converge to zero for an appropriate sequence of divisions by virtue of Theorem 3. Thus $G(\alpha) = G(\beta)$. Since this argument may be applied to any subarc we have established

THEOREM 6. *If* J *is a plane Jordan arc and the function* $G(v)$ *is uniformly of the Lipschitz class* x^2 *along* J, *then* $G(v)$ *is necessarily a constant.*

Let now J be a Jordan arc in the plane such that

$$\Lambda_\alpha J < \infty \quad (1 < \alpha < 2).$$

By Theorem 4 we see that Theorem 6 does not generalize to such arcs, for if J is an arc as described by Theorem 4 and $G(v) = \Lambda^\alpha J(0, v)$ then (6) implies that $G(v) \in \mathrm{U\ Lip}_J\, x^\alpha$ while $G(v)$ is not constant. However, a slightly weaker analogue of Theorem 6 holds which we state as

THEOREM 7. *If J is a plane Jordan arc of finite Λ^α-measure, $1 < \alpha < 2$, and $G(v)$ is of Lipschitz class $\phi(x)$ along J, then*

$$\phi(x) = o(x^\alpha) \quad as \quad x \to 0, \tag{9}$$

implies that $G(v)$ is a constant.

We conclude our Introduction with a few results when the Jordan arc J is in a space of dimension higher than two. There is a natural generalization of Theorem 7:

If $J \subset E_n$, $\Lambda^\alpha J < \infty$, $1 < \alpha \leqslant n$ and $G(v) \in \mathrm{Lip}_J\, \phi(x)$, then

$$\phi(x) = o(x^\alpha) \tag{10}$$

implies that $G(v)$ in a constant.

Theorem 7 and its generalization just stated suggest that if J is an arc of the real Hilbert space H, again the class $\mathrm{Lip}_J\, \phi(x)$ will contain only constants provided that the scale-function $\phi(x)$ tends to zero sufficiently fast as $x \to +0$. However, it is a curious fact that such is not the case and we state this as our last

THEOREM 8. *Let $\phi(x)$ be a given scale-function subject to the conditions of Definition 2. There are in the Hilbert space H Jordan arcs J such that the class* $\mathrm{U}\,\mathrm{Lip}_J\, \phi(x)$ *contains functions which are not constants.*

Observe that the scale-function $\phi(x)$ may tend to zero as fast as we wish.

§ 1. Proof of Theorem 1

1.1. THE CONSTRUCTION OF THE ARC J. Let S_0 be the unit-square, one side of which connects $x=0$ to $x=1$. This and all following squares will be assumed to be closed. We shall now construct a continuum J_1 as follows: Let

$$\theta_n = \frac{1}{2} - \frac{1}{16\,n^2} \quad (n = 1, 2, \ldots). \tag{1.1}$$

In S_0 we construct four corner squares $s_1^1, s_1^2, s_1^3, s_1^4$ of sides $=\theta_1$. We now connect these squares by three segments (or links) as shown in fig. 1, obserwing that two of these links lie along the two vertical sides of S_0 while the third link ab lies on the line which carries the two lower sides of s_1^2 and s_1^3.

On the link ab we consider its Cantor middle-third set γ and in particular its complementary set of intervals. On each of these intervals as side we construct a square, lying above ab, and denote by σ the set of squares so obtained. We now form the union $[a, b] \cup \sigma$ which is evidently a continuum joining a to b. We repeat

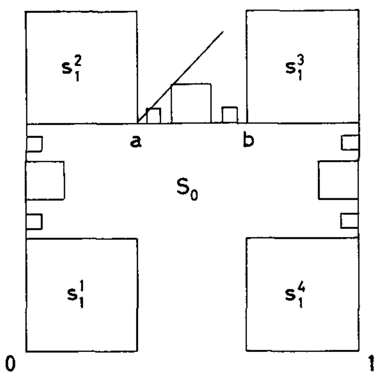

Fig. 1.

the same construction on each of the remaining two links placing the sets of squares as indicated in fig. 1. This completes the construction of the continuum J_1. Observe that J_1 is composed of four *corner* squares, enumerably many *intermediate* squares and finally three Cantor sets. Leaving out the Cantor sets we have a collection of squares s_1^i which we denote by S_1. We establish an order relation among the elements of $S_1 = \{s_1^i\}$ obtained by traversing J_1 from $x=0$ to $x=1$. Each square s_1 has an *entry point* and an *exit point* defined in an obvious way.

The second step of our construction is as follows: In each square s_1 $(s_1 \in S_1)$ we join its entry point to its exit point by a continuum similar in structure to J_1, the only difference being that the sides of its four corner squares are now $= \theta_2 \cdot \text{side } s_1$. Replacing in J_1 each square s_1 by its sub-continuum so constructed we obtain our second continuum J_2. It is composed of a set S_2 of squares $s_2 = s_2^i$ and enumerably many Cantor sets.

This construction is now repeated indefinitely by obtaining J_n from J_{n-1} by replacing each s_{n-1} $(\in S_{n-1})$ by a continuum similar in structure to J_1, having 4 corner squares of sides $= \theta_n \cdot \text{side } s_{n-1}$. $S_n = \{s_n\}$ will denote the set of squares of J_n.

Evidently
$$J_1 \supset J_2 \supset \dots$$
and
$$J = \bigcap_{\nu=1}^{\infty} J_\nu \tag{1.2}$$
is easily shown to be a Jordan arc joining the point $x=0$ to $x=1$.

Let us show that
$$m_2 J > 0. \tag{1.3}$$

To see this let $\sum_n$ $(n=1, 2, \dots)$ denote the set of those 4^n elements of S_n which are obtained by constructing, starting from S_0, only corner squares while omitting the intermediate squares altogether. These 4^n elements of $\sum_n$ are squares of sides $=\theta_1\theta_2\ldots\theta_n$ and $J_n\supset\sum_n$. By (1.1) and (1.2) we therefore find

$$m_2 J=\lim_{n\to\infty} m_2 J_n \geqslant \lim m_2 \sum_n = \lim 4^n(\theta_1\theta_2\ldots\theta_n)^2=\prod_{\nu=1}^{\infty}\left(1-\frac{1}{8\nu^2}\right)^2>0$$

and (1.3) is established.

A similar discussion shows easily that every subarc $J(v, v')$ of J has positive m_2-measure and this already establishes the first inequality (3). We now turn to a proof of the second inequality (3).

1.2. Proof of Theorem 1. A proof of the second inequality (3) will require a closer discussion of the relation between a subarc $J(v, v')$ and the squares s_n of the continuum J_n. The inclusion relation $J(v, v')\subset s_n$ requires no explanation; if $s_n\cap J\subset J(v, v')$ then we shall say that $J(v, v')$ contains the square s_n, or that s_n is contained in $J(v, v')$. The symbol s_n will also be used to denote the area of the square s_n. The square s_n contains four corner squares s_{n+1}; the least distance or the width of the corridor between two of these will be denoted by corr s_n, its value being

$$\text{corr } s_n=(1-2\theta_{n+1})\text{ side } s_n=\frac{1}{8(n+1)^2}\text{ side } s_n. \tag{1.4}$$

Our proof is based on the following preliminary remarks:

1. *The distance between two complementary intervals of the Cantor set is at least equal to the length of the smaller interval. The distance between a complementary interval and an endpoint of the Cantor set is never less than the length of the interval.*

2. *Given $\varepsilon>0$ there is a constant B_ε such that*

$$\frac{s_n}{(\text{corr } s_n)^{2-\varepsilon}}<B_\varepsilon \tag{1.5}$$

for all n and all squares s_n.

Omitting the simple proof of the first remark, we turn to the second. In view of (1.4) and the evident inequality side $s_n<2^{-n}$, we obtain

$$\frac{s_n}{(\text{corr } s_n)^{2-\varepsilon}}<8^{2-\varepsilon}(n+1)^{2(2-\varepsilon)}(\text{side } s_n)^{\varepsilon}<8^2(n+1)^4 2^{-\varepsilon n},$$

which is a bounded sequence and (1.5) is established.

Given the arc $J(v, v')$ we define the integer n such that $J(v, v')$ is contained in a square s_n but not in any s_{n+1}. We now distinguish three cases depending on the relation of $J(v, v')$ to the four corner squares of s_n.

1. *$J(v, v')$ contains points of at least two corner squares of s_n.* From the definition of corr s_n and the inequality (1.5) we obtain that

$$\frac{m_2 J(v, v')}{|v - v'|^{2-\varepsilon}} < \frac{s_n}{(\text{corr } s_n)^{2-\varepsilon}} < B_\varepsilon. \tag{1.6}$$

2. *$J(v, v')$ fully contains a corner square s_{n+1}, of s_n, but does not contain points of any of the other corner squares of s_n.* A glance at fig. 1 (where the large square now represents s_n) shows that

$$|v - v'| > \frac{1}{2} \text{ side } s_{n+1} = \frac{1}{2}\theta_{n+1} \text{ side } s_n > \frac{1}{8} \text{ side } s_n.$$

But then
$$\frac{m_2 J(v, v')}{|v - v'|^{2-\varepsilon}} < \frac{s_n}{(8^{-1} \text{ side } s_n)^{2-\varepsilon}} < 8^2 \, (\text{side } s_n)^\varepsilon \leqslant 8^2. \tag{1.7}$$

3. *In the remaining cases* (see fig. 1) all the squares s_{n+1} containing points of $J(v, v')$ are based on one and the same straight line. This will imply that the arc $J(v, v')$ is fairly stretched, in fact we shall prove the following: *If*

$$d = \text{diam } J(v, v') \tag{1.8}$$

then
$$|v - v'| \geqslant \frac{1}{13} d. \tag{1.9}$$

Indeed, let $v \in s^i_{n+1}$, $v' \in s^j_{n+1}$ and to fix the ideas we shall assume that the square s^i_{n+1} does not exceed s^j_{n+1} in size. Let v_0 be the orthogonal projection of v on to the common base line of our squares. Let v_1 be the exit point of s^i_{n+1} and v_2 the entry point of s^j_{n+1}.

We distinguish two cases depending on whether $|v_2 - v_1|$ is $\geqslant d/13$ or $< d/13$. In the first case when $|v_2 - v_1| \geqslant d/13$ it is evident that also

$$|v - v'| \geqslant \frac{1}{13} d. \tag{1.10}$$

Let us now assume
$$|v_2 - v_1| < \frac{1}{13} d.$$

The opening remark 1 of Section 1.2 implies that side $s^i_{n+1} \leqslant |v_2 - v_1| < d/13$ and *a fortiori*

$$|v - v_0| \leqslant \frac{1}{13} d, \tag{1.11}$$

as well as

$$\text{diam } J(v, v_2) \leqslant \text{diam } J(v, v_1) + \text{diam } J(v_1, v_2) < \frac{\sqrt{2}}{13} d + \frac{\sqrt{2}}{13} d < \frac{3}{13} d.$$

We now conclude from (1.8) that

$$\operatorname{diam} J(v_2, v') > \frac{10}{13} d. \tag{1.12}$$

Consider now the sequence of corner squares $s_{n+\nu} \subset s^j_{n+1}$ which have the common entry point v_2 ($\nu = 1, 2, \ldots$; $s_{n+1} = s^j_{n+1}$) and let p be such that

$$v' \in s_{n+p}, \quad v' \notin s_{n+p+1}.$$

By (1.12) $$\operatorname{diam} s_{n+p} \geqslant \operatorname{diam} J(v_2, v') \geqslant \frac{10}{13} d$$

and therefore

$$|v' - v_2| > \text{side } s_{n+p+1} > \frac{1}{3\sqrt{2}} \operatorname{diam} s_{n+p} \geqslant \frac{10}{3\sqrt{2}\cdot 13} d > \frac{2}{13} d.$$

But then *a fortiori* $$|v' - v_0| > \frac{2}{13} d.$$

This and (1.11) imply (1.9) which has now been shown to hold in any case.

Returning to our proof of (3) we observe (1.8) implies that $m_2 J(v, v') < d^2$ and now by (1.9)

$$\frac{m_2 J(v, v')}{|v - v'|^{2-\varepsilon}} < 13^2 d^{\varepsilon} \leqslant 13^2. \tag{1.13}$$

The estimates (1.6), (1.7) and (1.13) establish (3) and our proof is completed.

§ 2. The lower quadratic length of plane arcs

2.1. Proof of Theorem 2. The key to our discussion of quadratic length is Theorem 2 which we are now going to establish. Let $m_2 J > 0$. Denote by E the set of points v of J to which corresponds some v' ($\neq v$) such that

$$\frac{m_2 J(v, v')}{|v - v'|^2} > A > 8, \tag{2.1}$$

where A is a certain constant. E is open. Let E_1 be the complement of E on J. We shall show that, for any A, $m_2 E_1 = 0$.

Suppose that for a certain A this is not true, hence $m_2 E_1 > 0$, and let an interior point v_0 of J be a density point of E_1. Then to any $\eta > 0$ corresponds an $r_0 > 0$ such that

$$m_2 \{c(v_0, r) - E_1\} < \eta^2 r^2 \quad \text{if} \quad r < r_0, \tag{2.2}$$

where $c(v_0, r)$ denotes the circle having center v_0 and radius r; this circle and all circles of the present discussion will be considered to be closed. At this point we select positive quantities δ, η, r subject to the inequalities

$$\delta < \frac{2}{A}, \quad \eta \sqrt{32A} < \delta, \quad 4Ar < r_0, \tag{2.3}$$

and notice that the first two imply $\eta^2 32A < \delta^2 < \delta \cdot 2/A$ hence

$$16A^2\eta^2 < \delta. \tag{2.3'}$$

We shall use the order relation among the points of J using the symbol $\prec$, and shall speak of the first and last point of J in a given closed set, denoting both as the *extreme* points of J. Let now v_1 and v_2, $v_1 \prec v_0 \prec v_2$ be the extreme points of J belonging to the circle $c(v_0, r)$ so that no $v \prec v_1$, or $v \succ v_2$ belongs to the circle. Obviously $|v_1 - v_0| = |v_2 - v_0| = r$, while the arc $J(v_1, v_2)$ need not belong entirely to the circle $c(v_0, r)$. In fact the diameter of the arc $J(v_1, v_2)$ may well be large compared with $2r$. As v_0 does not belong to E we have

$$m_2 J(v_1, v_0) \leqslant A\,|v_1 - v_0|^2 = Ar^2,$$

$$m_2 J(v_0, v_2) \leqslant A\,|v_2 - v_0|^2 = Ar^2,$$

and therefore

$$m_2 J(v_1, v_2) \leqslant 2Ar^2. \tag{2.4}$$

Writing

$$U = E_1 J(v_1, v_2)$$

we have *a fortiori*

$$m_2 U \leqslant 2Ar^2. \tag{2.5}$$

We denote by $d(p; U)$ the distance from the point p to the set U and by $\{l, U\}$ the set of points p of the plane such that $d(p, U) \leqslant l$ and *not belonging to* U. We shall now study the set

$$V = \{\delta r, U\} \cdot c(v_0, 4Ar)$$

in its relation to the arc J. First we add to V such points of U which lie in $c(v_0, 4Ar)$ to obtain the closure $\overline{V}$. Let now v_3 and v_4 be the extreme points of J belonging to $\overline{V}$ and let us show that

$$v_3 \prec v_1, \quad v_2 \prec v_4. \tag{2.6}$$

To see this we have to show that $v_1 \in \overline{V}$ and $v_2 \in \overline{V}$. Suppose that $v_1 \notin \overline{V}$ so that $E_1 c(v_1, \delta r) = \emptyset$. But evidently

$$m_2 \{c(v_0, r) \cdot c(v_1, \delta r)\} > \delta^2 r^2,$$

while the set of point of $c(v_0, r)$ which are not in E_1 is of measure $<\eta^2 r^2$, which is $<\delta^2 r^2$. A similar argument shows that $v_2 \in \overline{V}$ and the relations (2.6) are established. We finally observe that v_3 and v_4 can not belong to $U = E_1 J(v_1, v_2)$ and therefore v_3 and v_4 lie in V. Thus v_3 and v_4 are also the extreme points of J belonging to V. Since v_3 and v_4 belong to $\{\delta r, U\}$, there are points v_1', v_2' of U such that

$$|v_3 - v_1'| \leqslant \delta r, \quad |v_4 - v_2'| \leqslant \delta r. \tag{2.7}$$

By (2.2) and the last condition (2.3)

$$m_2 \{c(v_0, 4Ar) - E_1\} < \eta^2 16 A^2 r^2$$

from which, in view of $V \subset c(v_0, 4Ar)$, we conclude that

$$m_2(V - VE_1) < \eta^2 16 A^2 r^2. \tag{2.8}$$

But the part of E_1 that belongs to V lies on the arc $J(v_3, v_4)$, and again the points of $E_1 J(v_1, v_2) = U$ do not belong to V. We conclude that

$$E_1 V \subset J(v_3, v_1) + J(v_2, v_4). \tag{2.9}$$

Now

$$V = (V - E_1 V) + E_1 V,$$

$$m_2 V = m_2(V - VE_1) + m_2 E_1 V,$$

and (2.8), (2.9) imply

$$m_2 V \leqslant \eta^2 16 A^2 r^2 + m_2 J(v_3, v_1) + m_2 J(v_2, v_4)$$

and *a fortiori*

$$m_2 J(v_3, v_1') + m_2 J(v_2', v_4) > m_2 V - \eta^2 16 A^2 r^2. \tag{2.10}$$

To estimate $m_2 V$ from below we shall introduce polar coordinates (ϱ, θ) with the origin at v_0, and we write

$$l(\theta) = \bigcup_{\varrho \geqslant 0} (\varrho, \theta), \quad (r_1, r_2, \theta) = \bigcup_{r_1 \leqslant \varrho \leqslant r_2} (\varrho, \theta).$$

We consider the set of directions

$$\Theta_1 [\theta \,|\, U \cdot ((1-\eta) r, r, \theta) = \emptyset].$$

Observing that all points of $E_1 c(v_0, r)$ lie on the arc $J(v_1, v_2)$, we have

$$E_1 c(v_0, r) = E_1 J(v_1, v_2) c(v_0, r) = U c(v_0, r)$$

and thus

$$c(v_0, r) - E_1 = c(v_0, r) - U.$$

By (2.2) $$m_2\{c(v_0, r) - U\} < \eta^2 r^2,$$

from which it follows at once that

$$m\,\Theta_1 < 2\,\eta. \tag{2.11}$$

Consider now the set

$$\Theta_2\,[\theta \,|\, m\{U\cdot(r,\, 4\,A\,r,\, \theta)\} > (4\,A - 1 - \delta)\,r].$$

By (2.5) and writing $S_\theta = (r,\, 4\,A\,r,\, \theta)$ we have

$$2\,A\,r^2 > m_2\,U \geqslant m_2\,U\cdot \bigcup_{\theta\in\Theta_2} (r,\, 4\,A\,r,\, \theta)$$

$$= \iint \varrho\, d\varrho\, d\theta = \int_{\Theta_2} d\theta \int_{U\cdot S_\theta} \varrho\, d\varrho > r \int_{\Theta_2} d\theta \int_{U\cdot S_\theta} d\varrho > r^2\,(4\,A - 1 - \delta)\, m\,\Theta_2.$$

Hence $$m\,\Theta_2 < \frac{2\,A}{4\,A - 1 - \delta} < \frac{8}{15}. \tag{2.12}$$

Consider finally the set Θ_3 which is the complement of $\Theta_1 + \Theta_2$.

By (2.11) and (2.12) $$m\,\Theta_3 > 2\,\pi - 1. \tag{2.13}$$

Let $C\,U$ denote the complement of U. For any $\theta \in \Theta_3$ the segment $((1-\eta)\,r,\, r,\, \theta)$ contains points of U while $m\{(r,\, 4\,A\,r,\, \theta)\,U\} \leqslant (4\,A - 1 - \delta)\,r$ and therefore

$$m\{(r,\, 4\,A\,r,\, \theta)\cdot C\,U\} > \delta\, r. \tag{2.14}$$

Consider now, for a fixed $\theta \in \Theta_3$, the intersections of the sets U and $C\,U$ with the closed segment $((1-\eta)\,r,\, 4\,A\,r,\, \theta)$: Its intersection with U is a closed non-void set while its intersection with $C\,U$ is an open set, i.e. a collection of non-overlapping open intervals of total measure $> \delta\, r$ by (2.14). If none of these intervals exceeds $\delta\, r$ in length then they belong to $\{\delta\, r,\, U\}$ by the definition of this set. If one of these intervals, I say, exceeds $\delta\, r$ in length then obviously the two sub-intervals of length $\delta\, r$, co-terminal with I, must belong to $\{\delta\, r,\, U\}$. In any case we have shown that

$$m\,(((1-\eta)\,r,\, 4\,A\,r,\, \theta)\,\{\delta\, r,\, U\}) \geqslant \delta\, r.$$

Now by (2.13)

$$m_2 \bigcup_{\theta\in\Theta_3} [((1-\eta)\,r,\, 4\,A\,r,\, \theta)\,\{\delta\, r,\, U\}] > (1-\eta)\,r\,\delta\, r\, m\,\Theta_3 > 5\,r^2\,\delta$$

and *a fortiori*, by the definition of V,

$$m_2\,V > 5\,r^2\,\delta.$$

By (2.10) and (2.3′)

$$m_2 J(v_3, v_1') + m_2 J(v_3', v_4) > 4 r^2 \delta$$

and at least one of the terms on the left side, say the first one, satisfies the inequality

$$m_2 J(v_3, v_1') > 2 r^2 \delta.$$

Now by (2.7) and (2.3)

$$\frac{m_2 J(v_3, v_1')}{|v_3 - v_1'|^2} > \frac{2 r^2 \delta}{\delta^2 r^2} = \frac{2}{\delta} > A$$

which is impossible because $v_1' \in E_1$. This contradiction establishes Theorem 2.

2.2. Proof of Theorem 3 when B is a Jordan arc. Let $J = J(0, 1)$ be a Jordan arc. Given ε we are to show that we can inscribe a polygon of vertices

$$0 = u_0 \prec u_1 \prec \ldots \prec u_s = 1, \tag{2.15}$$

such that

$$\sum_{i=1}^{s} |u_i - u_{i-1}|^2 < \varepsilon. \tag{2.16}$$

Suppose this to be already established; the additional requirement of the theorem, that (2.16) can be achieved while max $|u_i - u_{i-1}|$ is as small as we please, can now be satisfied in an obvious way. Indeed, we can first subdivide J into a finite sequence of arcs of sufficiently small diameters and then apply the result (2.16) to each of these arcs.

We may ignore the simple case when $m_2 J = 0$ for two reasons:

(1) It is easily disposed of by the second part of our proof which uses coverings U of small $\sum d^2$; (2) It is covered by A. Ville's theorem of 1936. We may therefore assume that $m_2 J > 0$.

Let $\varepsilon > 0$ be given. For $\delta_1 > 0$ denote by E_{δ_1}, the set of those points v, of J, to which correspond points v' with $|v - v'| > \delta_1$ and satisfying the condition

$$\frac{m_2 J(v, v')}{|v - v'|^2} > \frac{2 m_2 J}{\varepsilon}. \tag{2.17}$$

By Theorem 2

$$\lim_{\delta_1 \to 0} m_2 E_{\delta_1} = m_2 J.$$

We assume δ_1 so chosen that

$$m_2 E_{\delta_1} > \frac{2}{3} m_2 J.$$

Denote by $E_{\delta_1}^+$ and $E_{\delta_1}^-$ the disjoint sets of those points of E_{δ_1} to which correspond points $v' \succ v$ or $v' \prec v$ respectively: $E_{\delta_1} = E_{\delta_1}^+ + E_{\delta_1}^-$. Let E_{δ_1}' be one of the sets on the right hand side whose measure is $> \frac{1}{3} m_2 J$. Suppose it is $E_{\delta_1}^+$.

We can obviously select a sequence of disjoint arcs

$$J(v_{11}, v_{11}'), \quad J(v_{12}, v_{12}'), \ \ldots, \quad J(v_{1,n_1}, v_{1,n_1}'),$$

in natural order along J, where $v_{11}, v_{12}, \ldots, v_{1,n_1}$ are points of $E_{\delta_1}' = E_{\delta_1}^+$ and $v_{11}', v_{12}', \ldots, v_{1,n_1}'$ the corresponding points satisfying (2.17), so that the measure of E_{δ_1}' outside these n_1 arcs be as small as we please. These arcs are picked successively along J and their number n_1 is necessarily finite because the m_2-measure of each arc exceeds $2\delta_1^2 m_2 J/\varepsilon$. Writing

$$\Gamma_1 = J(v_{11}, v_{11}') + J(v_{12}, v_{12}') + \ldots + J(v_{1,n_1}, v_{1,n_1}'),$$

we may therefore assume that

$$m_2 \Gamma_1 > \frac{1}{3} m_2 J.$$

Let now $\delta_2 > 0$ and denote by E_{δ_2} the set of those v of $J - \Gamma_1$ to which correspond points v' satisfying (2.17), v' belonging to the same arc of $J - \Gamma_1$ as v, and such that $|v - v'| > \delta_2$. As before $m_2 E_{\delta_2} \to m_2 (J - \Gamma_1)$ as $\delta_2 \to 0$. Assume δ_2 so chosen that

$$m_2 E_{\delta_2} > \frac{2}{3} m_2 (J - \Gamma_1).$$

We now define the set E_{δ_2}' as E_{δ_1}' was defined before and a set Γ_2 of disjoint arcs in $J - \Gamma_1$, such that for each arc $J(v, v')$ of Γ_2 (2.17) holds, while

$$m_2 \Gamma_2 > \frac{1}{3} m_2 (J - \Gamma_1).$$

Similarly sets $\Gamma_3, \Gamma_4, \ldots, \Gamma_k$ are defined successively such that

$$m_2 \Gamma_i > \frac{1}{3} m_2 (J - \Gamma_1 - \ldots - \Gamma_{i-1}) \quad (i = 2, \ldots, k).$$

Since the measure of each Γ_1 exceeds a third of the remaining measure, we can reach a value k such that

$$m_2 (J - \Gamma_1 - \ldots - \Gamma_k) < \frac{\varepsilon}{2}. \tag{2.18}$$

Let $J(v_i, v_i')$, $(i=1, \dots, N)$, be all the arcs of $\Gamma_1 + \dots + \Gamma_k$ is ascending order. By (2.17)

$$\sum_1^N |v_i - v_i'|^2 < \frac{\varepsilon}{2 m_2 J} \sum_1^N m_2 J(v_i, v_i') \leq \frac{\varepsilon}{2}. \tag{2.19}$$

The distance between any pair of arcs of $J - \Gamma_1 - \dots - \Gamma_k$ being positive, let it be greater than $2\alpha\,(>0)$. Denote by $U = U(\alpha, \overline{J - \Gamma_1 - \dots - \Gamma_k})$ a collection of closed convex sets (e.g. squares with sides parallel to fixed directions), each set of diameter $<\alpha$ and such that every point of the closure $\overline{J - \Gamma_1 - \dots - \Gamma_k}$ is an interior point of at least one of the sets. U may always be assumed to consist of a finite number of sets. If we denote by d the diameter of the general set of U then, by (2.18), we can choose U so that

$$\sum_U d^2 < \frac{\varepsilon}{2}. \tag{2.20}$$

We may write

$$J - \Gamma_1 - \dots - \Gamma_k = \sum_{i=0}^N J(v_i', v_{i+1}),$$

where $v_0' = 0$ and $v_{N+1} = 1$, while the first and the last arc of this sum may not exist. Any element of U can cover points of one arc only. Thus we can write

$$U = \sum_{i=0}^N U_i,$$

where U_i consists of those sets of U which cover points of the arc $J(v_i', v_{i+1})$. Clearly, by (2.20),

$$\sum_{i=0}^N \sum_{U_i} d^2 = \sum_U d^2 < \frac{\varepsilon}{2}. \tag{2.21}$$

Take the general arc $J(v_i', v_{i+1})$ and define on it a finite sequence of points

$$v_i' = w_{i,0},\ w_{i,1},\ \dots,\ w_{i,p_i} = v_{i+1} \tag{2.22}$$

in the following way: Let $w_{i,0}$ be interior to the set $U_i^{(1)}$. If also v_{i+1} is in $U_1^{(i)}$ then $p_i = 1$ and we are through. If not, let $w_{i,1}$ be the last point of the arc $J(v_i', v_{i+1})$ which belongs to $U_i^{(1)}$. Clearly $w_{i,1}$ is on the boundary of $U_i^{(1)}$; let $w_{i,1}$ be interior to $U_i^{(2)}$. If also v_{i+1} belongs to $U_i^{(2)}$ then $p_i = 2$ and we stop the process. If not, let $w_{i,2}$ be the last point of $J(w_{i,1}, v_{i+1})$ belonging to $U_i^{(2)}$. Continuing in this way, we obtain the sequence of points (2.22) such that the points $w_{i,j-1}$ and $w_{i,j}$ belong to the same set $U_i^{(j)}$ $(j=1, \dots, p_i)$, where the p_i sets $U_i^{(j)}$ are distinct elements of the collection U_i. We conclude that

$$\sum_{j=1}^{p_i} |w_{i,j-1} - w_{i,j}|^2 \leq \sum_{U_i} d^2$$

and therefore
$$\sum_{i=0}^{N} \sum_{j=1}^{p_i} |w_{i,j-1} - w_{i,j}|^2 \leq \sum_{U} d^2 < \frac{\varepsilon}{2}. \tag{2.23}$$

We have thus obtained the following monotone sequence of points along J:

$$0 = w_{0,0},\ w_{0,1},\ \ldots,\ w_{0,p_0} = v_1,\ v_1' = w_{1,0},\ w_{1,1},\ \ldots,\ w_{1,p_1} = v_2,$$
$$v_2' = w_{2,0},\ \ldots,\ w_{N,p_N} = 1.$$

Denoting them in order by $0 = u_0, u_1, \ldots, u_s = 1$, we have

$$\sum |u_{i-1} - u_i|^2 = \sum |v_i - v_i'|^2 + \sum_{i=0}^{N} \sum_{j=1}^{p_i} |w_{i,j-1} - w_{i,j}|^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by (2.19) and (2.23), and the desired inequality (2.16) is established and therefore also the theorem for the case when B is a Jordan arc.

2.3. A lemma on continuous arcs and proof of Theorem 3. Let B be a non-closed continuous arc in the plane. By omitting from B subarcs with coincident endpoints (loops) we may reduce B to become a Jordan arc J joining the original endpoints of B. A precise description of this intuitive idea is given by (1)

Lemma 1. *Let $x = f(t)$ be an continuous complex-valued function of $t \in I = [0, 1]$ such that $f(0) \neq f(1)$. We can find in I a perfect set F such that the image $f(F)$ is a Jordan arc J, having as endpoints $f(0)$ and $f(1)$, in the sense that the relations*

$$a \in F,\ a' \in F,\ a < a'\ f(a) = f(a') \tag{2.24}$$

hold if and only if the open interval (a, a') is contiguous to F.

Remark 1. The set F is by no means always uniquely defined. An arc B in the shape of a pretzel, with its ends slightly extended, admits three distinct sets F obtained by removing from I appropriate single open intervals.

Remark 2. The lemma and its proof require nothing beyond the continuity of $f(t)$. The lemma therefore holds as stated if the values of $f(t)$ are in a Hausdorff space.

(1) Lemma 1 is a special case of the following Arcwise Connectedness Theorem: *Every two points a and b of a locally connected continuum M can be joined in M by a simple continuous arc.* (See [9], p. 36.) However, a simple proof of Lemma 1 is here included for the reader's convenience.

Proof: We call the open interval $S=(t, t')$ a *loopsegment* provided that $f(t)=f(t')$. Let L denote the totality of loopsegments. Since L is evidently compact, there exists a longest loopsegment which we denote by $S_1=(t_1, t_1')$ and define $F_1=I-S_1$. Observe that if $S \in L$, $S \subset F_1$, then S can not abut on S_1 since their union would give a longer loopsegment. Let S_2 be the longest among the $S \subset F_1$ and consider $F_2=I-S_1-S_2$. We repeat this operation successively obtaining the loopsegments $S_1, S_2, \ldots$ such that the closed segments $\bar{S}_1, \bar{S}_2, \ldots$ are pairwise disjoint and $l(S_1) \geqslant l(S_2) \geqslant \ldots$. Either the process terminates when $F_n=I-S_1-\ldots-S_n$ contains no further loopsegment, or else it continues indefinitely when evidently $l(S_n) \to 0$. In either case let $\Omega=\sum_i S_i$ and consider the perfect set $F=I-\Omega$.

Let (a, a') satisfy the conditions (2.24). We cannot have $[a, a'] \subset F$. Indeed, $(a, a') \in L$ and should have been removed before $l(S_n)$ has become $<a'-a$. Hence $[a, a'] \not\subset F$ and therefore $(a, a') \supset S_i=(t_i, t_i')$ for some i and where *we choose for i the least value which will do.* Now we must have $(a, a')=S_i=(t_i, t_i')$ for if $(a, a') \neq S_i$ then $a'-a>l(S_i)$ and (a, a') should have been removed *before* S_i. This proves our lemma except, perhaps, the main point that J is a Jordan arc. To see this, let $\tau=\tau(t)$ be a continuous non-decreasing function in the range I, $\tau(0)=0$, $\tau(1)=1$ and such that $\tau(t)=\tau(t')$ for $t<t'$ if and only if the interval (t, t') is contained in $\Omega=\sum S_i$. If we now identify the two endpoints t_i and t_i' of S_i for all i, we obtain a set F_1 which by $\tau=\tau(t)$ is homeomorphic with the range $0 \leqslant \tau \leqslant 1$. On the other hand, we have shown that $J=f(F)=f(F_1)$ is a homeomorph of F_1. It therefore follows that J is a homeomorph of the interval $0 \leqslant \tau \leqslant 1$ and our lemma is established.

A general proof of Theorem 3 now becomes obvious. Given $\varepsilon>0$ and applying Theorem 3 to the *Jordan* arc J just constructed we can find a division

$$0=t^{(0)}<t^{(1)}<\cdots<t^{(n)}=1$$

where all $t^{(i)} \in F$ and such that

$$\sum_{i=1}^{n} |f(t^{(i)})-f(t^{(i-1)})|^2<\varepsilon$$

and this already establishes the theorem.

§ 3. On plane Jordan arcs of finite and positive Λ^α-measure

3.1. Proof of Theorem 4. To obtain an arc J having the properties required by Theorem 4 we repeat with some simplifications the construction of § 1.1: We now choose $\theta_n=\theta$, independent of n, satisfying the equation

$$4\,\theta^{\alpha}=1 \quad (1<\alpha<2).$$

Starting as in § 1.1 with the unit square S_0, let the continuum J_1 consist of four corner squares of sides $=\theta$ and of three rectilinear links (fig. 1). J_2 is obtained from J_1, by replacing each square s_1 by a continuum geometrically similar to J_1 (because $\theta_2=\theta_1=\theta$) which joins its entry point to its exit point and so forth. Now $J=\bigcap J_n$ is our present Jordan arc. If we observe that J is covered by collection $\sum_n$ of 4^n squares having diameters $\theta^n\sqrt{2}$, we see that

$$\Lambda^{\alpha} J \leqslant (\sqrt{2})^{\alpha}$$

and we leave it to the reader to show that $\Lambda^{\alpha} J>0$. In terms of the notations of § 1 we can say that J consists of the set

$$\sum=\lim_{n\to\infty}\sum_n=\bigcap\sum_n$$

plus an enumerable set of links whose Λ^{α}-measure is 0. To any arc $J(v, v')$, which is not a rectilinear segment, corresponds a value n such that $J(v, v')\cap\sum$ belongs to one square of $\sum_n$ but to more than one square of $\sum_{n+1}$. From this it follows that

$$\Lambda^{\alpha} J(v, v')<\theta^{n\alpha}\,2^{\frac{1}{2}\alpha}.$$

On the other hand $|v-v'|$ is surely greater than or equal to the width of the corridors of s_n. Since corr $s_n=\theta^n(1-2\,\theta)$ we obtain

$$|v-v'|\geqslant\theta^n(1-2\,\theta).$$

Hence
$$\frac{\Lambda^{\alpha} J(v, v')}{|v-v'|^{\alpha}}<\left(\frac{\sqrt{2}}{1-2\,\theta}\right)^{\alpha}=K$$

which proves Theorem 4.

We might remark that there are plane Jordan arcs J of finite Λ^{α}-measure, $1<\alpha<2$, such that

$$\varlimsup_{v'\to v}\frac{\Lambda^{\alpha} J(v, v')}{|v-v'|^{\alpha}}=+\infty$$

at almost all points v in the sense of Λ^{α}-measure, but we do not dwell on giving an exemple here.

3.2. Proof of Theorem 5. We pick $\delta>0$ and A such that $0<A<1$ and denote by E the set of those points v of J, to which correspond v' satisfying the inequalities

$$\frac{\Lambda^\alpha J(v, v')}{|v-v'|^\alpha} > A, \quad |v-v'| < \delta. \tag{3.1}$$

The set E is either void or open; in any case its complement $E_1 = J - E$ is closed. Let us now show that if

$$\Lambda^\alpha E_1 = 0 \tag{3.2}$$

for every fractional A and every δ then our theorem follows. Indeed, let A_n, $0 < A_n < 1$, and δ_n $(n = 1, 2, \ldots)$ be such that

$$\lim A_n = 1, \quad \lim \delta_n = 0,$$

and let $E^{(n)}$, $E_1^{(n)}$ be the corresponding sets defined above. We assume that $\Lambda^\alpha E_1^{(n)} = 0$ for every n and therefore $F = \bigcap_1^\infty E_1^{(n)}$ also has the property $\Lambda^\alpha F = 0$. If $v \in J - F$ then

$$\frac{\Lambda^\alpha J(v, v_n')}{|v-v_n'|^\alpha} > A_n, \quad |v - v_n'| < \delta_n,$$

for appropriate points v_n', and the inequality (7) follows on letting $n \to \infty$.

To establish (3.2) let us assume that $\Lambda^\alpha E_1 > 0$ and see that we get a contradiction. By the definition of Λ^α-measure, for every $\varepsilon > 0$ we can find a closed convex set U of diameter dU as small as we please, in particular $dU < \delta$, and such that

$$\Lambda^\alpha (E_1 U) > (1-\varepsilon)(dU)^\alpha,$$

and in particular such that

$$\Lambda^\alpha (E_1 U) > A (dU)^\alpha. \tag{3.3}$$

Let v_1, v_2 be the extreme points of J belonging to the closed set $E_1 U$. In particular $J(v_1, v_2) \supset E_1 U$. But then $\Lambda^\alpha J(v_1, v_2) \geqslant \Lambda^\alpha (E_1 U)$ and (3.3) implies

$$\frac{\Lambda^\alpha J(v_1, v_2)}{(dU)^\alpha} > A$$

and *a fortiori*

$$\frac{\Lambda^\alpha J(v_1, v_2)}{|v_1 - v_2|^\alpha} > A, \quad |v_1 - v_2| \leqslant dU < \delta.$$

However, these inequalities imply that v_1 and v_2 belong to E while they actually belong to E_1 by construction. This contradiction establishes the theorem.

§ 4. On Lipschitz classes of functions defined on Jordan arcs

4.1. Proof of Theorem 7. It follows from the assumptions of Theorem 7, namely $G(v) \in \mathrm{Lip}_J\, \phi(x)$ and (9) that to any $a > 0$, however small, corresponds a function $\delta(v) > 0$ such that

$$|G(v) - G(v')| \leqslant a\,|v - v'|^\alpha \quad \text{if} \quad |v - v'| \leqslant \delta(v).$$

Let E_n be the set of points v of J, for which

$$|G(v) - G(v')| \leqslant a\,|v - v'|^\alpha \quad \text{if} \quad |v - v'| \leqslant 2^{-n}.$$

The set E_n is closed and $\lim E_n = J$. Take a sequence $\{\varepsilon_n\}$, $\varepsilon_n > \varepsilon_{n+1}$, $\varepsilon_n \to 0$, and such that $\sum \varepsilon_n < \Lambda^\alpha J$. Here we assume that $\Lambda^\alpha J > 0$, the proof for the case when $\Lambda^\alpha J = 0$ being a simplified version of the present one. For every n we can find a set $P_n = P_n(E_n - E_{n-1}, 2^{-n})$ of convex sets, each of diameter $< 2^{-n}$, and such that every point of $E_n - E_{n-1}$ is an interior point of at least one of the sets. We shall also assume that

$$\sum_{P_n} d^\alpha < \Lambda^\alpha (E_n - E_{n-1}) + \varepsilon_n,$$

where d denotes the diameter of a general set of the collection P_n. Every point of J is an interior point of at least one convex set of the collection $\sum P_n$ and by the Heine-Borel theorem there is a finite subcollection P of $\sum P_n$ with the same property. We obviously have

$$\sum_P d^\alpha < \Lambda^\alpha J + \sum \varepsilon_n < 2\,\Lambda^\alpha J. \tag{4.1}$$

Let $p^{(1)}, p^{(2)}, \ldots, p^{(k)}$ denote all the convex sets which are elements of P. If $p^{(i)} \in P_n$, then let $v^{(i)}$ be a point of $(E_n - E_{n-1}) \cap p^{(i)}$. Writing $r^{(i)} = d\,p^{(i)} < 2^{-n}$, we construct the circle $c^{(i)} = c(v^{(i)}, r^{(i)})$ which clearly contains $p^{(i)}$.

Thus $$J \subset C = c^{(1)} + c^{(2)} + \ldots + c^{(k)}$$

and by (4.1) $$\sum_{i=1}^{k} (r^{(i)})^\alpha < 2\,\Lambda^\alpha J. \tag{4.2}$$

By the definition of E_n we see that for any $v \in c^{(i)} \cap J$

$$|G(v) - G(v^{(i)})| < a\,|v - v^{(i)}|^\alpha < a\,(r^{(i)})^\alpha$$

and therefore for any pair v', v'' of points of $c^{(i)} \cap J$

$$|G(v') - G(v'')| < 2\,a\,(r^{(i)})^\alpha.$$

Let $v_0 \prec v^*$ by any pair of points of J and let v_0 be an interior point of $c^{(i_0)}$. Denote by v_1 the last point $v \succ v_0$, $v \preceq v^*$, such that $v \in c^{(i_0)}$. We have

$$|G(v_0) - G(v_1)| < 2a(r^{(i_0)})^\alpha.$$

Now v_1 is an interior point of one of the circles of C, say of $c^{(i_1)}$, $i_1 \neq i_0$, and let v_2 be the last point $v \succ v_1$, $v \preceq v^*$, such that $v \in c^{(i_1)}$. As before

$$|G(v_1) - G(v_2)| < 2a(r^{(i_1)})^\alpha$$

and so forth. After a finite number of steps, in fact after $k' \leqslant k$ steps, we shall reach the point v^*. By (4.2) we find

$$\begin{aligned}|G(v_0) - G(v^*)| &\leqslant |G(v_0) - G(v_1)| + |G(v_1) - G(v_2)| + \dots + |G(v_{k'}) - G(v^*)| \\ &< 2a((r^{(i_0)})^\alpha + (r^{(i_1)})^\alpha + \dots + (r^{(i_{k'})})^\alpha) < 4a\Lambda^\alpha J.\end{aligned}$$

Since a was arbitrary we conclude that

$$G(v_0) - G(v^*) = 0$$

which was to be proved.

4.2. Proof of Theorem 8. There remains as our last task to furnish a proof of Theorem 8. Let the positive monotone function $\phi(x)$ of that theorem be given. We select a function $\psi(x)$ which is *convex* and continuously differentiable in the range $[0, 1]$ and such that

$$0 < \psi(x) < \phi(\sqrt{x}) \quad (0 < x \leqslant 1) \quad \psi(0) = 0. \tag{4.3}$$

Let
$$t = A\,\psi(x), \quad A = \pi/\psi(1). \tag{4.4}$$

This is a relation which maps the range $0 \leqslant x \leqslant 1$ onto $0 \leqslant t \leqslant \pi$. We now invert (4.4) obtaining the *concave* increasing function

$$x = (F(t))^2 \quad (0 \leqslant t \leqslant \pi, \quad F(t) \geqslant 0). \tag{4.5}$$

We now consider the function

$$h(t) = 1 - (F(t))^2 \quad (0 \leqslant t \leqslant \pi), \tag{4.6}$$

which has the following properties: $h(0) = 1$, $h(\pi) = 0$, $h(t)$ is convex in $[0, \pi]$ and continuously differentiable in $(0, \pi)$. Notice in particular that in the range $(0, \pi)$, $h'(x) < 0$ and non-decreasing. We now extend the definition of $h(t)$ to the range $[-\pi, \pi]$

so as to be even, and expand it in cosine series

$$h(t)=\sum_{0}^{\infty} A_\nu \cos \nu t. \tag{4.7}$$

Clearly $A_0>0$. But also all $A_\nu>0$. Indeed

$$\frac{\pi}{2}A_\nu=\int_0^\pi h(t)\cos\nu t\,dt=\frac{1}{\nu}\int_0^\pi(-h'(t))\sin\nu t\,dt>0.$$

the last integral being positive, because $-h'(t)$ is positive and decreasing. (Compare Bochner [1], 76–77.)

We may therefore write the expression (4.7) as

$$h(t)=\sum_{0}^{\infty} 2a_\nu^2\cos\nu t,\quad (a_\nu>0),$$

and in particular, for $t=0$

$$1=\sum_{0}^{\infty} 2a_\nu^2.$$

Now (4.6) gives

$$F^2(t)=\sum_{1}^{\infty} 2a_\nu^2(1-\cos\nu t)=\sum_{1}^{\infty} 4a_\nu^2\sin^2\frac{\nu t}{2}. \tag{4.8}$$

This expansion implies that $F(t)$ is a *screw function* in Hilbert space which corresponds to a closed screw line of that space. We refer to von Neumann and Schoenberg [5] for further information on this subject; we, however, need none whatever, because what we need is perfectly elementary and explained in a few words: We mean that there is in the Hilbert space H a closed curve

$$C: x=f(t),\ (0\leqslant t\leqslant 2\pi;\ f(t) \text{ of period } 2\pi), \tag{4.9}$$

such that for all real t and t'

$$F(t-t')=\|f(t)-f(t')\|. \tag{4.10}$$

This curve is immediately constructed, for (4.8) gives

$$F^2(t-t')=\sum 4a_\nu^2\sin^2\frac{1}{2}\nu(t-t')$$

$$=\sum_{\nu=1}^{\infty}\{(a_\nu\cos\nu t-a_\nu\cos\nu t')^2+(a_\nu\sin\nu t-a_\nu\sin\nu t')^2\}. \tag{4.11}$$

In the space H of real sequences $\{x_n\}_0^\infty$ with $\sum x_n^2 < \infty$ and the usual norm $\|x\| = (\sum x_n^2)^{\frac{1}{2}}$ we indeed see by (4.11) that the closed curve C traced out by

$$f(t) = \{a_1 \cos t,\ a_1 \sin t,\ a_2 \cos 2t,\ a_2 \sin 2t,\ \ldots\} \quad (0 \leqslant t \leqslant 2\pi)$$

enjoys the property (4.10).

Along the Jordan arc

$$\Gamma : x = f(t) \quad (0 \leqslant t \leqslant \pi), \tag{4.12}$$

we now define the function

$$g(t) = t. \tag{4.13}$$

For any two values t, t' such that $0 \leqslant t < t' \leqslant \pi$

$$\frac{g(t') - g(t)}{\phi(\|f(t') - f(t)\|)} = \frac{t' - t}{\phi(F(t' - t))}$$

and by (4.3) this is

$$< \frac{t' - t}{\psi(F^2(t' - t))} = A,$$

the last equality relation holding because the relation (4.5) is the inverse of (4.4). We have therefore shown that

$$g(t') - g(t) < A\,\phi(\|f(t') - f(t)\|) \quad (0 \leqslant t < t' \leqslant \pi).$$

Returning to our old notation $v = f(t)$, $G(v) = g(t)$, this is precisely the relation

$$|G(v') - G(v)| < A\,\phi(\|v' - v\|)$$

which was to be established and which shows that $G(v) \in U$ $\mathrm{Lip}_J\ \phi(x)$. Since $G(v) = g(t) = t$ ist not a constant our Theorem 8 is thereby established.

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