A CRITICAL TOPOLOGY IN HARMONIC ANALYSIS ON SEMIGROUPS

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Introduction

Throughout this paper S shall denote a discrete Abelian semi-group with an irreducible unit, denoted 0, and with a law of cancellation. Spelled out explicitly the two last conditions read:

$$x_1 + x_2 = 0 \Rightarrow x_1 = x_2 = 0, \tag{1}$$

$$x_1 + y = x_2 + y \Rightarrow x_1 = x_2, \tag{2}$$

for elements x_1, x_2, y belonging to S. A semigroup of this kind possesses a natural partial ordering where $x_1 \leqslant x_2$ means that $y \in S$ exists such that $x_1 + y = x_2$. Since y is unique the notation $x_2 - x_1$ stands for an element in S well defined whenever $x_1 \leqslant x_2$.

On S we postulate the existence of a positive function $\omega(x)$, satisfying the following two conditions:

$$y \leqslant 2x \Rightarrow \omega(y) \leqslant 2\,\omega(x), \tag{3}$$

$$\sum e^{-\lambda_0 \omega(x)} \leqslant 1, \tag{4}$$

where λ_0 is a positive constant. In (4) as in all series in the sequel the summation is extended ower $x \in S$ if no other indication is given. The two previous conditions imply that

$$N(x, S) \equiv \sum_{y \leqslant x} 1 \leqslant e^{2\lambda_0 \omega(x)}. \tag{5}$$

The counting function $N(x, S)$ being finite expresses an intrinsic property of S not shared by all semigroups and particularly not by "half planes" of lattice points considered by Helson and Lowdenslager (cf. [3], [4]).

To S and ω we associate a locally convex topological space of functions $A = A(S, \omega)$ defined as follows: A contains all numeric functions $f(x)$ on S such that for some $\lambda > 0$, varying with f,

$$\|f\|_\lambda = \sup_{x \in S} |f(x)| e^{-\lambda \omega(x)} < \infty.$$

Bounded subsets of A are of the form

$$B_{\lambda, m} = \{f \mid \|f\|_\lambda \leq m\},$$

where λ and m are positive constants. A sequence $\{f_n\}_1^\infty$ converges in A if it is contained in a bounded set and converges pointwise, or equivalently expressed, if for some fixed λ, $\{f_n\}$ is a Cauchy sequence in the λ-norm.

Each continuous linear functional on A has the form

$$(f, \varphi) = \sum f(x) \varphi(x),$$

where φ is a function on S such that for all $\lambda > 0$,

$$\|\varphi\|'_\lambda = \sum |\varphi(x)| e^{\lambda \omega(x)} < \infty. \tag{6}$$

The topology of the dual space A' is determined by the family of norms defined by (6).

By $\chi_{A'}$ we shall denote all characters on S belonging to A', i.e. functions $\xi(x)$ satisfying (6) and the equations

$$\xi(0) = 1, \quad \xi(x+y) = \xi(x) \cdot \xi(y), \quad x, y \in S.$$

The character which equals 1 at $x = 0$ and vanishes elsewhere on S does always belong to $\chi_{A'}$ and shall be denoted e. The Laplace transform of an $f \in A$ is defined by the relation

$$\hat{f}(\xi) = (f, \xi) = \sum f(x) \xi(x), \quad \xi \in \chi_{A'}.$$

The shift operators $T_\tau, \tau \in S$, and their adjoints $T_{-\tau}$ are defined as follows:

$$T_\tau f(x) = \begin{cases} f(x-\tau), & \tau \leq x \\ 0, & \tau \not\leq x \end{cases}$$

$$T_{-\tau} f(x) = f(x+\tau), \quad x, \tau \in S.$$

If $\|f\|_\lambda$ is finite it follows by (3) that $\|T_\tau f\|_{2\lambda} \leq \|f\|_\lambda$ for all $\tau \in S$, so the set of shift operators is uniformly bounded in A. We should also notice that the transform of $T_\tau f$ equals $\xi(\tau) \hat{f}(\xi)$.

After these preliminaries our main problem can be stated. Let A_f denote the closed linear subset of A spanned by the set $\{T_\tau f \mid \tau \in S\}$. If $\hat{f}$ vanishes at a point

$\xi \in \chi_{A'}$, then the same is true of the transform of each $g \in A_f$. Hence, $A_f \neq A$ since e cannot belong to A_f, the transform $\hat{e}$ being everywhere equal to 1. The condition $\hat{f} \neq 0$ is thus necessary for A_f being the whole of A. If this condition is also sufficient we shall say that *the closure theorem holds in* A.

The first and most important result in this general field is Wiener's theorem on the translates in the space $L_1(R)$ which subsequently gave rise to the theory of Banach algebras. Without being precise we recall that the closure theorem is known to be true in a variety of Banach spaces provided the topology forces $f(x)$ to tend sufficiently fast to 0 at infinity. This study originates in the belief that the closure theorem would be true again in certain topological spaces on semigroups if the topology admits $f(x)$ to increase sufficiently fast. This conjecture is supported by results in some specific cases. In a previous paper [2] the closure theorem was shown to be false in the Hilbert space $A = L_2(S)$ on $S = Z^+$ (the additive semigroup of integers ≥ 0) with norm

$$\|f\| = \{\sum |f(x)|^2\}^{\frac{1}{2}}.$$

However, the necessary and sufficient condition given in [2] and implying the property $A_f = A$, permit us to derive the following conclusion: If $\omega(n) = n^\alpha$, $0 < \alpha < 1$, then the closure theorem in $A(Z^+, \omega)$ is false if $\alpha \leq {}^1/_3$ and true if $\alpha > {}^1/_2$.

In this paper we aim to show that the validity of the closure theorem in spaces $A(S, \omega)$ depends on a certain critical rate of growth of ω. To characterize that rate of growth shall be our main goal.

A necessary condition

THEOREM I. *If the closure theorem holds in $A(S, \omega)$ then the series*

$$\sum_{n=1}^{\infty} \frac{\omega(n x)}{n^{3/2}} \tag{7}$$

diverges for each $x \in S$, $x \neq 0$.

Under the assumption that (7) converges for an $x_0 \neq 0$ we shall form a counterexample showing that $\hat{f} \neq 0$ does not imply $A_f = A$. Let $f_0(z)$ be the analytic function

$$f_0(x) = \exp\left(-\frac{1+z}{1-z}\right) = \sum_0^{\infty} a_n z^n, \quad |z| < 1,$$

and define $f(x)$ on S by the conditions

$$f(x) = \begin{cases} a_n, & x = n x_0 \\ 0, & x \notin S_{x_0}, \end{cases}$$

where S_{x_0} denotes the subsemigroup $\{n x_0,\ n=0,1,\ldots\}$. This f does certainly belong to A because $f_0(z)$ is bounded in the unit disk, and consequently a_n is bounded. For each $\xi \in \chi_{A'}$, we have $\xi(x_0)=z$, $|z|<1$. Hence $\xi(n x_0)=z^n$ and

$$\hat{f}(\xi)=\sum_0^\infty a_n z^n = f_0(z) \neq 0.$$

In order to prove that $A_f \neq A$ it is now sufficient to exhibit an element $\varphi \in A'$, $\varphi \not\equiv 0$, such that for all $\tau \in S$

$$0=(T_\tau f, \varphi)=\sum f(x)\,\varphi(x+\tau). \tag{8}$$

We choose φ vanishing outside S_{x_0} and equal to c_n at $x=n x_0$, where the c_n shall be determined later. The relation (8) is automatically satisfied for $\tau \notin S_{x_0}$ because each term in the series will vanish. If $\varphi \in A'$, then the series

$$\varphi_0(z)=\sum_0^\infty \frac{\varphi(\nu)}{z^\nu} \tag{9}$$

converges absolutely for $|z|\geqslant 1$ and condition (6) takes the following form for

$$\tau = n x_0,\quad n \geqslant 0,$$

$$0=\sum_{\nu=0}^\infty a_\nu c_{\nu+n}=\frac{1}{2\pi}\int_0^{2\pi} f_0(e^{i\theta})\,\varphi_0(e^{i\theta})\,e^{in\theta}\,d\theta, \tag{10}$$

where $h_0(e^{i\theta})=f_0(e^{i\theta})\,\varphi_0(e^{i\theta})$ is a bounded function continuous for $\theta \neq 0 \pmod{2\pi}$. If therefore (8) is satisfied, then the Fourier coefficients of $h_0(e^{i\theta})$ vanishes for negative indices and $h_0(e^{i\theta})$ represents the boundary values of a function $h_0(z)$ analytic in $|z|<1$, bounded there, continuous for $|z|=1$, $z\neq 1$, and vanishing at $z=0$. This implies that $\varphi_0(z)$ can be continued analytically into the function $h_0(z)/f_0(z)$ across each point $\neq 1$ on $|z|=1$. Hence, $\varphi_0(z)$ is regular in the region $z \neq 1$. We now recall this classical theorem by Wigert: The series (9) represents a function $\varphi_0(z)$ regular for $z\neq 1$ and vanishing at $z=0$, if and only if there exists an entire function $\Phi(w)$ with the properties:

$$\Phi(\nu)=c_\nu,\quad \nu=0,1,\ldots,$$

$$\log|\Phi(w)|=o(|w|),\quad w\to\infty. \tag{11}$$

Wigert's theorem gives us good guidance concerning the choice of the c_ν, but our particular problem needs the following additional result. If (11) is strengthened to

$$\log|\Phi(w)|\leqslant |w|^{\frac{1}{2}}+\text{const} \tag{12}$$

and if
$$\sum_0^\infty |\Phi(\nu)| < \infty \tag{13}$$

then $\varphi_0(z)$ will satisfy the inequality

$$|\varphi_0(z)| \leqslant \text{const} \left| \exp\left(\frac{1}{8}\frac{1+z}{1-z}\right)\right|, \quad |z|<1. \tag{14}$$

The proof is elementary. The transformation

$$\psi(s) = \int_0^\infty e^{-w}\,\Phi(sw)\,dw$$

takes Φ into an entire function ψ satisfying $\log|\psi(s)| \leqslant |s|/4 + \text{const}$. The inversion formula

$$\Phi(w) = \frac{1}{2\pi i}\int_{|s|=r} \psi(s)\, e^{w/s}\,\frac{ds}{s}$$

holds for all finite w and for $r>0$. If the integral representation of $c_\nu = \Phi(\nu)$ is introduced in (9) and the order of integration and summation is reversed we obtain the formula

$$\varphi_0(z) = \frac{z}{2\pi i}\int_{|s|=r} \frac{\psi(s)\,ds}{(z-e^{1/s})\,s},$$

valid for z outside the contour described by $e^{1/s}$, $|s|=r$. The estimate on $\psi(s)$ together with an appropriate choice of r yield the inequality

$$|\varphi_0(z)| \leqslant \text{const}\,\frac{\exp\left(\dfrac{1}{4}\dfrac{1}{|z-1|}\right)}{|z-1|}. \tag{15}$$

By virtue of (13), $\varphi_0(z)$ is uniformly bounded for $|z|=1$, $z\neq 1$, and (14) now follows from (15) by an application of the Phragmen–Lindelöf principle to the function

$$\varphi_0(z)\,\exp\left(-\frac{1}{8}\frac{1+z}{1-z}\right) \quad \text{in } |z|<1.$$

In order to determine $\Phi(w)$ we set $k_n = \max\,(\omega(\nu x_0)$ for $\nu \leqslant 2n$. By (3) we have $k_n \leqslant 2\,\omega(n x_0)$ so the series $\sum n^{-3/2} k_n$ will converge. Since k_n is increasing with n there exists on $(0, \infty)$ a monotonic increasing function $\gamma(u)$ such that $\gamma(n)/k_n$ tends to ∞ with n while

$$\int_1^\infty \frac{\gamma(u)}{u^{3/2}}\,du = 2\int_1^\infty \frac{\gamma(u^2)}{u^2}\,du < \infty. \tag{16}$$

We now recall another well-known property of entire functions. If $\gamma(u)$ is increasing and (16) satisfied, then there exists even entire functions $F(w)$ with the properties: $F(0)=1$,

$$\log |F(w)| \leqslant |w| + \text{const}, \tag{17}$$

$$|F(u)| \leqslant \text{const}\, e^{-\gamma(u^2)}, \quad u \text{ real}. \tag{18}$$

We choose such a function $F(w)$ and define $\Phi(w)=F(\sqrt{w})$. Then $\Phi(w)$ is entire and satisfies (12). Moreover,

$$|\Phi(\nu)| \leqslant \text{const}\, e^{-\gamma(\nu)} \leqslant \text{const}\, e^{-\lambda_\nu\, \omega(\nu x_0)},$$

where λ_ν tends to ∞ as $\nu \to \infty$. The choice $c_\nu = \Phi(\nu)$ therefore yields a function $\varphi \in A'$ and the condition (10) is certainly satisfied since the product $\varphi_0(z) f_0(z)$ is a bounded analytic function in $|z|<1$, vanishing at $z=0$. Hence, $A_f \neq A$.

Multipliers

Assume $f, g \in A$, $\|f\|_\lambda < \infty$, $\|g\|_{\lambda'} < \infty$, then the pointwise product $f \cdot g$ belongs to A and $\|f\cdot g\|_{\lambda+\lambda'} \leqslant \|f\|_\lambda \|g\|_{\lambda'}$. The convolution $f \ast g$ is defined by the formula

$$f \ast g(x) = \sum_{x_1+x_2=x} f(x_1)\, g(x_2) = \sum_{y \leqslant x} f(y)\, g(x-y).$$

By virtue of (3), $\qquad |f(y)\, g(x-y)| \leqslant \|f\|_\lambda \|g\|_{\lambda'}\, e^{2(\lambda+\lambda')\omega(x)},$

which together with (5) implies

$$\|f \ast g\|_{\lambda''} \leqslant \|f\|_\lambda \|g\|_{\lambda'}$$

for $\lambda'' \geqslant 2(\lambda + \lambda' + \lambda_0)$. Both pointwise multiplication and convolution are therefore continuous mappings of $A \times A$ into A.

Of particular interest is the properties of A considered as a convolution algebra. By $J(f)$ we shall denote the ideal generated by f:

$$J(f) = \{f \ast g \mid g \in A\},$$

and by $\bar{J}(f)$ the closure of $J(f)$. Since each linear combination of the elements $T_\tau f$, $\tau \in S$, equals a convolution $f \ast k$ where k has finite support, it follows that $A_f = \bar{J}(f)$. We now introduce the following notion: *a function $\varrho \in A$ shall be called a converting multiplier if for each $f \in A$ with $\hat{f} \neq 0$ it holds that $\varrho \cdot f_1 \in \bar{J}(f)$ whenever $f_1 \in \bar{J}(f)$.*

The collection M of converting multipliers is obviously a closed linear subset of A with the property that $\varrho_1, \varrho_2 \in M$ implies $\varrho_1 \cdot \varrho_2 \in M$. This notion is connected with our main problem as follows: *The closure theorem holds in A if and only if $M = A$.*

The proof is trivial. If the closure theorem holds, then for each f with $\hat{f} \neq 0$ and for each $\varrho \in A$ we have $\varrho \cdot f_1 \in A = \bar{J}(f)$ and the conclusion $\varrho \in M$ follows. If, on the other hand $M = A$, then M contains the unit e of the convolution algebra, and $\hat{f} \neq 0$ implies $e \cdot f \in \bar{J}(f)$. But $e \cdot f = e \cdot f(0)$ and $f(0) = \hat{f}(e) \neq 0$, so $\bar{J}(f)$ contains e and is therefore equal to the whole of A.

The notion of converting multiplier is thus trivial whenever the closure theorem holds in the space. This is however not the case with a subset M_0 of M defined as follows: *An element $\varrho \in A$ is a proper converting multiplier and shall belong to M_0 if for each $f \in A$ with $\hat{f} \neq 0$ and for each $g \in A$ the relation $\varrho \cdot (f * g) = f * k$ is satisfied by some element $k \in A$.*

The closure $\bar{M}_0$ of M_0 is contained in M, and M_0 contains finite products of its elements. The question whether always $\bar{M}_0 = M$ has not been resolved in this paper.

The set M_0 derives its importance from the fact that it contains subsets which can be derived by a simple algebraic method, as will be shown in the following section.

Polynomials on S

By $H = H(S)$ we shall denote the set of all additive (and finite) functions $\theta(x)$ on S. Each mapping $x \to \theta(x)$ is thus an homomorphism of S into an additive semigroup of complex numbers. A function $p(x)$ shall be called a polynomial on S if it has a representation

$$p(x) = p(0) + \sum_n \prod_m \theta_{m,n}(x), \tag{19}$$

where the $\theta_{m,n}$ belong to H and where series and products are finite. In (19), $p(0)$ stands for the function equal to the constant $p(0)$ everywhere on S.

We shall first derive some properties valid, irrespective of the topology, for all functions $f(x)$ which are finite on S. Since the number $N(x, S)$ of elements $y \leqslant x$ is finite, it follows that the convolution $f * g$ is always well defined. We shall write h^{*n} for the n-fold convolution of h with itself, defined as e for $n = 0$. The value of h^{*n} at a point x does not change if h is replaced by the function $h_0(y)$ which equals $h(y)$ for $y \leqslant x$ and vanishes elsewhere on S. For h_0 we have the familiar inequality

$$\sum |h_0^{*n}(y)| \leqslant \{\sum |h_0(y)|\}^n.$$

Consequently
$$|h^{*n}(x)| = |h_0^{*n}(x)| \leqslant \{\sum_{y \leqslant x} |h(y)|\}^n.$$

If therefore $h(x)$ is finite on S, the series

$$f(x)=\sum_0^{\infty}\frac{h^{*n}(x)}{n!} \tag{20}$$

will always converge absolutely. We should also notice that if $h(0)=0$, then for fixed x, $h^{*n}(x)$ will vanish for all n sufficiently large. This follows from the fact that the equation $\sum_1^n x_\nu = x$, $x_\nu \neq 0$, has no solution if $n > (N(x, S)-1)^2$.

In order to avoid any confusion we denote by F the set of all finite numeric functions on S, by E the set of functions representable by the series (20) with $h \in F$, and by P the set of polynomials on S. The following lemma will play an important role in this study:

LEMMA I. *Assume $p \in P$, $f \in E$ and $g \in F$. Then the relation*

$$p\cdot(f * g)=f * k \tag{21}$$

is always satisfied by some element $k \in F$.

If $\theta \in H$, then the value of $\theta\cdot(g * h)$ at a point x can be written

$$\sum_{y\leqslant x}\{\theta(y)g(y)h(x-y)+g(y)\theta(x-y)h(x-y)\}.$$

Consequently

$$\theta\cdot(g * h)=(\theta\cdot g) * h+g * (\theta\cdot h). \tag{22}$$

By iteration of this formula we obtain

$$\theta\cdot h^{*n}=n\,h^{*n-1} * (\theta\cdot h). \tag{23}$$

It therefore (20) is multiplied by θ if follows that

$$\theta\cdot f=\sum_{n=0}^{\infty}\frac{n\,h^{*n-1}}{n!} * (\theta\cdot h)=f * (\theta\cdot h). \tag{24}$$

Another application of (22) yields the more general formula

$$\theta\cdot(f * g)=f * \{\theta\cdot g+g * (\theta\cdot h)\}. \tag{25}$$

For h and f fixed we denote by U_θ the linear operator: $g\to\theta\cdot g+g * (\theta\cdot h)$. If $\{\theta_\nu\}_1^q$ is a finite sequence $\in H$, then the relation

$$\prod_1^q\theta_\nu\cdot(f * g)=f * k \tag{26}$$

is satisfied by $k=U_{\theta_1}U_{\theta_2}\dots U_{\theta_q}g\in F$. Therefore (21) has a solution $k=p(0)g+\sum k_n$, where the k_n satisfy equations of the form (26).

We now return to the space A and denote by P_A the subset of polynomials generated by functions $\theta \in H \cap A$. An $f \in A$ possessing a representation (20) with $h \in A$ shall be called *exponential*. As a consequence of this definition we shall have

$$\hat{f}(\xi) = \sum_0^\infty \frac{\hat{h}(\xi)}{n!} = e^{\hat{h}(\xi)}, \quad \xi \in \chi_{A'},$$

so $\hat{f} \neq 0$ is a prerequisite for f being exponential.

Since A is an algebra both under multiplication and convolution, the operators denoted U_θ are bounded in A whenever h and θ belong to A, and Lemma I thus asserts that $p \cdot (f * g)$ belongs to $J(f)$ if f is exponential and $p \in P_A$.

We can now summarize: If $\hat{f} \neq 0$ implies that f is exponential then all polynomials $\in P_A$ are proper converting multipliers and the closure theorem holds in A if e is contained in the closure of P_A. The original problem has herewith branched out into two separate questions.

Exponential elements in A

Conclusive results on our main problem requires further information about S and ω. This should be obvious already by the fact that the conditions introduced so far do not imply that $\chi_{A'}$ contains any other character than $\xi = e$. In order to remedy this situation we observe that the topology of A remains unchanged if ω is replaced by a function ω_1 which is equivalent with ω in the sense that

$$k^{-1} \leqslant \frac{\omega_1(x)}{\omega(x)} \leqslant k \tag{27}$$

for some constant $k > 1$. Of particular significance for our problem is the subset H^+ of H consisting of real valued additive functions $\vartheta(x)$ tending to $+\infty$ as $x \to \infty$ in S. Such a function is obviously strictly positive for $x \neq 0$. Our new condition reads: H^+ *contains an element $\vartheta(x)$ such that $\omega(x)$ is equivalent with a function of the form $\psi(\vartheta(x))$, where $\psi(r)$ is positive and increasing for $r \geqslant 0$ with growth limited by the inequalities*

$$c_1 \log r \leqslant \psi(r) = o(r), \quad r \to \infty. \tag{28}$$

The first inequality implies $\vartheta \in A$, and the second together with (4) imply that the character $e^{-s\vartheta(x)}$ belongs to $\chi_{A'}$ if s is a complex number with positive real part.

LEMMA II. *Let S and ω satisfy the previously stated conditions. Then each $f \in A$ with non-vanishing transform is exponential.*

Let us first show that there exist constants k_1 and k_2 such that for $x, y \in S$.

$$\frac{\omega(y)}{\omega(x)} \leqslant k_1 \frac{\vartheta(y)}{\vartheta(x)} + k_2. \tag{29}$$

By virtue of (27) the inequality is satisfied for $\vartheta(y) \leqslant \vartheta(x)$ if $k_2 \geqslant k^2$. If $\vartheta(y) > \vartheta(x)$ we set $\vartheta(y) = r$, $\vartheta(x) = r_0$ and define $n \geqslant 1$ so that $2^{n-1} r_0 < r \leqslant 2^n r_0$. Consequently

$$\frac{\omega(y)}{\omega(x)} \leqslant k^2 \frac{\psi(2^n r_0)}{\psi(r_0)}.$$

By (3) we have $\omega(2^n x) \leqslant 2^n \omega(x)$. Hence, $\psi(2^n r_0)/\psi(r_0) \leqslant k^2 2^n$, and (29) is satisfied if we choose $k_1 \geqslant 2 k^4$.

Let G be the minimal extension of S to a group and let $\hat{G}$ be the compact Abelian group which is the dual of G. By $\beta = \beta(x)$ we denote characters on G of modulus 1, and by $d\beta$ Haar's measure on $\hat{G}$ normalized by the condition $\int d\beta = 1$. We shall use the notation

$$f_s(x) = e^{-s\vartheta(x)} f(x).$$

Each mapping $f \to f_s$ takes A into the space $L_1(S)$ with norm $\|f\|_{L_1} = \sum |f(x)|$. For the n-fold convolution of f_s we have

$$f_s^{*n}(x) = e^{-s\vartheta(x)} f^{*n}(x). \tag{30}$$

Without loss of generality we may assume $f(0) = 1$ and write $f = e + g$, with $g(0) = 0$. Let σ_0 be so large that $\|g_s\|_{L_1} \leqslant \frac{1}{2}$ for $s = \sigma + it$, $\sigma \geqslant \sigma_0$. Then

$$\hat{f}(e^{-s\vartheta} \beta) = 1 + \sum g_s(x) \beta(x) = 1 + \hat{g}_s(\beta),$$

where $|\hat{g}_s(\beta)| \leqslant \frac{1}{2}$ for each β. The logarithm of this function is now uniquely determined on $\hat{G}$ by the formula

$$\log \hat{f}(e^{-s\vartheta} \beta) = \sum_1^\infty \frac{(-1)^{n+1}}{n} \hat{g}_s^n(\beta), \quad \sigma \geqslant \sigma_0, \tag{31}$$

where
$$\hat{g}_s^n(\beta) = \sum g_s^{*n}(x) \beta(x).$$

Since $g(0) = 0$ we know that $g^{*n}(x) = 0$ for x fixed if n is sufficiently large. A function $h(x)$ is therefore well defined on S by the relation

$$\sum_1^\infty \frac{(-1)^{n+1}}{n} g_s^{*n}(x) = e^{-s\vartheta(x)} \sum_1^\infty \frac{(-1)^{n+1}}{n} g^{*n}(x) = e^{-s\vartheta(x)} h(x), \tag{32}$$

and $h(0)=0$. The left-hand side of (31) is a continuous function on $\hat{G}$, depending on a parameter s, with an absolutely convergent Fourier series:

$$\log \hat{f}(e^{-s\vartheta}\beta)=\sum_{x\in G} e^{-s\vartheta(x)}\, h(x)\,\beta(x), \quad \sigma \geqslant \sigma_0, \tag{33}$$

where h is defined $=0$ outside S. Hence

$$e^{-s\vartheta(x)}\, h(x)=\int_{\hat{G}} \log \hat{f}(e^{-s\vartheta}\beta)\,\bar{\beta}(x)\,d\beta, \quad \sigma \geqslant \sigma_0. \tag{34}$$

Since $\hat{f}\neq 0$ in $\chi_{A'}$ the logarithm has a unique analytic extension to the whole right half plane and the formula (34) holds there by analytic continuation. In particular, since $h(0)=0$,

$$0=\int_{\hat{G}} \log \hat{f}(e^{-s\vartheta}\beta)\,d\beta. \tag{35}$$

Hence,

$$e^{-\sigma\vartheta(x)}\,|h(x)| \leqslant 2 \max_{\beta} \log |\hat{f}(e^{-\sigma\vartheta}\beta)|, \quad \sigma > 0. \tag{36}$$

Let λ be so large that $\|f\|_\lambda \leqslant 1$. Then

$$|\hat{f}(e^{-\sigma\vartheta}\beta)| \leqslant \sum e^{-\sigma\vartheta(x)+\lambda\omega(x)}.$$

On defining

$$m(\varepsilon)=\sup_{y\in S}\,(-\varepsilon\vartheta(y)+\omega(y))$$

we obtain by virtue of (4)

$$\log |\hat{f}(e^{-\sigma\vartheta}\beta)| \leqslant (\lambda+\lambda_0)\, m\left(\frac{\sigma}{\lambda+\lambda_0}\right).$$

Hence, for all $\sigma>0$,

$$|h(x)| \leqslant 2(\lambda+\lambda_0)\, m\left(\frac{\sigma}{\lambda+\lambda_0}\right) e^{\sigma\vartheta(x)}.$$

In this relation we choose $\sigma=k_1(\lambda+\lambda_0)\,\omega(x)/\vartheta(x)$, where k_1 is the constant occurring in (29). We want to show that this choice yields

$$|h(x)| \leqslant c_1\, e^{\lambda_1\omega(x)}$$

with c_1 independent of x and with $\lambda_1=k_1(\lambda+\lambda_0)+1$. We have thus to show that for $x, y \in S$.

$$2(\lambda+\lambda_0)\left\{-k_1\frac{\vartheta(y)\,\omega(x)}{\vartheta(x)}+\omega(y)\right\} \leqslant c_1\, e^{\omega(x)},$$

or equivalently that

$$\frac{\omega(y)}{\omega(x)} \leqslant k_1 \frac{\vartheta(y)}{\vartheta(x)} + \frac{c_1}{2(\lambda+\lambda_0)} \cdot \frac{e^{\omega(x}}{\omega(x)}$$

and this inequality is satisfied by virtue of (29) if c_1 is chosen properly. We have thus shown that $h \in A$.

By the definition of h we have for $\sigma \geqslant \sigma_0$,

$$\hat{f}(e^{-s\vartheta}\beta) = \exp\{\sum_{x \in S} h_s(x)\beta(x)\}$$

implying

$$f_s(x) = \sum_0^\infty \frac{h_s^{*n}(x)}{n!}.$$

In view of (30) this yields the requested representation

$$f(x) = \sum_0^\infty \frac{h^{*n}(x)}{n!}.$$

It is well known from the elementary theory of Taylor series that the sole condition that ω is monotonic increasing is not sufficient in order to imply f exponential if $\hat{f} \neq 0$. Some additional property is required preventing the rate of growth of ω to change too erratically. We want to point out that the lemma remains true under the assumption that there exists a $\vartheta \in H^+$ and a positive constant ε such that for x outside some finite set,

$$\vartheta^\varepsilon(x) < \omega(x) < \vartheta^{1-\varepsilon}(x).$$

Polynomial approximation of e

THEOREM II. *Let S and ω satisfy the previously introduced condition, and assume in addition that $\psi(r)$ in* (28) *is a convex function of* $\log r$. *Then the following is true: The closure of the set of polynomials P_A contains the unit e and the closure theorem holds in A if and only if the divergence condition of Theorem* I *is satisfied.*

If $\vartheta \in H^+$ then the mapping $x \to \vartheta(x)$ takes S to a discrete set of numbers $\geqslant 0$, and $\vartheta(x)$ has a positive minimum r_0 for $x \neq 0$. In order to prove that e is contained in the closure of P_A it is thus sufficient to show the existence of a sequence of polynomials $Q_n(t)$ assuming the value 1 at $t=0$ and such that $Q_n(t)e^{-\psi(t)}$ converges uniformly to 0 for $t \geqslant r_0$. Then $p_n(x)$ defined as $Q_n(\vartheta(x))$ will belong to P_A and converge to e in the space A. The existence of Q_n can be considered as a special case of Bernstein's classical approximation problem, formulated for the positive real axis

R^+. Let $C_0(R^+)$ denote the space of functions continuous for $t \geq 0$ and vanishing at ∞, and let $\psi(t)$ be a given function continuous on R^+ and tending so fast to $+\infty$ that

$$A_n = \sup_{t \geq 0} t^n e^{-\psi(t)} < \infty, \quad n = 0, 1, 2 \dots . \tag{37}$$

The problem is to decide whether linear combinations of $t^n e^{-\psi(t)}$, $n = 0, 1, 2 \dots$, are dense in $C_0(R^+)$. Bernstein's original results imply that approximation is possible if $e^{\psi(t)}$ has a minorant for $t \geq 0$ of the form

$$F(t) = \sum_0^\infty c_\nu t^\nu, \qquad c_0 > 0, \ c_\nu \geq 0,$$

and with the property
$$\int_1^\infty \frac{\log F(t)}{t^{3/2}} \, dt = \infty. \tag{38}$$

This result applies immediately to the problem at hand. The divergence condition in Theorem I implies

$$\int_1^\infty \frac{\psi(t)}{t^{3/2}} \, dt = \infty, \tag{39}$$

where ψ is the function in condition (28). Moreover, if $\psi(t)$ is a convex function of $\log t$, then (39) implies

$$\lim_{t = \infty} \frac{\psi(t)}{\log t} = \infty \tag{40}$$

so (37) is satisfied. In order to show existence of minorants $F(t)$ it suffices to choose

$$F(t) = \sum_0^\infty \frac{t^n}{2^n A_n},$$

where A_n is defined by (37). Due to the convexity of ψ there exists for each $n \geq 0$ a number t_n such that $e^{\psi(t_n)} = t_n^n / A_n$. Hence, $\log F(2 t_n) > \psi(t_n)$ and a simple computation shows that (39) implies (38). A sequence $Q_n(t)$ with the requested properties does therefore exist since any continuous function equal to 1 at $t = 0$ and vanishing for $t \geq r_0$, can be approached uniformly on R^+ by functions $Q_n(t) e^{-\psi(t)}$. This finishes the proof of Theorem II since we already know that convergence in (7) implies that the closure theorem is false and consequently e not contained in the closure of P_A.

It should be pointed out that without the additional convexity condition the preceding analysis does not imply that the closure theorem is false in A if $e \notin \bar{P}_A$. This problem remains unsolved even in the case $S = Z^+$.

References

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