# ISOLATED SINGULARITIES OF SOLUTIONS OF QUASI-LINEAR EQUATIONS

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In a previous paper in this journal [1] we have studied the local behavior of solutions of non-linear second order partial differential equations having the form

$$\operatorname{div} \mathcal{A}(x, u, u_x) = \mathcal{B}(x, u, u_x). \tag{1}$$

Here $\mathcal{A}$ is a given vector function of the variables $x, u, u_x$, $\mathcal{B}$ is a given scalar function of the same variables, and $u_x = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ denotes the gradient of the dependent variable $u = u(x)$, where $x = (x_1, \ldots, x_n)$. The final chapter of [1] treated a variety of problems concerning the behavior of solutions at an isolated singularity. In that chapter I found it necessary to impose the condition $\mathcal{B} \equiv 0$, a condition which had not been required in the preceding parts of [1]. The main purpose of the present paper is to show that this additional condition can be removed, and thus to complete, in an important way, the theory of the earlier paper.

We assume as always that the functions $\mathcal{A}$ and $\mathcal{B}$ are defined for all points $x$ in some connected open set (domain) $\Omega$ of the Euclidean number space $E^n$, and for all values of $u$ and $u_x$. Furthermore, they are to satisfy inequalities of the general form

$$\begin{aligned} |\mathcal{A}(x,u,p)| &\leqslant a|p|^{\alpha-1} + b|u|^{\alpha-1} + e \\ |\mathcal{B}(x,u,p)| &\leqslant c|p|^{\alpha-1} + d|u|^{\alpha-1} + f \\ p \cdot \mathcal{A}(x,u,p) &\geqslant |p|^{\alpha} - d|u|^{\alpha} - g, \end{aligned} \tag{2}$$

where the exponent $\alpha$ is a fixed number in the range $1 < \alpha \leqslant n$, the coefficient $a$ is a positive real number, and the coefficients $b$ through $g$ are measurable functions of $x$ contained in the respective Lebesgue classes

$$b, e \in L_{n/(\alpha-1-\varepsilon)}; \quad c \in L_{n/(1-\varepsilon)}; \quad d, f, g \in L_{n/(\alpha-\varepsilon)},$$

where $\varepsilon > 0$, (if $\alpha < n$, the Lebesgue class of $b$ can be weakened to $L_{n/(\alpha-1)}$). We shall then prove the following result, generalizing Theorem 12 of [1] to the case of a non-vanishing right hand side $\mathcal{B}(x, u, u_x)$.

THEOREM 1. *Let $u$ be a continuous solution of* (1) *in the set $D - \{0\}$, where $D$ is a domain in $\Omega$. Suppose that $u \geqslant L$ for some constant $L$. Then either $u$ has a removable singularity at* 0, *or else*

$$u \approx \begin{cases} r^{(\alpha-n)/(\alpha-1)}, & \alpha < n, \\ \log 1/r, & \alpha = n, \end{cases} \tag{3}$$

*in the neighborhood of the origin.* (*Here $f \approx g$ means that $C' \leqslant f/g \leqslant C''$, where $C'$ and $C''$ are positive constants.*)

This result applies in particular to solutions of the variational problem

$$\delta \int F(x, u, u_x)\, dx = 0,$$

provided the integrand obeys certain natural conditions (cf. [2]). In the case of quadratic variational problems, for example, Theorem 1 implies that the order of growth at a positive isolated singularity is $r^{2-n}$ or $\log 1/r$, depending on whether $n > 2$ or $n = 2$. Earlier results of a similar nature are noted in the introductory paragraphs of [1].

Theorem 1 is proved in the following two sections. In contrast with the simple statement of the theorem, the intricacy of the proof comes as something of a surprise, lending its own interest to the result. The method of proof furthermore shows that a solution with a positive isolated singularity has certain of the attributes of a *fundamental solution* (cf. Theorem 3). In Section 4 we show under suitable conditions that there exists a solution of (1) with precisely the asymptotic behavior (3). This result is a generalization of Theorem 13 of [1], both in the equation treated and in the weaker structure required. For linear equations

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j} + b_i u + e_i\right) = c_i \frac{\partial u}{\partial x_i} + du + f \tag{4}$$

it is possible to obtain somewhat more detailed conclusions. In Section 5 we consider two results generalizing Theorems 14 and 15 of reference [1]. Although we shall not present the details here, it is also worth noting that the results of the paper may be used to construct a Green's function $G(x, y)$ for (4), having the usual properties of positivity and symmetry, and yelding a representation formula for solutions of the Dirichlet problem with zero boundary data.

The final section of the paper is more or less unrelated to the earlier part. Here we take the opportunity to correct an error in [1] occurring in the statement and proof of the maximum principle. We also point out several places in [1] where the results can be slightly improved.

It is assumed throughout that the reader is familiar with Chapters I and II of [1]. Moreover, the notation and terminology of that paper will be used as needed. In particular, we recall that a *continuous solution of* (1) *in a domain* $D$ is a function $u$ which is continuous and has strong derivatives which are locally of class $L_\alpha$ over $D$, and is such that

$$\int (\phi_x \cdot \mathcal{A} + \phi \mathcal{B})\, dx = 0$$

for any continously differentiable function $\phi = \phi(x)$ with compact support in $D$.

## 1. Proof of Theorem 1. First Part

We shall assume that the singularity at 0 is *not* removable. To prove the theorem it must therefore be shown that $u$ has the asymptotic behavior (3) in the neighborhood of the origin. We shall restrict our discussion, in fact, to the ball $S = \{|x| \leq R\}$ in $D$, where $R$ is chosen so that the Lebesgue norms $\|b\|$, $\|c\|$, and $\|d\|$ over $S$ are suitably small (how small will be determined in the course of the proof, but in any case will depend only on $\alpha$, $n$, and $\varepsilon$). It may be assumed without loss of generality that $u < 0$ on the circumference $|x| = R$; indeed, if this is not already the case, it can be accomplished simply by the subtraction of a suitable constant from $u$, a device which affects the structure of (1) only by increasing the relatively unimportant coefficients $e$, $f$, and $g$.

LEMMA 1. *Let* $\Theta$ *be a stongly* ($L_\alpha$) *differentiable function with compact support in* $|x| < R$, *which is identically* 1 *in some neighborhood of the origin. Then*

$$\int \{\Theta_x \cdot \mathcal{A} + (\Theta - 1)\mathcal{B}\}\, dx = \text{Const.} = \overline{K},$$

*where the constant is independent of the particular choice of* $\Theta$.

*Proof.* Let $\Theta$ and $\overline{\Theta}$ be two functions satisfying the conditions of the lemma. Then $\phi = \overline{\Theta} - \Theta$ has compact support in $0 < |x| < R$, and

$$\int \{(\overline{\Theta}-\Theta)_x \cdot \mathcal{A} + (\overline{\Theta}-\Theta)\mathcal{B}\}\, dx = 0.$$

Since $\Theta \equiv 1$ near 0 this can be rewritten

$$\int \{\overline{\Theta}_x \cdot \mathcal{A} + (\overline{\Theta}-1)\mathcal{B}\}\, dx = \int \{\Theta_x \cdot \mathcal{A} + (\Theta-1)\mathcal{B}\}\, dx,$$

proving the lemma.

Since we have assumed that the singularity at 0 is *not* removable, it is clear from the remark on page 278 of [1] that $u \to \infty$ as $x \to 0$. Hence there exists a constant $\sigma_0$ such that $u \geqslant 1$ for $0 < |x| \leqslant \sigma_0$. Let $m = m(\sigma)$ denote the minimum of $u$ on a given sphere $|x| = \sigma$, $(\sigma \leqslant \sigma_0)$. For $\sigma \leqslant |x| \leqslant R$ let us define

$$v = v(x, \sigma) = \begin{cases} 0 & \text{if } u(x) \leqslant 0 \\ u & \text{if } 0 < u(x) < m \\ m & \text{if } u(x) \geqslant m. \end{cases}$$

We suppose the definition of $v$ to be extended to the entire ball $S$ by setting $v \equiv m$ when $|x| \leqslant \sigma$. Then $v$ is a strongly differentiable function with compact support in $|x| < R$, and is identically equal to $m$ in a neighborhood of the origin.

Thus for fixed $\sigma \leqslant \sigma_0$, and $m = m(\sigma)$, $v = v(x, \sigma)$, we have by Lemma 1,

$$\begin{aligned} m\overline{K} &= \int \{v_x \cdot \mathcal{A}(x, u, u_x) + (v-m)\mathcal{B}(x, u, u_x)\}\, dx \\ &= \int \{v_x \cdot \mathcal{A}(x, v, v_x) + (v-m)\mathcal{B}(x, u, u_x)\}\, dx, \end{aligned} \tag{5}$$

since $v = u$ and $v_x = u_x$ almost everywhere in the set where $v_x \neq 0$. Now using inequality (2),

$$\int v_x \cdot \mathcal{A}(x, v, v_x)\, dx \geqslant \int (|v_x|^\alpha - dv^\alpha - g)\, dx. \tag{6}$$

The second term on the right may be estimated by the Hölder and Sobolev inequalities, thus (¹)

$$\int dv^\alpha dx \leqslant \|d\|_{n/\alpha} \|v\|_{\alpha^*}^\alpha \leqslant \text{Const. } \|d\|_{n/\alpha} \|v_x\|_\alpha^\alpha. \tag{7}$$

The radius $R$ introduced at the beginning of the proof can be chosen so small that the coefficient of $\|v_x\|_\alpha^\alpha$ is $\leqslant \frac{1}{2}$. Hence by (6) and (7)

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(¹) This calculation is given for the case $\alpha < n$. If $\alpha = n$ only slight changes are necessary.

$$\int v_x \cdot \mathcal{A}(x, v, v_x)\, dx \geqslant \tfrac{2}{3}\int |v_x|^\alpha\, dx - C, \tag{8}$$

where $C$ is an appropriate constant. Moreover, by Lemmas 9 and 10 of [1], since $v \equiv m$ for $|x| \leqslant \sigma$.

$$\int |v_x|^\alpha\, dx \geqslant \omega_n m^\alpha \begin{cases} \left(\dfrac{n-\alpha}{\alpha-1}\right)^{\alpha-1} \sigma^{n-\alpha}, & \alpha < n, \\ (\log R/\sigma)^{1-n}, & \alpha = n. \end{cases} \tag{9}$$

Inequalities (8) and (9) serve to estimate the first term on the right hand side of (5) from below. The remaining term can be estimated by setting

$$B(\sigma) \equiv \int_\sigma^R |\mathcal{B}(x, u, u_x)|\, dx \ (^1)$$

and noting that $|v-m| \leqslant m$. Thus we have

$$\int (v-m)\, \mathcal{B}(x, u, u_x)\, dx \geqslant -m\, B(\sigma). \tag{10}$$

It follows from (5), therefore, that

$$m^\alpha \leqslant \text{Const. } [(B(\sigma) + \bar{K})\, m + C] \begin{cases} \sigma^{\alpha-n}, & \alpha < n, \\ (\log R/\sigma)^{n-1}, & \alpha = n. \end{cases} \tag{11}$$

Since $m \geqslant 1$ we obtain finally

$$m^{\alpha-1} \leqslant \text{Const. } (B(\sigma) + 1) \begin{cases} \sigma^{\alpha-n}, & \alpha < n, \\ (\log R/\sigma)^{n-1}, & \alpha = n, \end{cases} \tag{12}$$

valid for $\sigma \leqslant \sigma_0$. (Many of the multiplicative factors which appear in the proof are denoted simply by Const. These factors are usually different from line to line, but they have this in common: they can be computed explicitly and they are independent of $\sigma$.)

LEMMA 2. *The function* $\mathcal{B}(x, u, u_x)$ *is integrable over* $0 < |x| \leqslant R$, *and*

$$\int_{r<\sigma} |\mathcal{B}(x, u, u_x)|\, dx \leqslant \text{Const.} \begin{cases} \sigma^{\varepsilon}, & \alpha < n, \\ \sigma^{\varepsilon/2}, & \alpha = n. \end{cases}$$

*Proof.* By the Harnack principle (cf. Theorem 7 of [1], and the corresponding remark in Section 6 of this paper) it is clear that for any fixed $\sigma$ less than $\sigma_0/4$ one has

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(1) We shall frequently write $\int_a^b$ instead of $\int_{a<r<b}$.

$$\max_{\sigma/2\leqslant r\leqslant 3\sigma} u(x) \leqslant C'(\min_{\sigma/2\leqslant r\leqslant 3\sigma} u(x) + k'), \tag{13}$$

where $C'$ and $k'$ are appropriate constants depending only on the structure of (1). Clearly

$$\min_{\sigma/2\leqslant r\leqslant 3\sigma} u(x) \leqslant m(\sigma) \quad \text{and} \quad m(\sigma)\geqslant 1,$$

so that (13) implies

$$\max_{\sigma/2\leqslant r\leqslant 3\sigma} u(x) \leqslant \text{Const. } m(\sigma), \quad (\sigma\leqslant \sigma_0/4).$$

This being established, we now have for $\sigma\leqslant\sigma_0/4$,

$$\begin{aligned}\int_\sigma^{2\sigma} (du^{\alpha-1}+f)\,dx &\leqslant \int_\sigma^{2\sigma} (d+f)\,u^{\alpha-1}\,dx \\ &\leqslant \max_{\sigma\leqslant r\leqslant 2\sigma} u^{\alpha-1}\cdot \|d+f\|_{n/(\alpha-\varepsilon)}\|1\|_{n/(n-\alpha+\varepsilon)} \\ &\leqslant \text{Const. } \sigma^{n-\alpha+\varepsilon} m(\sigma)^{\alpha-1}. \end{aligned} \tag{14}$$

Similarly

$$\begin{aligned}\int_\sigma^{2\sigma} c\,|u_x|^{\alpha-1}\,dx &\leqslant \|c\|_{n/(1-\varepsilon)}\|1\|_{\alpha n/(n-\alpha+\alpha\varepsilon)}\|u_x\|_\alpha^{\alpha-1} \\ &\leqslant \text{Const. } \sigma^{(n-\alpha+\alpha\varepsilon)/\alpha}[\sigma^{-1}\|u\|_\alpha + \sigma^{(n-\alpha)/\alpha}]^{\alpha-1},\end{aligned}$$

where we have used Theorem 1 of [1] at the second step; note that the norm $\|u\|_\alpha$ should be taken over the larger set $\sigma/2\leqslant r\leqslant 3\sigma$. Since

$$\|u\|_\alpha \leqslant \text{Const. } \sigma^{n/\alpha}\max u \leqslant \text{Const. } \sigma^{n/\alpha} m(\sigma),$$

there results finally

$$\int_\sigma^{2\sigma} c\,|u_x|^{\alpha-1}\,dx \leqslant \text{Const. } \sigma^{n-\alpha+\varepsilon} m(\sigma)^{\alpha-1}. \tag{15}$$

Since $|\mathcal{B}|\leqslant c\,|u_x|^{\alpha-1}+du^{\alpha-1}+f$ and since

$$\int_\sigma^{2\sigma} |\mathcal{B}|\,dx = B(\sigma)-B(2\sigma),$$

it follows from (14), (15), and (12) that (assuming $\alpha<n$)

$$B(\sigma)-B(2\sigma)\leqslant \beta\,\sigma^\varepsilon(B(\sigma)+1), \tag{16}$$

where $\beta$ is an appropriate constant.

To solve this difference inequality we first rewrite it in the form

$$(1-\beta\,\sigma^\varepsilon)\,B(\sigma)\leqslant \beta\,\sigma^\varepsilon + B(2\sigma).$$

Put $\hat\sigma=(2\beta)^{-\varepsilon}$. Then for $\sigma\leqslant\hat\sigma$,

$$(1-\beta\sigma^{\varepsilon})^{-1} \leqslant 2, \qquad (1-\beta\sigma^{\varepsilon})^{-1} \leqslant \exp(2\beta\sigma^{\varepsilon}),$$

and consequently

$$B(\sigma) \leqslant 2\beta\sigma^{\varepsilon} + \exp(2\beta\sigma^{\varepsilon}) \cdot B(2\sigma), \quad (\sigma \leqslant \hat{\sigma}).$$

By successive iteration

$$\begin{aligned} B(2^{-k}\hat{\sigma}) &\leqslant 2^{-k\varepsilon} + \exp(2^{-k\varepsilon}) \cdot B(2^{1-k}\hat{\sigma}) \\ &\leqslant 2^{-k\varepsilon} + \exp(2^{-k\varepsilon}) \cdot 2^{(1-k)\varepsilon} + \exp(2^{-k\varepsilon} + 2^{(1-k)\varepsilon}) \cdot B(2^{(2-k)}\hat{\sigma}) \\ &\leqslant \exp\left(\sum_1^k 2^{-j\varepsilon}\right) \cdot \left\{\sum_1^k 2^{-j\varepsilon} + B(\hat{\sigma})\right\} \\ &\leqslant \text{Const.}\,(B(\hat{\sigma})+1), \end{aligned}$$

since $\sum 2^{-j\varepsilon} < \infty$. This estimate holds for any $k = 1, 2, \ldots$, hence $B(\sigma)$ is uniformly bounded as $\sigma \to 0$. That is, $\mathcal{B}(x, u, u_x)$ is integrable over $0 < |x| \leqslant R$.

To complete the proof for the case $\alpha < n$, we observe that

$$\int_0^\sigma |\mathcal{B}(x, u, u_x)|\,dx = \sum_1^\infty \{B(2^{-j}\sigma) - B(2^{1-j}\sigma)\} \leqslant \beta\sigma^{\varepsilon}(B(0)+1)\sum_1^\infty 2^{-j\varepsilon} = \text{Const.}\ \sigma^{\varepsilon},$$

using (16) and the fact that $B(\sigma)$ is uniformly bounded. If $\alpha = n$, we obtain in place of (16)

$$B(\sigma) - B(2\sigma) \leqslant \beta\sigma^{\varepsilon}(B(\sigma)+1)(\log R/\sigma)^{n-1} \leqslant \text{Const.}\ \sigma^{\varepsilon/2}(B(\sigma)+1)$$

for some appropriate constant, and the required conclusion follows exactly as before. This proves the lemma.

We may now return to inequality (12). Since $B(\sigma)$ is uniformly bounded, (12) implies

$$m \leqslant \text{Const.} \begin{cases} \sigma^{(\alpha-n)/(\alpha-1)}, & \alpha < n, \\ \log R/\sigma, & \alpha = n, \end{cases} \tag{17}$$

valid for any $\sigma \leqslant \sigma_0$. In the next section we shall complete the proof of Theorem 1 by showing that a certain reverse inequality is also valid.

## 2. Completion of the proof

We begin with a result analogous to Lemma 1 of the preceding section.

LEMMA 3. *Let $\Theta$ be a strongly $(L_\alpha)$ differentiable function with compact support in $|x| < R$, which is identically 1 in some neighborhood of the origin. Then*

$$\int (\Theta_x \cdot \mathcal{A} + \Theta\mathcal{B})\,dx = \text{Const.} = K,$$

*where the constant is independent of the particular choice of $\Theta$.*

*Proof.* The integral $\int \Theta \mathcal{B}\, dx$ is well defined in view of Lemma 2. The result then follows by a repetition of the proof of Lemma 1.

LEMMA 4. $K > 0$.

*Proof.* Suppose for contradiction that $K \leqslant 0$. Then for fixed $\sigma \leqslant \sigma_0$, and $m = m(\sigma)$, $v = v(x, \sigma)$, we have from Lemma 3

$$0 \geqslant \int \{v_x \cdot \mathcal{A}(x, u, u_x) + v\mathcal{B}(x, u, u_x)\}\, dx$$
$$= \int v_x \cdot \mathcal{A}(x, v, v_x)\, dx + \int_{S_1} v\mathcal{B}(x, v, v_x)\, dx + \int_{S_2} m\mathcal{B}(x, u, u_x)\, dx, \tag{18}$$

where $S_1 = \{\sigma \leqslant |x| \leqslant R\} \cap \{u < m\}$ and $S_2 = \{0 < |x| < \sigma\} \cup \{u \geqslant m\}$, (this calculation should be compared with the corresponding one in Section 1).

As in Section 1, [1]

$$\int v_x \cdot \mathcal{A}(x, v, v_x)\, dx \geqslant \tfrac{2}{3} \int |v_x|^\alpha dx - C. \tag{19}$$

Similarly
$$\int_{S_1} v\, |\mathcal{B}(x, v, v_x)\, dx \leqslant \int (cv\, |v_x|^{\alpha-1} + dv^\alpha + vf)\, dx,$$

and if $R$ is suitably small

$$\int cv\, |v_x|^{\alpha-1} dx \leqslant \|c\|_n \|v\|_{\alpha*} \|v_x\|_\alpha^{\alpha-1} \leqslant \text{Const. } \|c\|_n \|v_x\|_\alpha^\alpha \leqslant \tfrac{1}{12} \|v_x\|_\alpha^\alpha,$$

and
$$\int dv^\alpha dx \leqslant \tfrac{1}{3} \|v_x\|_\alpha^\alpha, \quad \text{(cf. (7))}.$$

Finally,
$$\int vf\, dx \leqslant \|f\|_{\alpha n/(\alpha n + \alpha - n)} \|v\|_{\alpha*} \leqslant \text{Const. } \|v_x\|_\alpha \leqslant \tfrac{1}{12} \|v_x\|_\alpha^\alpha + \text{Const.}$$

using Young's inequality (with the trick). Combining the last four inequalities yields

$$\int_{S_1} v\, |\mathcal{B}(x, v, v_x)|\, dx \leqslant \tfrac{1}{2} \|v_x\|_\alpha^\alpha + C. \tag{20}$$

Hence from (18) (19), (20), and the capacity inequality (9), there results easily

$$m^\alpha \leqslant \text{Const. } [m \int_{S_2} |\mathcal{B}(x, u, u_x)|\, dx + C] \begin{cases} \sigma^{\alpha - n}, & \alpha < n, \\ (\log R/\sigma)^{n-1}, & \alpha = n, \end{cases} \tag{21}$$

valid for $\sigma \leqslant \sigma_0$. This should be compared with (11) in Section 1.

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[1] Cf. (8). We assume $\alpha < n$ for simplicity here.

On the basis of (21) we assert that for some appropriate constant $A \geqslant 1$,

$$m(\sigma) \leqslant A \begin{cases} \sigma^{\tau(1-\delta)}, & \alpha < n, \\ (\log R/\sigma)^{1-1/n}, & \alpha = n, \end{cases} \tag{22}$$

where $\tau = (\alpha - n)/(\alpha - 1)$ and $\delta = \varepsilon/(n - \alpha + \varepsilon)$. (1)

Consider first the case $\alpha < n$. Since (22) holds trivially for any $\sigma$ such that $m \leqslant \sigma^{\tau(1-\delta)}$, we need consider only those values of $\sigma$ for which $m > \sigma^{\tau(1-\delta)}$. Let $\theta$ be a fixed number, to be determined in a moment, and consider the point set

$$\theta\, \sigma^{1-\delta} \leqslant |x| \leqslant \sigma_0/2.$$

By application of the Harnack principle, we have for any point $x$ in this set,

$$u(x) \leqslant \text{Const. } m(|x|) \leqslant \text{Const. } |x|^{\tau}$$
$$\leqslant \text{Const. } \theta^{\tau}\sigma^{\tau(1-\delta)} < \text{Const. } \theta^{\tau}\, m(\sigma) = m(\sigma)$$

provided $\theta$ is chosen appropriately large (note that (17) was used at the second step in this chain of inequalities). It follows that the set $S_2$ is contained in the union of the sets

$$\{0 < |x| < \sigma\}, \quad \{0 < |x| < \theta\, \sigma^{1-\delta}\}, \quad \text{and} \quad \{\sigma_0/2 < |x| < R\}.$$

Since $m \to \infty$ as $\sigma \to 0$, there will be no points of $S_2$ in the last of these sets when $\sigma$ is small enough, say $\sigma \leqslant \sigma_1$. By making $\sigma_1$ even smaller, if necessary, the first set is contained in the second. In summary, then, if $\sigma_1$ is chosen appropriately small, the set $S_2$ will be contained in $\{0 < |x| < \theta\, \sigma^{1-\delta}\}$ whenever $\sigma \leqslant \sigma_1$. Thus for $\sigma \leqslant \sigma_1$, making use of Lemma 2,

$$\int_{S_2} |\mathcal{B}(x, u, u_x)|\, dx \leqslant \int_{r < \theta\sigma^{1-\delta}} |\mathcal{B}(x, u, u_x)|\, dx \leqslant \text{Const. } \sigma^{\varepsilon(1-\delta)}.$$

Substitution into (21) yields

$$m^{\alpha} \leqslant \text{Const. } (m\, \sigma^{\alpha - n + \varepsilon(1-\delta)} + \sigma^{\alpha - n}).$$

Hence after a short calculation (cf. Lemma 2 of [1]),

$$m \leqslant \text{Const. } (\sigma^{\tau(1-\delta)} + \sigma^{(\alpha - n)/\alpha}).$$

Since $(\alpha - n)/\alpha \geqslant \tau(1-\delta)$, (cf. the footnote below), the assertion follows for $\sigma \leqslant \sigma_1$. For $\sigma > \sigma_1$ it is clear that (22) holds for *some* constant $A$, whence in all cases (22) is verified.

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(1) In case $\alpha < n$ the assertion requires $\varepsilon \leqslant (n - \alpha)/(\alpha - 1)$. This restriction, however, clearly involves no loss of generality.

Before turning to the case $\alpha = n$ we observe that (22), together with Theorem 11 of [1], implies that the singularity at 0 is removable. This contradiction of our basic assumption establishes Lemma 4 in the case $\alpha < n$.

We now consider (22) in the case $\alpha = n$. Since it holds trivially for any $\sigma$ such that $m \leqslant (\log R/\sigma)^{1-1/n}$, we may suppose at the outset that $m > (\log R/\sigma)^{1-1/n}$. Put $y = (\log R/\sigma)^{1-1/n}$. Let $\theta$ be a fixed number, to be determined in a moment, and consider the point set

$$Re^{-2\theta y} \leqslant |x| \leqslant \sigma_0/2.$$

By application of the Harnack principle, we have for any point $x$ in this set,

$$u(x) \leqslant \text{Const. } m(|x|) \leqslant \text{Const. } \log R/|x| \leqslant \text{Const. } \theta y < \text{Const. } \theta m = m$$

provided $\theta$ is chosen appropriately small. Thus the set $S_2$ is contained in the union of the sets

$$\{0 < |x| < \sigma\}, \quad \{0 < |x| < Re^{-2\theta y}\}, \quad \text{and} \quad \{\sigma_0/2 < |x| < R\}.$$

As before, if $\sigma_1$ is chosen appropriately small the set $S_2$ will be contained in

$$\{0 < |x| < Re^{-2\theta y}\}$$

whenever $\sigma \leqslant \sigma_1$. Thus by Lemma 2,

$$\int_{S_2} |\mathcal{B}(x, u, u_x)|\, dx \leqslant \int_{r < Re^{-2\theta y}} |\mathcal{B}(x, u, u_x)|\, dx \leqslant \text{Const. } e^{-\varepsilon\theta y}.$$

Substitution into (21) yields

$$m^n \leqslant \text{Const. } (\log R/\sigma)^{n-1} (me^{-\varepsilon\theta y} + 1),$$

whence after a simple calculation

$$m \leqslant \text{Const. } (\log R/\sigma)^{1-1/n} \{y^{1/(n-1)} e^{-\varepsilon\theta y/(n-1)} + 1\}.$$

Since the expression in braces is uniformly bounded for $0 \leqslant y < \infty$ the assertion follows for $\sigma \leqslant \sigma_1$. For $\sigma > \sigma_1$ it is clear that (22) holds for *some* constant $A$, whence (22) is verified in all cases.

This being shown, it now follows from Theorem 11 of [1] that 0 is again a removable singularity. But this contradicts our basic assumption, and Lemma 4 is proved.

Now let $M = M(\sigma)$ denote the maximum value of $u$ on a given circumference $|x| = \sigma$, $(\sigma \leqslant \sigma_0)$. Our goal is to obtain the important inequality (29), reverse to (17). To this end we introduce a second auxiliary function $V = V(x, \sigma)$ according to the formula

$$V = \begin{cases} \mathrm{Max}\,(0,u) & \text{for } \sigma \leqslant |x| \leqslant R \\ \mathrm{Min}\,(M,u) & \text{for } 0 < |x| < \sigma \\ M & \text{at } x = 0. \end{cases}$$

Evidently $V$ is strongly differentiable, has compact support in $|x| < R$, and $V \equiv M$ in some neighborhood of the origin.

Let $\sigma_2$, $(\sigma_2 \leqslant \sigma_0)$, be chosen so that $\int_0^{\sigma_2} |\mathcal{B}(x, u, u_x)|\,dx \leqslant \frac{1}{2}K$; this choice is allowable by virtue of Lemmas 2 and 4. For fixed $\sigma < \sigma_2$, we define

$$\Theta = \Theta(x,\sigma) = \begin{cases} 0 & \text{for } \sigma_2 < |x| \leqslant R \\ \dfrac{r^\tau - \sigma_2^\tau}{\sigma^\tau - \sigma_2^\tau} & \text{for } \sigma \leqslant |x| \leqslant \sigma_2 \\ 1 & \text{for } |x| < \sigma, \end{cases}$$

where $\tau = (\alpha - n)/(\alpha - 1)$, (we restrict the discussion to the case $\alpha < n$, for simplicity). Clearly $\Theta$ is strongly differentiable, has compact support in $|x| < R$, and $\Theta \equiv 1$ in some neighborhood of the origin. Thus by Lemma 3, Hölder's inequality, and the choice of $\sigma_2$,

$$K = \int (\Theta_x \cdot \mathcal{A} + \Theta\, \mathcal{B})\,dx \leqslant \|\Theta_x\|_\alpha \|\mathcal{A}\|_{\alpha/(\alpha-1)} + \tfrac{1}{2}K.$$

Since $\|\Theta_x\|_\alpha^\alpha = \omega_n |\tau|^{\alpha-1} (\sigma^\tau - \sigma_2^\tau)^{1-\alpha}$, this yields at once

$$K'(\sigma^\tau - \sigma_2^\tau) \leqslant \int |\mathcal{A}|^{\alpha/(\alpha-1)}\,dx, \tag{23}$$

where $K'$ is a positive constant and the integral is to be taken over the set $\sigma \leqslant |x| \leqslant \sigma_2$.

The next problem is to estimate this integral. Since $\sigma_2 \leqslant \sigma_0$, it is clear that $V \equiv u$ in the domain of integration. Thus the integral will only be increased if it is extended over the entire set where $V = u$. Denoting this set by $\Delta$, and using inequality (2), we have

$$\begin{aligned} \int |\mathcal{A}|^{\alpha/(\alpha-1)}\,dx &\leqslant \text{Const.} \int_\Delta \{|u_x|^\alpha + b^{\alpha/(\alpha-1)} u^\alpha + e^{\alpha/(\alpha-1)}\}\,dx \\ &= \text{Const.} \int_\Delta \{(|u_x|^\alpha - 2du^\alpha - 2g) + (2d + b^{\alpha/(\alpha-1)})u^\alpha + (2g + e^{\alpha/(\alpha-1)})\}\,dx. \end{aligned} \tag{24}$$

Now by the Hölder and Sobolev inequalities,

$$\begin{aligned} \int_\Delta (2d + b^{\alpha/(\alpha-1)})\, u^\alpha\,dx &\leqslant \int (2d + b^{\alpha/(\alpha-1)})\, V^\alpha\,dx \\ &\leqslant \text{Const.}\,(\|d\|_{n/\alpha} + \|b\|_{n/(\alpha-1)}^{\alpha/(\alpha-1)})\, \|V_x\|_\alpha^\alpha \\ &= \text{Const.}\,(\|d\|_{n/\alpha} + \|b\|_{n/(\alpha-1)}^{\alpha/(\alpha-1)}) \int_\Delta |u_x|^\alpha\,dx, \end{aligned} \tag{25}$$

since $V_x = u_x$ almost everywhere in $\Delta$ and $V_x = 0$ almost everywhere in the complement of $\Delta$. Supposing that $R$ is suitably small, (25) implies

$$\int_\Delta (2d + b^{\alpha/(\alpha-1)})\, u^\alpha dx \leqslant \int_\Delta |u_x|^\alpha dx.$$

Substituting this into (24), and then using (2), yields

$$\int |\mathcal{A}|^{\alpha/(\alpha-1)} dx \leqslant \text{Const.}\left(1 + \int_\Delta u_x \cdot \mathcal{A}\, dx\right) = \text{Const.}\left(1 + \int V_x \cdot \mathcal{A}\, dx\right). \tag{26}$$

Now by Lemma 3

$$\int V_x \cdot \mathcal{A}\, dx = MK - \int V\mathcal{B}\, dx \leqslant MK + B(0) \max V, \tag{27}$$

where $B(0) = \int_0^R |\mathcal{B}(x, u, u_x)|\, dx$. We assert that

$$\max V \leqslant \text{ Const. } M. \tag{28}$$

Indeed this is obvious from the definition of $V$, except in the set $\sigma \leqslant |x| \leqslant R$. But by applying the maximum principle (cf. Theorem 8 below) to $u$ in this set, we find

$$\max_{\sigma \leqslant r \leqslant R} u \leqslant \text{Const. } M;$$

(application of Theorem 8 requires that the measure of the set $\sigma \leqslant |x| \leqslant R$ be suitably small, which in turn can be accomplished by choosing $R$ suitably small at the beginning of the proof; note also that $M \geqslant 1$, which allows us to absorb the additive constant appearing in Theorem 8). Inequality (28) now follows from the relation $V = \text{Max}\,(0, u)$ in $\sigma \leqslant |x| \leqslant R$. The required estimate for $\int |\mathcal{A}|^{\alpha/(\alpha-1)} dx$ finally results by combining (26), (27), and (28), thus

$$\int |\mathcal{A}|^{\alpha/(\alpha-1)} dx \leqslant \text{Const. } M.$$

Inserting this into (23) yields the estimate

$$M \geqslant 2K''(\sigma^\tau - \sigma_2^\tau), \quad (\sigma > \sigma_2),$$

where $K''$ is a *positive* constant. For suitably small $\sigma$ this implies

$$M \geqslant K'' \sigma^{(\alpha-n)/(\alpha-1)} \quad \text{if} \quad \alpha < n. \tag{29}$$

By a similar calculation, which may be omitted here, we find also

$$M \geqslant K'' \log R/\sigma \quad \text{if} \quad \alpha = n. \tag{29'}$$

The required asymptotic behavior of $u$ is a consequence of inequalities (17) and (29). Indeed by (17) and the Harnack principle we have

$$M \leqslant \text{Const. } m \leqslant \text{Const. } \sigma^{(\alpha-n)/(\alpha-1)}$$

valid for $\sigma \leqslant \sigma_0$, while by (29) and the Harnack principle,

$$m \geqslant \text{Const. } M \geqslant \text{Const. } \sigma^{(\alpha-n)/(\alpha-1)}$$

valid for all suitably small $\sigma$. Thus (3) holds in a neighborhood of the origin, when $\alpha < n$. The result for $\alpha = n$ is obtained in the same way, and Theorem 1 is completed.

## 3. Further results

The conclusion of Theorem 1 may be augmented by several further results concerning the behavior of a solution in the neighborhood of a positive isolated singularity.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if the singularity at* 0 *is* not *removable, then for all sufficiently small values of* $\sigma$.

$$\int_\sigma^{2\sigma} |u_x|^\alpha dx \leqslant \text{Const.} \begin{cases} \sigma^{(\alpha-n)/(\alpha-1)}, & \alpha < n, \\ (\log 1/\sigma)^n, & \alpha = n. \end{cases}$$

*Proof.* According to the argument preceding inequality (15),

$$\int_\sigma^{2\sigma} |u_x|^\alpha dx \leqslant \text{Const. } [\sigma^{-1} \|u\|_\alpha + \sigma^{(n-\alpha)/\alpha}]^\alpha \leqslant \text{Const. } \sigma^{n-\alpha} m(\sigma)^\alpha.$$

The required conclusion then follows from (17).

COROLLARY 1. *We have*

$$u_x \in L_{\theta-\delta}(S), \quad (S = \{|x| \leqslant R\}), \tag{30}$$

*where* $\theta = n(\alpha - 1)/(n-1)$ *and* $\delta$ *is any positive number. Moreover, if* $\theta > 1$ (*i.e. if* $\alpha > 2 - 1/n$) *then* $u \in W^1_{\theta-\delta}(S)$.

*Proof.* By Hölder's inequality and Theorem 2

$$\int_\sigma^{2\sigma} |u_x|^{\theta-\delta} dx \leqslant \text{Const.} \begin{cases} \sigma^{\delta(n-1)/(\alpha-1)}, & \alpha < n, \\ \sigma^\delta (\log 1/\sigma)^{n-\delta}, & \alpha = n. \end{cases}$$

Thus by the argument at the close of Lemma 2, $|u_x|^{\theta-\delta}$ is integrable over $S$, and (30) is proved.

Now let $\phi$ be a continuously differentiable function with compact support, and let $S_\sigma$ denote the ball of radius $\sigma$ about 0. Then obviously

$$\int_{D-S_\sigma}(u\phi_x+\phi u_x)\,dx=-\oint_{r=\sigma}u\phi\mathbf{n}ds, \tag{31}$$

where $\mathbf{n}$ denotes the unit normal to the sphere $r=\sigma$. By Theorem 1

$$\oint_{r=\sigma}u\phi\mathbf{n}ds=O(\sigma^{(\alpha-n)/(\alpha-1)+n-1}),$$

and this tends to zero with $\sigma$ when $\theta>1$. Consequently, if $\theta>1$ we may let $\sigma\to 0$ in (31) to obtain

$$\int(u\phi_x+\phi u_x)\,dx=0.$$

Thus $u\in W^1_{\theta-\delta}(S)$. (In fact, we have shown that $u$ is in $W^1_{\theta-\delta}$ over any proper subdomain of $D$.)

The following result should be compared with Lemma 2.

Corollary 2. *The function* $\mathcal{A}(x,u,u_x)$ *is integrable over* $0<|x|\leq R$, *and*

$$\int_{r<\sigma}|\mathcal{A}(x,u,u_x)|\,dx\leq \text{Const.}\begin{cases}\sigma & , \quad \alpha<n,\\ \sigma(\log 2R/\sigma)^{n-1}, & \alpha=n.\end{cases}$$

*Proof.* According to inequality (2), we have

$$|\mathcal{A}|\leq a|u_x|^{\alpha-1}+b|u|^{\alpha-1}+e.$$

Now by Hölder's inequality and Theorem 2

$$\int_\sigma^{2\sigma}|u_x|^{\alpha-1}dx\leq \text{Const.}\begin{cases}\sigma & , \quad \alpha<n,\\ \sigma(\log 1/\sigma)^{n-1}, & \alpha=n,\end{cases}$$

valid for all sufficiently small $\sigma$. Moreover, using Theorem 1 and the Lebesgue class conditions on $b$ and $e$ it is easy to see that

$$\int_\sigma^{2\sigma}(bu^{\alpha-1}+e)\,dx\leq \text{Const. }\sigma.$$

The required conclusion follows from the summation argument at the close of Lemma 2.

THEOREM 3. *Under the hypotheses of Theorem 1, if the singularity at* 0 *is* not *removable then*

$$\operatorname{div} \mathcal{A}(x, u, u_x) - \mathcal{B}(x, u, u_x) = -K\delta(0)$$

*in the sense of distributions; that is, for any continuously differentiable function* $\phi = \phi(x)$ *with compact support in* $D$,

$$\int (\phi_x \cdot \mathcal{A} + \phi \mathcal{B})\, dx = K\phi(0). \tag{32}$$

*Proof.* Let $\eta$ be a non-negative smooth function which vanishes outside the ball $S_\sigma$, is identically 1 in $S_{\sigma/2}$, and elsewhere satisfies $0 \leqslant \eta \leqslant 1$. We may obviously suppose that $|\eta_x| \leqslant 3/\sigma$. The function

$$\Theta = (1-\eta)\phi + \eta\phi(0)$$

has compact support in $D$ and is identically equal to $\phi(0)$ in a neighborhood of the origin. By Lemma 3,

$$\int (\Theta_x \cdot \mathcal{A} + \Theta \mathcal{B})\, dx = K\phi(0),$$

that is

$$\int \{(1-\eta)(\phi_x \cdot \mathcal{A} + \phi \mathcal{B}) + (\phi(0) - \phi)\, \eta_x \cdot \mathcal{A} + \phi(0)\, \eta \mathcal{B}\}\, dx = K\phi(0). \tag{33}$$

Now both $\mathcal{A}$ and $\mathcal{B}$ are locally summable in $D$. Hence if $\sigma \to 0$,

$$\int (1-\eta)(\phi_x \cdot \mathcal{A} + \phi \mathcal{B})\, dx \to \int (\phi_x \cdot \mathcal{A} + \phi \mathcal{B})\, dx.$$

Likewise $\int \eta \mathcal{B} dx \to 0$, while by Corollary 2 one easily obtains

$$\left| \int (\phi(0) - \phi)\, \eta_x \cdot \mathcal{A} dx \right| \leqslant \text{Const. } \sigma^{1-\varepsilon} \to 0.$$

Thus letting $\sigma \to 0$ in (33) we obtain (32), and the proof is complete.

*Remark.* Theorem 3 shows that every solution of (1) with a positive isolated singularity has the attributes of a fundamental solution, as in the case of linear equations.

## 4. Existence of solutions with isolated singularities

The very light hypotheses required for the proof of Theorem 1 do not seem strong enough to imply the general existence of solutions with isolated singularities. Accordingly, we shall suppose here that (1) is subject to certain further conditions. In particular, *it will be assumed that for any domain $D$ of sufficiently small diameter, the following two properties hold:*

*P1. The Dirichlet problem with continuous boundary data is uniquely solvable for any smoothly bounded domain in $D$, and the solutions are continuous functions of the data, in the uniform topology.*

*P2.* Let $\Gamma$ denote a smoothly bounded annular domain in $D$. By *P1* there exists a solution $v$ in $\Gamma$ assuming the continuous boundary data $\psi_0(x)$ on the outer boundary and $\psi_1(x)+m$ on the inner boundary. *We suppose that at each point $P$ in $\Gamma$, the value $v(P)$ tends to infinity with $m$.*

These conditions should be compared with the corresponding, but stronger, conditions imposed in [1].

A general discussion of these properties is beyond the scope of the paper. We may remark, however, that *P1* is satisfied for a wide variety of equations (including linear equations), provided that the domains in questions are suitably small. We believe, moreover, that *P2* is a consequence of the general structure of (1). Since a proof of this apparently involves an effort at least comparable to that of the preceding sections of the paper, we shall rest content here with imposing *P2* as an additional assumption on (1).

THEOREM 4. *Let continuous data $\psi(x)$ be assigned on the boundary of a smooth domain $D$, it being assumed that $D$ is small enough so that P1 and P2 are valid, and also so that the maximum principle (Theorem 8) holds.*

*Let 0 be a point of $D$. Then there exists a family of solutions $G=G(x)$ in $D-\{0\}$, taking on the given boundary values and satisfying*

$$G \approx \begin{cases} r^{(\alpha-n)/(\alpha-1)}, & \alpha<n, \\ \log 1/r \ , & \alpha=n, \end{cases}$$

*in the neighborhood of 0. The values of $G$ may be assigned arbitrarily at any point $P\neq 0$ in $D$, subject only to the restriction $G(P)>w(P)$, where $w$ denotes the unique solution of (1) in $D$ which takes on the assigned boundary values.*

*Proof.* This is for the most part a duplication of the proof of Theorem 13 of [1]. Applications of the (simple) maximum principle must be replaced with continuous dependence arguments, which may safely be left to the reader. The main difference lies in guaranteeing that each solution $v_\sigma$ is non-negative. Let $L$ denote the minimum of $w$ over $D$. Then each $v_\sigma$ has boundary values which are $\geqslant L$. Thus by the maximum principle we have $v_\sigma \geqslant \text{Const.}\ L - \text{Const.}$, where the coefficient of $L$ may even be negative. In any case, the functions $v_\sigma$ are uniformly bounded below. By making the change of variables $\bar{u} = u + \text{Const.}$, with the constant suitably chosen, it may therefore be supposed without loss of generality that $v_\sigma \geqslant 0$.

## 5. Linear equations

The results of the preceding sections can be sharpened somewhat in case (1) is linear, that is, of the form

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j} + b_i(x)\,u + e_i(x)\right) = c_i(x)\frac{\partial u}{\partial x_i} + d(x)\,u + f(x). \tag{34}$$

Here it is assumed that the coefficients $a_{ij}(x)$ are bounded measurable functions satisfying the ellipticity condition

$$a_{ij}\,\xi_i\,\xi_j \geqslant \lambda\,\xi^2, \quad \lambda = \text{Const.} > 0,$$

and that the coefficients $b_i$ through $f$ fall into the Lebesgue classes

$$b_i,\ c_i,\ e_i \in L_{n/(1-\varepsilon)}; \quad d,\ f \in L_{n/(2-\varepsilon)}.$$

The various conditions (2) are easily seen to be met with $\alpha = 2$, and it follows that Theorems 1 through 3 hold for (34), again with $\alpha = 2$.

Now let $\mathfrak{M}$ denote the class of smoothly bounded domains $D$ such $|D| \leqslant D_0$, $D_0$ being the constant in Theorem 8. We assert that properties *P1* and *P2* hold for any domain $D$ of class $\mathfrak{M}$.

Indeed, the uniqueness and continuous dependence of solutions of the Dirichlet problem is a direct consequence of Theorem 8 applied to the difference of two solutions. The existence of solutions is naturally a more difficult matter. For smooth data, the result can be obtained (in outline) by first smoothing the coefficients, then showing that the solution (known to exist by the classical Schauder theory) is uniformly continuous in the closure of the domain in question (this involves the usual straightening of the boundary), and finally tending to a limit. When the data is only continuous, existence can be obtained by a standard approximation process based on the existence theorem for smooth data. An alternate discussion based on functional analysis is given in reference [3]. Property *P2* is obvious, by the superposition property of linear equations.

This being the case, it follows that for any domain $D$ in the class $\mathfrak{M}$ there exists a family of solutions of (34) which satisfies the conclusion of Theorem 4. In addition, we have the following supplementary results, in which $(34)_0$ refers to equation (34) with $e_i \equiv f \equiv 0$.

THEOREM 5. *Let $G$ be a particular solution of $(34)_0$ in the set $D-\{0\}$, such that $G \approx r^{2-n}$ or $G \approx \log 1/r$ near 0, depending on whether $n > 2$ or $n = 2$. Then every solution of (34) in $D-\{0\}$ which in the neighborhood of 0 is bounded below by a multiple of $r^{2-n}$, or $\log 1/r$, has the form*

$$u = \text{Const. } G + w,$$

*where $w$ is a solution of (34) in the entire domain $D$.*

*Proof.* This result is an analogue of Theorem 5 of [4]. It is clearly enough to carry out the proof when $D$ is in the class $\mathfrak{M}$. This being the case, we may suppose by virtue of property *P1* that both $u$ and $G$ are zero on the boundary of $D$, and that $e_i \equiv f \equiv 0$. Moreover, by adding a suitable multiple of G to $u$ it can be assumed without loss of generality that $u \geq 0$ near 0.

Under these assumptions it follows from Theorem 8 that both $u$ and $G$ are non-negative in $D$. The proof of Theorem 5, reference [4], can now be taken over almost word for word (and indeed even simplifies a bit).

*Remark.* In view of Theorem 5, the family of solutions given by Theorem 4 is unique and depends continuously on a single multiplicative parameter.

The following result is a slight generalization of Theorem 5 in that the existence of a solution $G$ is not required.

THEOREM 6. *Let $u$ be a continuous solution of (34) in the set $D-\{0\}$, which in the neighborhood of 0 is bounded below by a multiple of $r^{2-n}$, or $\log 1/r$. Then either $u$ has a removable singularity at 0, or else (possibly after multiplication by $-1$)*

$$u \approx \begin{cases} r^{2-n}, & n > 2, \\ \log 1/r, & n = 2, \end{cases}$$

*in the neighborhood of the origin.*

*Proof.* Let $D'$ be a suitably small neighborhood of the origin. Then there exists a solution $G$ of $(34)_0$ in the set $D'-\{0\}$, such that $G \approx r^{2-n}$ or $G \approx \log 1/r$. Consequently, by Theorem 5,

$$u \equiv \text{Const. } G + w$$

in the set $D'-\{0\}$, and the conclusion follows at once.

## 6. Corrections and additions to reference [1]

1. As they are stated, Theorems 3, 4, 3′, and 4′ of [1] are valid only in case $M=0$. The correct versions of Theorems 3 and 4 are as follows:

THEOREM 7. *Let $u$ be a weak solution of* (1) *in a domain $D\subset\Omega$. Suppose that $u\leqslant M$ on the boundary of $D$, and that conditions* (2) *hold. Then*

$$\max u \leqslant M + C'|M| + C\{|D|^{-1/\alpha}\|\tilde{u}\|_{\alpha,D} + k\}, \quad (\tilde{u} = \mathrm{Max}\,(0, u-M)),$$

*where $C$, $C'$, and $k$ depend only on the structure of* (1). *In particular*

$$k = \{|D|^{\varepsilon/n}\|f\|\}^{1/(\alpha-1)} + \{|D|^{\varepsilon/n}\|g\|\}^{1/\alpha},$$

*while $C'$ tends to zero as $\|d\|$ tends to zero.*

THEOREM 8 (Maximum principle). *Let $u$ satisfy the hypotheses of Theorem 7. Then there exists a constant $D_0$, depending only on the structure of equation* (1), *such that if $|D|\leqslant D_0$ then*

$$\max u \leqslant M + C'|M| + Ck.$$

*Proof of Theorems 7 and 8.* The error in the original proof came in asserting the general validity of (25) on page 262. When $M=0$, $k>0$, however, (25) holds in the set $u\geqslant\varepsilon'$, and the proof as given is correct. The case $M=0$, $k=0$ may then be obtained by a trivial approximation argument. Thus, if $M=0$, we have shown that

$$\max u \leqslant C(|D|^{-1/\alpha}\|\tilde{u}\|_{\alpha,D} + k), \quad (\tilde{u} = \mathrm{Max}\,(0, u)), \tag{35}$$

or in the case of Theorem 8,

$$\max u \leqslant Ck. \tag{36}$$

The general result can be obtained by applying (35) and (36) to the new dependent variable $u^* = u - M$. This change affects the structure of (1) by replacing $f$ and $g$ respectively with

$$f + 2^{\alpha-1}M^{\alpha-1}d, \quad g + 2^{\alpha-1}M^{\alpha}d.$$

Hence $k$ must be replaced in (35) and (36) by

$$\{|D|^{\varepsilon/n}\|f + 2^{\alpha-1}M^{\alpha-1}d\|\}^{1/\alpha-1} + \{|D|^{\varepsilon/n}\|g + 2^{\alpha-1}M^{\alpha}d\|\}^{1/\alpha}.$$

The latter expression is easily seen to be less than

$$\text{Const.}\ k + C'|M|,$$

where $C'$ tends to zero as $|D|^{\varepsilon/n}\|d\|$ tends to zero. This completes the proof.

Theorems 3′ and 4′ are corrected in a similar way. We may omit the details.

Theorem 3 was not used in [1], and Theorem 4 was applied only at one point on page 278. Here the inequality

$$u(x) \geqslant \text{Min } (\mu_\nu, \mu_{\nu+1}) - Ck$$

must be replaced by

$$u(x) \geqslant (1 - C') \text{ Min } (\mu_\nu, \mu_{\nu+1}) - Ck,$$

according to Theorem 8. For sufficiently large $\nu$ we have $C' \leqslant \frac{1}{2}$, and the conclusion $u \to \infty$ as $x \to 0$ follows as before.

We note that the maximum principles above can be given wider validity in several directions. First, the results clearly apply not only to the equation div $\mathcal{A} = \mathcal{B}$, but also to the differential inequality div $\mathcal{A} \geqslant \mathcal{B}$. More generally, if

$$\text{div } \mathcal{A}(x, u, u_x) \geqslant \mathcal{B}(x, u, u_x)$$

whenever $u > M_0$, then Theorems 7 and 8 hold for all $M \geqslant M_0$, This result has particular application to the linear equation (34) when the coefficient $d$ is non-negative and $M \geqslant 0$. Indeed, in this case it is clear that we may omit the term $C'|M|$ in the conclusions of Theorems 7 and 8, and moreover neglect $\|d\|$ in the determination of the constants $C$ and $D_0$.

Also, the maximum principle holds for all exponents $\alpha$ in the range $1 \leqslant \alpha < \infty$, assuming for $\alpha > n$ that $b, e \in L_{\alpha/(\alpha-1)}$, $c \in L_\alpha$, and $d, f, g \in L_1$. This extension requires no essentially new ideas in the proofs already given.

2. The phrasing of Theorem 7 of [1] does not indicate the specific dependence of the coefficient $C'$ on the domains $D$ and $D'$. It is therefore worth pointing out that this dependence is completely expressed by a single number $N$, *the number of spheres* (*of radius* $\leqslant 1$) *required for the chaining argument.* In particular, for pairs of domains $D$, $D'$ and $\bar{D}$, $\bar{D}'$ which are geometrically similar and both contained in a bounded subset of $\Omega$, it may be assumed that the corresponding constants $C'$ and $\bar{C}'$ are the same. This fact plays an important part in our various applications of the Harnack principle over annular regions (cf., for example, the proof of Lemma 2).

3. In the same way, Theorem 8 of [1] is not stated in as sharp a form as could be desired. The following version is preferable.

THEOREM 9. *Let $u$ be a weak solution of* (1) *in a domain $D \subset \Omega$. Then $u$ is* (*essentially*) *Hölder continuous in $D$. Moreover, if $|u| \leqslant L$ then for $x$, $y \in D$*

$$|u(x)-u(y)| \leqslant H(L+k')\left|\frac{x-y}{R}\right|^{\lambda},$$

*where H and λ depend only on the structure of* (1), *and R is the maximum distance of x or y to the boundary of D*, (*or R* = 1 *if this is smaller*).

To see this, it is enough to replace the final sentence of the original proof by the following argument:

Let $x$ be the point further from the boundary of $D$. If $|x-y| \leqslant 2^{-(\gamma+1)\alpha/\varepsilon}R$, then by inequality (48)

$$|u(x)-u(y)| \leqslant C(L+k')\left|\frac{x-y}{R}\right|^{\lambda}.$$

On the other hand, if $|x-y| \geqslant 2^{-(\gamma+1)\alpha/\varepsilon}R$, then since $|u| \leqslant L$ we have

$$|u(x)-u(y)| \leqslant 2L \leqslant CL\left|\frac{x-y}{R}\right|^{\lambda}.$$

This completes the proof.

4. We observe (without proof) that assumption (8) in Chapter I of [1] can be replaced by the weaker condition (7). Condition (8), is required, however in Chapters II and III. Finally, though we shall not carry out the details, one can show that the hypothesis of Theorem 1 of the present paper can we weakened slightly to read

$$b, e \in L_{n/(\alpha-1)}; \quad c \in L_{n/(1-\varepsilon)}; \quad d, f, g \in L_{n/(\alpha-\varepsilon)},$$

except that for $\alpha = n$ we require $b \in L_{n/(n-1-\varepsilon)}$.

5. One further reference should be included in the bibliography of [1], namely

GEVREY, M., Sur certaines propriétés des fonctions harmoniques et leur extension aux équations aux dérivées partielles *C. R. Acad. Sci. Paris*, 183 (1926), 546–548.

In this paper there occurs for the first time a removable singularity theorem of interpolatory type for solutions of elliptic equations.

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