ON THE CLOSURE OF CHARACTERS AND THE ZEROS OF ENTIRE FUNCTIONS

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Introduction

The problem to be studied in this paper concerns the closure properties on an interval of a set of characters $\{e^{i\lambda_n x}\}_1^\infty$, where $\Lambda = \{\lambda_n\}_1^\infty$ is a given set of real or complex numbers without finite point of accumulation. This problem is for obvious reasons depending on the distribution of zeros of certain entire functions of exponential type.

The main problem of the paper is to determine *the closure radius* $\varrho = \varrho(\Lambda)$ defined as the upper bound of numbers r such that $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ span the space $L^2(-r, r)$. The value of $\varrho(\Lambda)$ does not change if a finite number of points are removed from or adjoined to Λ. Nor does $\varrho(\Lambda)$ change if the metric in the previous definition is replaced by any other L^p-metric, or by a variety of other topologies.

If Λ contains complex numbers we shall always assume

$$\sum_{\lambda\in\Lambda}\left|\Im m\left(\frac{1}{\lambda}\right)\right| < \infty, \tag{0.1}$$

thereby excluding the trivial case $\varrho(\Lambda) = \infty$ which occurs when the series diverges.

The problem to determine $\varrho(\Lambda)$ explicitly in terms of density properties of Λ was solved by the authors in 1961 for real sets Λ, and one of the main elements in the proof has been published earlier in this journal [2]. This paper will be concerned mainly with the distribution of zeros of entire functions of exponential type satisfying the condition

$$\int_{-\infty}^{\infty}\left|\log|f(x)|\right|\frac{dx}{1+x^2} < \infty. \tag{0.2}$$

The results in this field go beyond the preliminary report [3] and will be derived by essentially new methods.

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In order to explain the results we introduce the following notions concerning families Ω of intervals of the real axis. Without distinguishing between closed, open or semi-closed intervals we represent an interval ω by a point

$$T\omega = x(\omega) + iy(\omega)$$

in the upper half-plane, defining

$$x(\omega) = \text{center of } \omega; \qquad y(\omega) = |\omega| = \text{length of } \omega.$$

If the pointset $T\Omega$ is measurable we assign to Ω the measure

$$m(\Omega) = \int_{T\Omega} \frac{dx\,dy}{1+x^2+y^2}. \tag{0.3}$$

If ω is an interval, $\bar{\omega}$ shall denote the collection of all its subinterval and we define $\bar{\Omega} = \bigcup \bar{\omega}$ for $\omega \in \Omega$. We observe that $T\bar{\omega}$ consists of an isoscele with ω as base and of height $|\omega|$. As a union of such triangles a set $T(\bar{\Omega})$ is always measurable.

DEFINITIONS. I. *A set Ω of intervals is negligible if $m(\bar{\Omega}) < \infty$.*

II. *A positive measure $d\mu$ on the real axis is regular and of density $A(d\mu) = a$ if the family of intervals*

$$\Omega_\varepsilon = \left\{\omega;\ \left|\frac{1}{|\omega|}\int_\omega d\mu - a\right| \geqslant \varepsilon\right\} \tag{0.4}$$

is negligible for each $\varepsilon > 0$.

III. *The interior density $A_i(d\mu)$ of $d\mu$ is defined as the upper bound of $A(d\nu)$ for regular $d\nu \leqslant d\mu$. The exterior density $A_e(d\mu)$ is the lower bound of $A(d\nu)$ for regular $d\nu \geqslant d\mu$. If no such majorant $d\nu$ exists, $A_e(d\mu) = \infty$.*

An immediate consequence of these definitions is that $d\mu$ is regular if and only if its interior and exterior densities coincide and are finite.

To a sequence Λ of complex numbers without finite point of accumulation we assign the measure $dN = dN_\Lambda$ which vanishes off Λ and assumes at $\lambda \in \Lambda$ the value equal to the multiplicity of λ as element of the sequence.

By means of the notion of regular measures $d\mu$, we can define a notation of regularity of sets Λ.

DEFINITION IV. *A real set Λ without finite point of accumulation will be called regular if the measure dN_Λ is regular.*

This concept can be extended to complex sets Λ satisfying (0.1). We consider the mapping taking $\lambda = a + ib$ into the point $\lambda^* = |\lambda|^2/a$. The two open half planes

$x > 0$ and $x < 0$ are thus mapped on the postitive and the negative real axes respectively, and the line $x = 0$ is thrown out to $z = \infty$. We shall write $\Lambda^* = \{\lambda^*; \lambda \in \Lambda\}$ and always assume $0 \notin \Lambda$.

The importance of the projection $\Lambda \to \Lambda^*$ is due mainly to the relation

$$\frac{1}{\lambda^*} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right)$$

which implies that for all real x

$$\left|1 - \frac{x}{\lambda^*}\right| = \frac{1}{2}\left|\left(1 - \frac{x}{\lambda}\right) + \left(1 - \frac{x}{\bar{\lambda}}\right)\right| \leqslant \left|1 - \frac{x}{\lambda}\right|. \tag{0.5}$$

DEFINITION V. *A complex set Λ will be called regular if* (0.1) *holds and if the measure dN_{Λ^*} is regular.*

Expressed in the term of exterior density of a measure the solution to the closure problem reads as follows:

Let Λ satisfy condition (0.1). *Then*

$$\varrho(\Lambda) = \pi A_e(dN_{\Lambda^*}). \tag{0.6}$$

In order to describe the proof of this and further results we first recall some definitions and properties concerning entire functions $f(z)$. If $\log|f(z)| \leqslant O(|z|)$, $z \to \infty$, $f(z)$ is said to be of exponential type and the number

$$c = \limsup_{z \to \infty} \frac{\log|f(z)|}{|z|}$$

is referred to as the type of f. By E_a we will denote the set of all f of type $\leqslant a$ for which the summability (0.2) is fulfilled. Furthermore, $f \in E_a$ will be called normalized if $f(0) = 1$ and

$$a = \limsup_{y = +\infty} \frac{\log|f(iy)|}{y} = \limsup_{y = -\infty} \frac{\log|f(iy)|}{-y}.$$

For a normalized f of class $E_{k\pi}$ the following properties are well known:

$$\lim_{r = \infty} \frac{N_1(r)}{r} = \lim_{r = \infty} \frac{N_2(r)}{r} = k, \tag{0.7}$$

where $N_1(r)$ and $N_2(r)$ represent the number of zeros of f in the two half-circles having $(-ir, ir)$ as diameter. Moreover, the series (0.1) converges and f has the representation

$$f(z) = f(z; \Lambda) = \lim_{r \to \infty} \prod_{\substack{|\lambda| \leq r \\ \lambda \in \Lambda}} \left(1 - \frac{z}{\lambda}\right). \tag{0.8}$$

We shall find that the summability condition (0.2) implies a much more refined property of the distribution of the zeros than that described by (0.7). The following result is a consequence of a more precise theorem proved in Chapter I; it is also the key to the closure problem:

THEOREM I. *The zeros Λ of a normalized $f \in E_{k\pi}$ form a regular set and $A(dN_{\Lambda^*}) = k$.*

The solution of the closure problem relies also on the following result on multipliers proved in [2].

THEOREM A. *Let f belong to a class $E_{k\pi}$ and let ε and $\alpha < 1$ be given positive numbers. Then the class $E_{\varepsilon\pi}$ contains a function g with zeros $\Gamma = \{\gamma_n\}_1^\infty$ such that for all real x*

$$|g(x)\, f(x)| \leq \text{const}\; e^{-|x|^\alpha}.$$

The γ_n may be chosen real and so that for $m \neq n$, $|\gamma_m - \gamma_n| \geq 1/\varepsilon$.

The closure problem is herewith reduced to the question whether or not a given class $E_{k\pi}$ contains a function vanishing on a given set Λ. The answer reads as follows;

THEOREM II. *Let Λ satisfy* (0.1) *and assume $A_e(dN_{\Lambda^*}) = a < \infty$. Then the class $E_{k\pi}$ contains functions vanishing on Λ if $k > a$, whereas no such function exists in $E_{k\pi}$ if $k < a$.*

The closure theorem follows immediately on combining the two last theorems and will not be considered further in the text.

I. A class of subharmonic functions

We will be interested in functions $U(z) \leq O(|z|)$ which are subharmonic in the punctured plane $z \neq \infty$ and harmonic in the half planes $y > 0$ and $y < 0$. The real axis is thus the support of the positive measure $d\mu$ associated to U and defined as $\Delta U / 2\pi$ in the sense of distributions. In case U admits a representation

$$U(x + iy) = k\pi\, |y| + u(x + iy), \tag{1.1}$$

where k is a constant and where

$$\lim_{r=\infty} \frac{u(re^{i\theta})}{r} = 0, \quad \theta \neq 0, \pi, \tag{1.2}$$

we shall say that U belongs to the class $S_{k\pi}$. If in addition $U(x)/(1+x^2) \in L'(-\infty, \infty)$, U will be called *Poisson summable* and we will have the following representation for u figuring in (1.1)

$$u(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|y| U(\xi)\, d\xi}{y^2 + (x-\xi)^2} \equiv P(x+iy, U).$$

If U belongs to a class $S_{k\pi}$ and is constant on the real axis, then clearly $d\mu$ coincides with the measure $k\,dx$. It is therefore to be expected that the difference $\int_\omega d\mu - k|\omega|$ would be small whenever $U(x)$ is close to a constant in a neighborhood of an interval ω. As a measure of the "flatness" of $U(x)$ on $(x-y/2,\ x+y/2)$ we shall use the quantity

$$q(\omega) = \frac{w(x+iy)}{y}, \tag{1.3}$$

where

$$w(x+iy) = \inf_{-\infty<c<\infty} P(x+iy, |U-c|). \tag{1.4}$$

THEOREM I′. *Let U belong to the class $S_{k\pi}$ and be Poisson summable. Then the following holds:*

The set of intervals

$$Q_\delta = \{\omega;\ q(\omega) \geqslant \delta\} \tag{1.5}$$

is negligible for each $\delta > 0$.

The inequality

$$\left| \frac{1}{|\omega|} \int_\omega d\mu - k \right| < 5\sqrt{k\delta} \tag{1.6}$$

holds for each interval ω such that

$$q(\omega) < \delta \leqslant \tfrac{1}{4} \min\left(1, \frac{k}{6}\right).$$

Let us point out that unless U is constant the associated function w is a strictly positive continuous superharmonic function in $y > 0$, with the property that for fixed x, $w(x+iy)/y$ is strictly decreasing with increasing y. Define for $\delta > 0$

$$D_\delta = \{z = x+iy;\ 0 < w(z) < \delta \cdot y\}, \qquad B_\delta = \{z = x+iy;\ w(z) \geqslant \delta \cdot y \geqslant 0\}.$$

D_δ is a simply connected region with a boundary meeting each vertical line in a unique finite point. The set B_δ is the image under T of the family Q_δ of intervals defined in (1.5). As a first step in the proof that $m(\overline{Q}_\delta) < \infty$ we shall show that for all $\delta > 0$

$$\int_{B_\delta} \frac{dx\,dy}{1+|z|^2} < \infty. \tag{1.7}$$

To this purpose we observe that

$$v(x+iy) = \delta \cdot y - w(x+iy)$$

is a strictly positive subharmonic function in D_δ, tending to zero at all finite boundary points. Assume $z_0 = iy_0 \in D_\delta$ and let $D_{\delta,r}$ denote the component of the intersection of D_δ and the disc $C_r = \{z; |z+i| < r\}$ which contains z_0, $r > 1 + y_0$. Let $\theta(z, \gamma_r)$ be the harmonic measure for $D_{\delta,r}$ of its circular boundary arc γ_r. Then $\delta \cdot r\theta(z, \gamma_r)$ is a harmonic majorant of $v(z)$ and we shall have in particular

$$0 < v(z_0) < \delta \cdot r\theta(z_0, \gamma_r), \quad r > 1 + y_0.$$

In order to estimate θ we shall use the inequality

$$\theta(z_0, \gamma_r) < e^{-\pi L^2/A}$$

(cf. [3], p. 10, formula (26)) where A denotes the Dirichlet integral of any function ψ, harmonic in $D_{\delta,r}$ and satisfying the conditions

$$\psi(z_0) = 0, \quad \psi(z) \geq L, \quad z \in \gamma_r.$$

The choice $\psi(z) = \log|z+i| - \log|z_0+i|$ yields $L = \log(r+1) - \log(y_0+1)$, and

$$A = \int_{D_{\delta,r}} \frac{dx\,dy}{|z+i|^2} < \pi \log r - \int_{B_{\delta,r}} \frac{dx\,dy}{|z+i|^2}, \tag{1.8}$$

where $B_{\delta,r} = B_\delta \cap C_r$. If (1.7) diverges the last term in (1.8) would tend to $-\infty$ with increasing r. This implies $\theta(z_0, \gamma_r) = o(1/r)$ leading to the contradiction $v(z_0) = 0$. Therefore (1.7) is true.

The stronger property $m(\overline{Q}_\delta) < \infty$ will be proved next. To this purpose we recall the Harnack inequalities for positive harmonic functions in $y > 0$:

$$\frac{1}{k} \leq \frac{u(z)}{u(z_0)} \leq k, \qquad k = \frac{|z-\bar{z}_0| + |z-z_0|}{|z-\bar{z}_0| - |z-z_0|}.$$

These relations remain valid for w also, being the lower envelope of positive harmonic functions. Let $T\omega_0 = x_0 + iy_0$ and let R_{ω_0} denote the square with vertices at $x_0 \pm y_0/2$, $x_0 \pm y_0/2 + iy_0$. On the upper side of R_{ω_0} the Harnack inequality yields

$$w(z) > \tfrac{1}{2}\, w(z_0),$$

which implies
$$q(\omega) > \tfrac{1}{2}\, q(\omega_0), \quad T\omega \subset R_{\omega_0}, \tag{1.9}$$

proving that $T\bar{Q}_\delta \subset B_{\delta/2}$. Hence by (1.7) $m(\bar{Q}_\delta) < \infty$.

In the proof of (1.6) we need this elementary lemma; *Let $\varphi(z)$ be the Poisson integral of a function $\varphi(x)$ with the following properties:*

a) supp $\varphi \subset [-1, 1]$ *and* $0 \leqslant \varphi(x) \leqslant 1$;

b) $\int \varphi(x)\, dx \leqslant 3/2$;

c) φ *is of class* C^2 *and* $|\varphi''(x)| \leqslant M$, *where* $M \geqslant 16$.

Then on the real axis
$$\left|\frac{\partial \varphi}{\partial y}\right| < \frac{3\sqrt{M}}{1+x^2}. \tag{1.10}$$

For $|x| > 1$ derivation of the Poisson integral gives
$$\left|\frac{\partial \varphi}{\partial y}\right| < \frac{3}{2\pi} \frac{1}{(|x|-1)^2}. \tag{1.11}$$

We have also the representation
$$\frac{\partial \varphi}{\partial y} = \frac{1}{\pi} \int_0^\infty (\varphi(x+t) + \varphi(x-t) - 2\varphi(x)) \frac{dt}{t^2}$$

which, using the majoration
$$|\varphi(x+t) + \varphi(x-t) - 2\varphi(x)| \leqslant \min(2, Mt^2),$$

yields
$$\left|\frac{\partial \varphi}{\partial y}\right| \leqslant \frac{\sqrt{8M}}{\pi} < \sqrt{M}. \tag{1.12}$$

The stated result follows on combining (1.11) and (1.12).

For $0 < \varepsilon \leqslant \frac{1}{4}$ we define $\varphi_\varepsilon(x)$ as the even continuous function which vanishes for $x \geqslant \frac{1}{2} + 2\varepsilon$ and equals 1 on $[0, \frac{1}{2}]$, and furthermore such that on the first and second half of the interval $[\frac{1}{2}, \frac{1}{2} + 2\varepsilon]$, the second derivative φ_ε'' is $-1/\varepsilon^2$, $1/\varepsilon^2$ respectively. The function $\varphi_{-\varepsilon}(x)$ is defined similarly with the exception that the interval where $\varphi_{-\varepsilon}(x)$ decreases from 1 to 0 is now instead $[\frac{1}{2} - 2\varepsilon, \frac{1}{2}]$. Hence
$$\int \varphi_{\pm\varepsilon}(x)\, dx = 1 \pm 2\varepsilon \leqslant \tfrac{3}{2}.$$

The discontinuity of the second derivative does not impair the validity of (1.10) and we shall have

$$\left|\frac{\partial}{\partial y}\varphi_{\pm\varepsilon}\right| < \frac{3}{\varepsilon}\frac{1}{1+x^2}. \tag{1.13}$$

Let us first prove (1.6) for $\omega_0 = [-\frac{1}{2}, \frac{1}{2}]$. By virtue of the properties of u and φ_ε we can use the Greeen formula for a half plane writing

$$\frac{1}{\pi}\int \varphi_{\pm\varepsilon}\frac{\partial}{\partial y}(u-c_0)\,dx = \frac{1}{\pi}\int (u-c_0)\frac{\partial}{\partial y}\varphi_{\pm\varepsilon}\,dx,$$

where the integration is extended over the line $y=\eta>0$. When $\eta\to 0$ the measure $\partial u/\partial y\; dx/\pi$ converges weakly over finite intervals to $d\mu - k\,dx$. Hence

$$\int \varphi_{\pm\varepsilon}\,d\mu - k(1\pm 2\,\varepsilon) = \frac{1}{\pi}\int (u-c_0)\frac{\partial}{\partial y}\varphi_{\pm\varepsilon}\,dx.$$

By (1.13) the second member is in absolute value less than

$$\frac{3}{\varepsilon\pi}\int\frac{|U(x)-c_0|}{1+x^2}\,dx = \frac{3}{\varepsilon}\,w(i) = \frac{3}{\varepsilon}\,q(\omega_0),$$

where c_0 is the constant minimizing the integral.

Since
$$\int \varphi_{-\varepsilon}\,d\mu \leqslant \int_{\omega_0} d\mu \leqslant \int \varphi_\varepsilon\,d\mu$$

we get
$$\left|\int_{\omega_0} d\mu - k\right| \leqslant 2\,k\varepsilon + \frac{3}{\varepsilon}\,q(\omega_0)$$

and finally (1.6) by taking

$$\varepsilon = \sqrt{\frac{3\,q(\omega_0)}{2\,k}}.$$

In order to prove (1.6) for an arbitrary interval $[\xi-\eta/2, \xi+\eta/2]$ it is sufficient to apply the previous result to the function

$$U_0(z) = \frac{U(\eta z+\xi)}{\eta}$$

observing that
$$\int_{\omega_0} d\mu_0 = \frac{1}{|\omega|}\int_\omega d\mu,$$

$$q_0(\omega_0) = w_0(i) = \frac{w(\xi+i\eta)}{\eta} = q(\omega),$$

where q_0, w_0, $d\mu_0$, are derived from U_0.

II. Lemmas on summable Hilbert transforms

By L_0^1 we will denote the subset of the space $L^1 = L^1(-\infty, \infty)$ consisting of functions f with the properties;

$$|f(x)|^2 (1+x^2) \in L^1, \tag{2.1}$$

$$\int f(x)\,dx = 0. \tag{2.2}$$

For functions f satisfying (2.1) we introduce the translation invariant norm

$$p(f) = \min_{-\infty < x_0 < \infty} \sqrt[4]{\int |f(x)|^2\,dx \int (x-x_0)^2 |f(x)|^2\,dx}. \tag{2.3}$$

Lemma II.1. *For $f \in L_0^1$ the Hilbert transform*

$$f(x) \to \tilde{f}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|\xi - x| > \varepsilon} \frac{f(\xi)}{\xi - x}\,d\xi$$

is isometric in the norm p and

$$\int |\tilde{f}(x)\,dx| \leqslant \sqrt{2\pi}\, p(\tilde{f}) = \sqrt{2\pi}\, p(f). \tag{2.4}$$

The transformation $f \to \tilde{f}$ is isometric in the space L^2. It thus follows by (2.2) and the identity

$$\frac{\xi - x_0}{\xi - x} = 1 + \frac{x - x_0}{\xi - x}$$

that $(x-x_0)\tilde{f}(x)$ is the transform of $(x-x_0) f(x)$. Hence, $p(\tilde{f}) = p(f)$. By Schwarz' inequality

$$\left(\int |\tilde{f}(x)|\,dx\right)^2 \leqslant \frac{\pi}{t} \int |\tilde{f}(x)|^2 (1 + t^2 (x-x_0)^2)\,dx, \quad t > 0, \tag{2.5}$$

where the minimum of the right-hand side equals $2\pi p^2(\tilde{f})$. This proves (2.4).

Corollary of Lemma II.1. *Let $\{\omega_\nu\}$ be a set of non-overlapping intervals covering the support of a function $f(x)$ locally of summable square. Assume*

$$\int_{\omega_\nu} f(x)\,dx = 0, \tag{2.6}$$

$$\sum_\nu |\omega_\nu|^{\frac{1}{2}} \left\{\int_{\omega_\nu} |f|^2\,dx\right\}^{\frac{1}{2}} = A < \infty. \tag{2.7}$$

Then both f and $\hat{f}$ are summable and

$$\int |\hat{f}(x)|\, dx < \sqrt{\pi} A. \tag{2.8}$$

Set $f_\nu = f\chi_\nu$, where χ_ν is the characteristic function of ω_ν. Then $f_\nu \in L_0^1$ and

$$p(\hat{f}) = p(f) < \sqrt{\frac{|\omega_\nu|}{2}\int |f_\nu|^2\, dx}. \tag{2.9}$$

The inequality (2.8) follows on summation using (2.4).

Lemma II.2. *Let Λ be a real set with exterior density $A_e(\Lambda) = a$. Assume $0 \notin \Lambda$ and let the function $N(x) = N_\Lambda(x)$ be defined as*

$$N(x) = \int_0^x dN_\Lambda, \quad x \gtrless 0. \tag{2.10}$$

Then for each given $b > a$ there exists a function $\Phi(x) = \Phi_b(x)$ belonging to the class

$$\Gamma_b = \{\Phi(x);\ 0 \leqslant \Phi(x_2) - \Phi(x_1) \leqslant b(x_2 - x_1),\ x_1 < x_2\}$$

and such that $(\Phi(x) - N(x))/x^2$ as well as its Hilbert transform is summable.

Consider the variational problem

$$\inf \int_{-T}^{T} (\Phi(x) - N(x))^2 \frac{dx}{x^2} \equiv \inf J(\Phi), \tag{2.11}$$

where $\pm T \notin \Lambda$ and where Φ belongs to Γ_b over $[-T, T]$. Since Γ_b is convex we conclude that there exists a unique $\Phi = \Phi_T$ minimizing (2.11). We want to show that $[-T, T]$ is dissected into a finite sequence of intervals where alternately $\Phi' = 0$ and $\Phi' = b$.

Assume $\Phi - N < 0$ on an open interval (x_1, x_2). Then $\Phi' = b$ in (x_1, x_2) because otherwise there would exist a $\Phi_1 \in \Gamma_b$, equal to Φ off (x_1, x_2) and such that in (x_1, x_2) $\Phi \leqslant \Phi_1 \leqslant N$, where $\Phi < \Phi_1$ holds on a set of positive measure, contradictory to the minimal property of Φ. By similar reason the conclusion $\Phi' = b$ remains true under the assumption $\Phi - N > 0$ in (x, x_2), or more generally if

$$\Phi(x) - N(x) \neq 0, \quad \text{a.e. for } x \in (x_1, x_2). \tag{2.12}$$

If two points $\xi_1 < \xi_2$ are not separated by any point in Λ and if $\Phi(\xi_1) = \Phi(\xi_2)$ then clearly $\Phi - N = 0$ in $[\xi_1, \xi_2]$ and we shall have $\Phi' = 0$ there. An open subinterval of

$[-T, T]$ which is maximal with respect the property (2.12) will be called a beta-interval. A closed maximal interval where $\Phi' = 0$ will be referred to as an alpha-interval. We already know that at most one alpha-interval can be contained in $[\lambda, \lambda']$ where λ, λ' are adjacent points in Λ, and we may therefore conclude that $[-T, T]$ is divided into a finite sequence of intervals alternating between the α- and the β-type.

Our next aim is to show that for all beta-intervals

$$\int_\beta (\Phi - N) \frac{dx}{x^2} = 0. \tag{2.13}$$

To this purpose let $V_{\gamma, \varepsilon}$ be a local translation operator associated to the interval $\gamma = (x_1, x_2)$ and defined as follows. If $\varepsilon > 0$:

$$V_{\gamma,\varepsilon}\Phi(x) = \begin{cases} \Phi(x-\varepsilon), & x \in (x_1+\varepsilon, x_2+\varepsilon), \\ \Phi(x_1), & x \in (x_1, x_1+\varepsilon), \\ \Phi(x), & x \notin (x_1, x_2+\varepsilon), \end{cases} \tag{2.14}$$

and if $\varepsilon < 0$:

$$V_{\gamma,\varepsilon}\Phi(x) = \begin{cases} \Phi(x-\varepsilon), & x \in (x_1+\varepsilon, x_2+\varepsilon), \\ \Phi(x_2), & x \in (x_2+\varepsilon, x_2), \\ \Phi(x), & x \notin (x_1+\varepsilon, x_2). \end{cases} \tag{2.15}$$

If $\Phi' = b$ on γ, then for $\varepsilon \gtrless 0$

$$J(V_{\gamma,\varepsilon}\Phi) - J(\Phi) = -2b\varepsilon \int_\gamma (\Phi - N) \frac{dx}{x^2} + O(\varepsilon^2). \tag{2.16}$$

We should observe also that if $\Phi \in \Gamma_b$ and is constant on $(x_2, x_2+\delta)$, $\delta > 0$, then $V_{\gamma,\varepsilon}\Phi \in \Gamma_b$ for $0 < \varepsilon < \delta$. Similarly, if Φ is constant on $(x_1 - \delta, x_1)$, then $V_{\gamma,\varepsilon}\Phi \in \Gamma_b$ if $-\delta < \varepsilon < 0$. In order to prove (2.13) it is now sufficient to show that the endpoints of a beta-interval, $\beta = (\xi_1, \xi_2)$, do not belong to Λ, because then $V_{\beta,\varepsilon}\Phi \in \Gamma_b$ for $|\varepsilon|$ sufficiently small and (2.13) follows from (2.16).

Assume $\xi_2 \in \Lambda$. Then $\xi_2 \neq T$ and we must have $\Phi(\xi_2) = N(\xi_2 + 0)$ since the contrary assumption would imply that (2.12) holds on a neighborhood of ξ_2, which is inconsistent with the definition of a beta-interval. There exists therefore a ξ_0, $\xi_1 < \xi_0 < \xi_2$ such that $\Phi - N > 0$ in $\gamma = (\xi_0, \xi_2)$. Furthermore for small positive ε, $V_{\gamma,\varepsilon}\Phi \in \Gamma_b$, which leads to the contradiction $J(V_{\gamma,\varepsilon}\Phi) - J(\Phi) < 0$. The property $\xi_1 \notin \Lambda$ is proved similarly and we have thus shown that (2.13) holds for β-interval not containing, or limited by the origin.

It thus remains to be proved that the origin is not included in the closure of a beta-interval. Assume $\beta = (\xi_1, \xi_2)$. It is sufficient to consider the case $\xi_2 > 0$, $\xi_1 \leqslant 0$.

Since $N(x)$ by assumption vanishes on a neighborhood of $x=0$ and $\Phi(x)=bx$ on β, we can determine $\xi_0>0$ such that for $\gamma=(\xi_0,\xi_2)$

$$\int_\gamma (\Phi-N)\frac{dx}{x^2}>0, \tag{2.17}$$

which again leads to the contradiction $J(V_{\gamma,\varepsilon}\Phi)-J(\Phi)<0$. It is also easy to see that endpoints at the alpha-interval containing $x=0$ are bounded away from the origin as $T\to\infty$.

The previous discussion permits us to derive the following conclusions. There exists an infinite sequence $T_i\to\infty$ such that the corresponding solutions Φ_i of the minimum problem (2.11) converge uniformly over compact intervals to a function $\Phi\in\Gamma_b$. Moreover, Φ' is alternately $=0$ and $=b$ on the corresponding alpha- and beta-intervals as in the finite case. These intervals are discrete since their total number contained in $[-T,T]$ is limited by $2(N(T)-N(-T))+1$. For all β-intervals we will have

$$\frac{1}{|\beta|}\int_\beta d\Phi=\frac{1}{|\beta|}\int_\beta dN=b. \tag{2.18}$$

This relation together with the property $A_e(\Lambda)=a<b$ guarantees that no β-interval is infinite. In each β we have $|\Phi(x)-N(x)|\leqslant b|\beta|$, and the L^2-norm of $(\Phi-N)/x^2$ over β is therefore majorized by $b|\beta|^{\frac{3}{2}}/\xi^2(\beta)$, where $\xi(\beta)=\operatorname{dist}(\beta,0)$. Lemma II.2 now follows by virtue of Lemma II.1 and its Corollary, if we can show that

$$\sum\frac{|\beta|^2}{\xi^2(\beta)}<\infty. \tag{2.19}$$

By virtue of the definition of exterior density, dN has a majorant $d\nu$ which is regular and of density $A(d\nu)\leqslant a+\varepsilon$, where ε is given. According to (2.18) we have for the β-intervals corresponding to Φ,

$$\frac{1}{|\beta|}\int_\beta d\nu-a\geqslant\frac{1}{|\beta|}\int_\beta dN-a=b-a.$$

If $\varepsilon+\delta<b-a$, then the set B of all β-intervals is contained in

$$\Omega_\delta=\left\{\omega;\left|\frac{1}{|\omega|}\int_\omega d\nu-A(d\nu)\right|>\delta\right\}$$

and this set is by definition negligible and hence, $m(\overline{\Omega}_\delta)<\infty$. The mapping T considered in the introduction takes the collection of all subintervals of β into the isoscele having β as base and with height $|\beta|$, and thus of area $|\beta|^2/2$. By an elementary estimate

$$\frac{|\beta|^2}{1+x^2(\beta)} < 8\int_{T\beta} \frac{dx\,dy}{1+x^2+y^2}, \quad x(\beta) = \text{center of } \beta.$$

Hence,
$$\sum_{\beta \in B} \frac{|\beta|^2}{1+x^2(\beta)} < 8\,m(\overline{\Omega}_\delta) < \infty,$$

and this proves the convergence of (2.19) and finishes the proof of Lemma II.2.

III. Proofs of Theorems I and II

The results stated in the introduction concern sets Λ consisting either of real or of complex numbers. By means of the projection $\Lambda \to \Lambda^*$ and the inequality (0.5) it takes very little effort to pass from the real to the complex case. The vehicle to be used is

LEMMA III.1. *Let the sequence* $\Lambda = \{\lambda_n\}_1^\infty$ *satisfy* (0.1) *together with the conditions*

$$\lim_{r\to\infty} \sum_{|\lambda_n|<r} \frac{1}{\lambda_n} \text{ exists,} \tag{3.1}$$

$$\sum_1^\infty \frac{1}{|\lambda_n|^2} < \infty. \tag{3.2}$$

Then the function $f(z) = f(z;\Lambda)$ *and* $f^*(z) = f(z;\Lambda^*)$ *defined as in* (0.8), *have the property*

$$\int_{-\infty}^{\infty} \left| \log|f(x)| - \log|f^*(x)| \right| \frac{dx}{1+x^2} < \infty. \tag{3.3}$$

To begin with we observe that the conditions imposed on Λ imply that f and f^* are well-defined. By (0.5) we have $|f(x)| \geq |f^*(x)|$ for real x. If $\lambda = a+ib$, we define $\lambda' = a - i|b|$. The integrand written below is therefore positive and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log\frac{|1-x/\lambda|}{|1-x/\lambda^*|}\frac{dx}{1+x^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\frac{|1-x/\lambda'|}{|1-x/\lambda^*|}\frac{dx}{1+x^2}$$
$$= \log\frac{|1-i/\lambda'|}{|1-i/\lambda^*|} < \tfrac{1}{2}\log\left(1+\frac{2\,|b|+1}{|\lambda|^2}\right).$$

Relation (3.3) now follows on summation. We conclude also that if one of the functions f and f^* belongs to a certain class $E_{k\pi}$ and is normalized, then the same is true of the other.

Proof of Theorem I. By assumption $f(z)$ is a normalized function of class $E_{k\pi}$ and the same therefore holds true of f^*. The subharmonic function $U(z) = \log|f^*(z)|$ belongs to the class $S_{k\pi}$ considered in Chapter I and Theorem I is thus a special case of the more elaborate Theorem I'.

Proof of Theorem II. Let us show first that the assumptions $A_e(dN_{\Lambda^*}) = a$ and $k < a$ imply that $E_{k\pi}$ does not contain any g vanishing on Λ (with the prescribed multiplicities). Assume that $\Gamma = g(0)^{-1}$ does not contain the origin. Then $g(z) = g(z; \Gamma)$, and $\Delta^* \subset \Gamma^*$, if g vanishes on Λ. By Lemma III.1, $g(z; \Gamma^*) \in E_{k\pi}$, and dN_{Γ^*} is a regular majorant of dN_{Λ^*} leading to the contradiction

$$a = A_e(dN_{\Lambda^*}) \leqslant A(dN_{\Gamma^*}) \leqslant k.$$

In order to prove the remaining part of the theorem we have to construct a g vanishing on Λ and belonging to a given class $E_{k\pi}$, $k > a$. Denote by dN^* the measure dN_{Λ^*}. By virtue of Lemma II.2 there exists a function $\Phi(x)$ such that

$$0 \leqslant d\Phi(x) \leqslant k\,dx,$$

and such that $(\Phi(x) - N^*(x))/x^2$ and its Hilbert transform both belong to $L^1(-\infty, \infty)$, and

$$\int_{-\infty}^{\infty} (\Phi(x) - N^*(x)) \frac{dx}{x^2} = 0. \tag{3.4}$$

Due to these properties the Cauchy integral

$$h(z) = \int_{-\infty}^{\infty} \frac{\Phi(t) - N^*(t)}{t^2(t-z)}\, dt \tag{3.5}$$

represents a function in the Hardy class H^1 for the half-plane $y > 0$. Thus,

$$\int_{-\infty}^{\infty} |h(x+iy)|\, dx = O(1). \tag{3.6}$$

Since
$$\frac{1}{t^2(t-z)} = \frac{1}{zt(t-z)} - \frac{1}{zt^2},$$

we will have
$$h(z) = \frac{1}{z} \int_{-\infty}^{\infty} \frac{\Phi(t) - N^*(t)}{t(t-z)}\, dt. \tag{3.7}$$

After a partial integration we get

$$h(z) = \frac{1}{z^2} \int_{-\infty}^{\infty} \log\left(1 - \frac{z}{t}\right) d(N^*(t) - \Phi(t)), \quad y > 0. \tag{3.8}$$

Define for $y \geqslant 0$

$$U(z) = \int_{-\infty}^{\infty} \log \left| 1 - \frac{z}{t} \right| d(N^*(t) - \Phi(t)) = \lim_{T=\infty} \int_{-T}^{T}. \tag{3.9}$$

We have thus proved that $U(x)/x^2 \in L^1(-\infty, \infty)$. Define $d\nu = k\,dt - d\Phi$. Observe that $0 \leqslant d\nu \leqslant k\,dt$ and write

$$d(N^* - \Phi) = d(N^* + \nu) - k\,dt. \tag{3.10}$$

The logarithmic potential generated by the measure $d(N^* + \Phi)$ is by definition $U(x)$ on the real axis. Since the measure $k\,dt$ generates a potential equal to 0 on the real axis we shall have by (3.10)

$$U(x) = \int_{-\infty}^{\infty} \log \left| 1 - \frac{x}{t} \right| d(N^*(t) + \nu(t)). \tag{3.11}$$

Let $[\nu(t)]$ denote the integral part of $\nu(t) = kt - \Phi(t)$, and let $d[\nu]$ be the corresponding measure. A partial integration yields the estimate

$$\int_{-\infty}^{\infty} \log \left| 1 - \frac{x}{t} \right| d([\nu] - \nu) \leqslant O(\log |x|), \quad x \to \pm\infty. \tag{3.12}$$

Denote by $\Gamma = \{\gamma_n\}_1^{\infty}$, the support of $d[\nu]$ and set

$$g(z) = g(z; \Lambda \cup \Gamma), \qquad g^*(z) = g(z; \Lambda^* \cup \Gamma),$$

the notation being the same as in (0.8). Then

$$\frac{\log |g^*(x)|}{1 + x^2} \leqslant \frac{U(x)}{1 + x^2} + O\left(\frac{\log |x|}{1 + x^2}\right)$$

and it follows that (0.2) is satisfied by g^*. In addition

$$N^*(t) + \nu(t) = kt + (N^*(t) - \Phi(t)) = kt + o(|t|)$$

and we may therefore conclude that both g^* and g belong to the class $E_{k\pi}$. This finishes the proof.

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