ON SIMULTANEOUS APPROXIMATIONS OF TWO ALGEBRAIC NUMBERS BY RATIONALS

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1. Introduction

1.1. *Main results.* Throughout this paper, $\|\xi\|$ will denote the distance of the real number ξ from the nearest integer. We shall prove the following results which represent extensions to simultaneous approximations of Roth's famous theorem [5] on rational approximations to an algebraic irrational α.

THEOREM 1. *Let α, β be algebraic and $1, \alpha, \beta$ linearly independent over the field of rationals $\mathbf{Q}$. Then for every $\varepsilon > 0$ there are only finitely many positive integers q with*

$$\|q\alpha\| \cdot \|q\beta\| \cdot q^{1+\varepsilon} < 1. \tag{1}$$

COROLLARY. *Let $\alpha, \beta, \varepsilon$ be as before. There are only finitely many pairs of rationals $p_1/q, p_2/q$ satisfying*

$$\left|\alpha - \frac{p_1}{q}\right| < |q|^{-3/2-\varepsilon}, \qquad \left|\beta - \frac{p_2}{q}\right| < |q|^{-3/2-\varepsilon}. \tag{2}$$

A dual to Theorem 1 is

THEOREM 2. *Let $\alpha, \beta, \varepsilon$ be as in Theorem 1. There are only finitely many pairs of rational integers $q_1 \neq 0$, $q_2 \neq 0$ with*

$$\|q_1\alpha + q_2\beta\| \cdot |q_1 q_2|^{1+\varepsilon} < 1. \tag{3}$$

COROLLARY. *Again let $\alpha, \beta, \varepsilon$ be as in Theorem 1. There are only finitely many triples q_1, q_2, p of rational integers with $q = \max(|q_1|, |q_2|) > 0$ satisfying*

$$|q_1\alpha + q_2\beta + p| < q^{-2-\varepsilon}. \tag{4}$$

1.2. *Approximations by rationals or quadratic irrationals.* Let ω be either rational or a quadratic irrational. There is a polynomial $f(t) = xt^2 + yt + z \not\equiv 0$, unique up to a factor ± 1, whose coefficients x, y, z are coprime integers and which is irreducible over the rationals, such that $f(\omega)=0$. Define the *height* $H(\omega)$ of ω by

$$H(\omega) = \max(|x|, |y|, |z|). \tag{5}$$

THEOREM 3. *Let α be algebraic, but not rational or a quadratic irrational, and let $\varepsilon > 0$. There are at most finitely many numbers ω which are rationals or quadratic irrationals and which satisfy*

$$|\alpha - \omega| < H(\omega)^{-3-\varepsilon}. \tag{6}$$

This theorem should be compared with a recent result of Davenport and the author [3] which asserts the existence of infinitely many numbers ω of the type described above satisfying

$$|\alpha - \omega| < C(\alpha) H(\omega)^{-3}; \tag{7}$$

in fact in this latter result α can be any real number which is neither rational nor a quadratic irrational. (For results concerning approximations by algebraic numbers of degree $\leqslant k$, see Wirsing [7]. Wirsing (unpublished as yet) also proved a general result of the type of Theorem 3, but without best possible exponents.)

Theorem 3 follows easily from the corollary to Theorem 2; by Roth's Theorem, we may restrict ourselves to quadratic irrationals ω. Let

$$f(t) = xt^2 + yt + z = x(t-\omega)(t-\omega')$$

be the irreducible polynomial described above, and ω' the conjugate of ω. Then $|x| \leqslant H(\omega)$ and, as is easily seen, $|\omega' x| \leqslant 2H(\omega)$. If (6) holds, then

$$|x\alpha^2 + y\alpha + z| = |x\alpha - x\omega'|\,|\alpha - \omega| < (|\alpha| + 2) H(\omega) H(\omega)^{-3-\varepsilon} < H(\omega)^{-2-\varepsilon/2}$$

if $H(\omega)$ is large. Since $\quad H(\omega)^{-2-\varepsilon/2} \leqslant (\max(|x|, |y|))^{-2-\varepsilon/2},$

our inequality has only a finite number of solutions by the corollary of Theorem 2.

1.3. *Further results.*

THEOREM 4. *Let α, β, γ be algebraic, $1, \beta, \gamma$ linearly independent and $1, \alpha, \alpha\gamma - \beta$ linearly independent over $\mathbf{Q}$. Let $\varrho + \tau > 1$. There are only finitely many triples of rational integers q_1, q_2, q_3 with $q_1 > 0$ satisfying*

$$|\alpha q_1 + q_2| \leqslant q_1^{-\varrho}, \quad |\beta q_1 + \gamma q_2 + q_3| \leqslant q_1^{-\tau}. \tag{8}$$

This theorem appears to be more general than Theorems 1 or 2, since it involves three numbers α, β, γ; but actually it contains neither of them. Later in section 4.3 we shall prove a general but somewhat complicated theorem which contains Theorems 1, 2 and 4.

Notice that our conditions of linear independence are necessary: For $1, \beta, \gamma$ this is rather obvious. For $1, \alpha, \alpha\gamma-\beta$, assume $\varrho \geqslant \tau$. For sufficiently small C, the inequalities

$$|\alpha q_1+q_2| \leqslant C q_1^{-\varrho}, \quad |(\beta-\alpha\gamma)q_1+q_3| \leqslant C q_1^{-\tau} \tag{9}$$

imply (8), and (9) may have infinitely many solutions unless $1, \alpha, \alpha\gamma-\beta$ are linearly independent.

THEOREM 5. *Let α, β, γ be algebraic and $1, \alpha, \beta, \gamma$ linearly independent over* $\mathbf{Q}$, *and let $\varepsilon>0$.*

There are only finitely many triples of non-zero integers q_1, q_2, q_3 having

$$\|\alpha q_1+\beta q_2+\gamma q_3\| \cdot |q_1 q_2 q_3|^{5/3+\varepsilon}<1. \tag{10}$$

This theorem probably is not best possible; probably the exponent $5/3+\varepsilon$ in (10) may be replaced by $1+\varepsilon$. I am unable to prove the analogue of Theorem 1 or 2 for three numbers α, β, γ. I cannot prove any result in this direction for more than three numbers.

1.4. *Auxiliary results.* To prove the main results we shall derive some auxiliary theorems. Let $n \geqslant 1$,

$$\boxed{l=n+1}, \tag{11}$$

and let

$$L_1=\alpha_{11}X_1+\ldots+\alpha_{1l}X_l,$$
$$\ldots\ldots$$
$$L_l=\alpha_{l1}X_1+\ldots+\alpha_{ll}X_l$$

be linear forms. Denote the cofactor of α_{ij} in the matrix $(\alpha_{hk})(1 \leqslant h, k \leqslant l)$ by A_{ij}.

Definition. Let $L_1, \ldots, L_l$ be linear forms as above, and let S be a subset of $\{1, \ldots, l\}$. We say $L_1, \ldots, L_l$; S are *proper* if

(i) the α_{hk} are algebraic and $\det(\alpha_{hk}) \neq 0$

(ii) for every $i \in S$, the *non-zero* elements among $A_{i1}, \ldots, A_{il}$ are linearly independent over $\mathbf{Q}$.

(iii) for every k, $1 \leqslant k \leqslant l$, there is an $i \in S$ with $A_{ik} \neq 0$.

Of particular interest will be the following examples.

(1) $l=2$, $L_1=X_1-\alpha X_2$, $L_2=X_2$, $S=\{2\}$. L_1, L_2; S are proper if α is an algebraic irrational.

(2) $l=3$, $L_1=X_1-\alpha X_3$, $L_2=X_2-\beta X_3$, $L_3=X_3$, $S=\{3\}$. Now L_1, L_2, L_3; S are proper if α, β are algebraic and $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$.

(3) $l=3$, $L_1=X_1$, $L_2=X_2$, $L_3=\alpha X_1+\beta X_2+X_3$, $S=\{1, 2\}$. L_1, L_2, L_3; S are proper if α, β are both algebraic irrationals.

THEOREM 6. *Suppose $L_1, \dots, L_l$; S are proper, and $A_1, \dots, A_l$ are positive reals satisfying*

$$A_1 A_2 \dots A_l = 1, \tag{12}$$

$$A_i \geq 1 \quad \text{if} \quad i \in S. \tag{13}$$

The set defined by

$$|L_j(\mathfrak{x})| \leq A_j \quad (1 \leq j \leq l) \tag{14}$$

is a parallelopiped; denote its successive minima (in the sense of the Geometry of Numbers) by $\lambda_1, \dots, \lambda_n, \lambda_l$.

For every $\delta > 0$ there is then a $Q_0(\delta; L_1, \dots, L_l; S)$ such that

$$\lambda_n > Q^{-\delta} \tag{15}$$

if

$$Q \geq \max (A_1, \dots, A_l, Q_0(\delta)). \tag{16}$$

Applying this theorem to our example 1) we obtain a lower bound for λ_1. Hence it is easy to see that this particular case of Theorem 6 is equivalent to Roth's Theorem. Applying Theorem 6 to example 2) or 3) one only obtains a lower bound for λ_2 rather than for λ_1, and hence one does not immediately obtain Theorem 1 or 2. The following transference principle allows one in this case to proceed from the inequality for λ_2 to an inequality for λ_1.

THEOREM 7. *Let L_1, L_2, L_3 be three linear forms of determinant 1 in variables X_1, X_2, X_3, and let M_1, M_2, M_3 be the adjoint forms, i.e. the forms with*

$$L_1 M_1 + L_2 M_2 + L_3 M_3 \equiv X_1^2 + X_2^2 + X_3^2.$$

Let S, T be nonempty subsets of $\{1, 2, 3\}$ with empty intersection.

Suppose now the second minimum λ_2 of the parallelopiped (14) satisfies

$$\lambda_2 > Q^{-\delta} \tag{17}$$

provided (12), (13) and (16) are satisfied. Also suppose the second minimum μ_2 of

$$|M_i(\mathfrak{x})| \leq B_i \quad (i=1, 2, 3) \tag{18}$$

satisfies

$$\mu_2 > Q^{-\delta} \tag{19}$$

provided $B_1 B_2 B_3 = 1$, $B_i \geq 1$ if $i \in T$ and $Q \geq \max (B_1, B_2, B_3, Q_1(\delta))$.

Then the first minimum λ_1 *of* (14) *satisfies*

$$\lambda_1 > Q^{-\delta} \tag{20}$$

if $$A_1 A_2 A_3 = 1,$$

if $$A_i \geqslant 1 \quad \textit{for} \quad i \in S,\ A_j \leqslant 1 \quad \textit{for} \quad j \in T \tag{21}$$

and $$Q \geqslant \max (A_1, A_2, A_3, Q_2(\delta)). \tag{22}$$

We shall show in chapter 4 that Theorems 1, 2 and 4 are easy consequences of Theorems 6 and 7. Theorem 5 will be derived from Theorem 6 by a similar transference principle.

Our proof of Theorem 6 will follow the method of my previous paper [6] on this subject, where a weaker form of the theorem was proved. This method consists of a further development of the ideas involved in the proof of Roth's Theorem [5]. In the first draft of my manuscript I had derived the transference principles of chapter 4 by the methods of [6]. I am indebted to Professor H. Davenport for suggesting the much more lucid method of the present version.

2. The index of a polynomial

2.1. *The index.* $\mathfrak{R}$ will denote the ring of polynomials in ml variables

$$X_{11}, \ldots, X_{1l}; \ldots; X_{m1}, \ldots, X_{ml}$$

with real coefficients. Let $L_1, \ldots, L_m$ be linear forms, none of them identically zero, of the special type

$$L_h = L_h(X_{h1}, \ldots, X_{hl}) \quad (1 \leqslant h \leqslant m).$$

Also let positive integers $r_1, \ldots, r_m$ be given. For $c \geqslant 0$ we denote by

$$I(c)$$

the ideal in $\mathfrak{R}$ generated by the polynomials

$$L_1^{i_1} L_2^{i_2} \ldots L_m^{i_m} \tag{1}$$

with $$\sum_{h=1}^{m} i_h r_h^{-1} \geqslant c. \tag{2}$$

$I(c) \supset I(c')$ if $c \leqslant c'$. One has $I(0) = \mathfrak{R}$ and $\bigcap_{c \geqslant 0} I(c) = (0)$.

Definition. The *index* of a polynomial $P \in \mathfrak{R}$ with respect to $(L_1, \ldots, L_m; r_1, \ldots, r_m)$ is defined as the largest c with $P \in I(c)$ if $P \not\equiv 0$, and it is $+\infty$ if $P \equiv 0$.

Remark. Since the set of numbers $\sum_{h=1}^{m} i_h r_h^{-1}$ is discrete, there is for a polynomial $P \not\equiv 0$ always such a maximal c. For given $L_1, \ldots, L_m$ and $r_1, \ldots, r_m$ we denote the index of P by ind P.

By Hilfssatz 6 of [6],

$$\operatorname{ind}(P+Q) \geqslant \min(\operatorname{ind} P, \operatorname{ind} Q), \tag{3}$$

$$\operatorname{ind}(PQ) = \operatorname{ind} P + \operatorname{ind} Q. \tag{4}$$

In what follows, $\mathfrak{r}$ will always denote an m-tuple of *positive* integers $(r_1, \ldots, r_m)$ and $\mathfrak{J}$ will denote an lm-tuple of *nonnegative* integers $(i_{11}, \ldots, i_{1l}; \ldots; i_{m1}, \ldots, i_{ml})$. We put

$$(\mathfrak{J}/\mathfrak{r}) = \sum_{h=1}^{m} (i_{h1} + \ldots + i_{hl})\, r_{hl}^{-1}. \tag{5}$$

Given a polynomial $P \in \mathfrak{R}$, set

$$P^{\mathfrak{J}} = (i_{11}! \ldots i_{ml}!)^{-1} \frac{\partial^{i_{11}+\ldots+i_{ml}}}{\partial X_{11}^{i_{11}} \ldots X_{ml}^{i_{ml}}} P. \tag{6}$$

The inequality

$$\operatorname{ind} P^{\mathfrak{J}} \geqslant \operatorname{ind} P - (\mathfrak{J}/\mathfrak{r}) \tag{7}$$

follows easily from our definitions.

LEMMA 1. *Suppose the polynomial P has index $c \neq \infty$ with respect to $(L_1, \ldots, L_m; r_1, \ldots, r_m)$. Let T be the $(ml-m)$-dimensional subspace of ml-dimensional space R^{ml} defined by*

$$L_1(X_{11}, \ldots, X_{1l}) = \ldots = L_m(X_{m1}, \ldots, X_{ml}) = 0.$$

There is an $\mathfrak{J}$ with $(\mathfrak{J}/\mathfrak{r}) = c$ such that $P^{\mathfrak{J}}$ does not vanish identically on T.

Proof. This is a weakened version of one of the assertions of Hilfssatz 7 in [6].

Given a polynomial $P \in \mathfrak{R}$, write $|P|$ for the maximum of the absolute values of its coefficients. If P has integral coefficients, then so does $P^{\mathfrak{J}}$.

LEMMA 2. *Let $P \in \mathfrak{R}$ be homogeneous in $X_{h1}, \ldots, X_{hl}$ of degree r_h $(1 \leqslant h \leqslant m)$. (That is, P is a sum of monomials $cX_{11}^{j_{11}} \ldots X_{ml}^{j_{ml}}$ having $j_{h1} + \ldots + j_{hl} = r_h (1 \leqslant h \leqslant m)$.) Then for any $\mathfrak{J}$,*

$$|P^{\mathfrak{J}}| \leqslant 2^{r_1+\ldots+r_m} |P|. \tag{8}$$

Proof. It will suffice to prove this estimate for monomials. Now

$$(X_{11}^{j_{11}} \ldots X_{ml}^{j_{ml}})^{\mathfrak{J}} = \binom{j_{11}}{i_{11}} \ldots \binom{j_{ml}}{i_{ml}} X_{11}^{j_{11}-i_{11}} \ldots X_{ml}^{j_{ml}-i_{ml}}.$$

Since
$$\binom{j_{11}}{i_{11}} \cdots \binom{j_{ml}}{i_{ml}} \leqslant 2^{j_{11}+\cdots+j_{ml}},$$

the desired inequality follows.

2.2. *Existence of certain polynomials.*

THEOREM 8. *Let l, t be positive integers and*

$$L_j = \alpha_{j1} X_1 + \ldots + \alpha_{jl} X_l \quad (1 \leqslant j \leqslant t)$$

linear forms, none identically zero, whose coefficients are algebraic integers. Construct new linear forms

$$L_{hj} = \alpha_{j1} X_{h1} + \ldots + \alpha_{jl} X_{hl} \quad (1 \leqslant j \leqslant t)$$

in variables $X_{h1}, \ldots, X_{hl}$ $(1 \leqslant h \leqslant m)$. Set Δ_j for the degree of $K_j = \mathbf{Q}(\alpha_{j1}, \ldots, \alpha_{jl})$ and $\Delta = \max(\Delta_1, \ldots, \Delta_t)$.

Let $\varepsilon > 0$ and assume m to be so large that

$$m \geqslant 4\varepsilon^{-2} \log (2t\Delta). \tag{9}$$

Let $r_1, \ldots, r_m$ be positive integers.

There is a polynomial $P \in \mathfrak{R}$ with rational integral coefficients, not vanishing identically and satisfying

(i) *P is homogeneous in $X_{h1}, \ldots, X_{hl}$ of degree r_h $(1 \leqslant h \leqslant m)$,*

(ii) *P has index $\geqslant (l^{-1} - \varepsilon) m$ with respect to*

$$(L_{1j}, \ldots, L_{mj}; r_1, \ldots, r_m) \quad (1 \leqslant j \leqslant t),$$

(iii) $|P| \leqslant D^{r_1 + \cdots + r_m}$,

where D is a constant depending only on the coefficients α_{jk}.

Proof. This is Satz 7 (Indexsatz) of [6].

The following theorem is almost but not quite identical with Satz 8 of [6].

THEOREM 9. *Let*

$$L_j = \alpha_{j1} X_1 + \ldots + \alpha_{jl} X_l \quad (1 \leqslant j \leqslant l)$$

be linear forms with nonvanishing determinant whose coefficients are algebraic integers. Define Δ and the linear forms L_{hj} as in Theorem 8. Let $\varepsilon > 0$ and assume

$$m \geqslant 4\varepsilon^{-2} \log (2l\Delta). \tag{10}$$

Let $r_1, \ldots, r_m$ be positive integers.

There is a polynomial $P \not\equiv 0$ in $\mathfrak{R}$ with rational integral coefficients such that

(i) *P is homogeneous in $X_{h1}, \ldots, X_{hl}$ of degree r_h $(1 \leqslant h \leqslant m)$,*

(ii) $|P| \leqslant D^{r_1+\cdots+r_m}$,

(iii) *writing (uniquely!)*

$$P^{\mathfrak{J}} = \sum d^{\mathfrak{J}}(j_{11}, \ldots, j_{ml}) L_{11}^{j_{11}} \ldots L_{1l}^{j_{1l}} \ldots L_{m1}^{j_{m1}} \ldots L_{ml}^{j_{ml}}, \tag{11}$$

one has

$$|d^{\mathfrak{J}}(j_{11}, \ldots, j_{ml})| \leqslant E^{r_1+\cdots+r_m}$$

for arbitrary $\mathfrak{J}$ and $j_{11}, \ldots, j_{ml}$.

(iv) *If*

$$(\mathfrak{J}/\mathfrak{r}) \leqslant 2\varepsilon m, \tag{12}$$

then $d^{\mathfrak{J}}(j_{11}, \ldots, j_{ml}) = 0$ unless

$$\left| \sum_{h=1}^{m} j_{hk} r_h^{-1} - ml^{-1} \right| \leqslant 3lm\varepsilon \quad (1 \leqslant k \leqslant l). \tag{13}$$

Here D, E depend only on the coefficients α_{hk}.

Proof. As we shall see, the polynomial P constructed in Theorem 8 satisfies everything. This is clear as far as (i) and (ii) are concerned. As for (iii),

let

$$X_i = \sum_{k=1}^{l} \beta_{ik} L_k \quad (1 \leqslant i \leqslant l),$$

whence

$$X_{hi} = \sum_{k=1}^{l} \beta_{ik} L_{hk} \quad (1 \leqslant i \leqslant l;\ 1 \leqslant h \leqslant m). \tag{14}$$

Let

$$G = \max(1, |\beta_{11}|, \ldots, |\beta_{ll}|).$$

One obtains $P^{\mathfrak{J}}$ in the form (11) by substituting the right-hand side of (14) for each X_{hi} in

$$P^{\mathfrak{J}} = \sum c^{\mathfrak{J}}(j_{11}, \ldots, j_{ml}) X_{11}^{j_{11}} \ldots X_{ml}^{j_{ml}}. \tag{15}$$

A typical product in (15), namely $X_{11}^{j_{11}} \ldots X_{ml}^{j_{ml}}$, then becomes

$$\left(\sum_{k=1}^{l} \beta_{1k} L_{1k}\right)^{j_{11}} \ldots \left(\sum_{k=1}^{l} \beta_{lk} L_{mk}\right)^{j_{ml}}$$

and as a polynomial in $L_{11}, \ldots, L_{ml}$ has coefficients of absolute value

$$\leqslant (lG)^{j_{11}+\cdots+j_{ml}} \leqslant (lG)^{r_1+\cdots+r_m}.$$

By (ii) and by Lemma 2,

$$|c^{\mathfrak{J}}(j_{11}, \ldots, j_{ml})| \leqslant (2D)^{r_1+\cdots+r_m}.$$

Therefore $P^{\mathfrak{J}}$ as a polynomial in $L_{11}, \ldots, L_{ml}$ has coefficients of absolute value $\leqslant (2lDG)^{r_1+\cdots+r_m}$. This proves (iii).

The index of P with respect to $(L_{1j}, \ldots, L_{mj}; r_1, \ldots, r_m)$ is at least $(l^{-1}-\varepsilon)$ by the previous theorem. Hence if (12) holds, the index of $P^{\mathfrak{J}}$ is at least

$$(l^{-1}-\varepsilon)m-(\mathfrak{J}/\mathfrak{r}) \geqslant (l^{-1}-3\varepsilon)m$$

by (7). Hence any lm-tuple $(j_{11}, \ldots, j_{ml})$ having $d^{\mathfrak{J}}(j_{11}, \ldots, j_{ml}) \neq 0$ satisfies

$$\sum_{h=1}^{m} j_{hk} r_h^{-1} - ml^{-1} \geqslant -3m\varepsilon \quad (1 \leqslant k \leqslant l). \tag{16}$$

Since $P^{\mathfrak{J}}$ is homogeneous in $L_{h1}, \ldots, L_{hl}$ of degree $\leqslant r_h$, one obtains

$$\sum_{k=1}^{l} j_{hk} r_h^{-1} \leqslant 1, \qquad \sum_{k=1}^{l} \left(\sum_{h=1}^{m} j_{hk} r_h^{-1} - ml^{-1} \right) \leqslant 0,$$

whence by (16),

$$\sum_{h=1}^{m} j_{hk} r_h^{-1} - ml^{-1} \leqslant 3m(l-1)\varepsilon. \tag{17}$$

The inequalities (16) and (17) give (iv).

2.3. *Grids.* Now as always let

$$l = n+1, \tag{18}$$

and let $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$ be n linearly independent vectors of R^l, spanning a subspace H. Let s be a positive integer. Write

$$\varrho = \varrho(s; \mathfrak{w}_1, \ldots, \mathfrak{w}_n)$$

for the set of all vectors

$$\mathfrak{w} = h_1 \mathfrak{w}_1 + \ldots + h_n \mathfrak{w}_n,$$

where $h_1, \ldots, h_m$ are integers in the interval $1 \leqslant h_i \leqslant s$. ϱ will be called a *grid of size s* on H, and $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$ are *basis vectors* of the grid.

In what follows a polynomial in $X_1, \ldots, X_l$ will be interpreted as a function on R^l. The next lemma contains the idea which will enable us to improve upon the results of [6].

LEMMA 3. *Let $P(X_1, \ldots, X_l)$ be a polynomial in $X_1, \ldots, X_l$ with real coefficients of total degree $\leqslant r$, and let s, t be positive integers satisfying*

$$s(t+1) > r. \tag{19}$$

Suppose ϱ is a grid of size s on a subspace H of R^l such that P and all the partial derivatives

$$\frac{\partial^{t_1+\ldots+t_l}}{\partial X_1^{t_1} \ldots \partial X_l^{t_l}} P \quad \textit{with} \quad t_1 + \ldots + t_l \leqslant t$$

vanish on ϱ. Then P vanishes identically on H.

Proof. After a linear transformation we may assume the basis vectors of the grid to be $\mathfrak{w}_1=(1, 0, ..., 0), ..., \mathfrak{w}_n=(0, ..., 0, 1, 0)$. Putting $P(X_1, ..., X_n, 0)=Q(X_1, ..., X_n)$, we see: *It will suffice to show that a polynomial Q of total degree $\leqslant r$ is identically zero, if Q and its mixed partial derivatives of order $\leqslant t$ vanish in the s^n integer points $(h_1, ..., h_n)$ where $1 \leqslant h_i \leqslant s$ $(1 \leqslant i \leqslant n)$.*

If $n=1$, Q has zeros of order $\geqslant t+1$ at $X_1=1, 2, ..., s$, hence altogether counting multiplicities Q has at least $s(t+1)>r\geqslant \deg Q$ zeros, and Q is identically zero.

Now comes the induction from $n=1$ to n. It will suffice to show that $(X_1-h)^{t+1}$ divides $Q(X_1, ..., X_n)$ for $h=1, 2, ..., s$, because this implies that the product $(X_1-1)^{t+1} ... (X_1-s)^{t+1}$ divides $Q(...)$, and since here the divisor has degree $s(t+1)>r\geqslant \deg Q$, $Q\equiv 0$ follows.

Let e_h be the largest exponent with $(X_1-h)^{e_h}|Q$ (that is, $(X_1-h)^{e_h}$ divides Q), and put $e=\min(e_1, ..., e_s)$. We have to show that $e \geqslant t+1$.

Assume now $e \leqslant t$, and without loss of generality assume $e=e_1\leqslant t$. We may write

$$Q(X_1, ..., X_n) = (X_1-1)^{e_1} ... (X_1-s)^{e_s} R(X_1, ..., X_n). \tag{20}$$

The degree of R is at most $r-e_1-...-e_s\leqslant r-es$. After taking the partial derivative with respect to X_1 of order $e=e_1$ and putting $X_1=1$ afterwards, the right-hand side of (20) becomes

$$e!\,(1-2)^{e_2} ... (1-s)^{e_s} R(1, X_2, ..., X_n).$$

Now every mixed partial derivative of the polynomial $R(1, X_2, ..., X_n)$ in $n-1$ variables of order $\leqslant t-e$ vanishes in each of the integer points $(h_2, ..., h_n)$ where $1\leqslant h_i \leqslant s$ $(2\leqslant i\leqslant n)$. Since

$$s(t-e+1)>r-es,$$

our inductive assumption gives $R(1, X_2, ..., X_n)\equiv 0$. This can only be so if

$$(X_1-1)\,|\,R(X_1, ..., X_n),$$

whence $(X_1-1)^{e_1+1}|Q$, and this contradicts our choice of e_1.

LEMMA 4. *Let the polynomial $P\in\mathfrak{R}$ be of total degree $\leqslant r_h$ in $X_{h1}, ..., X_{hl}$ $(1\leqslant h\leqslant m)$. We may write $P(X_{11}, ..., X_{1l}; ...; X_{m1}, ..., X_{ml})=P(\mathfrak{X}_1, ..., \mathfrak{X}_m)$ where $\mathfrak{X}_h=(X_{h1}, ..., X_{hl})$, and interprete P as a function on the m-fold product space $R^l\times ... \times R^l$.*

Now let $H_1, ..., H_m$ be subspaces of dimension $n=l-1$ of R^l, and let ϱ_h on H_h be a grid of size s_h $(1\leqslant h\leqslant m)$. Let $T=H_1\times ... \times H_m$ be the subspace of $R^l\times ... \times R^l$ consisting of all $(\mathfrak{X}_1, ..., \mathfrak{X}_m)$ with $\mathfrak{X}_h\in H_h$ $(1\leqslant h\leqslant m)$, and let $\varrho^*=\varrho_1\times ... \times\varrho_m$ consist of all $(\mathfrak{X}_1, ..., \mathfrak{X}_m)$ with $\mathfrak{X}_h\in\varrho_h$ $(1\leqslant h\leqslant m)$. Let $t_1, ..., t_m$ be integers with

$$s_h(t_h+1)>r_h \tag{21}$$

such that P and the partial derivatives (more precisely, partial derivatives except for constant factors)

$$P^{\mathfrak{T}} \quad \textit{where} \quad \mathfrak{T} = (t_{11}, \ldots, t_{ml}) \quad \textit{with} \quad t_{h1} + \ldots + t_{hl} \leqslant t_h \ (1 \leqslant h \leqslant m)$$

vanish on ϱ^*. *Then P is identically zero on T.*

Proof. This lemma is easily proved by using Lemma 3 and induction on m.

2.4. *The index with respect to certain rational linear forms.* Suppose $n \geqslant 1$ and $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$ are linearly independent integer points in R^l where $l = n+1$. Except for a factor ± 1, there is exactly one linear form $M = m_1 X_1 + \ldots + m_l X_l \not\equiv 0$ where $m_1, \ldots, m_l$ are coprime rational integers, having

$$M(\mathfrak{w}_i) = m_1 w_{i1} + \ldots + m_l w_{il} = 0 \quad (1 \leqslant i \leqslant n).$$

Write

$$M = M\{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}. \tag{22}$$

Put

$$|M| = \max(|m_1|, \ldots, |m_l|).$$

THEOREM 10. *Let* $c_1, \ldots, c_l$ *be reals having*

$$|c_i| \leqslant 1 \quad (i = 1, \ldots, l); \qquad c_1 + \ldots + c_l = 0. \tag{23}$$

Let $\varepsilon > 0$, $0 < \delta < 1$ *and*

$$\delta > 16 l^2 \varepsilon. \tag{24}$$

Let $L_1, \ldots, L_l$ *be linear forms and* m; $r_1, \ldots, r_m$ *integers satisfying the hypothesis of Theorem 9. Let E be the constant of part* (iii) *of that theorem, and P the polynomial described there.*

Let $Q_1, \ldots, Q_m$ *be reals satisfying the inequalities*

(a) $$Q_h^{\varepsilon} > 2^l E, \quad Q_h^{\varepsilon} > l(\varepsilon^{-1} + 1) \quad (1 \leqslant h \leqslant m),$$

(b) $$r_1 \log Q_1 \leqslant r_h \log Q_h \leqslant (1+\varepsilon) r_1 \log Q_1 \quad (1 \leqslant h \leqslant m).$$

Finally, for $h = 1, \ldots, m$, *let* $\mathfrak{w}_{h1}, \ldots, \mathfrak{w}_{hn}$ *be linearly independent integer points of* R^l *satisfying*

(c) $$|L_j(\mathfrak{w}_{hk})| \leqslant Q_h^{c_j - \delta} \quad (1 \leqslant j \leqslant l;\ 1 \leqslant k \leqslant n;\ 1 \leqslant h \leqslant m).$$

Then P has index at least

$$m\varepsilon$$

with respect to $(M_1, \ldots, M_m; r_1, \ldots, r_m)$ *where* M_h $(h = 1, \ldots, m)$ *is the linear form in* $X_{h1}, \ldots, X_{hl}$ *given by* $M_h = M\{\mathfrak{w}_{h1}, \ldots, \mathfrak{w}_{hn}\}$.

Remark. The advantage of this theorem as compared with the corresponding Satz 9 in [6] is the absence of a condition $Q_h^{\varepsilon} \geqslant (r_h + 1) l$ $(1 \leqslant h \leqslant m)$. Such a condition is a serious disadvantage, since in the applications r_1 has order of magnitude $\log Q_m$, so the condition would require that Q_1 is not too small compared to Q_m.

Proof. By Lemma 1 it will suffice to show that $P^{\mathfrak{J}}$ is identically zero on T provided $(\mathfrak{J}/\mathfrak{r}) < \varepsilon m$. Putting

$$\varrho_h = \varrho([\varepsilon^{-1}]+1;\, \mathfrak{w}_{h1}, \ldots, \mathfrak{w}_{hn}),$$

it will be enough by Lemma 4 to prove that

$$(P^{\mathfrak{J}})^{\mathfrak{T}}(\mathfrak{v}_1, \ldots, \mathfrak{v}_m) = 0$$

for $\mathfrak{v}_h \in \varrho_h$ and $\mathfrak{T} = (t_{11}, \ldots, t_{ml})$ satisfying $t_{h1} + \ldots + t_{hl} \leqslant [r_h \varepsilon]$, because $s_h = [\varepsilon^{-1}] + 1$ and $t_h = [r_h \varepsilon]$ satisfy the inequality (21). Since

$$\varepsilon m + [r_1 \varepsilon]/r_1 + \ldots [r_m \varepsilon]/r_m \leqslant 2\varepsilon m,$$

it will suffice to verify that
$$P^{\mathfrak{J}}(\mathfrak{v}_1, \ldots, \mathfrak{v}_m) = 0 \tag{25}$$

for $\mathfrak{v}_h \in \varrho_h$ $(1 \leqslant h \leqslant m)$ and $(\mathfrak{J}/\mathfrak{r}) < 2\varepsilon m$.

The left-hand side of (25) may be written

$$\sum_{j_{11} \ldots j_{ml}} d^{\mathfrak{J}}(j_{11}, \ldots, j_{ml})\, L_1(\mathfrak{v}_1)^{j_{11}} \ldots L_l(\mathfrak{v}_1)^{j_{1l}} \ldots L_1(\mathfrak{v}_m)^{j_{m1}} \ldots L_l(\mathfrak{v}_m)^{j_{ml}}. \tag{26}$$

By (24), (a) and (c),

$$|L_k(\mathfrak{v}_h)| \leqslant Q_h^{c_k - \delta} l(\varepsilon^{-1} + 1) \leqslant Q_h^{c_k - \delta + \varepsilon} \leqslant Q_h^{c_k - 15 l^2 \varepsilon} \quad (1 \leqslant k \leqslant l;\ 1 \leqslant h \leqslant m). \tag{27}$$

Furthermore, by part (iv) of Theorem 9 and by (b), indices $j_{11}, \ldots, j_{ml}$ having $d^{\mathfrak{J}}(j_{11}, \ldots, j_{ml}) \neq 0$ satisfy

$$\sum_{h=1}^{m} j_{hk} \log Q_h \geqslant r_1 \log Q_1 \sum_{h=1}^{m} j_{hk} r_h^{-1} \geqslant r_1 \log Q_1 (l^{-1} - 3l\varepsilon)\, m,$$

$$\sum_{h=1}^{m} j_{hk} \log Q_h \leqslant (1+\varepsilon)\, r_1 \log Q_1 \sum_{h=1}^{m} j_{hk} r_h^{-1}$$

$$\leqslant r_1 \log Q_1 (1+\varepsilon)(l^{-1} + 3l\varepsilon)\, m \leqslant r_1 \log Q_1 (l^{-1} + 7l\varepsilon)\, m,$$

whence
$$\left| \sum_{h=1}^{m} j_{hk} \log Q_h - r_1 \log Q_1 l^{-1} m \right| \leqslant 7 l m \varepsilon r_1 \log Q_1 \quad (1 \leqslant k \leqslant l).$$

Combining this with (27) we get

$$|L_k(\mathfrak{v}_1)^{j_{1k}} \ldots L_k(\mathfrak{v}_m)^{j_{mk}}| \leqslant Q_1^{r_1 l^{-1} m (c_k - 15 l^2 \varepsilon) + 14 l m \varepsilon r_1} = Q_1^{r_1 l^{-1} m c_k - r_1 l m \varepsilon},$$

and each summand of (26) has absolute value

$$\leqslant E^{r_1 + \ldots + r_m} Q_1^{r_1 m l^{-1}(c_1 + \ldots + c_l) - r_1 m l^2 \varepsilon} = E^{r_1 + \ldots + r_m} Q_1^{-r_1 m l^2 \varepsilon} \leqslant E^{r_1 + \ldots + r_m} (Q_1^{-r_1 \varepsilon} \ldots Q_m^{-r_m \varepsilon})^{l^2/(1+\varepsilon)}$$

$$\leqslant (E Q_1^{-\varepsilon})^{r_1} \ldots (E Q_m^{-\varepsilon})^{r_m}$$

by virtue of part (iii) of Theorem 10 and by (b). Since (26) has at most $2^{l(r_1+\dots+r_m)}$ summands, we get

$$|P^{\mathfrak{J}}(\mathfrak{v}_1, \dots, \mathfrak{v}_m)| \leq \prod_{h=1}^{m} (2^l E Q_h^{-\varepsilon})^{r_h} < 1$$

by (a). The left-hand side of this inequality is a rational integer, hence is zero.

This proves Theorem 10.

2.5. *A variant of Roth's Lemma.*

THEOREM 11. *Let*

$$\omega = \omega(m, \varepsilon) = 24 \cdot 2^{-m} (\varepsilon/12)^{2^{m-1}}, \tag{28}$$

where m is a positive integer and

$$0 < \varepsilon < 1/12. \tag{29}$$

Let $r_1, \dots, r_m$ be positive integers such that

$$\omega r_h \geq r_{h+1} \quad (1 \leq h \leq m). \tag{30}$$

Let $M_h = m_{h1} X_{h1} + \dots + m_{hl} X_{hl}$ $(1 \leq h \leq m)$ be linear forms whose coefficients are relatively prime integers. Let $0 < \tau \leq n$ and assume

$$|M_h|^{r_h} \geq |M_1|^{r_1 \tau} \quad (1 \leq h \leq m), \tag{31}$$

$$|M_h|^{\omega \tau} \geq 2^{3mn^2} \quad (1 \leq h \leq m). \tag{32}$$

Let $P(X_{11}, \dots, X_{1l}; \dots; X_{m1}, \dots, X_{ml}) \not\equiv 0$ be a polynomial with rational integral coefficients which is a form in $X_{h1}, \dots, X_{hl}$ of degree r_h $(1 \leq h \leq m)$ and which satisfies

$$|P|^{n^2} \leq |M_1|^{\omega r_1 \tau}. \tag{33}$$

Then the index of P with respect to $(M_1, \dots, M_m; r_1, \dots, r_m)$ is at most ε.

Proof. This is Satz 11 of [6].

3. Proof of Theorem 6

3.1. *Two lemmas.*

LEMMA 5. *Let $l = n+1$, let $\mathfrak{u}_1, \dots, \mathfrak{u}_n$ be vectors of R^l, $\mathfrak{u}_i = (u_{i1}, \dots, u_{il})$ $(1 \leq i \leq n)$ and let $U_1, \dots, U_l$ be the $n \times n$ subdeterminants of the matrix (u_{ij}) $(1 \leq i \leq n,\ 1 \leq j \leq l)$. Similarly, let $\mathfrak{v}_1, \dots, \mathfrak{v}_n$ be vectors and $V_1, \dots, V_l$ subdeterminants of (v_{ij}). Then*

$$\begin{vmatrix} \mathfrak{u}_1\mathfrak{v}_1 & \dots & \mathfrak{u}_1\mathfrak{v}_n \\ & \dots & \\ \mathfrak{u}_n\mathfrak{v}_1 & \dots & \mathfrak{u}_n\mathfrak{v}_n \end{vmatrix} = U_1 V_1 + \dots + U_l V_l. \tag{1}$$

Proof. Without doubt, this lemma in some disguised form may be found in the literature. A simple proof is as follows.

Both the left and right-hand side of (1) are linear functions in each of the vectors $\mathfrak{u}_i$ and each of the vectors $\mathfrak{v}_j$. It therefore suffices to verify the equation if the $\mathfrak{u}$'s as well as the $\mathfrak{v}$'s are taken from a fixed orthonormal basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_l$ of R^l. Both sides are zero unless $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ consist of n distinct basis vectors, the $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ consist of n basis vectors and furthermore the set $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ is identical with the set $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$. Now both sides of (1) are $+1$ or -1 depending on whether $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ is an even or odd permutation of $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$.

LEMMA 6. *Let $l=n+1$, and let $L_1, \ldots, L_l$; S be proper in the sense explained in* § 1.4. *Let $c_1, \ldots, c_l$ be real numbers having*

$$c_1+\ldots+c_l=0 \tag{2}$$

and

$$|c_i| \leqslant 1 \text{ for } i=1, \ldots, l \text{ and } c_i \geqslant 0 \text{ for } i \in S. \tag{3}$$

Let $\delta>0$, $Q>0$ and let $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$ be linearly independent integer points of R^l satisfying

$$|L_i(\mathfrak{w}_j)| \leqslant Q^{c_i-\delta} \quad (1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant n). \tag{4}$$

Then

$$M = M\{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}$$

satisfies

$$Q^{C_1} \leqslant |M| \leqslant Q^{C_2} \tag{5}$$

provided $Q \geqslant C_3$.

Here $C_i=C_i(\delta, L_1, \ldots, L_l)>0$ $(i=1, 2, 3)$.

Proof. Using the vectors $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$, construct the determinants $W_1, \ldots, W_l$ as in Lemma 5. Then putting $M_k=W_k/W$ $(1 \leqslant k \leqslant l)$ where W is the greatest common divisor of $W_1, \ldots, W_l$, one has

$$M = M_1X_1+\ldots+M_lX_l.$$

By (4) and since $|c_i| \leqslant 1$, each component w_{jk} of $\mathfrak{w}_j$ satisfies $|w_{jk}| \leqslant C_4Q$, whence $|W_k| \leqslant C_5Q^n$ and $|M| \leqslant C_5Q^n \leqslant Q^{n+1}$ if Q is large.

As for the lower bound, suppose a particular M_k is $\neq 0$. By condition (iii) of proper systems, there is an $i \in S$ with $A_{ik} \neq 0$, and by (ii), the non-zero elements among

$$A_{i1}, \ldots, A_{ik}, \ldots, A_{il}$$

are linearly independent over $\mathbf{Q}$. For this particular i, $c_1+\ldots+c_{i-1}+c_{i+1}+\ldots+c_l \leqslant 0$ by (2) and (3), and by (4),

$$\left\| \begin{matrix} L_1(\mathfrak{w}_1) \ldots L_{i-1}(\mathfrak{w}_1)\, L_{i+1}(\mathfrak{w}_1) \ldots L_l(\mathfrak{w}_1) \\ \ldots \\ L_1(\mathfrak{w}_n) \ldots L_{i-1}(\mathfrak{w}_n)\, L_{i+1}(\mathfrak{w}_n) \ldots L_l(\mathfrak{w}_n) \end{matrix} \right\| \leqslant n!\, Q^{-n\delta}. \tag{6}$$

On the other hand, by Lemma 5, the left-hand side of (6) equals

$$|W_1 A_{i1} + \dots + W_l A_{il}|,$$

whence
$$|M_1 A_{i1} + \dots + M_k A_{ik} + \dots + M_l A_{il}| \leqslant n!\, Q^{-n\delta}. \tag{7}$$

Let $K_i = \mathbf{Q}(A_{i1}, \dots, A_{il})$ have degree d_i, and let d be $\max d_i$, taken over all $i \in S$. Since $M_k \neq 0$, $A_{ik} \neq 0$, and since the non-zero elements among $A_{i1}, \dots, A_{il}$ are linearly independent over $\mathbf{Q}$, $M_1 A_{i1} + \dots + M_l A_{il}$ is not zero, and in fact its norm (from K_i to $\mathbf{Q}$) has absolute value $\geqslant C_6$. Since each conjugate has absolute value $\leqslant C_7 |M|$, we may conclude that $|M_1 A_{i1} + \dots + M_l A_{il}| \geqslant C_8 |M|^{1-d_i} \geqslant C_8 |M|^{1-d}$. For large Q, the last inequality combined with (7) yields $d > 1$ and $|M| \geqslant C_9 Q^{n\delta/(d-1)} \geqslant Q^{\delta/d}$.

3.2. *Reductions of the problem.*

It suffices to prove Theorem 6 in the special case where

$$A_i = Q^{c_i} \quad (1 \leqslant i \leqslant l) \tag{8}$$

and $c_1, \dots, c_l$ are fixed constants subject to the conditions (2) *and* (3).

To prove this statement, we remark that because of $A_1 A_2 \dots A_l = 1$, we may restrict ourselves to numbers Q satisfying not only $Q \geqslant \max(A_1, \dots, A_l)$ but also

$$Q \geqslant \max(A_1^{-1}, \dots, A_l^{-1}).$$

Then $A_i = Q^{c_i}$ $(i = 1, \dots, l)$ where $c_1, \dots, c_l$ satisfy (2) and (3), but of course these $c_1, \dots, c_l$ will in general depend on $A_1, \dots, A_l$.

Now let N be an integer $> 2/\delta$, and put $\eta = N^{-1}$; then $0 < \eta < \delta/2$. Write $\mathbf{Z}\eta$ for the set of integral multiples of η. There are $c_1', \dots, c_l'$, all lying in $\mathbf{Z}\eta$, such that

$$c_1' + \dots + c_l' = 0, \quad |c_i' - c_i| < \eta \quad (i = 1, \dots, l).$$

Since all integers are in $\mathbf{Z}\eta$, one has again

$$|c_i'| \leqslant 1 \ (i = 1, \dots, l) \quad \text{and } c_i' \geqslant 0 \text{ if } i \in S.$$

Put $A_i' = Q^{c_i'}$ $(i = 1, \dots, l)$. Then $A_1' \dots A_l' = 1$, $A_i' \geqslant 1$ if $i \in S$. Furthermore, if the nth successive minimum λ_n' of $|L_i(x)| \leqslant A_i'$ $(i = 1, \dots, l)$ has $\lambda_n' > Q^{-\delta/2}$, then the nth minimum λ_n of $|L_i(\mathfrak{x})| \leqslant A_i$ $(i = 1, \dots, l)$ satisfies $\lambda_n > Q^{-\delta}$. It therefore suffices to prove the theorem with δ replaced by $\delta/2$ and with $c_1, \dots, c_l$ in the finite set of l-tuples having $c_i \in \mathbf{Z}\eta$ and $|c_i| \leqslant 1$. Hence it is enough to prove the theorem for a particular such l-tuple.

It suffices to prove Theorem 6 when the coefficients α_{ij} of $L_1, \dots, L_l$ are algebraic integers.

Namely, there is always a rational integer $q>0$ such that the forms $qL_1, \ldots, qL_l$ all have integral coefficients. If $L_1, \ldots, L_l; S$ are proper, then so are $qL_1, \ldots, qL_l; S$. Our reduction now follows from the remark that the successive minima of $|qL_i(\mathfrak{x})| \leqslant A_i$ $(i=1, \ldots, l)$ are q^{-1} times the successive minima of $|L_i(\mathfrak{x})| \leqslant A_i$ $(i=1, \ldots, l)$.

3.3. *Proof of Theorem 6.* Let $c_1, \ldots, c_l$ be constants satisfying (2) and (3). Let $\mathfrak{M}$ be the set of reals $Q>1$ such that there are n linearly independent integer points $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$ having

$$|L_i(\mathfrak{w}_j)| \leqslant Q^{c_i-\delta} \quad (1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant n). \tag{9}$$

We have to show that the set $\mathfrak{M}$ is bounded.

We may clearly assume $0<\delta<1/12$. Pick $\varepsilon>0$ small enough to satisfy

$$\delta > 16\, l^2 \varepsilon. \tag{10}$$

Then also $0<\varepsilon<1/12$. Next, pick an integer m so large that

$$m \geqslant 4\varepsilon^{-2} \log (2l\Delta), \tag{11}$$

where Δ is the maximum of the degrees Δ_i of $K_i = \mathbf{Q}(\alpha_{i1}, \ldots, \alpha_{il})$. Further set

$$\omega = 24 \cdot 2^{-m} (\varepsilon/12)^{2^m-1}. \tag{12}$$

Now $\varepsilon<1$ and $m \geqslant 1$ implies $\omega<1$.

In what follows, D, E will be the constants of parts (ii), (iii) of Theorem 9, and C_1, C_2, C_3 the constants of Lemma 6.

We argue indirectly and assume that $\mathfrak{M}$ is unbounded. There is then a Q_1 in $\mathfrak{M}$ such that

$$Q_1^{\varepsilon} > 2^l E, \quad Q_1^{\varepsilon} > l(\varepsilon^{-1}+1), \tag{13–14}$$

$$Q_1 > C_3, \quad Q_1^{C_1^2 \omega} > 2^{3mn^2 C_2}, \quad Q_1^{C_1^2 \omega} > D^{mn^2 C_2}. \tag{15–17}$$

We also may pick $Q_2, \ldots, Q_m$ in $\mathfrak{M}$ satisfying

$$\tfrac{1}{2}\omega \log Q_{h+1} > \log Q_h \quad (1 \leqslant h < m). \tag{18}$$

In particular this implies

$$Q_1 < \ldots < Q_m. \tag{19}$$

Let r_1 be an integer so large that

$$\varepsilon r_1 \log Q_1 \geqslant \log Q_m$$

and for $h=2, 3, \ldots, m$ put

$$r_h = [r_1 \log Q_1 / \log Q_h] + 1.$$

This choice of $r_1, \ldots, r_m$ implies

$$r_1 \log Q_1 \leqslant r_h \log Q_h \leqslant (1+\varepsilon) r_1 \log Q_1 \quad (1 \leqslant h \leqslant m). \tag{20}$$

By virtue of (18) and (20),

$$\omega r_h \geqslant 2(1+\varepsilon)^{-1} r_{h+1} \geqslant r_{h+1}. \tag{21}$$

Our linear forms $L_1, \ldots, L_l$ as well as ε, m and $r_1, \ldots, r_m$ satisfy all the hypotheses of Theorem 9. Let $P(X_{11}, \ldots, X_{ml})$ be the polynomial described in that theorem. Now the hypotheses of Theorem 10 are also satisfied.

Conditions (2.23), (2.24) (i.e. formulae (23), (24) of chapter 2) of Theorem 10 follow from (2), (3) and (10), while (a), (b) follow from (13), (14), (19) and (20). By definition of $\mathfrak{M}$ and since $Q_h \in \mathfrak{M}$, there exist for each h, $1 \leqslant h \leqslant m$, linearly independent integer points $\mathfrak{w}_{h1}, \ldots, \mathfrak{w}_{hn}$ such that (c) holds. Let $M_1, \ldots, M_m$ be the linear forms of Theorem 10. Then we have

P has index at least mε with respect to $(M_1, \ldots, M_m; r_1, \ldots, r_m)$.

By Lemma 6 and since $Q_h \geqslant C_3$,

$$Q_h^{C_1} \leqslant |M_h| \leqslant Q_h^{C_2} \quad (1 \leqslant h \leqslant m), \tag{22}$$

whence

$$|M_h|^{r_h} \geqslant Q_h^{r_h C_1} \geqslant Q_1^{r_1 C_1} \geqslant |M_1|^{r_1 C_1/C_2},$$

and this gives

$$|M_h|^{r_h} \geqslant |M_1|^{r_1 \tau} \quad (1 \leqslant h \leqslant m) \tag{23}$$

with $\tau = C_1/C_2$. Furthermore,

$$|M_h|^{\omega\tau} \geqslant Q_h^{\omega\tau C_1} \geqslant 2^{3mn^2} \quad (1 \leqslant h \leqslant m) \tag{24}$$

by (16), (19) and (22). By Theorem 9,

$$|P| \leqslant D^{r_1 + \cdots + r_m} \leqslant D^{mr_1}$$

and because of (17) and (22) this implies

$$|P|^{n^2} \leqslant D^{mn^2 r_1} \leqslant Q_1^{\omega r_1 C_1^2/C_2} \leqslant |M_1|^{\omega r_1 C_1/C_2} = |M_1|^{\omega r_1 \tau}. \tag{25}$$

By our choice of ε and ω and by (21), $\varepsilon, m, \omega, r_1, \ldots, r_m$ satisfy the hypotheses of Theorem 11. Also τ, the linear forms $M_1, \ldots, M_m$ and the polynomial P satisfy the conditions. The inequalities (2.31), (2.32), (2.33) of Theorem 11 are our inequalities (23), (24) and (25). We therefore conclude:

P has index at most ε with respect to $(M_1, \ldots, M_m; r_1, \ldots, r_m)$.

Since $m > 1$, this contradicts the lower bound for the index given earlier. The assumption that $\mathfrak{M}$ is unbounded was therefore wrong, and Theorem 6 holds.

4. Proof of the main theorems

4.1. *Davenport's Lemma.*

LEMMA 7. *Let* $L_1, \ldots, L_l$ *be linear forms of determinant* 1, *and let* $\lambda_1, \ldots, \lambda_l$ *denote the successive minima of the parallelopiped defined by*

$$|L_j(\mathfrak{x})| \leqslant 1 \quad (1 \leqslant j \leqslant l). \tag{1}$$

Suppose $\varrho_1, \ldots, \varrho_l$ *satisfy*

$$\varrho_1 \varrho_2 \cdots \varrho_l = 1, \tag{2}$$

$$\varrho_1 \geqslant \varrho_2 \geqslant \ldots \geqslant \varrho_l > 0, \tag{3}$$

$$\varrho_1 \lambda_1 \leqslant \varrho_2 \lambda_2 \leqslant \ldots \leqslant \varrho_l \lambda_l. \tag{4}$$

Then, after a suitable permutation of $L_1, \ldots, L_l$, *the successive minima* $\lambda_1', \ldots, \lambda_l'$ *of the new parallelopiped*

$$\varrho_j |L_j(\mathfrak{x})| \leqslant 1 \quad (1 \leqslant j \leqslant l) \tag{5}$$

satisfy

$$2^{-l} \varrho_j \lambda_j \leqslant \lambda_j' \leqslant 2^{l^2} l! \, \varrho_j \lambda_j \quad (1 \leqslant j \leqslant l). \tag{6}$$

Proof. We shall use the ideas of [2]. By a well-known Theorem of Minkowski (for example, see [1], chapter VIII, Theorem V),

$$\frac{1}{l!} \leqslant \lambda_1 \ldots \lambda_l \leqslant 1, \quad \frac{1}{l!} \leqslant \lambda_1' \ldots \lambda_l' \leqslant 1. \tag{7}$$

Set $N(\mathfrak{x}) = \max(|L_1(\mathfrak{x})|, \ldots, |L_l(\mathfrak{x})|)$ and let $\mathfrak{x}_1, \ldots, \mathfrak{x}_l$ be linearly independent integer points such that $N(\mathfrak{x}_i) = \lambda_i$ $(i = 1, \ldots, l)$. If $\mathfrak{x}$ lies in the subspace S_i generated by $0, \mathfrak{x}_1, \ldots, \mathfrak{x}_i$, then $L_1(\mathfrak{x}), \ldots, L_l(\mathfrak{x})$ satisfy $l - i$ independent linear conditions, the coefficients in which depend only on $\mathfrak{x}_1, \ldots, \mathfrak{x}_l$.

We order $L_1, \ldots, L_l$ in the following way. In the condition

$$U_1 L_1 + \ldots + U_l L_l = 0 \tag{8}$$

implied by $\mathfrak{x} \in S_{l-1}$, U_l is the largest coefficient in absolute value. In the additional linear relation implied by $\mathfrak{x} \in S_{l-2}$, which we can take in the form

$$V_1 L_1 + \ldots + V_{l-1} L_{l-1} = 0, \tag{9}$$

V_{l-1} is to be the largest coefficient in absolute value, and so on.

Then if $L_1, \ldots, L_l$ satisfy (8), we have

$$|U_l L_l| \leqslant |U_1 L_1| + \ldots + |U_{l-1} L_{l-1}|$$

and so

$$|L_l| \leqslant |L_1| + \ldots + |L_{l-1}|,$$

whence

$$|L_1| + \ldots + |L_{l-1}| \geqslant \tfrac{1}{2}(|L_1| + \ldots + |L_l|).$$

If $L_1, \ldots, L_l$ satisfy both (8) and (9), we have, similarly,

$$|L_1| + \ldots + |L_{l-2}| \geqslant \tfrac{1}{2}(|L_1| + \ldots + |L_{l-1}|) \geqslant \tfrac{1}{4}(|L_1| + \ldots + |L_l|),$$

and so on generally.

Now suppose $\mathfrak{x}$ lies in S_i but not in S_{i-1} ($1 \leqslant i \leqslant l$). Then $N(\mathfrak{x}) \geqslant \lambda_i$ and $L_1(\mathfrak{x}), \ldots, L_l(\mathfrak{x})$ satisfy $l-i$ linear relations, whence

$$|L_1| + \ldots + |L_i| \geqslant 2^{i-l}(|L_1| + \ldots + |L_l|) \geqslant 2^{i-l}\lambda_i.$$

By (3) this yields

$$\max(\varrho_1|L_1|, \ldots, \varrho_l|L_l|) \geqslant \max(\varrho_1|L_1|, \ldots, \varrho_i|L_i|) \geqslant \frac{1}{i} 2^{i-l}\varrho_i\lambda_i \geqslant 2^{-l}\varrho_i\lambda_i.$$

By (4), this inequality in fact holds for any $\mathfrak{x}$ which is not in S_{i-1}. This shows $\lambda_i' \geqslant 2^{-l}\varrho_i\lambda_i$. The lower bound for λ_j' now follows from (2) and (7).

4.2. *Proof of Theorem 7.* Let the forms $L_1, L_2, L_3, M_1, M_2, M_3$ and the sets S, T satisfy the hypotheses of Theorem 7. If $S = \{1, 2, 3\}$, condition (1.21) implies $A_1 = A_2 = A_3 = 1$, the set (1.14) is a fixed set, and (1.20) certainly holds if Q is large. We may therefore assume that neither S nor T contains all three elements 1, 2, 3. Since S and T are not empty and since $S \cap T$ is, S contains either one or two elements, and similarly for T.

There exist integers c_1, c_2, c_3 with

$$c_1 + c_2 + c_3 = 0, \quad |c_i| \leqslant 2 \ (i = 1, 2, 3), \quad c_i \geqslant 1 \text{ if } i \in S, \quad c_j \leqslant -1 \text{ if } j \in T. \tag{10}$$

Throughout this section, A_1, A_2, A_3 will be positive reals with $A_1A_2A_3 = 1$ and

$$A_i \geqslant 1 \text{ if } i \in S, \quad A_j \leqslant 1 \text{ if } j \in T, \tag{11}$$

i.e. (1.21). $\lambda_1, \lambda_2, \lambda_3$ will denote the successive minima of

$$|L_i(\mathfrak{x})| \leqslant A_i \quad (i = 1, 2, 3) \tag{12}$$

and μ_1, μ_2, μ_3 the successive minima of

$$|M_i(\mathfrak{x})| \leqslant A_i^{-1} \quad (i = 1, 2, 3). \tag{13}$$

The convex bodies defined by (12), (13), respectively, are polar to each other. By a well-known Theorem of Mahler ([4], or see [1], chapter VIII, Theorem VI),

$$1 \leqslant \lambda_j \mu_{4-j} \leqslant 3! \quad (j = 1, 2, 3). \tag{14}$$

By the hypothesis of Theorem 7, one has

$$\lambda_2 > Q^{-\delta} \tag{15}$$

provided $Q \geqslant \max(A_1, A_2, A_3, C_1(\delta))$. Similarly,

$$\mu_2 > Q^{-\delta} \tag{16}$$

provided $Q \geqslant \max(A_1^{-1}, A_2^{-1}, A_3^{-1}, C_2(\delta))$. By virtue of (14), applied for $j=2$, and since $Q \geqslant \max(A_1, A_2, A_3)$ implies $Q^2 \geqslant \max(A_1^{-1}, A_2^{-1}, A_3^{-1})$, we therefore have

$$Q^{-\delta} < \lambda_2 < Q^{\delta} \tag{17}$$

if $Q \geqslant \max(A_1, A_2, A_3, C_3(\delta))$.

Suppose now for some A_1, A_2, A_3 satisfying all our conditions one has

$$\lambda_1 \leqslant Q^{-15\delta} \tag{18}$$

where $Q \geqslant \max(A_1, A_2, A_3)$. Put

$$\bar{A}_i = A_i Q^{4\delta c_i} \quad (i=1, 2, 3). \tag{19}$$

Then $\bar{A}_1 \bar{A}_2 \bar{A}_3 = 1$ and

$$\bar{A}_i \geqslant Q^{4\delta} \text{ if } i \in S, \qquad \bar{A}_j \leqslant Q^{-4\delta} \text{ if } j \in T, \tag{20}$$

and $Q^{1+8\delta} \geqslant \max(\bar{A}_1, \bar{A}_2, \bar{A}_3)$. The first minimum $\bar{\lambda}_1$ of the set

$$|L_i(\mathfrak{x})| \leqslant \bar{A}_i \quad (i=1, 2, 3) \tag{21}$$

satisfies $\bar{\lambda}_1 \leqslant Q^{-7\delta}$. By an inequality of the type (17), applied to $\bar{A}_1, \bar{A}_2, \bar{A}_3$, one has $Q^{-\delta} < \bar{\lambda}_2 < Q^{\delta}$ if $Q \geqslant C_4(\delta)$.

Set
$$\varrho_1 = Q^{4\delta}, \quad \varrho_2 = Q^{-2\delta}, \quad \varrho_3 = Q^{-2\delta}.$$

Since $\varrho_1 \bar{\lambda}_1 \leqslant Q^{-3\delta} \leqslant \varrho_2 \bar{\lambda}_2 \leqslant \varrho_3 \bar{\lambda}_3$, Lemma 7 is applicable to the parallelopiped defined by (21). There is a permutation j_1, j_2, j_3 of 1, 2, 3 such that the successive minima $\lambda_1', \lambda_2', \lambda_3'$ of the parallelopiped

$$|L_i(\mathfrak{x})| \leqslant \bar{A}_i \varrho_{j_i}^{-1} = A_i' \quad (i=1, 2, 3) \tag{22}$$

satisfy

$$2^{-12} \varrho_j \bar{\lambda}_j \leqslant \lambda_j' \leqslant 2^{12} \varrho_j \bar{\lambda}_j \quad (j=1, 2, 3). \tag{23}$$

In particular, $\lambda_2' \leqslant 2^{12} Q^{-\delta}$.

One has $A_1' A_2' A_3' = 1$, and by (20) and the construction of $\varrho_1, \varrho_2, \varrho_3$, $A_i' \geqslant 1$ if $i \in S$ and $A_j' \leqslant 1$ if $j \in T$. Also note $Q^{1+12\delta} \geqslant \max(A_1', A_2', A_3')$. Now suppose $Q > C_3(\delta/2)$ and put $Q' = Q^{1+12\delta}$. Inequality (17) applied to the parallelopiped (22) and to Q' yields $\lambda_2' \geqslant Q'^{-\delta/2} > 2^{12} Q^{-\delta}$ if Q is large. We thus have reached a contradiction, and (18) cannot hold if Q is large. Since $\delta > 0$ was arbitrary, Theorem 7 is proved.

4.3. *A general theorem.*

THEOREM 12. *Let $L_i = \alpha_{i1} X_1 + \alpha_{i2} X_2 + \alpha_{i3} X_3$ $(i=1, 2, 3)$ be linear forms with algebraic coefficients and with determinant $\neq 0$, and let $M_i = \beta_{i1} X_1 + \beta_{i2} X_2 + \beta_{i3} X_3$ be the adjoint forms. Let A, B be subsets of $\{1, 2, 3\}$ such that $A \cap B = \emptyset$ and assume that*

(i) *for $i \in A$, the non-zero elements among $(\alpha_{i1}, \alpha_{i2}, \alpha_{i3})$ are linearly independent over $\mathbf{Q}$;*
(ii) *for $j \in B$, the non-zero elements among $(\beta_{i1}, \beta_{i2}, \beta_{i3})$ are linearly independent over $\mathbf{Q}$;*
(iii) *for $k=1, 2, 3$, there is an $i \in A$ and a $j \in B$ such that $\alpha_{ik} \neq 0$, $\beta_{jk} \neq 0$.*

Let $\varepsilon>0$, $\eta>0$. There are only finitely many integer points $\mathfrak{q}\neq 0$ such that

$$|L_1(\mathfrak{q})L_2(\mathfrak{q})L_3(\mathfrak{q})|<|\mathfrak{q}|^{-\varepsilon}, \tag{24}$$

$$|L_i(\mathfrak{q})|<|\mathfrak{q}|^{-\varepsilon} \text{ for } i\in A, \tag{25}$$

$$|L_j(\mathfrak{q})|\geqslant\eta \text{ for } j\in B, \tag{26}$$

where $|\mathfrak{q}|=\max(|q_1|, |q_2|, |q_3|)$ *if* $\mathfrak{q}=(q_1, q_2, q_3)$.

Proof. L_1, L_2, L_3; B and M_1, M_2, M_3; A are proper. Without loss of generality we may assume that L_1, L_2, L_3 have determinant 1. We may apply Theorem 6 and then Theorem 7 with $S=B$, $T=A$. As before we may assume that S, T contain one or two elements each.

Suppose now (24), (25), (26) hold. Set

$$A_i=\max(|L_i(\mathfrak{q})|\,|\mathfrak{q}|^{\varepsilon/3}, |\mathfrak{q}|^{-5}) \quad \text{if } i\notin S=B. \tag{27}$$

Now if $i\in T=A$, then $i\notin S$, and by (25) one has $A_i<1$. By (24), if $|\mathfrak{q}|$ is sufficiently large and if $\varepsilon<1$,

$$\prod_{j\in S}(|L_j(\mathfrak{q})|\,|\mathfrak{q}|^{\varepsilon/6})<\min(|\mathfrak{q}|^3, |\mathfrak{q}|^{-2\varepsilon/3}\prod_{i\notin S}|L_i(\mathfrak{q})|^{-1})\leqslant\prod_{i\notin S}A_i^{-1}.$$

Hence one may for each $j\in S=B$ choose A_j such that $|L_j(\mathfrak{q})|\,|\mathfrak{q}|^{\varepsilon/6}<A_j$ and that $A_1A_2A_3=1$. By (26), $A_j>1$ if $j\in S=B$, at least if $|\mathfrak{q}|$ is large.

We have

$$|L_i(\mathfrak{q})|\leqslant A_i|\mathfrak{q}|^{-\varepsilon/6} \quad (i=1, 2, 3). \tag{28}$$

By (27), we have $|\mathfrak{q}|^{-5}\leqslant A_i\leqslant|\mathfrak{q}|^2$ if $i\notin S$. Since $A_j\geqslant 1$ if $j\in S$ and since $A_1A_2A_3=1$, one obtains $|\mathfrak{q}|^{10}\geqslant\max(A_1, A_2, A_3)$. Put $Q=|\mathfrak{q}|^{10}$. By virtue of (28), the first minimum λ_1 of

$$|L_i(\mathfrak{x})|\leqslant A_i \quad (i=1, 2, 3) \tag{29}$$

satisfies $\lambda_1\leqslant|\mathfrak{q}|^{-\varepsilon/6}=Q^{-\varepsilon/60}$. By Theorem 7 this cannot happens if $|\mathfrak{q}|$ and hence Q is large.

4.4. *Proof of Theorem 1, 2 and 4.*

Proof of Theorem 1. Let α, β be algebraic and $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$. Set

$$L_1=X_1-\alpha X_3, \quad L_2=X_2-\beta X_3, \quad L_3=X_3.$$

Theorem 12 applies with $A=\{1, 2\}$, $B=\{3\}$. Now suppose $q>0$ and

$$\|\alpha q\|\cdot\|\beta q\|\cdot q^{1+\varepsilon}<1. \tag{30}$$

Choose $\mathfrak{q}=(p_1, p_2, q)$ such that $|L_1(\mathfrak{q})|=\|\alpha q\|$, $|L_2(\mathfrak{q})|=\|\beta q\|$, $|L_3(\mathfrak{q})|=q$. By Roth's Theorem, $\|\alpha q\|>q^{-1-\varepsilon/3}$, whence by (30),

$$|L_2(\mathfrak{q})|=\|\beta q\|<q^{-2\varepsilon/3}<|\mathfrak{q}|^{-\varepsilon/2} \text{ if } q \text{ is large.}$$

Since also $|L_1(\mathfrak{q})| < |\mathfrak{q}|^{-\varepsilon/2}$, a relation of the type (25) but with exponent $-\varepsilon/2$ instead of $-\varepsilon$ holds. Similarly, (30) implies a relation of the type (24). Since (26) with $\eta=1$ is obvious, there can be only a finite number of such integer points $\mathfrak{q}$, hence a finite number of positive integers q satisfying (30).

Proof of Theorem 2. Let α, β be as before and let $L_1=X_1$, $L_2=X_2$, $L_3=\alpha X_1+\beta X_2+X_3$. Theorem 13 applies with $A=\{3\}$, $B=\{1, 2\}$.

Now suppose $q_1 \neq 0$, $q_2 \neq 0$ and

$$\|\alpha q_1+\beta q_2\| \cdot |q_1 q_2|^{1+\varepsilon} < 1. \tag{31}$$

Choose $\mathfrak{q}=(q_1, q_2, q_3)$ such that $L_1(\mathfrak{q})=q_1$, $L_2(\mathfrak{q})=q_2$, $|L_3(\mathfrak{q})| = \|\alpha q_1+\beta q_2\|$. (26) with $\eta=1$ is obvious, and for large $|\mathfrak{q}|$, (31) implies relations of the type (24), (25). Hence by Theorem 12 there are only finitely many solutions.

Proof of Theorem 4. Suppose α, β, γ satisfy the hypotheses of Theorem 4, and let

$$L_1 = X_1, \quad L_2 = \alpha X_1 + X_2, \quad L_3 = \beta X_1 + \gamma X_2 + X_3.$$

The adjoint forms are now

$$M_1 = X_1 + \alpha' X_2 + \beta' X_3, \quad M_2 = X_2 + \gamma' X_3, \quad M_3 = X_3$$

where $\alpha' = -\alpha$, $\gamma' = -\gamma$, $\beta' = \alpha\gamma - \beta$. Theorem 12 applies with $A=\{3\}$, $B=\{1\}$.

Now suppose $\mathfrak{q}=(q_1, q_2, q_3)$, $q_1>0$ and

$$|\alpha q_1 + q_2| < q_1^{-\varrho}, \quad |\beta q_1 + \gamma q_2 + q_3| < q_1^{-\tau}, \tag{32}$$

where $\varrho+\tau=1+\varepsilon>1$. For large q_1 whence large $|\mathfrak{q}|$, (32) implies a relation of the type (24), but with exponent $-\varepsilon/(1+|\varrho|+|\tau|)$ instead of $-\varepsilon$. By Roth's Theorem, the number of solutions is finite unless $\varrho \leqslant 1$. Hence $\tau \geqslant \varepsilon$ and (32) implies a relation of the type (25). Obviously (26) holds with $\eta=1$. Hence by Theorem 12 there are only finitely many $\mathfrak{q}$ satisfying our inequalities.

4.5. *Proof of Theorem 5.* Let α, β, γ be numbers satisfying the conditions of Theorem 5. Let

$$L_1 = X_1 - \alpha X_4, \quad L_2 = X_2 - \beta X_4, \quad L_3 = X_3 - \gamma X_4, \quad L_4 = X_4,$$

and let $M_1, \ldots, M_4$ be the adjoint forms, i.e.

$$M_1 = X_1, \quad M_2 = X_2, \quad M_3 = X_3, \quad M_4 = \alpha X_1 + \beta X_2 + \gamma X_3 + X_4.$$

Set $S=\{4\}$, $T=\{1, 2, 3\}$. $L_1, \ldots, L_4$; S and $M_1, \ldots, M_4$; T are proper and Theorem 6 applies.

Let $A_1, \ldots, A_4$ be positive reals with $A_1A_2A_3A_4=1$ and

$$A_i \leqslant 1 \text{ for } i=1, 2, 3; \quad A_4 \geqslant 1. \tag{33}$$

Let $\lambda_1, \ldots, \lambda_4$ and $\mu_1, \ldots, \mu_4$ denote the successive minima of the parallelopipeds

$$|L_i(\mathfrak{x})| \leqslant A_i \quad (i=1, \ldots, 4) \tag{34}$$

and

$$|M_i(\mathfrak{x})| \leqslant A_i^{-1} \quad (i=1, \ldots, 4), \tag{35}$$

respectively. By Mahler's Theorem,

$$1 \leqslant \lambda_j \mu_{5-j} \leqslant 4! \quad (j=1, \ldots, 4). \tag{36}$$

We claim that

$$\lambda_1 > A_4^{-1/9-\varepsilon} \tag{37}$$

if A_4 is large. Otherwise there is a $\mathfrak{q}=(\ldots, q) \neq 0$ having $|L_i(\mathfrak{q})| \leqslant A_i A_4^{-1/9-\varepsilon}$ $(i=1, \ldots, 4)$. Since $A_1A_2A_3=A_4^{-1}$, we may assume $A_1A_2 \leqslant A_4^{-2/3}$.

$$\|\alpha q\| \cdot \|\beta q\| \cdot q^{1+\varepsilon} \leqslant A_1 A_4^{-1/9-\varepsilon} A_2 A_4^{-1/9-\varepsilon} A_4^{(1+\varepsilon)(8/9-\varepsilon)} < A_4^{-2/3-2/9-2\varepsilon+8/9} < 1.$$

By Theorem 1 this cannot happen if A_4 hence q is large.

By Theorem 6,

$$\lambda_3 > A_4^{-\varepsilon}, \quad \mu_3 > A_4^{-\varepsilon} \tag{38}$$

if A_4 is large. Combining the inequalities written down so far with Minkowski's well-known inequality $1/4! \leqslant \mu_1 \ldots \mu_4 \leqslant 1$ ([1], chapter VIII, Theorem V), we see that

$$A_4^{-1/9-\varepsilon} < \mu_2 < A_4^{\varepsilon}, \quad A_4^{-\varepsilon} < \mu_3 < A_4^{1/9+\varepsilon}, \quad \mu_4 < A_4^{1/9+\varepsilon} \tag{39}$$

if A_4 is large.

LEMMA 8. *Suppose A_4 is large and*

$$A_i \leqslant A_4^{-1/9-\varepsilon} \quad (i=1, 2, 3). \tag{40}$$

Then

$$\mu_1 > A_4^{-1/9-4\varepsilon}. \tag{41}$$

Proof. Set

$$\varrho_1 = \varrho_2 = (\mu_3/\mu_2)^{1/2}, \quad \varrho_3 = \varrho_4 = (\mu_2/\mu_3)^{1/2}.$$

Since

$$\varrho_1\mu_1 \leqslant \varrho_2\mu_2 = \varrho_3\mu_3 \leqslant \varrho_4\mu_4, \tag{42}$$

we may apply Lemma 7. There is a permutation $j_1, \ldots, j_4$ of $1, \ldots, 4$ such that the successive minima $\mu_1', \ldots, \mu_4'$ of

$$|M_i(\mathfrak{x})| \leqslant A_i^{-1}\varrho_{j_i}^{-1} = A_i'^{-1} \quad (i=1, \ldots, 4) \tag{43}$$

satisfy

$$2^{-70}\varrho_j\mu_j \leqslant \mu_j' \leqslant 2^{70}\varrho_j\mu_j \quad (j=1, \ldots, 4). \tag{44}$$

By (39), $\mu_3/\mu_2 \leqslant A_4^{2/9+2\varepsilon}$, and therefore by (40), $A_i' = A_i\varrho_{j_i} \leqslant A_4^{-1/9-\varepsilon}A_4^{1/9+\varepsilon} = 1$ $(i=1, 2, 3)$. Also $A_4^{8/9-\varepsilon} \leqslant A_4' \leqslant A_4^{10/9+\varepsilon}$.

What we said earlier about (35) therefore also applies to the parallelopiped (43). Since by (42) and (44) μ_2' and μ_3' are of the same order of magnitude, we have $\mu_2' < A_4^{\varepsilon}$, $\mu_3' < A_4^{\varepsilon}$ if A_4 is large. Using (44) again we obtain

$$1/4! \leqslant \mu_1'\mu_2'\mu_3'\mu_4' < A_4^{2\varepsilon}\mu_1'\mu_4' \leqslant 2^{140} A_4^{2\varepsilon}\mu_1\mu_4 < 2^{140} A_4^{2\varepsilon} A_4^{1/9+\varepsilon}\mu_1.$$

For large A_4, (41) follows.

Proof of Theorem 5. Let α, β, γ satisfy the conditions of Theorem 5. Suppose integers $q_1 \neq 0$, $q_2 \neq 0$, $q_3 \neq 0$ satisfy

$$\|\alpha q_1 + \beta q_2 + \gamma q_3\| \cdot |q_1 q_2 q_3|^{5/3+\varepsilon} < 1, \tag{45}$$

where $\varepsilon > 0$. Choose $\mathfrak{q} = (q_1, q_2, q_3, q_4)$ such that $|M_i(\mathfrak{q})| = |q_i|$ $(i = 1, 2, 3)$ and $|M_4(\mathfrak{q})| = \|\alpha q_1 + \beta q_2 + \gamma q_3\|$.

Put $\quad A_i = |q_i|^{-1} |q_1 q_2 q_3|^{-1/6-\varepsilon/9} \quad (i = 1, 2, 3), \quad A_4 = |q_1 q_2 q_3|^{3/2+\varepsilon/3}.$

Then
$$A_4^{1/9+\varepsilon/30} < |q_1 q_2 q_3|^{1/6+\varepsilon/9} \leqslant A_i^{-1} \quad (i = 1, 2, 3),$$

whence
$$A_i \leqslant A_4^{-1/9-\varepsilon/30}. \tag{46}$$

$$A_4^{1/9+\varepsilon/30} |q_i| = A_i^{-1} |q_1 q_2 q_3|^{-1/6-\varepsilon/9+(3/2+\varepsilon/3)(1/9+\varepsilon/30)} < A_i^{-1} \quad (i = 1, 2, 3),$$

and therefore

$$|M_i(\mathfrak{q})| = |q_i| < A_i^{-1} A_4^{-1/9-\varepsilon/30} \quad (i = 1, 2, 3), \quad |M_4(\mathfrak{q})| < |q_1 q_2 q_3|^{-5/3-\varepsilon} < A_4^{-1-1/9-\varepsilon/30}.$$

Therefore $\mu_1 < A_4^{-1/9-\varepsilon/30}$. On the other hand, since (46) is an inequality of the type (40), Lemma 8 implies $\mu_1 > A_4^{-1/9-\varepsilon/30}$.

Hence there are no solutions of our inequalities having large A_4, and (45) has only a finite number of solutions.

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