ON INVERSE PROBLEMS ASSOCIATED WITH SECOND-ORDER DIFFERENTIAL OPERATORS

BY

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In a by now classical theorem G. Borg [1] proved the following:

THEOREM A. *Consider the two Sturm–Liouville problems*

$$y'' + [\lambda - q(x)]\, y = 0 \tag{1}$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \tag{2}$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad y(\pi) \cos \gamma + y'(\pi) \sin \gamma = 0, \tag{3}$$

where $q(x)$ is real and integrable on $(0, \pi]$ and sin $(\gamma - \beta) \neq 0$. Then the two spectra corresponding to the boundary conditions (2) *and* (3) *uniquely determine $q(x)$, almost everywhere.*

More recently Li [3] proved the following theorem.

THEOREM B. *Consider the boundary value problem*

$$y'' + [\lambda^2 - q(x)]\, y = 0 \tag{4}$$

$$y(0) = 0, \quad a\, y'(\pi) + \lambda\, y(\pi) = 0, \tag{5}$$

where $a \neq 0$ is real and $q(x)$ is integrable on $[0, \pi]$. The spectrum of the problem (4), (5) *uniquely determines $q(x)$, almost everywhere.*

At first glance it seems paradoxical that the determination of $q(x)$ depends on two spectra in Theorem A and only one spectrum in Theorem B. It is our purpose to

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discuss the relationship between these two theorems, to generalize Theorem B, and to investigate the expansion theorems associated with the operator in Theorem B.

Borg's proof of Theorem A is quite long and a much simpler proof has since been given by N. Levinson [2]. Li's proof of Theorem B is different from either of these and is related to techniques developed in the quantum theory of scattering. Furthermore, in a key step he refers to a result that has appeared only in the Chinese literature.

Although Borg's method is quite involved one can provide a rather simple heuristic argument based on it. For the sake of simplicity let $\alpha=\beta=0$, and $\gamma=\pi/2$ in (2) and (3). Then the asymptotic forms of the solutions of (1), (2) are $y_n\approx\sin nx$ and those of (1), (3) are $y_n\approx\sin(n+\frac{1}{2})x$. Suppose that

$$u''+[\lambda-p(x)]u=0 \tag{a}$$

has the same spectra as (1) corresponding to the boundary conditions (2) and (3) respectively. Then, using (1) and (a) one finds

$$\int_0^\pi (p-q)\, y_n u_n\, dx=0$$

for all eigenfunctions. Using their asymptotic form one finds that

$$\int_0^\pi (p-q)\sin^2 nx\, dx=0,\quad n=1,2,\ldots$$

and

$$\int_0^\pi (p-q)\sin^2 (n+\tfrac{1}{2})x\, dx=0,\quad n=1,2,\ldots$$

from which if follows that

$$\int_0^\pi (p-q)\cos nx\, dx=0,\quad n=0,1,2,\ldots$$

so that $p-q=0$, almost everywhere.

The eigenvalues of (4), (5) have the asymptotic form

$$\lambda_n=n-\frac{1}{\pi}\tan^{-1}a+O\left(\frac{1}{n}\right),$$

where $n=0,\pm1,\pm2,\ldots$. Note that the spectrum, in this case, stretches from $-\infty$ to $+\infty$, whereas the Sturm–Liouville operators are semibounded. The eigenfunctions have the asymptotic form $y_n=\sin(n-(1/\pi)\tan^{-1}a)$. Again using (7) we have

$$\int_0^{\pi} (p-q) \sin^2 \left(n - \frac{1}{\pi} \tan^{-1} a\right) dx = 0$$

from which one can show that

$$\int_0^{\pi} (p-q) \cos \left(2n - \frac{2}{\pi} \tan^{-1} a\right) dx = 0, n = 0, \pm 1, \pm 2, \ldots.$$

The above leads to

$$\int_0^{\pi} (p-q) \cos \left(2n \pm \frac{2}{\pi} \tan^{-1} a\right) dx = 0, n = 0, 1, 2, \ldots$$

from which we have again $p-q=0$ almost everywhere.

The proofs in the literature do not indicate how Theorems A and B are related and their relationship will be discussed in the sequel. A simpler proof of Theorem B will be given, using the method of [2], and also a number of generalizations will be proved. In a second part of this paper the expansion theorems associated with (4), (5) will be discussed fully.

Part I

Let $\lambda = \sigma + i\tau$, and define two solutions of (4) by the initial conditions

$$\left.\begin{aligned} y_1(0) = 1, \quad y_1'(0) = 0 \\ y_2(0) = 0, \quad y_2'(0) = 1. \end{aligned}\right\} \tag{6}$$

It is well known that $y_1(\pi, \lambda)$ and $y_2(\pi, \lambda)$ are entire functions of order 1, in terms of λ. As a matter of fact, they are entire functions of order $\frac{1}{2}$ in terms of λ^2. This follows from expansions of the type (55). Detailed proofs may be found in the treatise by Titchmarsh [4]. We denote the eigenvalues corresponding to the boundary conditions

$$y(0) = 0, \quad y(\pi) = 0$$

by $\{\zeta_n\}$, and those corresponding to

$$y(0) = 0, \quad y'(\pi) = 0$$

by $\{\zeta_n'\}$. Then

$$y_2(\pi, \lambda) = k_1 \prod_{n=0}^{\infty} \left(1 - \frac{\lambda^2}{\zeta_n}\right) \tag{7}$$

$$y_2'(\pi, \lambda) = k_2 \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\zeta_n'}\right). \tag{8}$$

It is also well known that for large n

$$\zeta_n \approx n^2 \tag{9}$$

$$\zeta_n' \approx (n+\tfrac{1}{2})^2. \tag{10}$$

Now we let
$$\omega(\lambda) = ak_2 \prod_{n=1}^{\infty}\left(1-\frac{\lambda^2}{\zeta_n'}\right) + \lambda k_1 \prod_{n=0}^{\infty}\left(1-\frac{\lambda^2}{\zeta_n}\right). \tag{11}$$

The zeros of $\omega(\lambda)$ represent the eigenvalues of problem (4), (5). $\omega(\lambda)$ is an entire function of λ of order 1. No general theorem will guarantee the existence of zeros. But if we recall that, according to standard oscillation theorems, the sets $\{\zeta_n\}$ and $\{\zeta_n'\}$ interlace we observe from (11) that for large n $\omega(\lambda)$ has precisely one zero between $\sqrt{\zeta_n'}$ and $\sqrt{\zeta_{n+1}'}$. A more precise analysis carried out in part II shows that all zeros of $\omega(\lambda)$ are real, simple and asymptotically

$$\lambda_n = n - \frac{1}{\pi}\tan^{-1} a + \frac{1}{2n}\int_0^{\pi} q\,dx + o\left(\frac{1}{n}\right), \tag{12}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Note that the zeros accumulate both at $+\infty$ as well as $-\infty$.

The following asymptotic estimates are well-known [2], [3], [4].

$$\left.\begin{aligned}
y_2 &= \frac{\sin \lambda x}{\lambda} + O\left(\frac{e^{|\tau| x}}{\lambda^2}\right)\\
y_2' &= \cos \lambda x + O\left(\frac{e^{|\tau| x}}{\lambda}\right)\\
y_1(x) &= \cos \lambda x + O\left(\frac{e^{|\tau| x}}{\lambda}\right)\\
y_1'(x) &= -\lambda \sin \lambda x + O(e^{|\tau| x}).
\end{aligned}\right\} \tag{13}$$

Consider a second problem of type (4), (5).

$$u'' + [\lambda^2 - p(x)]\,u = 0 \tag{14}$$

$$u(0) = 0, \quad au'(\pi) + \lambda u(\pi) = 0, \tag{15}$$

and we suppose that the eigenvalues of (14), (15) coincide with those of (4), (5). We now define a third solution of (4) using the initial conditions

$$y_3(\pi) = -a, \quad y_3'(\pi) = \lambda. \tag{16}$$

Similarly we define u_1, u_2, u_3 as in (6) and (16). Then

$$y_2 y_3' - y_2' y_3 = \omega(\lambda) \tag{17}$$

and also

$$u_2 u_3' - u_2' u_3 = \omega(\lambda). \tag{18}$$

That the Wronskians (17) and (18) coincide is a consequence of the fact that both have the same zeros and the same asymptotic form. Being functions of order 1 and having the same zeros implies that they differ by at most an exponential factor. Both are real for real λ and have the asymptotic form

$$\omega(\lambda) \approx a \cos \lambda\pi + \sin \lambda\pi + O\left(\frac{1}{\lambda}\right)$$

so that such an exponential factor has to reduce to unity. By evaluating the Wronskian at $x=0$ and $x=\pi$ we have

$$\omega(\lambda) = -y_3(0) = a y_2'(\pi) + \lambda y_2(\pi). \tag{19}$$

Similarly we have

$$\omega(\lambda) = -u_3(0) = a u_2'(\pi) + \lambda u_2(\pi). \tag{20}$$

When $\lambda = \lambda_n$, at some eigenvalue, y_2 and y_3 are linearly dependent so that

$$y_2 = C_n y_3$$

and also

$$u_2 = D_n u_3.$$

We shall show that necessarily $C_n = D_n$. At $x=\pi$ we have

$$C_n = \frac{y_2(\pi)}{y_3(\pi)} = \frac{y_2(\pi)}{-a}, \qquad D_n = \frac{u_2(\pi)}{u_3(\pi)} = \frac{u_2(\pi)}{-a}.$$

Recall that $y_2(\pi), u_2(\pi), y_2'(\pi), u_2'(\pi)$ are even functions of λ. Then by comparing the odd parts of (19) and (20) we find that

$$u_2(\pi) = y_2(\pi).$$

It follows from the above expressions for C_n and D_n that now $C_n = D_n$. It is in this step that we use the fact that (4), (5) and (14), (15) have the same spectrum.

Now we define the two functions

$$\Phi = \frac{y_3(x)\int_0^x y_2(t)\,f(t)\,dt + y_2(x)\int_x^\pi y_3(t)\,f(t)\,dt}{\omega(\lambda)} \tag{21}$$

and
$$\Psi = \frac{u_3(x)\int_0^x y_2(t)\,f(t)\,dt + u_2(x)\int_x^\pi y_3(t)\,f(t)\,dt}{\omega(\lambda)}, \tag{22}$$

where $f(t)$ is an arbitrary square integrable function on $[0, \pi]$.

Let $\{R_n\}$ denote a sequence of squares with vertices at

$$\left(\pm\left[n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}a\right],\ \pm i\left[n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}a\right]\right).$$

These are uniformly bounded away from the zeros of $\omega(\lambda)$ by virtue of (12) for large n. We shall show that

$$\lim_{n\to\infty}\int_{R_n}(\Phi-\Psi)\,d\lambda = 0. \tag{23}$$

A typical term in the integrand of (23) is

$$\frac{(y_3-u_3)\int_0^x y_2(t)\,f(t)\,dt}{\omega(\lambda)}.$$

Using estimates of the type (13) we see that the above has the asymptotic form

$$\frac{O\left(\frac{e^{|\tau|(\pi-x)}}{\lambda^2}\right)\int_0^x \sin\lambda t\,f(t)\,dt}{a\cos\lambda\pi+\sin\lambda\pi}.$$

Using the latter we see that

$$\int_{R_n}(\Phi-\Psi)\,d\lambda = O\left(\frac{1}{n}\right)$$

from which (23) follows. Now use residue integration and the fact that all zeros of $\omega(\lambda)$ are simple and also that at λ_n

$$y_3 = C_n y_2, \quad u_3 = C_n u_2.$$

Then from (23)
$$\sum_{n=-\infty}^{\infty}\frac{C_n[y_2(x,\lambda_n)-u_2(x,\lambda_n)]\int_0^\pi y_2(t,\lambda_n)\,f(t)\,dt}{\omega'(\lambda_n)} = 0. \tag{24}$$

In part II it is shown that the eigenfunctions $y_2(t, \lambda_n)$ are independent. Note that we do not require their completeness here. We can, therefore, select $f(t)$ so that

$$\int_0^\pi y_2(t, \lambda_n) f(t)\, dt = 0 \quad n \neq k$$
$$= 1 \quad n = k$$

and from (24) it follows that

$$y_2(x, \lambda_n) = u_2(x, \lambda_n).$$

The latter implies that y_2 and u_2 satisfy the same differential equation. This concludes our proof of Theorem B.

Theorem B can be generalized in the following direction.

THEOREM 1. *Consider*

$$y'' + [\lambda^2 - q(x)]\, y = 0 \tag{25}$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad y(\pi) \cos \beta + y'(\pi) \sin \beta + f(\lambda)\, [y(\pi) \cos \gamma + y'(\pi) \sin \gamma] = 0, \tag{26}$$

where $f(\lambda)$ is an odd real entire function of λ of order less than 1 *and* $\sin(\beta - \gamma) \neq 0$. *The eigenvalues of (25) and (26) uniquely determine $q(x)$.*

Let y satisfy the initial conditions

$$y(0) = -\sin \alpha, \quad y'(0) = \cos \alpha$$

and

$$\omega(\lambda) = S(\lambda) + f(\lambda)\, T(\lambda),$$

where

$$S(\lambda) = y(\pi) \cos \beta + y'(\pi) \sin \beta$$

$$T(\lambda) = y(\pi) \cos \gamma + y'(\pi) \sin \gamma.$$

The zeros and asymptotic form of $\omega(\lambda)$ uniquely determine $\omega(\lambda)$. The even part of $\omega(\lambda)$ is $S(\lambda)$ and its odd part is $f(\lambda)\, T(\lambda)$, since clearly $S(\lambda)$ and $T(\lambda)$ are even functions of λ. Then knowing $\omega(\lambda)$ we know the zeros of $S(\lambda)$ and $T(\lambda)$. But these determine the eigenvalues of (1), (2) and (1), (3). By Theorem A these uniquely determine $q(x)$.

Theorem 1 with $f(\lambda) = a\lambda$, $a \neq 0$ can be proved directly by the same technique as Theorem B in the preceding. Except for some details there is no difference. This leads us to the following theorem.

THEOREM 2. *Theorems A and* 1 *are fully equivalent.*

In the proof of Theorem 1 it was shown that Theorem A implies Theorem 1. To prove the converse we assume Theorem 1 to be true. Suppose $S_1(\lambda), T_1(\lambda), S_2(\lambda),$

$T_2,(\lambda)$ correspond to two different boundary value problems of type (1), (2) and (1), (3), where $S(\lambda)$ and $T(\lambda)$ are defined as in the preceding proof. Then we form

$$\omega_1(\lambda) = S_1(\lambda) + a\lambda T_1(\lambda)$$
$$\omega_2(\lambda) = S_2(\lambda) + a\lambda T_2(\lambda).$$

If $S_1(\lambda) = S_2(\lambda)$ and $T_1(\lambda) = T_2(\lambda)$, then $\omega_1(\lambda) = \omega_2(\lambda)$. By Theorem 1 the latter fact shows that both differential equations are the same, thereby establishing Theorem A.

Part II

We now turn our attention to the problem

$$y'' + [\lambda^2 + \mu - q(x)]\, y = 0 \tag{27}$$

$$y(0) = 0, \quad ay'(\pi) + \lambda y(\pi) = 0, \tag{28}$$

where $q(x)$ is real and integrable on $[0, \pi]$ and μ and $a \neq 0$ are real parameters. To study the problem (27), (28) we shall relate it to a different problem. We introduce the function $M(x)$ defined by

$$\left.\begin{aligned} M'' + [\mu - q(x)]\, M = 0 \\ M(\pi) = 1,\ M'(\pi) = 0, \end{aligned}\right\} \tag{29}$$

and now restrict μ so that $M > 0$ on $[0, \pi]$. The operator corresponding to the eigenvalue problem

$$M'' + [\mu - q(x)]\, M = 0$$

and the boundary conditions

$$M(0) = 0, \quad M'(\pi) = 0 \tag{30}$$

is lower semibounded. If we denote the smallest eigenvalue of the above problem by μ_0 then for all $\mu < \mu_0$, the solution M of (29) remain positive. This is an immediate consequence of Sturm's oscillation theorem.

We now transform (27), (28) by introducing new dependent and independent variables η, ξ by means of

$$y = M\eta, \quad \xi = \int_0^x \frac{dx}{M^2}. \tag{31}$$

This results in the new boudary value problem

$$\eta'' + \lambda^2 M^4 \eta = 0 \tag{32}$$

$$\eta(0) = 0, \quad a\eta'(\varrho) + \lambda\eta(\varrho) = 0, \tag{33}$$

where $\varrho = \int_0^\pi 1/M^2 dx$. To study the problem (32), (33) we introduce the new functions

$$\left.\begin{aligned} x_1 &= -\lambda\eta \\ x_2 &= \eta'. \end{aligned}\right\} \tag{34}$$

These satisfy the differential equaitions

$$\left.\begin{aligned} x_1' &= -\lambda x_2 \\ x_2' &= \lambda M^4 x_1 \end{aligned}\right\} \tag{35}$$

and the boundary conditions

$$x_1(0) = 0, \quad a x_2(\varrho) - x_1(\varrho) = 0. \tag{36}$$

(35) can be rewritten in the form

$$L_0 X = \lambda X, \tag{37}$$

where $$L_0 = \begin{pmatrix} 0 & \dfrac{1}{M^4}\dfrac{d}{d\xi} \\ -\dfrac{d}{d\xi} & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The effect of all these substitutions is to linearize problem (27), (28) in terms of the parameter λ. L_0 can be inverted by means of an integral operator. Then we obtain

$$\left.\begin{aligned} x_1 &= -\lambda \int_0^\xi x_2\, d\xi \\ x_2 &= -\frac{\lambda}{a}\int_0^\varrho x_2\, d\xi - \lambda \int_\xi^\varrho M^4 x_1\, d\xi. \end{aligned}\right\} \tag{38}$$

We shall denote the integral operator defined in (38) by $\mathcal{G}_0$ so that (38) can be rewritten as

$$X = \lambda \mathcal{G}_0 X.$$

We now introduce the Hilbert space H consisting of all vectors

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ for which } \int_0^\varrho [|x_1|^2 + |x_2|^2]\, d\xi < \infty.$$

As a suitable inner product we introduce

$$(X, Y) = \int_0^\varrho (M^4 x_1 \bar{y}_1 + x_2 \bar{y}_2)\, d\xi \tag{39}$$

and the norm
$$\|X\| = \sqrt{\int_0^\varrho [M^4|x_1|^2 + |x_2|^2]\,d\xi}. \tag{40}$$

In view of the fact that M is a continuous and positive function on $[0, \pi]$, $\|X\|$ as defined in (40) is equivalent to

$$\sqrt{\int_0^\varrho [|x_1|^2 + |x_2|]^2\,d\xi}.$$

A simple exercise shows that
$$(L_0 X, Y) = (X, L_0 Y) \tag{41}$$

if X and Y are absolutely continuous and it follows also that

$$(\mathcal{G}_0 X, Y) = (X, \mathcal{G}_0 Y), \tag{42}$$

for general X, Y in H. From (38) and (42) it follows that $\mathcal{G}_0$ is a compact, selfadjoint operator defined on H. From (38) it is also evident that the nullspace of $\mathcal{G}_0$ is empty. In other words, the only solution of $\mathcal{G}_0 X = 0$ is $X = 0$.

Using the standard theory of compact, selfadjoint operators we can conclude that $\mathcal{G}_0$ has real eigenvalues, its eigenfunctions form a complete orthonormal set in H. Since H is infinite dimensional $\mathcal{G}_0$ must have an infinity of eigenvalues. One can also show that all eigenvalues are simple. If we had two eigenfunctions corresponding to (36) and (37) we could form a linear combination satisfying (37) and the boundary conditions
$$x_1(0) = 0, \quad x_2(0) = 0.$$

Using (38) we can then show that x_1 satisfies

$$x_1'' + \lambda^2 M^4 x_1 = 0$$
$$x_1(0) = x_1'(0) = 0.$$

It follows that $x_1 = 0$ and also $x_2 = 0$. Hence the two eigenfunctions are identical.

Let F be any function in H and let X_n denote the normalized eigenfunctions and λ_n the eigenvalues of L_0. Then

$$F = \sum_{n=1}^{\infty} \alpha_n X_n \tag{43}$$

$$\alpha_n = (F, X_n) = \int_0^\varrho [M^4 f_1 \overline{x_1^{(n)}} + f_2 \overline{x_2^{(n)}}]\,d\xi. \tag{44}$$

If, in particular, we select $f_2 = 0$ we obtain from (43)

$$f_1 = \sum_{n=1}^{\infty} \alpha_n x_1^{(n)}$$

$$0 = \sum_{n=1}^{\infty} \alpha_n x_2^{(n)}.$$

If in the first of these we return to our original variables as defined in (27), (28), we obtain

$$f_1 = \sum_{n=1}^{\infty} \alpha_n \frac{(-\lambda_n)}{M} y_n, \qquad \alpha_n = \int_0^{\pi} f_1(-\lambda_n) M y_n \, dx. \tag{45}$$

We shall summarize these results in the following theorems.

THEOREM 3. *The operator L_0, defined by (36), (37), or equivalently $\mathcal{G}_0$, defined by (38), acting on the space H has an infinity of simple, real eigenvalues. The eigenfunctions form a complete orthonormal set with respect to the inner product (39).*

THEOREM 4. *The operator defined by (27), (28) has an infinity of simple real eigenvalues. The associated eigenfunctions are complete and functions that are square integrable on $[0, \pi]$ can be expanded in series of the type (45). It is assumed that $\mu < \mu_0$, which is defined by (29), (30).*

Note that the eigenfunctions associated with (27), (28) are not orthonormal.

We now turn to the problem

$$y'' + [\lambda^2 - q(x)]\, y = 0 \tag{46}$$

$$y(0) = 0, \quad a y'(\pi) + \lambda y(\pi) = 0. \tag{47}$$

If the value $\mu_0 > 0$ we can set $\mu = 0$ in (27), (28) and then the problem fits directly into the framework of Theorem 3, 4. If $\mu_0 \leqslant 0$ the preceding results no longer apply. Nevertheless we define M as in (29) and introduce the change of variables (31). This leads to the equation

$$\eta'' + [\lambda^2 - \mu]\, M^4 \eta = 0 \tag{48}$$

with the boundary conditions

$$\eta(0) = 0, \quad a\eta'(\varrho) + \lambda \eta(\varrho) = 0. \tag{49}$$

To linearize the problem in terms of λ we now let

$$\left.\begin{aligned} x_1 &= -\lambda \eta \\ x_2 &= \eta' + \mu \int_0^{\varrho} M^4 \eta \, d\xi. \end{aligned}\right\} \tag{50}$$

These satisfy the system

$$\left.\begin{aligned} &L_\mu X = \lambda X \\ &x_1(0) = 0, \quad a x_2(\varrho) - x_1(\varrho) = 0 \end{aligned}\right\}, \tag{51}$$

where

$$L_\mu X = \begin{pmatrix} \dfrac{x_2'}{M^4} \\ -x_1' - \mu \displaystyle\int_\xi^\varrho M^4 x_1 \, d\xi \end{pmatrix}$$

Note that

$$L_\mu = L_0 + \mu L_1,$$

where L_0 was defined in (37) and

$$L_1 X = \begin{pmatrix} 0 \\ -\displaystyle\int_\xi^\varrho M^4 x_1 \, d\xi \end{pmatrix} \tag{52}$$

is compact. The adjoint of L_1 is given by

$$L_1^* X = \begin{pmatrix} -\displaystyle\int_0^\xi x_2 \, d\xi \\ 0 \end{pmatrix}$$

so that

$$L_\mu^* = L_0 + \mu L_1^*,$$

with the same boundary conditions as in (51), since L_0 is selfadjoint.

We shall now assume that $\lambda = 0$ is not an eigenvalue of (51). In that case (50) can be rewritten as an integral equation. The case where $\lambda = 0$ leads to no fundamental complication, except that one does not work in the whole space H, but in the subspace orthogonal to the eigenfunction corresponding to $\lambda = 0$. Functions u and v are now defined as solutions of

$$x'' - \mu M^4 x = 0$$

satisfying the initial conditions

$$u(0) = 0, \quad u'(0) = 1$$

$$v(0) = 1, \quad v'(0) = 0.$$

For a suitable choice of μ $u'(\varrho) \neq 0$. Then we obtain the integral equation

$$\left.\begin{aligned} x_1 &= -\lambda u \int_\xi^\varrho v' x_2 \, d\xi + \frac{\lambda u v'(\varrho)}{u'(\varrho)} \int_0^\varrho u' x_2 \, d\xi - \lambda v \int_0^\xi u' x_2 \, d\xi \\ x_2 &= \frac{-\lambda}{a u'(\varrho)} \int_0^\varrho u' x_2 \, d\xi - \lambda \int_\xi^\varrho M^4 x_1 \, d\xi. \end{aligned}\right\} \tag{53}$$

(53) replaces (51) and for $\mu=0$ it reduces to (38). We shall denote the operator defined in (53) by

$$X=\lambda \mathcal{G}_\mu X. \tag{54}$$

Note that $\mathcal{G}_\mu$ is a compact operator on H, but in general it will not be selfadjoint.

LEMMA 1. *The eigenvalues of L_μ are real.*

Suppose $L_\mu X_i=\lambda_i X_i$, where λ_i is not real and is an eigenvalue of L_μ. Since the operator is real, if λ_i is an eigenvalue so is $\bar\lambda_i$. Then there exists F_i such that $L_\mu F_i=\bar\lambda_i F_i$ and since λ_i is an eigenvalue of L_μ, $\bar\lambda_i$ will be an eigenvalue of L_μ^*. Let G_i be such that $L_\mu^* G_i=\bar\lambda_i G_i$. It also follows from

$$\bar\lambda(F_i,G_i)=(L_\mu F_i,G_i)=(F_i,L_\mu^* G_i)=\lambda_i(F_i,G_i)$$

that $(F_i,G_i)=0$. In that case, according to the Fredholm alternative the equation

$$L_\mu Z_i-\lambda_i Z_i=F_i$$

must have a solution. Using the fact that $L_\mu F_i=\bar\lambda_i F_i$ the latter can be reduced to the following scalar second order equation.

$$z_1''+(\lambda_i^2-\mu)M^4 z_1=-(\lambda_i+\bar\lambda_i)M^4 f_1$$

$$z_1(0)=0,\quad a z_1'(\varrho)+\lambda_i z_1(\varrho)=0,$$

where z_1 and f_1 are the first components of Z_i and F_i respectively. Now consider $\bar f_1$ which satisfies

$$\bar f_1''+(\lambda_i^2-\mu)M^4\bar f_1=0$$

$$\bar f_1(0)=0,\quad a\bar f_1'(\varrho)+\lambda_i\bar f_1(\varrho)=0.$$

By multiplying the equation for z_1 by $\bar f_1$ and the one for $\bar f_1$ by z_1, subtracting and integrating we are led to

$$(\lambda_i+\bar\lambda_i)\int_0^\varrho M^4|f_1|^2\,dt=0.$$

We conclude, therefore, that

$$\lambda_i+\bar\lambda_i=0$$

so that λ_i is pure imaginary. But in that case we have

$$x''+[\lambda_i^2-\mu]M^4x=0$$

$$x(0)=0,\quad a x'(\varrho)+\lambda_i x(\varrho)=0$$

and also
$$ax'(\varrho) - \lambda_i x(\varrho) = 0$$
so that
$$x'(\varrho) = x(\varrho) = 0.$$

The latter implies that $x \equiv 0$, which is certainly not true. It follows that all eigenvalues are real.

LEMMA 2. *The eigenspace associated with any eigenvalue is one dimensional.*

To prove this we note first that to every eigenvalue there corresponds precisely one eigenfunction. If there were two we could form a linear combination such that $x(0) = x'(0) = 0$, which implies that $x \equiv 0$. Now if the subspace were more than one dimensional and if
$$L_\mu F_i = \lambda_i F_i$$
then the equation
$$L_\mu X_i - \lambda_i X_i = F_i$$
would have to have a solution. But as in the proof of Lemma 1 we can show that the above has no solution.

LEMMA 3. *The eigenfunctions of L_μ and L_μ^* form a biorthogonal set. It follows that both consist of linearly independent elements.*

Let $L_\mu X_i = \lambda_i X_i$ and $L_\mu^* Y_i = \lambda_i Y_i$. Clearly
$$\lambda_i (X_i, Y_j) = (L_\mu X_i, Y_j) = (X_i, L_\mu^* Y_j) = \lambda_j (X_i, Y_j)$$
so that
$$(X_i, Y_j) = 0 \quad \text{for} \quad i \neq j.$$
If $(X_i, Y_i) = 0$ for some i, the equation
$$L_\mu Z_i - \lambda_i Z_i = X_i$$
would have a solution. But as in the proof of Lemma 1, this cannot be so that $(X_i, Y_i) \neq 0$.

We shall now turn to a consideration of the asymptotic structure of the eigenvalues of L_μ. We shall seek a solution of (27) in the form
$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} y_n \\ y_0 &= \sin \lambda x \\ y_{n+1} &= \int_0^x \frac{\sin \lambda (x-t)}{\lambda} [q(t) - \mu]\, y_n(t)\, dt. \end{aligned}\right\} \tag{55}$$

By induction we note that

$$|y_n| \leq \frac{1}{|\lambda|^n n!} \int_0^x |q(t)-\mu|^n \, dt$$

so that (55) converges uniformly for real λ. We see that

$$y = \sin \lambda x + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} (q-\mu) \sin \lambda t \, dt + o\left(\frac{1}{\lambda^2}\right)$$

$$y' = \lambda \cos \lambda x + \int_0^x \cos \lambda(x-t)\,(q-\mu) \sin \lambda t \, dt + o\left(\frac{1}{\lambda}\right)$$

so that

$$\begin{aligned} ay'(\pi) + \lambda y(\pi) &= \lambda(a \cos \lambda\pi + \sin \lambda\pi) \\ &\quad + \int_0^\pi [a \cos \lambda(\pi - t) + \sin \lambda(\pi - t)]\,(q-\mu) \sin \lambda t \, dt + o\left(\frac{1}{\lambda}\right) \\ &= \lambda(a \cos \lambda\pi + \sin \lambda\pi) + \frac{a \sin \lambda\pi - \cos \lambda\pi}{2} \int_0^\pi (q-\mu)\, dt + o(1). \end{aligned}$$

From the above we note that

$$\lambda_n = n + \lambda_0 + \frac{1}{2n} \int_0^\pi (q-\mu)\, dt + o\left(\frac{1}{n}\right), \tag{56}$$

where $\tan \lambda_0 \pi = -a$ and $|\lambda_0 \pi| < \pi/2$. Note that $n = 0, \pm 1, \pm 2, \ldots$, so that the operator is not semibounded as is the case with Sturm–Liouville operators. We also observe that the μ dependence enters into the terms vanishing like $1/n$.

We now wish to show that the eigenfunctions associated with the operator L_μ, defined in (51), are complete. It will turn out to be more advantageous to work with L_μ^2, rather than L_μ. Note that formally

$$L_\mu^2 X = \begin{pmatrix} -\dfrac{x_1''}{M^4} + \mu x_1 \\ -\left(\dfrac{x_2'}{M^4}\right)' + \mu x_2 - \mu x_2(\varrho) \end{pmatrix} \tag{57}$$

and the boundary conditions associated with L_μ^2 are

$$\begin{gathered} x_1(0) = 0, \quad x_2'(0) = 0, \\ a x_2(\varrho) - x_1(\varrho) = 0, \quad a x_1'(\varrho) + x_2'(\varrho) = 0. \end{gathered} \tag{58}$$

The eigenvalues of L_μ^2 are given by λ_n^2, and the eigenfunctions are the same as those of L_μ. By the preceding results all eigenvalues of L_μ^2 are real and simple. Note that we can write

$$L_\mu^2 X = (L_0^2 + \mu) X - \mu N X, \tag{59}$$

where

$$NX = \begin{pmatrix} 0 \\ x_2(\varrho) \end{pmatrix}. \tag{60}$$

Theorem 3 tells us that $L_0^2 + \mu$ is a selfadjoint operator and that its eigenfunctions form a complete orthonormal set. The operator N, given by (60) is not selfadjoint and also unbounded.

We now consider the equation

$$(\lambda - L_\mu^2) X = F, \tag{61}$$

with boundary conditions (58). The solution of (61) will formally be denoted by

$$X = \mathcal{G}_2 F.$$

From the preceding results we know that the operator $\mathcal{G}_2$ can be expressed as an integral operator where the kernel is a meromorphic function of λ. For all regular values of λ $\mathcal{G}_2$ is compact. Similarly we associate with the differential operator $\lambda - L_0^2 - \mu$ the integral operator $\mathcal{G}_1$. The solution of

$$(\lambda - L_0^2 - \mu)\, Y = F \tag{62}$$

with boundary conditions (58) is given by

$$Y = \mathcal{G}_1 F.$$

If we express F in the form

$$F = \sum_{n=-\infty}^{\infty} \alpha_n X_n,$$

where the X_n are the eigenfunctions of L_0, then

$$Y = \sum_{n=-\infty}^{\infty} \frac{\alpha_n X_n}{\lambda - \lambda_n^2 - \mu}. \tag{63}$$

We then can write

$$\frac{1}{2\pi i} \int \mathcal{G}_1 F \, d\lambda = \sum_{n=-\infty}^{\infty} \alpha_n X_n = F, \tag{64}$$

where the integral is taken over a sufficiently large contour in the λ plane. We can also use (63) to determine $\|\mathcal{G}_1\|$. Then

$$\| \mathcal{G}_1 F \| = \sqrt{\sum_{-\infty}^{\infty} \frac{|\alpha_n|^2}{|\lambda - \lambda_n^2 - \mu|^2}}$$

and

$$\frac{\| \mathcal{G}_1 F \|}{\| F \|} = \sqrt{\frac{\sum_{-\infty}^{\infty} \frac{|\alpha_n|^2}{|\lambda - \lambda_n^2 - \mu|^2}}{\sum_{-\infty}^{\infty} |\alpha_n|^2}}.$$

From the above we note that

$$\| \mathcal{G}_1 \| = \sup_{F \neq 0} \frac{\| \mathcal{G}_1 F \|}{\| F \|} = \sup_n \frac{1}{|\lambda - \lambda_n^2 - \mu|}, \tag{65}$$

(64) is a consequence of the completeness of the eigenfunctions of L_0. If we succeed in establishing

$$\frac{1}{2\pi i} \int \mathcal{G}_2 F \, d\lambda = F \tag{66}$$

we will have proved that the eigenfunctions of L_μ^2 and correspondingly those of L_μ are complete. Note the residue of $\mathcal{G}_2 F$ at $\lambda = \lambda_n$ represents the projection of F into the n'th eigenspace. The left side of (66) then yields the expansion of F in terms of the eigenfunctions of L_μ.

Using (59) we can rewrite (61) in the form

$$(\lambda - L_0^2 - \mu) X = F - \mu N X$$

and, using (62), we obtain

$$X = \mathcal{G}_1 [F - \mu N X]$$

or equivalently

$$\mathcal{G}_2 F = \mathcal{G}_1 F - \mu \mathcal{G}_1 N \mathcal{G}_2 F. \tag{67}$$

An immediate solution of (67) is given by

$$\mathcal{G}_2 F = \mathcal{G}_1 \sum_{S=0}^{\infty} [-\mu N \mathcal{G}_1]^S F \tag{68}$$

assuming that the series in (68) converges. To prove the convergence of (68) we shall estimate $\| N \mathcal{G}_1 \|$.

By means of (63) we see that

$$N \mathcal{G}_1 F = \sum_n \frac{\alpha_n}{\lambda - \lambda_n^2 - \mu} \begin{pmatrix} 0 \\ x_2^{(n)}(\varrho) \end{pmatrix}$$

so that

$$\| N \mathcal{G}_1 F \| \leqslant \sqrt{\varrho \left| \sum_n \frac{\alpha_n x_2^{(n)}(\varrho)}{\lambda - \lambda_n^2 - \mu} \right|^2} \leqslant \sqrt{\varrho \sum_n [\alpha_n x_2^{(n)}(\varrho)]^2 \sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|^2}}.$$

Using (13) one can deduce for the orthonormalized eigenfunctions the asymptotic formulas

$$X_n = \begin{pmatrix} \dfrac{\sin \lambda_n x(\xi)}{M\sqrt{\pi}} \\ \dfrac{-M \cos \lambda_n x(\xi)}{\sqrt{\pi}} \end{pmatrix} + O\left(\frac{1}{\lambda_n}\right),$$

where
$$\xi = \int_0^{x(\xi)} \frac{dx}{M^2(x)}.$$

Combining the latter with (12) we find

$$x_2^{(n)}(\varrho) = \frac{-\cos \lambda_n \pi}{\sqrt{\pi}} + O\left(\frac{1}{n}\right) = \frac{(-1)^{n+1}}{\sqrt{\pi(1+a^2)}} + O\left(\frac{1}{n}\right).$$

Then
$$\|N G_1 F\| \leqslant K \|F\| \sqrt{\sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|^2}}$$
$$\leqslant K \|F\| \, \|G_1\|^{\frac{1}{2}} \sqrt{\sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|}},$$

if we use the explicit estimate (65) for $\|G_1\|$. If follows that

$$\|N G_1\| \leqslant K \|G_1\|^{\frac{1}{2}} \sqrt{\sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|}}. \tag{69}$$

We now consider a sequence of squares $\{R_k\}$ with vertices at

$$\left(k^2 + \mu + \int_0^\pi q\,dt\right)(\pm 1 \pm i),$$

and estimate the sum under the radical in (69) for λ on R_k. From (56) we have

$$\lambda_n^2 = (n + \lambda_0)^2 + \int_0^\pi q\,dt + o(1). \tag{70}$$

We decompose the sum into three terms

$$\sum_{n=0}^{\infty} = \sum_{n=0}^{[\frac{1}{2}k]} + \sum_{n=[\frac{1}{2}k]+1}^{[\frac{3}{2}k]} + \sum_{n=[\frac{3}{2}k]+1}^{\infty}$$

so that,
$$\sum_{n=0}^{[\frac{1}{2}k]} \frac{1}{|\lambda - \lambda_n^2 - \mu|} = O\left(\frac{1}{k}\right)$$

$$\sum_{n=[\frac{1}{2}k]+1}^{[\frac{3}{2}k]} \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant k \sup_{[\frac{1}{2}k]+1\leqslant n\leqslant[\frac{3}{2}k]} \frac{1}{|\lambda-\lambda_n^2-\mu|} = o(1)$$

$$\sum_{n=[\frac{3}{2}k]+1}^{\infty} \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant \sum_{n=[\frac{3}{2}k]+1}^{\infty} \frac{1}{(n+\lambda_0)^2+o(1)} = O\left(\frac{1}{k}\right).$$

Finally we have for all λ on R_k, with sufficiently large k

$$\|N\mathcal{G}_1\| \leqslant K_1 \|\mathcal{G}_1\|^{\frac{1}{2}}. \tag{71}$$

A more delicate analysis allows us to show that for the constant K_1 in (71) we actually have

$$K_1 = O\left(\frac{1}{k}\ln k\right)$$

but this is not necessary for our subsequent results. Now

$$\|\mathcal{G}_1\| = \sup_n \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant \frac{K}{k}. \tag{72}$$

It follows, therefore, that the series in (68) converges on R_k for sufficiently large k. Note that this also establishes the existence of $\mathcal{G}_2 F$, knowing only the structure of $\mathcal{G}_1$ and N.

Finally we consider the integral

$$\frac{1}{2\pi i}\int_{R_k} \mathcal{G}_2 F\, d\lambda.$$

To estimate the above we consider the general term in (68),

$$\frac{1}{2\pi i}\int \mathcal{G}_1 (N\mathcal{G}_1)^S F\, d\lambda.$$

Using (71) we see that

$$\|\mathcal{G}_1(N\mathcal{G}_1)^S\| \leqslant K_1 \|\mathcal{G}_1\|^{\frac{1}{2}S+1} = K_1 \sup_n \frac{1}{|\lambda-\lambda_n^2-\mu|^{\frac{1}{2}S+1}},$$

and at first we restrict our attention to $\lambda = k^2+\mu+\int_0^\pi q\,dt + iy$, where $|y| \leqslant k^2+\mu+\int_0^\pi q\,dt$, so that

$$\sup_n \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant \frac{K}{\sqrt{4k^2\lambda_0^2+y^2}}$$

for a suitable constant K. Then

$$\|\mathcal{G}_1(N\mathcal{G}_1)^S\| \leqslant \frac{K_2}{[4k^2\lambda_0^2+y^2]^{\frac{1}{4}S+\frac{1}{2}}}$$

and $$\left\| \frac{1}{2\pi i} \int_{-k^2-\mu-\int_0^\pi q\,dt}^{k^2+\mu+\int_0^\pi q\,dt} G_1 (N G_1)^S F\,dy \right\| \leqslant K_3 \|F\| \int_0^{k^2+\mu+\int_0^\pi q\,dt} \frac{dy}{[4k^2\lambda_0^2+y^2]^{\frac{1}{2}S+\frac{1}{2}}}$$

$$\leqslant \frac{K_3 \|F\|}{(2k\lambda_0)^{\frac{1}{2}S}} \int_0^{\frac{1}{2}\pi} \cos^{\frac{1}{2}S-1}\theta\,d\theta \leqslant \frac{K_4}{k^{\frac{1}{2}S}} \|F\|.$$

Applying similar, and even simpler estimates, to the other three sides of R_k, we see that

$$\frac{1}{2\pi i} \int_{R_k} G_1 (N G_1)^S F\,d\lambda = O\left(\frac{1}{k^{\frac{1}{2}S}}\right).$$

Returning to (68) we have

$$\frac{1}{2\pi i} \int_{R_k} G_2 F\,d\lambda = \frac{1}{2\pi i} \int_{R_k} G_1 F\,d\lambda + O\left(\frac{1}{k^{\frac{1}{2}}}\right)$$

so that $$\lim_{k\to\infty} \frac{1}{2\pi i} \int_{R_k} G_2 F\,d\lambda = F. \tag{73}$$

We can now state the following theorem.

THEOREM 5. *The operator L_μ, defined by (51), acting on the Hilbert space H, has an infinity of real, simple eigenvalues. The eigenfunctions associated with it form a complete set. The eigenfunctions associated with its adjoint operator L_μ^* are also complete and biorthogonal to those of L_μ.*

In analogy to Theorem 4, we obtain expansion theorems associated with the eigenfunctions of (46), (47). We restate this result as follows:

THEOREM 6. *Theorem 4 applies to the problem (27), (28) without any restrictions placed on the parameter μ.*

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