ON THE UNIFORMIZATION OF SETS IN TOPOLOGICAL SPACES

BY

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1. Introduction

Given a set E in the cartesian product $X \times Y$ of two spaces X and Y, a set U is said to uniformize E, if the projections $\pi_X E$, $\pi_X U$ of E and U through Y onto X coincide, and if, for each point x of $\pi_X E$, the set

$$\{(x) \times Y\} \cap U \tag{1}$$

of points of U lying above x consists of a single point. The existence of such a uniformizing set U follows immediately from the axiom of choice. But, if X and Y are topological spaces, and E is, in some sense, topologically respectable, for example if E belongs to some Borel class, it is natural to seek a uniformizing set U that is equally respectable, or at any rate not much worse. Usually there is no way of controlling the topological respectability of sets obtained by use of the axiom of choice, and quite different methods have to be used in obtaining topologically respectable uniformizing sets.

The earlier work of N. Lusin (see [19]) on problems of this nature was confined to the case when, for each x in $\pi_X E$, the set (1) of points of E lying above x is at most countable. The first general result seems to have been the result obtained independently by N. Lusin [20] and by W. Sierpinski [27] showing that, when X and Y are Euclidean spaces and E is a Borel set in $X \times Y$, the uniformizing set U can be taken to be the complement of an analytic set. Following work by N. Lusin and P. Novikoff [21] on the effective choice of a point from a complement of an analytic set defined by a given sieve, M. Kondô [15] showed that, in the Euclidean case, the complement of an analytic set could be uniformized by a complement of an analytic set. Since any Borel set in a Euclidean space is the complement of an analytic set, this provided a most satisfactory generalization of the result of Lusin and

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Sierpinski. Kondô's proof has been greatly simplified by Y. Sampei [26] and by Y. Suzuki [37].

A year before Kondô's work S. Braun [1], by a much simpler method, showed, in the Euclidean plane, that any closed set E can be uniformized by a $\mathcal{G}_\delta$ $-$set, and that any $\mathcal{F}_\sigma$-set can be uniformized by a $\mathcal{G}_{\delta\sigma}$-set.

It is easy to extend the results of Lusin and Sierpinski and of Kondô to the case when X and Y are complete separable metric spaces by the following mapping technique. If X is a complete separable metric space there will be a continuous function f that maps a relatively closed subset I_0 of the set I of irrational numbers between 0 and 1, regarded as a subset of the set R_1, of real numbers, one-one onto X. Similarly there will be a continuous function g that maps a relatively closed subset J_0 of the set J of irrational numbers between 0 and 1, regarded as a subset of the set S_1, of real numbers, one-one onto Y. Then the product map $f \times g$ maps $I_0 \times J_0$ continuously and one-one onto $X \times Y$. Hence the inverse map $f^{-1} \times g^{-1}$ maps Borel sets and complements of analytic sets in X and Y into Borel sets and complements of analytic sets in $I_0 \times J_0$, which remain Borel sets and complements of analytic sets in $R_1 \times S_1$. So the results of Lusin and Sierpinski or of Kondô can be applied in $R_1 \times S_1$; when the uniformizing set is intersected with $I_0 \times J_0$ and mapped by $f \times g$ back to $X \times Y$ there results in $X \times Y$ a uniformizing set that is the complement of an analytic set as required. These results will not hold, in general, when we merely take X and Y to be separable metric spaces; but it is easy to verify (by passing to the completions of the spaces) that they hold, if X and Y are separable metric spaces that are absolutely Borel relative to metric spaces. We recall that a space is absolutely Borel, if it is a Borel set in every metric space in which it can be embedded; a necessary and sufficient condition for this is that it be a Borel set in its completion under its metric.

Although other authors, see for example [21, 36] have obtained results that have some similarity with these results, the only further work, with which we are acquainted, that lies in the main line of development is a generalization of Sierpinski's result by K. Kunugui [17] and further developments due to M. Sion [28]. As Sion adopts a slightly different point of view, it will be convenient to defer consideration of his main work until we have stated some of our results.

Although Braun's work is based on a very simple idea we have not discovered any direct generalization in the literature. But a parenthetic remark in Corollory 4.2 to Theorem 4.1 of Sion [28], suggests that he had a generalization of Braun's work covering the case of a compact set in the cartesian product of two metric spaces. This suggestion is reinforced by the fact that such a result can be obtained by a minor development of his method. For sake of completeness we use his method to prove:

THEOREM 1. *Let X be any topological space. Let Y be any σ-compact metric space. Let $\mathcal{D}$ be the family of finite unions of differences of closed sets in $X \times Y$. Then a closed set in $X \times Y$ can be uniformized by a $\mathcal{D}_\delta$-set, and a $\mathcal{F}_\sigma$-set in $X \times Y$ can be uniformized by a $\mathcal{D}_{\delta\sigma}$-set.*

Before proceeding further we need to introduce generalized analytic sets. The natural generalization of analytic sets to general Hausdorff spaces seems to have been first given by V. E. Šnejder [33, 34, 35]; this was rediscovered and a considerable theory developed by G. Choquet [3, 4, 5]. It was further developed by M. Sion [28, 29, 30, 31, 32] (in part independently of Choquet) and by Z. Frolík [7, 8, 9, 10, 11, 12, 13]. Further results are due to C. A. Rogers [23, 24], J. D. Knowles and C. A. Rogers [14] and C. A. Rogers and R. C. Willmott [25]. For useful summaries see Z. Frolík [9] and D. W. Bressler and M. Sion [2]. We will use the following terminology in the introduction; further concepts will be necessary later.

$\mathbf{I}$ will denote the space of all infinite sequences or vectors

$$\mathbf{i} = i_1, i_2, \ldots$$

of positive integers with the metric

$$\varrho(\mathbf{i}, \mathbf{j}) = \begin{cases} 0, & \text{if} \quad \mathbf{i} = \mathbf{j}, \\ 2^{-k}, & \text{if} \quad i_\nu = j_\nu \ \text{ for } \ \nu < k, \, i_k \neq j_k. \end{cases}$$

If n is a positive integer and $\mathbf{i} \in \mathbf{I}$ the symbol $\mathbf{i}|n$ will denote the finite sequence $i_1, i_2, \ldots, i_n$ of the first n components of the vector $\mathbf{i}$. We will use $\mathbf{I}_{\mathbf{i}|n}$ or sometimes $\mathbf{I}(\mathbf{i}|n)$ to denote the set of $\mathbf{j}$ in $\mathbf{I}$ with $\mathbf{j}|n = \mathbf{i}|n$. These sets $\mathbf{I}(\mathbf{i}|n)$ are called Baire intervals.

A function K from $\mathbf{I}$ to the space $\mathcal{K}$ of compact subsets of a Hausdorff space X will be said to be semi-continuous, if, given any $\mathbf{i}_0$ in $\mathbf{I}$ and any open set G in X with $K(\mathbf{i}_0) \subset G$, there is a positive integer $n = n(\mathbf{i}_0, G)$ such that

$$K(\mathbf{I}_{\mathbf{i}_0|n}) \subset G,$$

i.e. such that $K(\mathbf{i}) \subset G$ for $\mathbf{i}$ in $\mathbf{I}$ sufficiently close to $\mathbf{i}_0$.

A set A in a Hausdorff space X will be said to be analytic if it is of the form $A = K(\mathbf{I})$ where K is a semi-continuous function from $\mathbf{I}$ to $\mathcal{K}$. A set B in a Hausdorff space will be said to be a descriptive Borel set if it is of the form $B = K(\mathbf{I})$, where K is a semi-continuous function from $\mathbf{I}$ to $\mathcal{K}$ with the property

$$K(\mathbf{i}) \cap K(\mathbf{j}) = \emptyset$$

whenever $\mathbf{i} \neq \mathbf{j}$ in $\mathbf{I}$. A set A will be called a Souslin set if it is of the form

$$A = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

where each set $F(\mathbf{i}|n)$ is closed. A set B will be said to have a disjoint Souslin representation if it has the form

$$B = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

where each set $F(\mathbf{i}|n)$ is closed and

$$F(\mathbf{i}) \cap F(\mathbf{j}) = \emptyset,$$

whenever $\mathbf{i} \neq \mathbf{j}$ in $\mathbf{I}$. Such sets were introduced and studied by K. Kunugui in [17] under the name 'ensemble d'unicité'.

We remark that is is possible to show that any analytic set is a Souslin set and that any descriptive Borel set is a Borel set and has a disjoint Souslin representation.

We can state our main result in terms of these concepts.

THEOREM 19. *Let X be a topological space. Suppose that Y is a Hausdorff space with a representation $Y = K(\mathbf{I})$ as a descriptive Borel set. Suppose that each open set in $X \times Y$ has a disjoint Souslin representation. Let E be the complement of a Souslin set in $X \times Y$. Then there is a set U that is the complement of a Souslin set in $X \times Y$ and that satisfies*

(a) $U \subset E$,

(b) $\pi_X U = \pi_X E$, *and*

(c) *for each x in $\pi_X E$ the set*

$$\pi_Y\{\pi_X^{-1}(x) \cap U\}$$

is compact, and is contained in some set $K(\mathbf{i})$ with $\mathbf{i} \in \mathbf{I}$.

In this result the set U will not in general be a set uniformizing E, but it may well be a suitable substitute for such a set, more especially as a given space Y can often be fragmented into a descriptive Borel representation $Y = K(\mathbf{I})$ where the sets $K(\mathbf{i})$, $\mathbf{i} \in \mathbf{I}$ are chosen to be small from some point of view. For an example of such a decomposition see Theorem 7 below. Of course, we obtain a genuine uniformization in the case when each set $K(\mathbf{i})$ contains at most a single point; this can, naturally always be arranged when Y is a complete separable metric space (see Theorem 18 for a slight refinement of the result in this case). The condition that each open set in $X \times Y$ has a disjoint Souslin representation arises naturally in our proof, but it is not a particularly convenient condition to verify. We remark that it will be satisfied if:

(a) each open set in X is an $\mathfrak{F}_\sigma$-set;

(b) each open set in Y is an $\mathfrak{F}_\sigma$-set; and

(c) either:

(c_1) X has a countable base for its topology, or

(c_2) Y has a countable base for its topology, or

(c_3) each open set in $X \times Y$ is the union of a countable sequence of rectangles $U \times V$ with U and V open.

The proof will depend on the mapping technique we have already explained. This enables us to reduce the general theorem to the special case when $Y = \mathbf{I}$. The proof in this special case is closely modelled on Sampei's simplified version of that of Kondô (see Theorem 17).

Till now we have been concerned with the problem of finding a 'respectable' uniformizing set U for a given set E in $X \times Y$. Associated with this set U is a function f defined on $\pi_X E$ mapping a point x of $\pi_X E$ onto the unique point y in Y lying in the set

$$\pi_Y\{\pi_X^{-1}(x) \cap U\}.$$

We say that such a function uniformizes the set E. In his work on uniformization M. Sion transfers his attention from the problem of the 'respectability' of U to that of f. He proves a theorem, which is in its general form very similar to Theorem 19 above. He allows X and Y to be arbitrary Hausdorff spaces, but insists on E being an analytic set. He proves the existence of a set U satisfying (a), (b) and the first part of (c) of Theorem 19, and with the property that, if F is any closed set of $X \times Y$, the set $\pi_X(U \cap F)$ belongs to the smallest system of sets $\mathfrak{M}$ that contains the analytic sets in X and is closed under the operations of countable union and set difference. By applying a second theorem he shows that, provided the space Y has certain properties, he is able to define a uniformizing function f on $\pi_X E$ with the property that $f^{-1}(V)$ lies in the above system $\mathfrak{M}$ for each open V in Y. The conditions on Y are that it should be a regular Hausdorff space, with a base whose power is at most that of the first uncountable cardinal, and that every family of open sets should have a countable subfamily with the same cover.

By using some of Sion's arguments in conjunction with a transfinite inductive application of Theorem 19 we obtain

THEOREM 20. *Let X be a topological space. Let Y be a Hausdorff space that is descriptive Borel, suppose that open sets in Y are $\mathfrak{F}_\sigma$-sets, that the topology has a base whose power is at most that of the first uncountable cardinal and that every open set is a countable union of base elements. Suppose that each open set in $X \times Y$ has a disjoint Souslin representation. Let E be the complement of a Souslin set in $X \times Y$. Then there is a uniformizing function f from $\pi_X E$ to Y, with the property*

$$(x \times f(x)) \in E$$

for each x in $\pi_X E$, and such that for each open set V in Y the set $f^{-1}[V]$ is the projection on X of the complement of a Souslin set in $X \times Y$.

Note that the example $V = Y$ and $f^{-1}[Y] = \pi_X E$ shows that it would be unreasonable to expect $f^{-1}[V]$ to satisfy any stronger condition.

In § 2 we give a summary of definitions and notational conventions. in § 3 we prove the generalization (Theorem 1) of Braun's theorems; the remaining sections do not depend on this section. In §§ 4–7 we develop some preliminary results. In particular in § 4 we study sets that have disjoint Souslin representations. In § 5 we study some procedures for the decomposition of spaces with the property that each open set has a disjoint Souslin representation. In § 6 we study a rather special mapping that can in certain circumstances be found mapping a space $X \times Y$ into the space $X \times \mathbf{I}$; the results will be used to reduce the general Theorem 19 to the more special Theorem 17. In § 7 we study sets defined in terms of sieves and make comparisons between different sieves; the results are vital for the sequel. In § 8 we digress to use results of § 7 to prove a generalization of Lusin's second separation theorem due to Kunugui [17] and to obtain a form of the first separation principle due to Kondô [16]. In § 9 we give the main proof, that of the 'reduced' Theorem 17. In §§ 10 and 11 we use the results of § 6 to establish the more general theorems 18 and 19. In § 12 we use Theorem 19 together with Sion's methods to obtain the final Theorem 20 above.

2. Definitions and notational conventions

The notations and definitions are all stated elsewhere in this paper at the places where they are first needed; they are repeated here for ease of reference.

The space of sequences of positive integers. We use lower case bold letters, typically the letter $\mathbf{i}$, to denote a corresponding sequence, such as $i_1, i_2, \ldots$ of positive integers. We use $\mathbf{I}$ to denote the space of all such sequences $\mathbf{i}$. When $\mathbf{i} \in \mathbf{I}$ and n is a positive integer we use $\mathbf{i}|n$ to denote the finite sequence $i_1, i_2, \ldots, i_n$ formed by truncating the sequencee $i_1, i_2, \ldots$ after n terms. For each positive integer n we use $\mathbf{I}_{\mathbf{i}|n}$ or $\mathbf{I}(\mathbf{i}|n)$ to denote the set of all $\mathbf{j}$ in $\mathbf{I}$ with $\mathbf{j}|n = \mathbf{i}|n$. When we have occasion to use the space $\mathbf{I}$ in two different contexts within a single argument we use $\mathbf{J}$ and $\mathbf{H}$ to denote copies of $\mathbf{I}$ and use similar notations $\mathbf{J}_{\mathbf{j}|n}$, $\mathbf{J}(\mathbf{j}|n)$, $\mathbf{H}_{\mathbf{h}|n}$ and $\mathbf{H}(\mathbf{h}|n)$.

Classes of sets. If X is a topological space we use $\mathcal{G}(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$ and $\mathcal{D}(X)$ to denote respectively its classes of open sets, closed sets, compact sets and set differences between open sets. When no ambiguity can arise we drop the '(X)' from this notation. If $\mathcal{H}$ is any class of sets we use $\mathcal{H}_\sigma$ and $\mathcal{H}_\delta$ to denote the class of all countable unions of sets of

$\mathcal{H}$ and the class of all countable intersections of sets of $\mathcal{H}$. We use Souslin-$\mathcal{H}$ to denote the class of all sets of the form

$$H(\mathbf{I}), \quad H(\mathbf{i}) = \bigcap_{n=1}^{\infty} H(\mathbf{i}|n),$$

where all sets $\mathbf{H}(\mathbf{i}|n)$ belong to $\mathcal{H}$ for $\mathbf{i} \in \mathbf{I}$ and n a positive integer.

We say that a set is a Souslin set in X if it belongs to the family Souslin-$\mathcal{F}(X)$ and that a set is bi-Souslin if both it and its complement are Souslin. We say that a set has a disjoint Souslin representation if it is of the form

$$F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n)$$

where all the sets $F(\mathbf{i}|n)$ are closed and $F(\mathbf{i}) \cap F(\mathbf{j}) = \emptyset$, whenever $\mathbf{i}$, $\mathbf{j}$ are distinct sequences in $\mathbf{I}$; such sets are called 'ensemble d'unicité' in Kunugui [17].

A function K from I to the space $\mathcal{K}(X)$ of compact subsets of a Hausdorff space X will be called semi-continuous, if, given any $\mathbf{i}_0$ in $\mathbf{I}$ and any open set G in X with $K(\mathbf{i}_0) \subset G$, there is a positive integer $n = n(\mathbf{i}_0, G)$ such that

$$K(\mathbf{I}_{\mathbf{i}_0|n}) \subset G.$$

A function F from $\mathbf{I}$ to the space $\mathcal{F}(X)$ of closed subsets of a topological space X will be called weakly semi-continuous if given any $\mathbf{i}_0$ in $\mathbf{I}$ and any point e of X not in $F(\mathbf{i}_0)$ there is an open set V containing e and a positive integer n with

$$V \cap F(\mathbf{I}_{\mathbf{i}_0|n}) = \emptyset.$$

Both these semi-continuities are discussed in G. Choquet's paper [6].

A set in a Hausdorff space X will be said to be analytic if it is of the form $K(\mathbf{I})$ where K is a semi-continuous function from $\mathbf{I}$ to $\mathcal{K}(X)$. A set in a Hausdorff space X will be said to be descriptive Borel if it is of this form $K(\mathbf{I})$ where K is a semi-continuous function from $\mathbf{I}$ to $\mathcal{K}(X)$ that carries distinct elements of $\mathbf{I}$ into disjoint compact sets of X.

When we have a disjoint Souslin representation $F(\mathbf{I})$ or a descriptive Borel set $K(\mathbf{I})$ the disjoint sets $\{F(\mathbf{i})\}_{\mathbf{i}\in\mathbf{I}}$ or $\{K(\mathbf{i})\}_{\mathbf{i}\in\mathbf{I}}$ will be called the fragments of the representations.

Cartesian product spaces. When we study a cartesian product $X \times Y$ of two spaces we use π_X, π_Y to denote the projection operators onto X and Y respectively, so that

$$\pi_X(x \times y) = x, \quad \pi_Y(x \times y) = y,$$

for $x \in X$ and $y \in Y$. If C is any set in $X \times Y$ we use $C^{(x)}$, for $x \in X$, to denote the set of y in Y with $x \times y \in C$; and we use $C^{[y]}$, for $y \in Y$, to denote the set of x in X with $x \times y \in C$. We also define the cylinder on a set E in $X \times Y$ to be the set

$$(\pi_X E) \times Y$$

and use cy E to denote this cylinder.

A set U will be said to uniformize a set E in $X \times Y$, if

$$U \subset E,$$
$$\pi_X U = \pi_X E,$$

and $U^{(x)}$ contains a single point for each x in $\pi_X E$. A function f will be said to uniformize a set E in $X \times Y$, if f is defined on $\pi_X E$ and maps $\pi_X E$ into Y so that

$$x \times f(x) \in E,$$

for all x in $\pi_X E$.

Sieves. Let X be any space and let Y be any subset (not necessarily proper) of the space of real numbers. Any set in $X \times Y$ is called a sieve. The set sifted by a sieve C in $X \times Y$ is the set E of those points x in X for which $C^{(x)}$ (which is a set of real numbers) contains an infinite strictly decreasing sequence. The complementary set $\tilde{E} = X \setminus E$ will be called the complementary set determined by the sieve.

Ordinal functions τ and σ. If R is any set of real numbers we associate two ordinals τR and σR with R, taking

$$\tau R = \sigma R = \Omega,$$

where Ω denotes the first ordinal with uncountable cardinal, when R contains an infinite descending sequence, and taking $\sigma R = \tau T + 1$ and τR to be the ordinal similar to R, when R is well-ordered.

3. Braun's uniformization theorems

Our aim in this section is to prove Theorem 1. We first introduce some notation and prove a lemma.

We will call a set Z in $X \times Y$ a cylinder parallel to Y if it is of the form $Z = P \times Y$ for some subset P of X. If E is any set in $X \times Y$, we use cy E to denote the cylinder $(\pi_X E) \times Y$.

LEMMA 1. *Let Y^* be a compact subset of a Hausdorff space Y. Let J be a set of the form $F \setminus Z$ in $X \times Y$ with F closed and Z a closed cylinder parallel to Y. Then the sets*

$$J_0 = J \cap \{X \times Y^*\},$$
$$J_1 = J \setminus \mathrm{cy}\{J_0\},$$

are both of the same form as J *and*

$$(\pi_X J_0) \cap (\pi_X J_1) = \varnothing,$$

$$(\pi_X J_0) \cup (\pi_X J_1) = \pi_X J.$$

Proof. Let $J = F \backslash Z$ with F closed and Z a closed cylinder parallel to Y. Then

$$J_0 = \{F \backslash Z\} \cap \{X \times Y^*\} = \{F \cap (X \times Y^*)\} \backslash Z$$

is of the required form, as $F \cap (X \times Y^*)$ is closed. As Y is compact, it follows easily that $\pi_X\{F \cap (X \times Y^*)\}$ is a closed set in X. Hence the set

$$Z_1 = \text{cy}\ \{F \cap (X \times Y^*)\} = [\pi_X\{F \cap (X \times Y^*)\}] \times Y$$

is a closed cylinder parallel to Y. Further

$$\text{cy}\ J_0 = \text{cy}\ [\{F \cap (X \times Y^*)\} \backslash Z] = Z_1 \backslash Z.$$

So
$$J_1 = J \backslash \text{cy}\ J_0 = \{F \backslash Z\} \backslash \{Z_1 \backslash Z\} = F \backslash \{Z \cup Z_1\},$$

and J_1 has the required form.

The formulae for the intersection and union of the projections of J_0 and J_1 follow immediately from the facts that J_0 is a subset of J and that $J_1 = J \backslash \text{cy}\ J_0$. This proves the lemma.

Proof of Theorem 1. Let E be a closed set in $X \times Y$. As Y is σ-compact and metric, we can choose a sequence $Y_1, Y_2, \ldots$ of compact sub-sets of Y with the properties:

(a) the diameter of Y_i tends to 0 as i tends to infinity;

(b) each point of Y belongs to infinitely many sets of the sequence.

We define sets $D_0, D_1, D_2, \ldots$ inductively by taking $D_0 = E$,

$$D_{n+1} = [D_n \cap (X \times Y_{n+1})] \cup [D_n \backslash \text{cy}\ \{D_n \cap (X \times Y_{n+1})\}] \tag{2}$$

for $n = 0, 1, 2, \ldots$. It follows inductively, by use of the lemma that D_n is the union of 2^n sets, each of the form $F \backslash Z$ with F closed and Z a closed cylinder parallel to Y, having disjoint projections with union $\pi_X E$.

Write
$$U = \bigcap_{n=0}^{\infty} D_n.$$

Then each set D_n belongs to $\mathcal{D}$ and U is a $\mathcal{D}_\delta$set. As

$$\pi_X U \subset \pi_X D_0 = \pi_X E,$$

it will suffice to show that, given any x in $\pi_X E$, there is a unique point y_x with $x \times y_x \in U$.

If x is any point of X and A is any set of $X \times Y$ we use $A^{(x)}$ to denote the set of points y of Y with $x \times y \in A$. So, if x is given in $\pi_X E$, the set $D_0^{(x)} = E_0^{(x)}$ is closed and non-empty. Our aim is to show that the sets $D_n^{(x)}$, $n = 0, 1, 2, \ldots$, are closed, decreasing, non-empty, compact for n sufficiently large and with diameter tending to zero. This will ensure that the set

$$U^{(x)} = \bigcap_{n=0}^{\infty} D_n^{(x)}$$

consists of a single point, as required.

It follows from the formula (2) that

$$D_{n+1}^{(x)} = D_n^{(x)} \cap Y_{n+1}, \quad \text{if} \quad D_n^{(x)} \cap Y_{n+1} \neq \emptyset,$$

$$D_{n+1}^{(x)} = D_n^{(x)}, \qquad\qquad \text{if} \quad D_n^{(x)} \cap Y_{n+1} = \emptyset,$$

for $n = 0, 1, 2, \ldots$. It follows immediately that the sets $D_n^{(x)}$, $n = 0, 1, 2, \ldots$, are closed, decreasing and non-empty. As the sequence $Y_1, Y_2, \ldots$ covers Y, it follows that for some first integer $n(x)$, the set $D_{n(x)}^{(x)} \cap Y_{n(x)+1}$ is non-empty. This implies that $D_n^{(x)}$ is compact for $n \geqslant n(x) + 1$. As each point of Y lies in infinitely many sets of the sequence $Y_1, Y_2, \ldots$, and as the diameters of these sets tend to zero, it follows that the diameter of $D_n^{(x)}$ also tends to zero. This completes the proof of the case when E is closed.

Now consider the case when E is an $\mathcal{F}_\sigma$-set. As Y is σ-compact we can express E in the form $E = \bigcup_{n=1}^{\infty} E_n$, where each set E_n is closed and each set $\pi_Y E_n$ is a subset of a compact set in Y. Then each set $\pi_X E_n$ is closed (being effectively the projection of a closed set through a compact space). Let U_n be a $\mathcal{D}_\delta$-set uniformizing E_n, for $n = 1, 2, \ldots$.

Let
$$U_n = \bigcap_{m=1}^{\infty} D_{nm}$$

where each set D_{nm} belongs to $\mathcal{D}$. It is clear that the set

$$U = \bigcup_{n=1}^{\infty} \{U_n \backslash \mathrm{cy} \, (\bigcup_{\nu<n} U_\nu)\}$$

uniformizes E. But
$$U_n \backslash \mathrm{cy} \, (\bigcup_{\nu<n} U_\nu) = \bigcap_{m=1}^{\infty} \{D_{nm} \backslash \mathrm{cy} \, (\bigcup_{\nu<n} E_\nu)\}.$$

As $\pi_x E_\nu$ is closed for $\nu < n$, the set

$$D_{nm} \backslash \text{cy} \left(\bigcup_{\nu<n} E_\nu\right)$$

belongs to $\mathcal{D}$. Hence U is a $\mathcal{D}_{\delta\sigma}$-set as required.

COROLLARY. *If each open set in X is a $\mathcal{F}_\sigma$-set, the uniformizing set for a closed set can be taken to be a $\mathcal{G}_\delta$-set and that for a $\mathcal{F}_\sigma$-set can be taken to be a $\mathcal{G}_{\delta\sigma}$-set.*

Proof. In this case each open set in $X \times Y$ is a countable union of open rectangles $U \times V$. Then, as each open set U in X and V in Y is an $\mathcal{F}_\sigma$-set, each open set in $X \times Y$ is an $\mathcal{F}_\sigma$-set. So each closed set and so each set of $\mathcal{D}$ is a $\mathcal{G}_\delta$-set in $X \times Y$. The result follows.

4. Sets with disjoint Souslin representations

In this section we develop some of the properties of this class of sets, proving a little more than will be essential in the sequel.

We first prove a result, noted by Frolík [13], showing that the Souslin sets can be characterised as images of $\mathbf{I}$ under certain mappings. We say that a function F from $\mathbf{I}$ to the space $\mathcal{F}$ of closed subsets of a space X is weakly semi-continuous if given any $\mathbf{i}_0$ in $\mathbf{I}$ and any point e of X not in $F(\mathbf{i}_0)$ there is an open set V containing e and an integer n so large that $V \cap F(\mathbf{I}_{\mathbf{i}_0|n}) = \emptyset$. In terms of this definition we obtain Frolík's result:

THEOREM 2. *A set A is a Souslin set if and only if $A = F(\mathbf{I})$ for some weakly semi-continuous map F from $\mathbf{I}$ to $\mathcal{F}$. A set B has a disjoint Souslin representation if and only if $B = F(\mathbf{I})$ for some weakly semi-continuous map F from $\mathbf{I}$ to $\mathcal{F}$ that satisfies the condition*

$$F(\mathbf{i}) \cap F(\mathbf{j}) = \emptyset$$

whenever $\mathbf{i} \neq \mathbf{j}$ and $\mathbf{i}$, $\mathbf{j}$ are in $\mathbf{I}$.

Proof. Suppose A is a Souslin set. Then

$$A = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

with all the sets $F(\mathbf{i}|n)$ closed. Now $F(\mathbf{i})$ is closed for each $\mathbf{i}$ in $\mathbf{I}$. So we need to prove that F is weakly semi-continuous. Given $\mathbf{i}_0$ in $\mathbf{I}$ and e not in

$$F(\mathbf{i}_0) = \bigcap_{n=1}^{\infty} F(\mathbf{i}_0|n)$$

we can choose n_0 with
$$e \notin F(\mathbf{i}_0 | n_0);$$
and e belongs to the open set $V = X \backslash F(\mathbf{i}_0 | n_0)$. Then for all $\mathbf{i}$ with $\mathbf{i}|n_0 = \mathbf{i}_0|n_0$ we have
$$F(\mathbf{i}) \subset F(\mathbf{i}|n_0) = F(\mathbf{i}_0|n_0) = X \backslash V$$
and $V \cap F(\mathbf{i}) = \emptyset$. Thus F is weakly semi-continuous.

Now suppose that $A = F^*(\mathbf{I})$ where F^* is a weakly semi-continuous map from $\mathbf{I}$ to $\mathcal{F}$. We define the set $F(\mathbf{i}|n)$ to be the closure
$$\text{cl } F^*(\mathbf{I}_{\mathbf{i}|n}),$$
for each $\mathbf{i} \in \mathbf{I}$ and each n. To prove that A is Souslin it suffices to prove that
$$F^*(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n)$$
for each $\mathbf{i}$ in $\mathbf{I}$. Clearly
$$F^*(\mathbf{i}) \subset \bigcap_{n=1}^{\infty} F^*(\mathbf{I}_{\mathbf{i}|n}) \subset \bigcap_{n=1}^{\infty} \text{cl } F^*(\mathbf{I}_{\mathbf{i}|n}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n).$$
Thus it suffices to prove that
$$\bigcap_{n=1}^{\infty} F(\mathbf{i}|n) \subset F^*(\mathbf{i}).$$
Suppose $e \notin F^*(\mathbf{i})$. Then we can choose an open set V containing e and an integer n_0 so that
$$V \cap F^*(\mathbf{I}_{\mathbf{i}|n_0}) = \emptyset.$$
Then
$$F^*(\mathbf{I}_{\mathbf{i}|n_0}) \subset X \backslash V.$$
As $X \backslash V$ is closed it follows that
$$F(\mathbf{i}|n_0) \subset X \backslash V$$
so that
$$\bigcap_{n=1}^{\infty} F(\mathbf{i}|n) \subset X \backslash V.$$
Hence
$$e \notin \bigcap_{n=1}^{\infty} F(\mathbf{i}|n).$$
This proves the required result
$$F^*(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n).$$

The corresponding result for a set B with a disjoint Souslin representation follows in the same way.

COROLLARY. *The relationship between the Souslin representation* $A = F(\mathbf{I})$, $F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n)$ *and the representation* $A = F^*(\mathbf{I})$ *in terms of a weakly semi-continuous function* F^* *is provided by the formulae*

$$F^*(\mathbf{i}) = F(\mathbf{i}),$$

$$F(\mathbf{i}|n) = \text{cl } F^*(\mathbf{I}_{\mathbf{i}|n}).$$

Our next result shows that the intersection of a countable sequence of sets with disjoint Souslin representations is again a set with a disjoint Souslin representation. To state the result in a refined form it is convenient to introduce a further definition. If the set B has a disjoint Souslin representation

$$B = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

with each $F(\mathbf{i}|n)$ closed, we will call the sets $F(\mathbf{i})$ the fragments of the representation.

THEOREM 3. *Let* $B_1, B_2, \ldots$ *be sets having Souslin representations*

$$B_r = F^{(r)}(\mathbf{I}), \quad F^{(r)}(\mathbf{i}) = \bigcap_{n=1}^{\infty} F^{(r)}(\mathbf{i}|n), \quad r = 1, 2, \ldots, \tag{3}$$

the sets $F^{(r)}(\mathbf{i}|n)$ *being closed. Then the intersection* $B = \bigcap_{r=1}^{\infty} B_r$ *has a Souslin representation*

$$B = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n), \tag{4}$$

each $F(\mathbf{i}|n)$ *being closed, with the property that, for each* $r \geq 1$, *each set* $F(\mathbf{i})$, $\mathbf{i} \in \mathbf{I}$ *is contained in some set* $F^{(r)}(\mathbf{j})$, $\mathbf{j} \in \mathbf{I}$. *Further if the representation (3) is disjoint, the representation (4) will be disjoint and, for each* $r \geq 1$, *each non-empty fragment* $F(\mathbf{i})$ *meets just one fragment of the representation (3).*

Proof. Let $\mathbf{I}^{(r)}$ denote a copy of $\mathbf{I}$ for $r = 1, 2, \ldots$. Let $\boldsymbol{\varphi}$ be one of the standard homeomorphisms (see for example, [19] or [18] mapping $\mathbf{I}$ onto the cartesian product

$$\mathop{\times}_{r=1}^{\infty} \mathbf{I}^{(r)}.$$

We may write

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{(1)}, \boldsymbol{\varphi}^{(2)}, \ldots,$$

where $\boldsymbol{\varphi}^{(r)}$ maps $\mathbf{I}$ onto $\mathbf{I}^{(r)}$ for $r = 1, 2, \ldots$. Define a function F from $\mathbf{I}$ to $\mathfrak{F}$ by taking

$$F(\mathbf{i}) = \bigcap_{r=1}^{\infty} F^{(r)}(\boldsymbol{\varphi}^{(r)}(\mathbf{i})).$$

Then, for each r,
$$F(\mathbf{I}) \subset F^{(r)}(\mathbf{I}) = B_r,$$
so that
$$F(\mathbf{I}) \subset B.$$
Conversely if $b \in B$, then $b \in B_r$, $r = 1, 2, \ldots$, and so, for some sequence $\mathbf{j}^{(1)}, \mathbf{j}^{(2)}, \ldots$,
$$b \in F^{(r)}(\mathbf{j}^{(r)}).$$
Now we can choose $\mathbf{i} \in \mathbf{I}$ with
$$\boldsymbol{\varphi}^{(r)}(\mathbf{i}) = \mathbf{j}^{(r)}, \quad r = 1, 2, \ldots.$$
Hence
$$b \in \bigcap_{r=1}^{\infty} F^{(r)}(\mathbf{j}^{(r)}) = \bigcap_{r=1}^{\infty} F^{(r)}(\boldsymbol{\varphi}^{(r)}(\mathbf{i})) = F(\mathbf{i}) \subset F(\mathbf{I}).$$
Thus $B = F(\mathbf{I})$.

It is now clear, by virtue of Theorem 2, that it will suffice to prove that the function F is weakly semi-continuous. By virtue of Theorem 2, we may suppose that the functions $F^{(r)}$ are weakly semicontinuous for $r = 1, 2, \ldots$. Let $\mathbf{i}_0$ be given in $\mathbf{I}$ and suppose that e is a point not in
$$F(\mathbf{i}_0) = \bigcap_{r=1}^{\infty} F^{(r)}(\boldsymbol{\varphi}^{(r)}(\mathbf{i}_0)).$$
We can consequently choose an integer r_0 so that e is not in
$$F^{(r_0)}(\boldsymbol{\varphi}^{(r_0)}(\mathbf{i}_0)).$$
By the weak semi-continuity, we can choose an open set V and an integer m_0 such that $e \in V$ and
$$V \cap F^{(r_0)}(\mathbf{i}^{(r_0)}) = \varnothing$$
for all $\mathbf{i}^{(r_0)}$ in $\mathbf{I}^{(r_0)}$ with
$$\mathbf{i}^{(r_0)} \mid m_0 = \boldsymbol{\varphi}^{(r_0)}(\mathbf{i}_0) \mid m_0.$$
Then we can choose n_0 so large that
$$\boldsymbol{\varphi}^{(r_0)}(\mathbf{i}) \mid m_0 = \boldsymbol{\varphi}^{(r_0)}(\mathbf{i}_0) \mid m_0,$$
for all $\mathbf{i}$ with $\mathbf{i} \mid n_0 = \mathbf{i}_0 \mid n_0$. Then for $\mathbf{i}$ in $\mathbf{I}$ with $\mathbf{i} \mid n_0 = \mathbf{i}_0 \mid n_0$ we have
$$V \cap F(\mathbf{i}) \subset V \cap F^{(r_0)}(\boldsymbol{\varphi}^{(r_0)}(\mathbf{i})) = \varnothing,$$
as
$$\boldsymbol{\varphi}^{(r_0)}(\mathbf{i}) \mid m_0 = \boldsymbol{\varphi}^{(r_0)}(\mathbf{i}_0) \mid m_0 = \mathbf{i}^{(r_0)} \mid m_0.$$
Thus F is weakly semi-continuous.

When the representation (3) is disjoint the representation
$$B = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{r=1}^{\infty} F^{(r)}(\boldsymbol{\varphi}^{(r)}(\mathbf{i}))$$

is disjoint (but is not a Souslin representation). It follows that the Souslin representation provided by Theorem 2 is a disjoint Souslin representation. It is now clear that the fragments satisfy our requirements.

This proof is only a minor modification of a standard proof that the intersection of a countable sequence of analytic sets is analytic.

COROLLARY. *Given* $\mathbf{i}^*$ *in* $\mathbf{I}$ *and an integer* $r \geq 1$, *there is a* $\mathbf{j}^*$ *in* $\mathbf{I}$ *with the property that for each positive n there is an integer m with*

$$F(\mathbf{I}_{\mathbf{i}^*|m}) \subset F^{(r)}(\mathbf{I}_{\mathbf{j}^*|n}).$$

Proof. Take

$$\mathbf{j}^* = \boldsymbol{\varphi}^{(r)}(\mathbf{i}^*).$$

Then, for any integer $n \geq 1$, we can choose m so large that

$$\boldsymbol{\varphi}^{(r)}(\mathbf{i})\,|\,n = \mathbf{j}^*\,|\,n$$

for all $\mathbf{i}$ in $\mathbf{I}_{\mathbf{i}^*|m}$. Then we have

$$F(\mathbf{I}_{\mathbf{i}^*|m}) = \bigcup_{\mathbf{i}\in\mathbf{I}(\mathbf{i}^*|m)} F(\mathbf{i}) = \bigcup_{\mathbf{i}\in\mathbf{I}(\mathbf{i}^*|m)} \bigcap_{\varrho=1}^{\infty} F^{(\varrho)}(\boldsymbol{\varphi}^{(\varrho)}(\mathbf{i})) \subset F^{(r)}(\mathbf{I}_{\mathbf{j}^*|n}),$$

as required.

THEOREM 4. *Let B be a descriptive Borel set of the form* $B = K(\mathbf{I})$ *where K is a semi-continuous function from* $\mathbf{I}$ *to* $\mathcal{K}$ *and the sets* $K(\mathbf{i})$, $\mathbf{i} \in \mathbf{I}$ *are disjoint. Let E have a disjoint Souslin representation*

$$E = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}\,|\,n),$$

the sets $F(\mathbf{i}\,|\,n)$ *being closed. Then* $B \cap E$ *has a descriptive Borel representation* $B \cap E = L(\mathbf{I})$, L *being a semi-continuous function from* $\mathbf{I}$ *to* $\mathcal{K}$, *the sets* $L(\mathbf{i})$, $\mathbf{i} \in \mathbf{I}$, *being disjoint and each such set being of the form* $K(\mathbf{j}) \cap F(\mathbf{h})$ *for some* $\mathbf{j}$, $\mathbf{h}$ *in* $\mathbf{I}$.

Proof. Let $\mathbf{J}$ and $\mathbf{H}$ be two copies of $\mathbf{I}$. Let $\boldsymbol{\varphi}$ be a homeomorphism mapping $\mathbf{I}$ onto the cartesian product $\mathbf{J} \times \mathbf{H}$. Write

$$\boldsymbol{\varphi}(\mathbf{i}) = \mathbf{j}(\mathbf{i}) \times \mathbf{h}(\mathbf{i}),$$

with $\mathbf{j}(\mathbf{i}) \in \mathbf{J}$, $\mathbf{h}(\mathbf{i}) \in \mathbf{H}$. Define

$$L(\mathbf{i}) = K(\mathbf{j}(\mathbf{i})) \cap F(\mathbf{h}(\mathbf{i})).$$

As in the proof of Theorem 3, it is clear that $B \cap E = L(\mathbf{I})$ and that L maps $\mathbf{I}$ into $\mathcal{K}$. So it remains to prove that L is semi-continuous.

Let $\mathbf{i}_0$ be given in $\mathbf{I}$ and suppose that G is an open set with

$$L(\mathbf{i}_0) \subset G.$$

Then
$$K(\mathbf{j}(\mathbf{i}_0)) \cap F(\mathbf{h}(\mathbf{i}_0)) \subset G.$$

So
$$\{K(\mathbf{j}(\mathbf{i}_0)) \backslash G\} \cap F(\mathbf{h}(\mathbf{i}_0)) = \varnothing.$$

By the weak semi-continuity of F, for each point x of $K(\mathbf{j}(\mathbf{i}_0)) \backslash G$ we can choose an open set $V(x)$ containing x and an integer $n(x)$ such that

$$V(x) \cap F(\mathbf{h}) = \varnothing$$

for all $\mathbf{h}$ with $\mathbf{h} | n(x) = \mathbf{h}(\mathbf{i}_0) | n(x)$.

As $K(\mathbf{j}(\mathbf{i}_0)) \backslash G$ is compact we can choose $x_1, x_2, \ldots, x_r$ so that

$$K(\mathbf{j}(\mathbf{i}_0)) \backslash G \subset \bigcup_{\varrho=1}^{r} V(x_\varrho). \tag{5}$$

We can then choose m so large that

$$\mathbf{h}(\mathbf{i}) | n(x_\varrho) = \mathbf{h}(\mathbf{i}_0) | n(x_\varrho), \quad \varrho = 1, 2, \ldots, r,$$

for all $\mathbf{i}$ with $\mathbf{i} | m = \mathbf{i}_0 | m$. This ensures that

$$\bigcup_{\varrho=1}^{r} V(x_\varrho) \cap F(\mathbf{h}(\mathbf{i})) = \varnothing,$$

for all $\mathbf{i}$ in $\mathbf{I}$ with $\mathbf{i} | m = \mathbf{i}_0 | m$. By (5) and the semicontinuity of K we can choose m' so that

$$K(\mathbf{j}(\mathbf{i})) \subset G \cup \{\bigcup_{\varrho=1}^{r} V(x_\varrho)\}$$

for all $\mathbf{i}$ in $\mathbf{I}$ with $\mathbf{i} | m' = \mathbf{i}_0 | m'$. Taking $m'' = \max\{m, m'\}$ we have

$$L(\mathbf{i}) = K(\mathbf{j}(\mathbf{i})) \cap F(\mathbf{h}(\mathbf{i})) \subset G,$$

for all $\mathbf{i}$ in $\mathbf{I}$ with $\mathbf{i} | m'' = \mathbf{i}_0 | m''$. This proves the semi-continuity as required.

COROLLARY. *Given* $\mathbf{i}^*$ *in* $\mathbf{I}$ *there are* $\mathbf{j}^*$ *in* $\mathbf{J}$ *and* $\mathbf{h}^*$ *in* $\mathbf{H}$ *with the property that for each* $n \geq 1$ *there is an* $m \geq 1$ *with*

$$L(\mathbf{I}_{\mathbf{i}^* | m}) \subset K(\mathbf{J}_{\mathbf{j}^* | n}) \cap F(\mathbf{H}_{\mathbf{h}^* | n}).$$

Proof. The corollary follows by the argument used to prove the Corollary to Theorem 3.

To justify one of the assertions of the Introduction we need

THEOREM 5. *If each open set in a space is an $\mathfrak{F}_\sigma$-set then each open set in the space has a disjoint Souslin representation.*

Proof. This theorem is merely a restatement of the result that is actually proved in the first two paragraphs of Lemma 2 of Rogers [24].

We conclude this section on sets with disjoint Souslin representations with the remark that, if each open set of X is a Souslin set, then each set A with a disjoint Souslin representation is bi-Souslin in the sense that both A and $X \setminus A$ are Souslin. This result, due to K. Kunugui [17], is established later as a Corollary to Theorem 16.

5. Fragmentation of a space

In this section we consider spaces with the property that each open set has a disjoint Souslin representation. We show that such a space has disjoint Souslin representations whose fragments can be made sufficiently 'small' to ensure that certain sets are unions of fragments.

THEOREM 6. *Suppose that each open set in a toplogical space X has a disjoint Souslin representation. Let $F_1, F_2, \ldots$ be a countable sequence of closed sets in X. Then X has a disjoint Souslin representation*

$$X = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i} \mid n),$$

with the sets $F(\mathbf{i} \mid n)$ all closed, such that each set F_r, $r = 1, 2, \ldots$, is the union of those fragments $F(\mathbf{i})$ that it meets.

Proof. For each r, the set $X \setminus F_r$ has a disjoint Souslin representation

$$X \setminus F_r = E_r(\mathbf{I}), \quad E_r(\mathbf{i}) = \bigcap_{n=1}^{\infty} E_r(\mathbf{i} \mid n),$$

with the sets $E_r(\mathbf{i} \mid n)$ closed. Define closed sets $F_r^*(\mathbf{i} \mid n)$ by taking

$$F_r^*(1) = F_r^*(1, 1) = F_r^*(1, 1, 1) = \ldots = F_r,$$

$$F_r^*(2, i_1, i_2, \ldots, i_n) = E_r(i_1, i_2, \ldots, i_n),$$

and

$$F_r^*(\mathbf{i} \mid n) = \emptyset$$

for all other finite vectors $\mathbf{i} \mid n$. Write

$$F_r^*(\mathbf{i}) = \bigcap_{n=1}^{\infty} F_r^*(\mathbf{i} \mid n).$$

It follows immediately that

$$X = F_r^*(\mathbf{I}), \quad F_r^*(\mathbf{i}) = \bigcap_{n=1}^{\infty} F_r^*(\mathbf{i}\,|\,n),$$

gives a disjoint Souslin representation of the space X. Further in this representation, depending on r, both F_r and $X \backslash F_r$ are unions of fragments.

The required result now follows immediately from Theorem 3 with $B_1 = B_2 = \ldots = X$.

COROLLARY. *For each* $\mathbf{i}$ *in* $\mathbf{I}$ *and each* $r \geqslant 1$ *there is an integer* n *with either*

$$F(\mathbf{I}_{\mathbf{i}\,|\,n}) \subset F_r$$

or

$$F(\mathbf{I}_{\mathbf{i}\,|\,n}) \subset X \backslash F_r.$$

Proof. This follows from the Corollary to Theorem 3 and the choice of the sets $F_r^*(\mathbf{i}\,|\,n)$ in the proof of Theorem 6.

Theorem 6 generalizes to yield

THEOREM 7. *Suppose that each open set in a topological space* X *has a disjoint Souslin representation. Let* $A_1, A_2, \ldots$ *be a countable sequence of sets that are either Souslin sets or complements of Souslin sets. Then* X *has a disjoint Souslin representation*

$$X = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}\,|\,n),$$

with the sets $F(\mathbf{i}\,|\,n)$ *all closed, such that each set* A_r *is the union of those of the fragments* $F(\mathbf{i})$ *that it meets.*

Proof. We first remark that given a disjoint Souslin representation of X, a set will be the union of those fragments that it meets if and only if its complement is the union of those fragments that it (the complement) meets. Hence we may suppose that each of the sets $A_1, A_2, \ldots$ is a Souslin set.

Suppose

$$A_r = F_r(\mathbf{I}), \quad F_r(\mathbf{i}) = \bigcap_{n=1}^{\infty} F_r(\mathbf{i}\,|\,n),$$

the sets $F_r(\mathbf{i}\,|\,n)$ being closed. Let $F_1, F_2, \ldots$ be an enumeration of this countable system of closed sets

$$F_r(\mathbf{i}\,|\,n), \quad r = 1, 2, \ldots, \; \mathbf{i} \in \mathbf{I}, \; n = 1, 2, \ldots. \tag{6}$$

Let

$$X = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}\,|\,n),$$

with $F(\mathbf{i}|n)$ closed, be the disjoint Souslin representation provided by Theorem 6. Then each set of the system (6) is the union of those fragments that it meets. Hence each set

$$F_r(\mathbf{i}), \quad \mathbf{i} \in \mathbf{I}, \ r = 1, 2, \ldots,$$

is the union of those fragments that it meets. Hence each set A_r is the union of those fragments that it meets.

COROLLARY. *In this construction each of the closed sets F involved in the Souslin representation of the Souslin sets of the sequence or in the Souslin representations of the complements of the sets of the sequence that are complements of Souslin sets (i.e. the sets $F_r(\mathbf{i}|n)$) have the property that for each $\mathbf{i}$ in $\mathbf{I}$ there is an $n \geqslant 1$ with either*

$$F(\mathbf{I}_{\mathbf{i}|n}) \subset F$$

or

$$F(\mathbf{I}_{\mathbf{i}|n}) \subset X \backslash F.$$

Proof. This follows by the corollary to Theorem 6.

6. A mapping from X×Y to X×I

In this section we discuss circumstances when a certain map can be set up from $X \times Y$ to $X \times \mathbf{I}$ with the property that it and its inverse take certain Souslin sets into Souslin sets. Clearly such a result is potentially useful in reducing Theorem 19 to Theorem 17, we discuss this in more detail in some remarks after we have stated

THEOREM 8. *Let Y be a descriptive Borel space of the form $Y = K(\mathbf{J})$ where K is semi-continuous from $\mathbf{J}$ to $\mathcal{K}(Y)$ and the sets $K(\mathbf{j})$, $\mathbf{j} \in \mathbf{J}$ are disjoint. Suppose that the space $X \times Y$ has a disjoint Souslin representation*

$$X \times Y = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

the sets $F(\mathbf{i}|n)$ being closed. Suppose that for each $\mathbf{i}$ in $\mathbf{I}$ there is a $\mathbf{j}$ in $\mathbf{J}$ such that for each integer $m \geqslant 1$ there is an integer n with

$$F(\mathbf{I}_{\mathbf{i}|n}) \subset \{X \times K(\mathbf{J}_{\mathbf{j}|m})\}.$$

Let the map ω: $X \times Y \to X \times \mathbf{I}$ be defined by

$$\omega(x \times y) = x \times \mathbf{i},$$

where $\mathbf{i} = \mathbf{i}(x \times y)$ is the unique $\mathbf{i}$ in $\mathbf{I}$ with $x \times y \in F(\mathbf{i})$.

If A is a closed set in $X \times Y$ that is the union of those fragments $F(\mathbf{i})$ that it meets, then ωA is closed in $X \times \mathbf{I}$.

If A is a Souslin set in $X \times Y$ with a representation

$$A = A(\mathbf{I}), \quad A(\mathbf{i}) = \bigcap_{n=1}^{\infty} A(\mathbf{i}\,|\,n),$$

the sets $A(\mathbf{i}\,|\,n)$ being such closed sets (that are the unions of those fragments $F(\mathbf{i})$ that they meet), then ωA is a Souslin set in $X \times \mathbf{I}$.

Further, if B in $X \times \mathbf{I}$ is Souslin, then $\omega^{-1}B$ is Souslin in $X \times Y$.

Remarks. Our plans for the reduction of Theorem 19 about $X \times Y$ to the more special Theorem 17 about $X \times \mathbf{I}$ should now be becoming clearer. We shall be able to apply Theorem 8 to the situation of Theorem 19 provided we can introduce an appropriate disjoint Souslin representation for $X \times Y$. To obtain such a representation we need to apply Theorem 7 and Theorem 4 and its Corollary. This explains why we need the condition on $X \times Y$ in Theorem 19 asserting that each open set has a disjoint Souslin representation.

The proof of this mapping theorem is based on a recent result [25] of ours on the projection of Souslin sets.

Proof. Let A be a closed set in $X \times Y$ that is the union of those fragments $F(\mathbf{i})$ that it meets. Consider any point $x \times \mathbf{i}^*$ of $X \times \mathbf{I}$ that is not in ωA. Then

$$A \cap (\{x\} \times Y) \cap F(\mathbf{i}^*) = \varnothing.$$

We can choose $\mathbf{j}^*$ in $\mathbf{J}$ with the property that, for each $m \geqslant 1$, there is an integer n with

$$F(\mathbf{I}_{\mathbf{i}^*|n}) \subset X \times K(\mathbf{J}_{\mathbf{j}^*|m}).$$

By the semi-continuity of K we have

$$K(\mathbf{j}^*) = \bigcap_{m=1}^{\infty} K(\mathbf{J}_{\mathbf{j}^*|m})$$

so that
$$F(\mathbf{i}^*) = \bigcap_{n=1}^{\infty} F(\mathbf{I}_{\mathbf{i}^*|n}) \subset \bigcap_{m=1}^{\infty} \{X \times K(\mathbf{J}_{\mathbf{j}^*|m})\} = X \times K(\mathbf{j}^*). \tag{8}$$

As
$$D = (\{x\} \times Y) \cap F(\mathbf{i}^*) = (\{x\} \times K(\mathbf{j}^*)) \cap F(\mathbf{i}^*)$$

is the intersection of a compact set and a closed set, it is compact. By (7) this compact set D is contained in the open set $X \backslash A$. By the definition of the product topology, the set D is covered by a union of open rectangles not meeting A. So D is covered by a finite system, say

$$U_\varrho \times V_\varrho, \quad \varrho = 1, 2, \ldots, r,$$

with U_ϱ open in X and V_ϱ open in Y for $\varrho=1, 2, ..., r$. If $D \neq \emptyset$, we have $r \geq 1$, and we may suppose that none of the rectangles forming the covering are redundant. Then, taking

$$U=\bigcap_{\varrho=1}^{r} U_\varrho, \quad V=\bigcup_{\varrho=1}^{r} V_\varrho,$$

we obtain open sets, U, V in X and Y satisfying the conditions

$$\begin{aligned} &x \in U, \\ &D \subset U \times V, \\ &(U \times V) \cap A = \emptyset. \end{aligned} \tag{9}$$

When $D=\emptyset$, we satisfy these three conditions trivially by taking

$$U=X, \quad V=\emptyset.$$

Now
$$\begin{aligned} (\{x\} \times K(\mathbf{j}^*)) \backslash (U \times V) &\subset (\{x\} \times K(\mathbf{j}^*)) \backslash D \\ &= (\{x\} \times K(\mathbf{j}^*)) \backslash [(\{x\} \times Y) \cap F(\mathbf{i}^*)] \\ &\subset (X \times Y) \backslash F(\mathbf{i}^*). \end{aligned}$$

So
$$E = (\{x\} \times K(\mathbf{j}^*)) \backslash (U \times V)$$

is a compact set that does not meet $F(\mathbf{i}^*)$. Hence, by the weak semicontinuity of F, we can for each e in E choose open sets $U(e)$ and $V(e)$ in X and Y and an integer $n(e)$ so that

$$\begin{aligned} &e \in U(e) \times V(e), \\ &\{U(e) \times V(e)\} \cap F(\mathbf{I}_{\mathbf{i}^* | n(e)}) = \emptyset. \end{aligned}$$

When the compact set E is non-empty we can reduce this cover of the separate points e of E to a non-redundant finite cover and as before we can construct open sets U^*, V^* in X, Y and choose an integer n^* to ensure that

$$\begin{aligned} &x \in U^*, \\ &E \subset U^* \times V^*, \\ &\{U^* \times V^*\} \cap F(\mathbf{I}_{\mathbf{i}^* | n^*}) = \emptyset. \end{aligned} \tag{10}$$

When E is empty we satisfy these conditions by taking

$$U^*=X, \quad V^*=\emptyset, \quad n^*=1.$$

We now have
$$\{x\} \times K(\mathbf{j}^*) \subset E \cup (U \times V) \subset (U \times V) \cup (U^* \times V^*).$$

Hence
$$K(\mathbf{j}^*) \subset V \cup V^*.$$

By the semi-continuity of K we can choose an integer M so that
$$K(\mathbf{J}_{\mathbf{j}^*|M}) \subset V \cup V^*.$$

Hence we can choose an integer $N \geqslant n^*$ so that
$$F(\mathbf{I}_{\mathbf{i}^*|N}) \subset X \times K(\mathbf{J}_{\mathbf{j}^*|M}) \subset X \times (V \cup V^*). \tag{11}$$

Now consider any point $p \times \mathbf{i}$ in the open set
$$(U \cap U^*) \times \mathbf{I}_{\mathbf{i}^*|N}$$
in $X \times I$ containing the original point $x \times \mathbf{i}^*$. We have
$$\begin{aligned} & A \cap (\{p\} \times Y) \cap F(\mathbf{i}) \\ & \quad \subset A \cap [(U \cap U^*) \times Y] \cap F(\mathbf{I}_{\mathbf{i}^*|N}) \\ & \quad \subset A \cap [(U \cap U^*) \times \{V \cup V^* \cup (Y \backslash (V \cup V^*))\}] \cap F(\mathbf{I}_{\mathbf{i}^*|N}) \\ & \quad \subset \{A \cap (U \times V)\} \cup \{(U^* \times V^*) \cap F(\mathbf{I}_{\mathbf{i}^*|n^*})\} \cup \{(X \times \{Y \backslash (V \cup V^*)\}) \cap F(\mathbf{I}_{\mathbf{i}^*|N})\} \\ & \quad = \emptyset, \end{aligned}$$
on using (9), (10) and (11). Thus the open set
$$(U \cap U^*) \times \mathbf{I}_{\mathbf{i}^*|N}$$
does not meet ωA. As $x \times \mathbf{i}^*$ was any point, not in ωA, this proves that ωA is closed.

We note, in particular, that $\omega(X \times Y)$ is a closed set in $X \times \mathbf{I}$. Now, as
$$\omega(F(\mathbf{i})) \subset X \times \{\mathbf{i}\},$$
the sets
$$\omega F(\mathbf{i}), \quad \mathbf{i} \in \mathbf{I},$$
are all disjoint and together form the closed set $\omega(X \times Y)$. So as long as we take unions or intersections of sets that are made up as unions of the fragments $F(\mathbf{i})$, $\mathbf{i} \in \mathbf{I}$ of $X \times Y$, the mapping ω will commute with the union and intersection operators. It follows that if ω is applied to any set A in $X \times Y$ that has a Souslin representation
$$A = A(\mathbf{I}), \quad A(\mathbf{i}) = \bigcap_{n=1}^{\infty} A(\mathbf{i}|n),$$
the sets $A(\mathbf{i}|n)$ being closed sets that are the unions of those fragments $F(\mathbf{i})$ that they meet, then ωA has the Souslin representation
$$\omega A = A^{\omega}(\mathbf{I}), \quad A^{\omega}(\mathbf{i}) = \bigcap_{n=1}^{\infty} \omega A(\mathbf{i}|n)$$
the sets $\omega A(\mathbf{i}|n)$ being closed in $X \times \mathbf{I}$.

Now suppose that B is a Souslin set in $X \times \mathbf{I}$. We prove that $\omega^{-1}B$ is a Souslin set in $X \times Y$. As $\omega(X \times Y)$ is closed, the set $B \cap w(X \times Y)$ is Souslin in $X \times \mathbf{I}$. Since

$$\omega^{-1}B = \omega^{-1}[B \cap \omega(X \times Y)],$$

it is clear that we may suppose that $B \subset \omega(X \times Y)$. Using Theorem 3 and its Corollary, we may suppose that B has a Souslin representation

$$B = B(\mathbf{H}), \quad B(\mathbf{h}) = \bigcap_{n=1}^{\infty} B(\mathbf{h} \mid n),$$

the sets $B(\mathbf{h} \mid n)$ being closed in $X \times \mathbf{I}$, with the property that, for each $\mathbf{h}^*$ in $\mathbf{H}$, there is an $\mathbf{i}(\mathbf{h}^*)$ in $\mathbf{I}$ with

$$B(\mathbf{h}^*) \subset X \times \{\mathbf{i}(\mathbf{h}^*)\},$$

and for each positive n there is an integer m with

$$B(\mathbf{H}_{\mathbf{h}^* \mid m}) \subset X \times \mathbf{I}_{\mathbf{i}(\mathbf{h}^*) \mid n}. \tag{12}$$

We define sets

$$A(\mathbf{h}) = \omega^{-1}B(\mathbf{h})$$

in $X \times Y$ for each $\mathbf{h}$ in $\mathbf{H}$. Since

$$B(\mathbf{h}) \subset X \times \{\mathbf{i}(\mathbf{h})\},$$

it follows that

$$A(\mathbf{h}) = \omega^{-1}B(\mathbf{h}) = F(\mathbf{i}(\mathbf{h})) \cap [\{\pi_X B(\mathbf{h})\} \times Y].$$

Thus $A(\mathbf{h})$ is closed for each $\mathbf{h}$ in $\mathbf{H}$.

Since

$$\omega^{-1}B = \bigcup_{\mathbf{h} \in \mathbf{H}} A(\mathbf{h}),$$

the required conclusion that $\omega^{-1}B$ is a Souslin set in $X \times Y$ will follow from Theorem 2, provided we prove that A is a weakly semi-continuous function.

Suppose $\mathbf{h}^* \in \mathbf{H}$ and $e \notin A(\mathbf{h}^*)$. If $e \notin F(\mathbf{i}(\mathbf{h}^*))$, by the weak semi-continuity of F, we can choose an open rectangle $U \times V$ in $X \times Y$ and an integer $n \geqslant 1$ with

$$e \in U \times Y,$$

$$(U \times V) \cap F(\mathbf{I}_{\mathbf{i}(\mathbf{h}^*) \mid n}) = \emptyset.$$

By (12), we can choose m so that

$$B(\mathbf{H}_{\mathbf{h}^* \mid m}) \subset X \times \mathbf{I}_{\mathbf{i}(\mathbf{h}^*) \mid n}.$$

Then
$$e \in U \times V$$
and
$$(U \times V) \cap A(\mathbf{H}_{\mathbf{h}^*|m}) \subset (U \times V) \cap \{F(\mathbf{I}_{\mathbf{i}(\mathbf{h}^*)|n}) \cap [\{\pi_X B(\mathbf{H}_{\mathbf{h}^*|m})\} \times Y]\} = \emptyset.$$

On the other hand, if $e \in F(\mathbf{i}(\mathbf{h}^*))$ then the point $(\pi_X e) \times \mathbf{i}(\mathbf{h}^*)$ in $X \times \mathbf{I}$ is not in $B(\mathbf{h}^*)$, as otherwise e would be in $A(\mathbf{h}^*)$. By the weak semi-continuity of B we can choose an open rectangle $U \times \mathbf{I}_{\mathbf{i}(\mathbf{h}^*)|n}$ and an $m \geqslant 1$, with

$$e \in U, \quad \mathbf{i}(\mathbf{h}^*) \in \mathbf{I}_{\mathbf{i}(\mathbf{h}^*)|n},$$
$$(U \times \mathbf{I}_{\mathbf{i}(\mathbf{h}^*)|n}) \cap B(\mathbf{H}_{\mathbf{h}^*|m}) = \emptyset. \tag{13}$$

Here it is clear that m may be chosen so large that

$$B(\mathbf{H}_{\mathbf{h}^*|m}) \subset X \times \mathbf{I}_{\mathbf{i}(\mathbf{h}^*)|n}. \tag{14}$$

Combining (13) and (14)
$$U \cap (\pi_X B(\mathbf{H}_{\mathbf{h}^*|m})) = \emptyset,$$
so that
$$e \in U \times Y$$
and
$$(U \times Y) \cap A(\mathbf{H}_{\mathbf{h}^*|m}) = \emptyset.$$

This shows that the criterion for weak semi-continuity is satisfied in each case. The result follows. This completes the proof.

7. Properties of sieves

In this section we develop those properties of sieves that will be essential for our uniformization theorems. Although the methods we use are very similar to the classical methods we give the proofs in some detail to make the section accessible to those not already familiar with the subject.

If X is a space, a sieve is a set in a space $X \times Y$ where Y is a space ordered by a relation '$<$'. The set sifted by the sieve C in $X \times Y$ is the set E of those points x in X for which the set $C^{(x)}$, of those points y in Y, with $x \times y \in C$, contains an infinite strictly decreasing sequence. The complementary set $\tilde{E}$ determined by C is the set of those points x for which the set $C^{(x)}$ in Y is well-ordered by $<$. We shall only consider such sieves when Y is taken to be either the real line R_1 or the set Q of rational numbers lying strictly between 0 and 1.

Although we shall not change the accepted terminology, we have not found the analogy with the sieves used by gardeners and cooks very helpful. It may help to think in terms of a game to be played by the points x of X. At each of a countable sequence of turns each player x is forced to choose a point y of his set $C^{(x)}$ that is strictly smaller than any of his previous choices or to loose if he can make no such choice. Then assuming that all players play to their best advantage, E is the set of winners and $\tilde{E}$ is the set of loosers.

THEOREM 9. *If A is a Souslin set in X there is an $\mathfrak{F}_\sigma$-set C in $X \times Q$ whose sifted set is A.*

Proof. We may suppose that

$$A = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}\,|\,n)$$

the sets $F(\mathbf{i}\,|\,n)$ being closed in X. We suppose as we may that each sequence $F(\mathbf{i}\,|\,n)$, $n = 1, 2, \ldots$, is monotonic decreasing. Define rationals $r(\mathbf{i}\,|\,n)$ in Q by taking

$$r(\mathbf{i}\,|\,n) = 1 - 2^{-i_1} - 2^{-i_1-i_2} - \ldots - 2^{-i_1-i_2-\cdots-i_n},$$

for all $\mathbf{i} \in \mathbf{I}$ and $n \geqslant 1$. Take

$$C = \bigcup_{\mathbf{i}|n} [F(\mathbf{i}\,|\,n) \times \{r(\mathbf{i}\,|\,n)\}].$$

Clearly C is an $\mathfrak{F}_\sigma$-set in $X \times Q$. It is easy to verify that A is the set sifted by C.

COROLLARY. *Let $\mathcal{H}$ be any class of sets that is closed under finite intersections. If A is a Souslin-$\mathcal{H}$ set in X there is a set C of the form*

$$C = \bigcup_{n=1}^{\infty} H_n \times \{q_n\},$$

with $H_n \in \mathcal{H}$ and $q_n \in Q$ for $n = 1, 2, \ldots$, whose sifted set is A.

Proof. This follows by the same method.

The next result is essentially due to Kunugui [17]; he took X to be a T_1-space and used his 'projection' theorem 4 in place of our more general result [25].

THEOREM 10. *If C is a Souslin set in $X \times R_1$ the set sifted by C is a Souslin set in X.*

Proof. As we can map R_1 by a continuous order preserving map into the open interval $(0, 1)$ it is clear that we may suppose that C is contained in the cylinder

$$X \times (0, 1).$$

Let

$$r(1),\ r(2), \ldots,$$

be an enumeration of the rational numbers strictly between 0 and 1. Define a system of half-open half-closed intervals $R(\mathbf{i}\,|\,n)$ by taking $R(\mathbf{i}\,|\,n)$ to be the set of all y with

$$r(i_1)r(i_2) \ldots r(i_n) \leqslant y < r(i_1)r(i_2) \ldots r(i_{n-1}),$$

with the natural convention

$$r(i_1)r(i_2) \ldots r(i_{n-1}) = 1 \quad \text{when} \quad n = 1.$$

Write
$$A(\mathbf{i}|n) = \pi_X[C \cap \{X \times R(\mathbf{i}|n)\}].$$

By [25] this set $A(\mathbf{i}|n)$ is a Souslin set in X. Hence

$$A(\mathbf{I}), \quad \text{where} \quad A(\mathbf{i}) = \bigcap_{n=1}^{\infty} A(\mathbf{i}|n),$$

is a Souslin–Souslin set and so is a Souslin set. Thus it remains to show that $A(\mathbf{I})$ is the set A sifted by C.

If $x \in A$ we can choose first an infinite decreasing sequence $y_1, y_2, \ldots$, in $C^{(x)}$ and then a sequence of positive integers $i_1, i_2, \ldots$, defining a vector $\mathbf{i}$ with

$$y_n \in R(\mathbf{i}|n), \quad n = 1, 2, \ldots. \tag{15}$$

This ensures that
$$x \in A(\mathbf{i}|n), \; n = 1, 2, \ldots, \tag{16}$$

so that $x \in A(\mathbf{i}) \subset A(\mathbf{I})$. On the other hand, if $x \in A(\mathbf{I})$ we can first choose $\mathbf{i}$ in $\mathbf{I}$ with $x \in A(\mathbf{i})$ so that (16) holds and then choose points $y_1, y_2, \ldots$ in $C^{(x)}$ so that (15) holds. Then $y_1, y_2, \ldots$ is strictly decreasing and $x \in A$. This proves that $A = A(\mathbf{I})$ as required.

Our next result is essentially the main lemma that Lusin [19] uses in the proof of his second separation principle. We follow the proof given by Kuratowski [18] rather than that of Lusin. We recall that two sets of real numbers are said to be similar if there is a one-one order-preserving map from one set onto the other.

THEOREM 11. *Let A and B be two sets in $X \times Q$. Suppose that the sets*

$$(X \times Q) \backslash A \quad \textit{and} \quad B$$

are Souslin sets in $X \times Q$. Then the set C of points x in X with

$$A^{(x)} \textit{ similar to a subset of } B^{(x)}$$

is a Souslin set in X.

Proof. Let $r(1), r(2), \ldots$ and $s(1), s(2), \ldots$ be enumerations of the points of Q. Let $\mathbf{J}$ be a copy of $\mathbf{I}$.

With each $\mathbf{i}$ in $\mathbf{I}$ we associate the sequence

$$R(\mathbf{i}): r(i_1), r(i_2), \ldots,$$

and with each $\mathbf{j}$ in $\mathbf{J}$ we associate the sequence

$$S(\mathbf{j}): s(j_1), s(j_2), \ldots.$$

We ask the nature of the set T in $\mathbf{I} \times \mathbf{J}$ of those pairs $\mathbf{i}$, $\mathbf{j}$ such that the sequences $R(\mathbf{i})$ and $S(\mathbf{j})$ are similar. Let T be the set of all pairs $\mathbf{i}$, $\mathbf{j}$ in $\mathbf{I} \times \mathbf{J}$ with the properties:

for all integers n, m with $n \neq m$ we have either

(a) $r(i_n) < r(i_m)$ and $s(j_n) < s(j_m)$, or

(b) $r(i_n) > r(i_m)$ and $s(j_n) > s(j_m)$.

Let L be the set of those sets of positive integers i, i^*, j, j^* with either

(a) $r(i) < r(i^*)$ and $s(j) < s(j^*)$, or

(b) $r(i) > r(i^*)$ and $s(j) > s(j^*)$.

For each set of positive integers i, i^*, j, j^* let $\mathbf{H}_{n,m}(i, i^*, j, j^*)$ denote the set of those points $\mathbf{i} \times \mathbf{j}$ of $\mathbf{I} \times \mathbf{J}$ with

$$i_n = i, \quad i_m = i^*, \quad j_n = j, \quad j_m = j^*.$$

Then $\mathbf{H}_{n,m}(i, i^*, j, j^*)$ is a closed set in $\mathbf{I} \times \mathbf{J}$. So the set

$$\bigcup_{i, i^*, j, j^* \in L} \mathbf{H}_{n,m}(i, i^*, j, j^*)$$

is an $\mathfrak{F}_\sigma$-set in $\mathbf{I} \times \mathbf{J}$. So the set

$$T = \bigcap_{n \neq m} \bigcup_{i, i^*, j, j^* \in L} \mathbf{H}_{n,m}(i, i^*, j, j^*)$$

is an $\mathfrak{F}_{\sigma\delta}$-set in $\mathbf{I} \times \mathbf{J}$.

Now consider the set U of points $x \times \mathbf{i}$ in $X \times \mathbf{I}$ for which $A^{(x)} \subset \{R(\mathbf{i})\}$. Let $A^{[r]}$ denote the set of points x in X with $x \times r \in A$. Then $x \times \mathbf{i}$ belongs to U, if and only if, for all r in Q either

$$x \in X \backslash A^{[r]}$$

or

$$r \in \{R(\mathbf{i})\}.$$

Let W_r be the set of $\mathbf{i}$ in $\mathbf{I}$ with

$$r \in \{R(\mathbf{i})\},$$

i.e. the set of $\mathbf{i}$ in $\mathbf{I}$ with $r(i_n) = r$ for some n.
This set W_r is clearly a $\mathfrak{F}_\sigma$-set in $\mathbf{I}$.

Further
$$U = \bigcap_{r \in Q} [[\{X \backslash A^{[r]}\} \times \mathbf{I}] \cup [X \times W_r]].$$

As
$$X \backslash A^{[r]} = [(X \times Q) \backslash A]^{[r]},$$

in the obvious notation, the set $X \backslash A^{[r]}$ is a Souslin set in X and so U is a Souslin set in $X \times \mathbf{I}$.

Let V be the set of points $x \times \mathbf{j}$ in $X \times \mathbf{J}$ with $\{S(\mathbf{j})\} \subset B^{(x)}$. Let $B^{[s]}$ denote the set of points x in X with $x \times s \in B$. Then $x \times \mathbf{j}$ is in V, if and only if, for each positive integer m, there is an s in Q with

$$x \in B^{[s]} \text{ and } s(j_m) = s.$$

Let $\mathbf{Y}_{sm}$ be the set of all points $\mathbf{j}$ of $\mathbf{J}$ with $s(j_m) = s$. Then $\mathbf{Y}_{sm}$ is a closed set in $\mathbf{J}$, and

$$V = \bigcap_{m=1}^{\infty} \bigcup_{s \in Q} [\{B^{[s]} \times \mathbf{J}\} \cap \{X \times Y_{sm}\}].$$

As $B^{[s]}$ is a Souslin set in X, for each s in Q, it follows that V is a Souslin set in $X \times \mathbf{J}$. Now put

$$\mathcal{T} = X \times T, \quad \mathcal{U} = U \times \mathbf{J},$$

and let $\mathcal{V}$ be the set of all points $x \times \mathbf{i} \times \mathbf{j}$ in $X \times \mathbf{I} \times \mathbf{J}$ with

$$x \times \mathbf{j} \in V, \quad \mathbf{i} \in \mathbf{I}.$$

This last set $\mathcal{V}$ is, of course, a Cartesian product like $\mathcal{T}$ and $\mathcal{U}$, but our notation does not enable us to define it so simply. Then the set

$$\mathcal{T} \cap \mathcal{U} \cap \mathcal{V}$$

is a Souslin set in $X \times \mathbf{I} \times \mathbf{J}$.

Now a point x belongs to C if and only if there is an $\mathbf{i}$ in $\mathbf{I}$ and a $\mathbf{j}$ in $\mathbf{J}$ such that:

(a) the sequences $R(\mathbf{i})$ and $S(\mathbf{j})$ are similar;

(b) $A^{(x)} \subset \{R(\mathbf{i})\}$;

(c) $\{S(\mathbf{j})\} \subset B^{(x)}$.

Hence
$$C = \pi_X\{\mathcal{T} \cap \mathcal{U} \cap \mathcal{V}\}$$

and so is a Souslin set, as required by [25] or by [17].

We now associate an ordinal τR with each subset R of Q. Let Ω be the first uncountable ordinal. If R contains an infinite descending sequence we write $\tau R = \Omega$, otherwise R is well-ordered and we take τR to be the ordinal similar to R. Here we allow 0 as the ordinal similar to the empty set. Thus either $\tau R < \Omega$ or $\tau R = \Omega$ and R contains an infinite descending sequence. We also define σR by

$$\sigma R = \tau R + 1, \quad \text{if} \quad \tau R < \Omega,$$

$$\sigma R = \tau R, \quad \text{if} \quad \tau R = \Omega.$$

We recall that a set is said to be bi-Souslin if both it and its complement are Souslin sets. Clearly the class of bi-Souslin sets is closed under the operations of complementation, countable union and countable intersection. This class always contains the empty set and the whole space, but it need not necessarily contain any other set (consider, for example, the case of any uncountable space where the open sets are taken to be the complements of countable sets). We prove

THEOREM 12. *Let A and B be bi-Souslin sets in $X \times Q$. Then the four subsets of X defined respectively by the conditions:*

C: $\tau A^{(x)} \geqslant \tau B^{(x)}$,

D: $\tau A^{(x)} = \tau B^{(x)}$,

E: $\tau A^{(x)} \geqslant \sigma B^{(x)}$,

F: $\sigma A^{(x)} \geqslant \tau B^{(x)}$,

are Souslin sets.

Proof. By Theorem 11 the set G of points x with

$$B^{(x)} \text{ similar to a subset of } A^{(x)}$$

is a Souslin set. Further the set H of points sifted by A is a Souslin set by Theorem 10. Hence the set $G \cup H$ is a Souslin set. We prove that $C = G \cup H$. If $x \in C$, and $x \notin H$, then $\tau A^{(x)} < \Omega$, so that

$$\tau B^{(x)} \leqslant \tau A^{(x)} < \Omega,$$

which implies that $B^{(x)}$ is necessarily similar to an initial segment of $A^{(x)}$ and $x \in G$. Hence $C \subset G \cup H$. On the other hand, if $x \in G \cup H$ then, when $x \in H$ we have

$$\tau B^{(x)} \leqslant \Omega = \tau A^{(x)},$$

and when $x \in G$, $x \notin H$ we have $B^{(x)}$ similar to a subset of the well-ordered set $A^{(x)}$ so that

$$\tau B^{(x)} \leqslant \tau A^{(x)} < \Omega.$$

In either case $x \in C$. Thus $C = G \cup H$ and is a Souslin set.

Similarly the set C' of x with

$$\tau B^{(x)} \geqslant \tau A^{(x)}$$

is a Souslin set so that the set $D = C \cap C'$ is also a Souslin set.

Now let B_1 be the set of all points $x \times \frac{1}{2}q$ with $x \times q$ in B, and write

$$B_2 = B_1 \cup [X \times \{\tfrac{3}{4}\}].$$

Then for all x in X we have
$$\sigma B^{(x)} = \tau B_2^{(x)}.$$
Thus E is the set of points x with
$$\tau A^{(x)} \geqslant \tau B_2^{(x)},$$
and so is a Souslin set, as B_2 is clearly bi-Souslin in $X \times Q$.

Defining A_1, A_2 in the same way we see that F is the set of points x with
$$\tau A_2^{(x)} \geqslant \tau B^{(x)}$$
and so is a Souslin set.

COROLLARY. *Let A be a bi-Souslin set in $X \times Q$. If τ is a countable ordinal, the set A_τ of points x of X with*
$$\tau A^{(x)} = \tau$$
is a bi-Souslin set in X.

Proof. If τ is a countable ordinal we can choose a countable set T in Q similar to τ. Then the set
$$B = X \times T$$
is a bi-Souslin set in $X \times Q$. Applying the theorem to the sets A and B we see that A_τ is a Souslin set. Further the set $B_{\tau+1}$ of all points x with
$$\tau A^{(x)} \geqslant \tau + 1$$
is a Souslin set. So the set
$$\bigcup_{\alpha \leqslant \tau} A_\alpha = X \backslash B_{\tau+1}$$
is a bi-Souslin set, for each countable ordinal τ. So the set
$$A_\tau = [\bigcup_{\alpha \leqslant \tau} A_\alpha] \backslash [\bigcup_{\beta < \tau} \{\bigcup_{\alpha \leqslant \beta} A_\alpha\}],$$
being the difference of two bi-Souslin sets, is a bi-Souslin set.

Our next result is one of the key lemmas used in the proof of the uniformization theorem. It is a generalization of lemmas 1 and 2 of Sampei's paper [26]. Its relevance can perhaps be seen most clearly at this stage by considering the special case when $k = 1$; in this case it enables us to find in the complement of a Souslin set determined by the sieve $\mathcal{A}_1$ in $X \times \mathbf{I} \times Q$ a smaller complement, $\tilde{C}_1$, of a Souslin set having the same projection on X.

THEOREM 13. *Let $\mathcal{A}_l$, $l = 1, 2, \ldots, k$, be bi-Souslin sieves in $X \times \mathbf{I} \times Q$. Let $\tilde{C}_0 = X \times \mathbf{I}$, and let $\tilde{C}_1, \ldots, \tilde{C}_k$ be defined inductively, by writing*
$$\alpha_l(x) = \min_{\substack{\mathbf{i} \in \mathbf{I} \\ x \times \mathbf{i} \in \tilde{C}_{l-1}}} \tau \mathcal{A}_l^{(x \times \mathbf{i})},$$

with the convention that a minimum taken over $\emptyset$ is given the value Ω, and by taking $\tilde{C}_l$ to be the set of points $x \times \mathbf{i}$ in $\tilde{C}_{l-1}$ with

$$\tau\mathcal{A}_l^{(x\times\mathbf{i})} = \alpha_l(x) < \Omega,$$

all for $l = 1, 2, \ldots, k$. Then $\tilde{C}_k$ is the complement of a Souslin set in $X \times \mathbf{I}$.

Proof. Suppose that for some h with $1 \leqslant h \leqslant k$ we know that $\tilde{C}_{h-1}$ is the complement of a Souslin set C_{h-1} in $X \times \mathbf{I}$. Now $\tilde{C}_h$ is the set of all points $x \times \mathbf{i}$ satisfying the condition

$$x \times \mathbf{i} \in \tilde{C}_{h-1},$$

the condition

$$\tau\mathcal{A}_h^{(x\times\mathbf{i})} < \Omega,$$

and the condition that for all $\mathbf{j}$ in $\mathbf{J}$ with

$$x \times \mathbf{j} \in \tilde{C}_{h-1},$$

we have

$$\tau\mathcal{A}_h^{(x\times\mathbf{j})} \geqslant \tau\mathcal{A}_h^{(x\times\mathbf{i})}.$$

Now, given that $x \times \mathbf{i} \in \tilde{C}_{h-1}$ the condition that $x \times \mathbf{j} \in \tilde{C}_{h-1}$ is equivalent to the condition that $x \times \mathbf{j}$ satisfies

$$\tau\mathcal{A}_l^{(x\times\mathbf{j})} = \tau\mathcal{A}_l^{(x\times\mathbf{i})}, \quad l = 1, 2, \ldots, h-1.$$

Further, given that

$$\tau\mathcal{A}_h^{(x\times\mathbf{i})} < \Omega,$$

the condition that $x \times \mathbf{j}$ satisfies

$$\tau\mathcal{A}_h^{(x\times\mathbf{j})} \geqslant \tau\mathcal{A}_h^{(x\times\mathbf{i})}$$

is equivalent to the condition that $x \times \mathbf{j}$ satisfies

$$\sigma\mathcal{A}_h^{(x\times\mathbf{j})} > \tau\mathcal{A}_h^{(x\times\mathbf{i})}.$$

Hence $\tilde{C}_h$ is the set of points $x \times \mathbf{i}$ satisfying the condition

$$x \times \mathbf{i} \in \tilde{C}_{h-1},$$

the condition

$$\tau\mathcal{A}_h^{(x\times\mathbf{i})} < \Omega.$$

and the condition that for all $\mathbf{j}$ in $\mathbf{J}$ with

$$\tau\mathcal{A}_l^{(x\times\mathbf{j})} = \tau\mathcal{A}_l^{(x\times\mathbf{i})}, \quad l = 1, 2, \ldots, h-1,$$

we have

$$\sigma\mathcal{A}_h^{(x\times\mathbf{j})} > \tau\mathcal{A}_h^{(x\times\mathbf{i})}.$$

The third condition here is that for all $\mathbf{j}$ in $\mathbf{J}$ we have either

$$\tau\mathcal{A}_l^{(x\times\mathbf{j})} \neq \tau\mathcal{A}_l^{(x\times\mathbf{i})}$$

for some l with $1 \leqslant l \leqslant h-1$ or

$$\sigma\mathcal{A}_h^{(x\times\mathbf{j})} > \tau\mathcal{A}_h^{(x\times\mathbf{i})}.$$

Hence $\tilde{C}_h$ is the complement of the set C_h of points $x \times \mathbf{i}$ satisfying either the condition

$$x \times \mathbf{i} \in C_{h-1},$$

or the condition

$$\tau \mathcal{A}_h^{(x \times \mathbf{i})} = \Omega,$$

or the condition that for some $\mathbf{j}$ in $\mathbf{J}$ we have

$$\tau \mathcal{A}_l^{(x \times \mathbf{j})} = \tau \mathcal{A}_l^{(x \times \mathbf{i})}, \quad l = 1, 2, \ldots, h-1,$$

and

$$\sigma \mathcal{A}_h^{(x \times \mathbf{j})} \leqslant \tau \mathcal{A}_h^{(x \times \mathbf{i})}.$$

Thus

$$C_h = C_{h-1} \cap A_h \cap \pi_{X \times \mathbf{I}} D, \tag{17}$$

where A_h is the set sifted by the sieve $\mathcal{A}_h$, and D is the set of points $x \times \mathbf{i} \times \mathbf{j}$ in $X \times \mathbf{I} \times \mathbf{J}$ with

$$\tau \mathcal{A}_l^{(x \times \mathbf{j})} = \tau \mathcal{A}_l^{(x \times \mathbf{i})}, \quad l = 1, 2, \ldots, h-1,$$

and

$$\sigma \mathcal{A}_h^{(x \times \mathbf{j})} \leqslant \tau \mathcal{A}_h^{(x \times \mathbf{i})}.$$

Now let $\mathcal{E}_l$, $\mathfrak{F}_l$, $l = 1, 2, \ldots, h$, be the sets of points $x \times \mathbf{i} \times \mathbf{j} \times q$ in $X \times \mathbf{I} \times \mathbf{J} \times Q$ satisfying the conditions

$\mathcal{E}_l$: $$x \times \mathbf{j} \times q \in \mathcal{A}_l, \quad \mathbf{i} \in \mathbf{I},$$

$\mathfrak{F}_l$: $$x \times \mathbf{i} \times q \in \mathcal{A}_l, \quad \mathbf{j} \in \mathbf{J},$$

$l = 1, 2, \ldots, h$. Then $\mathcal{E}_l$ and $\mathfrak{F}_l$ are two different cartesian products of $\mathcal{A}_l$ with $\mathbf{I}$ and are both bi-Souslin sets in $X \times \mathbf{I} \times \mathbf{J} \times Q$, for $l = 1, 2, \ldots, h$. Now D is the set of points $x \times \mathbf{i} \times \mathbf{j}$ in $X \times \mathbf{I} \times \mathbf{J}$ with the properties

$$\tau \mathcal{E}_l^{(x \times \mathbf{i} \times \mathbf{j})} = \tau \mathfrak{F}_l^{(x \times \mathbf{i} \times \mathbf{j})}, \quad l = 1, 2, \ldots, h-1,$$

$$\sigma \mathcal{E}_h^{(x \times \mathbf{i} \times \mathbf{j})} \leqslant \tau \mathfrak{F}_h^{(x \times \mathbf{i} \times \mathbf{j})}.$$

By Theorem 12, the set D is a Souslin set in $X \times \mathbf{I} \times \mathbf{J}$. Hence by [25], or by [17], the set

$$\pi_{X \times \mathbf{I}} D$$

is a Souslin set in $X \times \mathbf{I}$. Now, it follows from (17) and our inductive hypothesis that C_h is a Souslin set in $X \times \mathbf{I}$. So the required result follows by induction.

Remark. No very special properties of the space $\mathbf{I}$ are used in this proof; the result would clearly hold with $\mathbf{I}$ replaced by any analytic space as the results of [25] hold for such spaces.

8. Separation Theorems

In this section we digress from the main purpose of this paper to use the results obtained in the last section to prove two separation theorems due to Kunugui [17] and Kondô [16] and to establish a result on sets having disjoint Souslin representations due to Kunugui [17]. The first result [Kunugui] generalizes Lusin's second separation principle, the proof follows his closely. As it is no more difficult, we state the result in terms of the class of Souslin-(bi-Souslin) sets (i.e. the class of sets obtained by applying the Souslin operation to the bi-Souslin sets); we remark that, if we know that each open set of a space is a Souslin set, then every closed set is bi-Souslin, so that every Souslin set is a Souslin-(bi-Souslin) set.

THEOREM 14. *Let A and B be Souslin-(bi-Souslin) sets. Then there are sets C, D that are complements of Souslin sets in X and that satisfy*

$$A \setminus B \subset C, \quad B \setminus A \subset D, \quad C \cap D = \emptyset.$$

Proof. By Theorem 9 we can construct bi-Souslin sets $\mathcal{A}$ and $\mathcal{B}$ in $X \times Q$ so that A and B are the sets sifted by $\mathcal{A}$ and $\mathcal{B}$ respectively.

By Theorem 12, the set C of points x with

$$\tau \mathcal{A}^{(x)} > \tau \mathcal{B}^{(x)}, \tag{18}$$

and the set D of points x with

$$\tau \mathcal{B}^{(x)} > \tau \mathcal{A}^{(x)} \tag{19}$$

are complements of Souslin sets. Further for each x in $A \setminus B$ we have

$$\tau \mathcal{A}^{(x)} = \Omega > \tau \mathcal{B}^{(x)},$$

so that $A \setminus B \subset C$. Similarly $B \setminus A \subset D$. Finally $C \cap D = \emptyset$ as the conditions (18) and (19) are incompatible.

We now prove Kondô's result in [16] which is in some ways a substitute for the first separation principle, by use of the methods usually reserved for the proof of the second separation theorem.

THEOREM 15. *Let $A_1, A_2, \ldots$ be a sequence of pairwise disjoint Souslin-(bi-Souslin) sets. Then there is a sequence $B_1, B_2, \ldots$ of pairwise disjoint bi-Souslin sets with $A_i \subset B_i$, $i = 1, 2, \ldots$.*

Proof. By Theorem 9, we can construct bi-Souslin sets $\mathcal{A}_i$, $i = 1, 2, \ldots$, in $X \times Q$ so that A_i is the set sifted by $\mathcal{A}_i$ for $i = 1, 2, \ldots$. For each i we define a set $\mathcal{B}_i$ by taking $\mathcal{B}_i$ to be the set of all points

$$x \times \tfrac{1}{2} q$$

with
$$x \times q \in \mathcal{A}_i,$$
together with all points
$$x \times \{\tfrac{3}{4} - 2^{-j-2}\}$$
with $x \in X$ and j a positive integer, together with all points
$$x \times \{1 - 2^{-k-2}\}$$
with $x \in X$ and k positive integer not exceeding i. This ensures that A_i is the set sifted by $\mathcal{B}_i$ as well as by $\mathcal{A}_i$ and that for each x in X
$$\tau \mathcal{B}_i^{(x)}$$
is of the form $\sigma + i$ with σ a limit ordinal, or is the ordinal Ω.

As the sets A_i are pairwise disjoint we cannot have
$$\tau \mathcal{B}_i^{(x)} = \Omega, \quad \tau \mathcal{B}_j^{(x)} = \Omega$$
with $i \neq j$. Hence, when $i \neq j$,
$$\tau \mathcal{B}_i^{(x)} \neq \tau \mathcal{B}_j^{(x)}.$$

Let B_i be defined to be the set of all x in X with
$$\tau \mathcal{B}_i^{(x)} > \tau \mathcal{B}_j^{(x)} \quad \text{for all} \quad j \neq i.$$
Then, in fact, B_i is the set of all x in X with
$$\tau \mathcal{B}_i^{(x)} \geq \tau \mathcal{B}_j^{(x)} \quad \text{for all} \quad j \neq i.$$
It follows immediately that
$$B_i \supset A_i,$$
and that the sets B_i are pairwise disjoint. Further, by Theorem 12, the sets B_i are bi-Souslin, as required.

By use of one of the standard techniques that leads from the first separation theorem to a proof that a continuous one-one image of the irrationals in a metric space is a Borel set, we use the last theorem to prove

THEOREM 16. *Suppose that a set A in a space X has a disjoint representation as a Souslin-(bi-Souslin) set. Then A is bi-Souslin itself.*

Proof. We know that A has a representation
$$A = A(\mathbf{I}), \quad A(\mathbf{i}) = \bigcap_{n=1}^{\infty} A(\mathbf{i}|n),$$
where each set $A(\mathbf{i}|n)$ is a bi-Souslin set, and where the sets $A(\mathbf{i})$, $\mathbf{i} \in \mathbf{I}$ are all mutually disjoint.

For each fixed $n \geqslant 1$, the system of sets

$$A(\mathbf{I}_{\mathbf{i}|n}), \quad \mathbf{i} \in \mathbf{I},$$

form a countable family of disjoint Souslin-(bi-Souslin) sets. By Theorem 15 we can choose a corresponding family

$$B(\mathbf{i}|n), \quad \mathbf{i} \in \mathbf{I},$$

of mutually disjoint bi-Souslin sets with

$$A(\mathbf{I}_{\mathbf{i}|n}) \subset B(\mathbf{i}|n), \quad \text{for all} \quad \mathbf{i} \in \mathbf{I}. \tag{20}$$

We suppose such sets chosen for each $n \geqslant 1$. We define sets $C(\mathbf{i}|n)$ inductively by taking

$$c(\mathbf{i}|0) = X,$$

and

$$C(\mathbf{i}|n) = B(\mathbf{i}|n) \cap A(\mathbf{i}|n) \cap C(\mathbf{i}|n-1), \tag{21}$$

for $n \geqslant 1$. Then for all $\mathbf{i}$ and n the set $C(\mathbf{i}|n)$ is a bi-Souslin set, and so is the set

$$C = \bigcap_{n=1}^{\infty} [\bigcup_{\mathbf{i} \in \mathbf{I}} C(\mathbf{i}|n)].$$

Thus to prove that A is a bi-Souslin set it will suffice to prove that $A = C$.

Now if $a \in A$ there is an $\mathbf{i}$ with $a \in A(\mathbf{i}|n)$ for $n = 1, 2, \ldots$. It follows by induction, using (20) and (21) that $a \in C(\mathbf{i}|n)$ for $n = 1, 2, \ldots$. Hence $a \in C$. Thus $A \subset C$. On the other hand, if $c \in C$ it follows from the properties of disjointness and inclusion of the sets $C(\mathbf{i}|n)$ that a sequence of integers $i_1, i_2, \ldots$ is determined uniquely by the condition

$$c \in C(\mathbf{i}|n), \quad n = 1, 2, \ldots,$$

the integers being determined one at a time, i_n being fixed by the relation $c \in C(\mathbf{i}|n)$. Then $c \in A(\mathbf{i}) \subset A$. Thus $C \subset A$ and $C = A$ as required.

COROLLARY. (*Kunugui, Theorem 11*) *If each open set in X is a Souslin set then each set in X having a disjoint Souslin representation is a bi-Souslin set.*

9. Uniformization on X×I

We can now state our uniformization for the complement of a Souslin set in a space of the form $X \times \mathbf{I}$. After the first stage, showing that a Souslin set in $X \times \mathbf{I}$ can be expressed as a Souslin-$\mathcal{R}$ set the proof is closely modeled on Sampei's [26] simplified form of Kondô's proof.

THEOREM 17. *Suppose that each open set in a topological space X is a Souslin set. Let E be the complement of a Souslin set in $X \times \mathbf{I}$. Then there is a set U, that is a complement of a Souslin set in $X \times \mathbf{I}$ and that satisfies:*

(a) $U \subset E$;

(b) $\pi_X U = \pi_X E$; *and*

(c) *for each x in $\pi_X E$ the set*

$$(\{x\} \times \mathbf{I}) \cap U$$

consists of a single point.

Proof. Let $\mathcal{R}$ denote the system of rectangles in $X \times \mathbf{I}$ of the form $F \times \mathbf{J}$ where F is closed in X and $\mathbf{J}$ is a Baire interval in $\mathbf{I}$. Our first aim is to show that the Souslin set $A = (X \times \mathbf{I}) \setminus E$ belongs to the system of Souslin-$\mathcal{R}$ sets.

We have

$$A = F(\mathbf{I}), \quad F(\mathbf{i}) = F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

the sets $F(\mathbf{i}|n)$ being all closed in $X \times \mathbf{I}$. We may further suppose that

$$F(\mathbf{i}|m) \subset F(\mathbf{i}|n)$$

whenever $\mathbf{i} \in \mathbf{I}$ and $m > n \geq 1$.

For each $n \geq 1$, let $\mathbf{J}_n(1)$, $\mathbf{J}_n(2)$, ... be an enumeration of the Baire intervals

$$\mathbf{I}_{\mathbf{i}|n}, \quad \mathbf{i} \in \mathbf{I},$$

of order n.

We define a system of sets $R(\mathbf{i}|n)$ of $\mathcal{R}$ by taking, for each $\mathbf{i}$ in $\mathbf{I}$,

$$R(\mathbf{i}|1) = R(\mathbf{i}|2) = X \times \mathbf{J}_2(i_1),$$

and

$$R(\mathbf{i}|2n-1) = R(\mathbf{i}|2n) = \mathrm{Cl}\,[\pi_X\{F(i_2, i_4, \ldots, i_{2n-2}) \cap (X \times \mathbf{J}_{2n}(i_{2n-1}))\}] \times \mathbf{J}_{2n}(i_{2n-1}),$$

for $n = 1, 2, \ldots$. It is clear that these sets belong to $\mathcal{R}$ so that the set

$$B = \bigcup_{\mathbf{i} \in \mathbf{I}} \bigcap_{n=1}^{\infty} R(\mathbf{i}|n)$$

is a Souslin-$\mathcal{R}$ set. Thus we will have achieved our first aim if we can show that $A = B$.

First consider any point $a \times \mathbf{a}$ in A. Then for some $\mathbf{i}^*$ in $\mathbf{I}$ and some $\mathbf{j}^*$ in $\mathbf{I}$ we have

$$a \in F(\mathbf{i}^*) = \bigcap_{n=1}^{\infty} F(\mathbf{i}^*|n),$$

and
$$\{\mathbf{a}\} = \bigcap_{n=1}^{\infty} \mathbf{J}_{2n}(j_n^*).$$

Write
$$\mathbf{h}^* = j_1^*, i_1^*, j_2^*, i_2^*, \ldots.$$

Then, for each $n \geq 1$ we have
$$a \in F(h_2^*, h_4^*, \ldots, h_{2n-2}^*),$$
$$\mathbf{a} \in \mathbf{J}_{2n}(h_{2n-1}^*),$$

so that
$$a \times \mathbf{a} \in \bigcap_{n=1}^{\infty} R(\mathbf{h}^* | n) \subset B.$$

Thus $A \subset B$.

Now suppose that $b \times \mathfrak{b} \in B$. Then there is an $\mathbf{i}^*$ in $\mathbf{I}$ with
$$b \times \mathfrak{b} \in \bigcap_{n=1}^{\infty} R(\mathbf{i}^* | n).$$

Suppose that $b \times \mathfrak{b}$ does not lie in the corresponding set
$$\bigcap_{n=1}^{\infty} F(i_2^*, i_4^*, \ldots, i_{2n}^*).$$

Then for some positive integer N we have
$$b \times \mathfrak{b} \notin F(i_2^*, i_4^*, \ldots, i_{2N}^*).$$

As $F(i_2^*, i_4^*, \ldots, i_{2N}^*)$ is closed there are open sets G and $\mathfrak{G}$ in X and $\mathbf{I}$ with $b \in G$, $\mathfrak{b} \in \mathfrak{G}$ and
$$F(i_2^*, i_4^*, \ldots, i_{2N}^*) \cap (G \times \mathfrak{G}) = \emptyset.$$

As
$$b \times \mathfrak{b} \in \bigcap_{n=1}^{\infty} R(\mathbf{i}^* | n)$$

we have
$$\mathfrak{b} \in \bigcap_{n=1}^{\infty} \mathbf{J}_{2n}(\mathbf{i}_{2n-1}^*).$$

Since $\mathbf{J}_{2n}(i_{2n-1}^*)$ is a Baire interval of order $2n$, we can choose a positive integer M with $\mathfrak{b} \in \mathbf{J}_{2M}(i_{2M-1}^*) \subset \mathfrak{G}$. Write $L = \max\{N+1, M\}$. Then
$$F(i_2^*, i_4^*, \ldots, i_{2L-2}^*) \subset F(i_2^*, i_4^*, \ldots, i_{2N}^*)$$

and, as
$$\mathbf{J}_{2L}(\mathbf{i}_{2L-1}^*) \cap \mathbf{J}_{2M}(i_{2M-1}^*) \supset \{\mathfrak{b}\} \neq \emptyset,$$

we have
$$\mathbf{J}_{2L}(i_{2L-1}^*) \subset \mathbf{J}_{2M}(i_{2M-1}^*) \subset \mathfrak{G}.$$

Hence
$$F(i_2^*, i_4^*, \ldots, i_{2L-2}) \cap [G \times \mathbf{J}_{2L}(i_{2L-1}^*)] = \emptyset,$$

so that
$$[\pi_X\{F(i_2^*, i_4^*, \ldots, i_{2L-2}^*) \cap (X \times \mathbf{J}_{2L}(i_{2L-1}^*))\}] \cap G = \emptyset.$$

Since $b \in G$, and G is open, it follows that

$$f \notin \mathrm{cl}\,[\pi_X\{F(i_2^*, i_4^*, \ldots, i_{2L-2}^*) \cap (X \times \mathbf{J}_{2L}(i_{L2-1}^*))\}],$$

so that
$$b \times \mathbf{b} \notin R(\mathbf{i}^* \,|\, 2L),$$

contrary to our supposition. This shows that $b \times \mathbf{b}$ must lie in the set

$$\bigcap_{n=1}^{\infty} F(i_2^*, i_4^*, \ldots, i_{2n}^*)$$

and so must lie in A. Hence $B \subset A$. As we have already $A \subset B$ it follows that $A = B$ and that A is a Souslin-$\mathcal{R}$ set.

Now the system $\mathcal{R}$ of rectangles of the form $F \times \mathbf{J}$ with F closed in X and $\mathbf{J}$ a Baire interval in $\mathbf{I}$ is closed under finite intersections. Hence, by the corollary to Theorem 9, we can choose a sequence of sets $R_1, R_2, \ldots$ in $\mathcal{R}$ and a sequence of points $q_1, q_2, \ldots$ in Q so that A is the set sifted by the sieve C in $X \times \mathbf{I} \times Q$ given by

$$C = \bigcup_{n=1}^{\infty} R_n \times \{q_n\}.$$

We need to modify this sieve to ensure that it has certain special properties. We form a sieve C' from C by taking $x \times \mathbf{i} \times q$ to be in C' if either $x \times \mathbf{i} \times (2q)$ is in C or q has one of the values $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, 1-2^{-n}, \ldots$. Then C' can be expressed in the same form

$$C' = \bigcup_{n=1}^{\infty} R_n' \times \{q_n'\},$$

with $R_n' \in \mathcal{R}$ and $q_n' \in Q$ for $n = 1, 2, \ldots$, and A is still the set sifted by C'. This ensures that for each $x \times \mathbf{i}$ in $X \times \mathbf{I}$ the set

$$C'^{(x \times \mathbf{i})} = \pi_Q[C' \cap (\{x \times \mathbf{i}\} \times Q)],$$

has infinitely many elements.

Now, for each n,
$$R_n' = F_n \times \mathbf{J}_n$$

for some closed set F_n in X and for some Baire interval $\mathbf{J}_n$ in $\mathbf{I}_n$. Let $k(n)$ be the order of the Baire interval $\mathbf{J}_n$. Write

$$k''(n) = n + \max_{1 \leqslant r \leqslant n} \{k(r)\}.$$

Then $\mathbf{J}_n$ can be expressed as a disjoint union

$$\mathbf{J}_n = \bigcup_{m=1}^{\infty} \mathbf{H}_{nm},$$

where each set $\mathbf{H}_{nm}$ is a Baire interval of order $k^*(n)$. As the unions are disjoint, it follows that if $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_r$ are Baire intervals choosen from the system

$$\mathbf{H}_{nm}, \quad n, m = 1, 2, \ldots,$$

so that each pair has at least one point in common, then they must be sets $\mathbf{H}_{nm}$ corresponding to different values of n and so at least one is a Baire interval of order at least r. Now the system of triples

$$F_n, \mathbf{H}_{nm}, q'_n, \quad n, m = 1, 2, \ldots,$$

is countable; let

$$F''_n, \mathbf{J}''_n, q''_n, \quad n = 1, 2, \ldots,$$

be an enumeration of these triples. Put

$$R''_n = F''_n \times \mathbf{J}''_n, \quad n = 1, 2, \ldots,$$

and

$$C'' = \bigcup_{n=1}^{\infty} R''_n \times \{q''_n\}.$$

Then

$$C'' = C',$$

so that A remains the set sifted by C''. All the sets

$$[\{x \times \mathbf{i}\} \times Q] \cap C''$$

are infinite. Further, if $\mathbf{J}''_{n(1)}, \mathbf{J}''_{n(2)}, \ldots$ is any subsequence of the sequence of Baire intervals, with the property that

$$\mathbf{J}''_{n(i)} \cap \mathbf{J}''_{n(j)} \neq \emptyset, \quad i, j, = 1, 2, \ldots,$$

then the order of $\mathbf{J}''_{n(i)}$ tends to infinity with i. These are the properties we will require. We shall in the remainder of this proof work with this sieve C'' but, for convenience, we shall drop the double dashes (double primes).

Since each open set in X is a Souslin set, the closed sets in X are bi-Souslin sets. Similarly each Baire interval in $\mathbf{I}$ is a bi-Souslin set in $\mathbf{I}$. Hence each set

$$R_n \times \{q_n\} = F_n \times \mathbf{J}_n \times \{q_n\}, \quad n = 1, 2, \ldots,$$

is a bi-Souslin set in $X \times \mathbf{I} \times Q$, so that the sieve

$$C = \bigcup_{n=1}^{\infty} [F_n \times \mathbf{J}_n \times \{q_n\}] \tag{22}$$

is a bi-Souslin set in $X \times \mathbf{I} \times Q$.

We now introduce two sequences of sieves. We write

$$C_0 = C,$$

and we define C_n for $n \geqslant 1$, by taking

$$C_n = C \cap \{X \times \mathbf{J}_n \times Q_n\}, \tag{23}$$

where we take Q_n to be the set of rationals q in Q with $0 < q < q_n$.

For each $n \geqslant 1$ we take

$$\mathbf{K}_{nm} = \mathbf{J}_{n+m-1} \setminus \bigcup_{n \leqslant l < n+m-1} \mathbf{J}_l, \quad m = 1, 2, \ldots, \tag{24}$$

and define D_n to be the set of all points of the form $x \times \mathbf{i} \times q$ with

$$x \in X, \quad \mathbf{i} \in \mathbf{K}_{nm}, \quad q = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^l},$$

with $m \geqslant 1$ and $1 \leqslant l \leqslant m$. Then D_n is a bi-Souslin set in $X \times \mathbf{I} \times Q$ and $\pi_{X \times \mathbf{I}} D_n = X \times \mathbf{I}$ for all $n \geqslant 1$, as the sequence $\mathbf{J}_1, \mathbf{J}_2, \ldots$ covers $\mathbf{I}$ infinitely often.

We define sets $H_0, H_1, H_2, \ldots,$

$E_0, E_1, E_2, \ldots,$

integral valued functions $d_1, d_2, \ldots,$

on X, and ordinal valued functions

$$\gamma_0, \gamma_1, \gamma_2, \ldots,$$

on X, inductively as follows:

$$H_0 = X \times \mathbf{I};$$

$$\gamma_n(x) = \min_{\substack{\mathbf{i} \in \mathbf{I} \\ x \times \mathbf{i} \in H_n}} \tau C_n^{(x \times \mathbf{i})}, \quad n = 0, 1, 2, \ldots; \tag{25}$$

$$E_n = \{x \times \mathbf{i} \mid x \times \mathbf{i} \in H_n \text{ and } \tau C_n^{(x \times \mathbf{i})} = \gamma_n(x) < \Omega\}, \quad n = 0, 1, 2, \ldots; \tag{26}$$

$$d_n(x) = \min_{\substack{\mathbf{i} \in \mathbf{I} \\ x \times \mathbf{i} \in E_{n-1}}} \tau D_n^{(x \times \mathbf{i})}, \quad n = 1, 2, \ldots; \tag{27}$$

$$H_n = \{x \times \mathbf{i} \mid x \times \mathbf{i} \in E_{n-1} \text{ and } \tau D_n^{(x \times \mathbf{i})} = d_n(x)\}, \quad n = 1, 2, \ldots. \tag{28}$$

This ensures that the sequence

$$E_0, H_1, E_1, H_2, \ldots$$

is a non-increasing sequence of sets that coincides with the sequence of sets generated, as in the definition in the statement of Theorem 13, from the sequence of sieves

$$C = C_0,\ D_1,\ C_1,\ D_2,\ \dots .$$

As all these sieves are bi-Souslin sets, it follows from Theorem 13, that all the sets

$$E_0,\ H_1,\ E_1,\ H_2,\ \dots$$

are complements of Souslin sets. Hence the set

$$U = \bigcap_{n=0}^{\infty} E_n$$

is a complement of a Souslin set.

Our aim will be to show that the set U defined in this way is a set uniformizing E. As E is the complementary set determined by the sieve $C = C_0$ it is clear from (25) and (26) that $E_0 \subset E$. Hence $U \subset E$. So it will suffice to prove that, for each point x of $\pi_X E$, the set $U^{(x)}$ is a one point set.

For the remainder of this proof we can regard x as a given point of $\pi_X E$. Where no confusion can arise we shall not show explicitly the dependence of our parameters on x. As $x \in \pi_X E$, it follows from (25) and (26) that $E^{(x)} \neq \emptyset$. Once we know that $E^{(x)}_{n-1} \neq \emptyset$ for some $n \geqslant 1$, the minimum

$$\min_{\substack{\mathbf{i} \in \mathbf{I} \\ x \times \mathbf{i} \in E_{n-1}}} \tau D_n^{(x \times \mathbf{i})}$$

is necessarily attained for some $\mathbf{i}$ in $E^{(x)}_{n-1}$ so that $H^{(x)}_n \neq \emptyset$ on using (27) and (28). Given that $H^{(x)}_n \neq \emptyset$ for some $n \geqslant 1$, the minimum

$$\min_{\substack{\mathbf{i} \in \mathbf{I} \\ x \times \mathbf{i} \in H_n}} \tau C_n^{(x \times \mathbf{i})}$$

is a countable ordinal, as $H^{(x)}_n \subset E^{(x)}$, and is necessarily attained for some $\mathbf{i}$ in $H^{(x)}_n$, so that $E^{(x)}_n \neq \emptyset$ on using (25) and (26). It follows inductively that all the sets

$$E_0^{(x)},\ H_1^{(x)},\ E_1^{(x)},\ H_2^{(x)},\ \dots,$$

are non-empty.

Now for each integer $n \geqslant 1$, the set $H^{(x)}_n$ is the set of all $\mathbf{i}$ in $E^{(x)}_{n-1}$ with

$$\tau D_n^{(x \times \mathbf{i})} = d_n(x).$$

By the construction of the sieve D_n the points $\mathbf{i}$ satisfying this condition, are those in the set

$$\mathbf{K}_{n d_n(x)} = \mathbf{J}_{n+d_n(x)-1} \setminus \bigcup_{n \leqslant l < n+d_n(x)-1} \mathbf{J}_l,$$

i.e. those in the set $\mathbf{J}_{d(n)}$ that lie in none of the sets

$$\mathbf{J}_n, \mathbf{J}_{n+1}, \ldots, \mathbf{J}_{d(n)-1},$$

where we write

$$d(n) = n + d_n(x) - 1.$$

Thus for $n \geqslant 1$,

$$E_n^{(x)} \subset H_n^{(x)} \subset E_{n-1}^{(x)} \cap \mathbf{J}_{d(n)}, \tag{29}$$

$$H_n^{(x)} \cap \mathbf{J}_l = \varnothing, \quad n \leqslant l < d(n). \tag{30}$$

Comparing (30) with the formula

$$\varnothing \neq H_{n+1}^{(x)} = H_{n+1}^{(x)} \cap \mathbf{J}_{d(n+1)} \subset H_n^{(x)} \cap \mathbf{J}_{d(n+1)},$$

which follows from (29), and noting that $d(n+1) \geqslant n+1$, we see that we must have $d(n+1) \geqslant d(n)$ for $n \geqslant 1$.

Now, for any positive integers m, n, there is an integer l with $l \geqslant m$, $l \geqslant n$, so that

$$E_l^{(x)} \subset E_n^{(x)} \subset \mathbf{J}_{d(n)}, \quad E_l^{(x)} \subset E_m^{(x)} \subset \mathbf{J}_{d(m)}.$$

Since $E_l^{(x)} \neq \varnothing$ this implies that

$$\mathbf{J}_{d(n)} \cap \mathbf{J}_{d(m)} \neq \varnothing.$$

As $d(n) \to \infty$, as $n \to \infty$, we have an infinite sequence of Baire intervals

$$\mathbf{J}_{d(n)}, \quad n = 1, 2, \ldots, \tag{31}$$

with the property that any two have at least one point in common. By our original choice of the sequence $\mathbf{J}_1, \mathbf{J}_2, \ldots$, it follows that the orders of the Baire intervals of this sequence (31) tend to infinity. Hence the set

$$\bigcap_{n=1}^{\infty} \mathbf{J}_{d(n)}$$

consists of a single point. Let $\mathbf{u} = \mathbf{u}(x)$ denote this point. As

$$U^{(x)} = \bigcap_{n=0}^{\infty} E_n^{(x)} \subset \bigcap_{n=1}^{\infty} \mathbf{J}_{d(n)},$$

it follows that either $U^{(x)} = \{\mathbf{u}(x)\}$ or $U^{(x)} = \varnothing$.

It remains to prove that $\mathbf{u}(x) \in U^{(x)}$. To this end we first study the set of integers d with

$$\mathbf{u} \in \mathbf{J}_d.$$

By the definition of $\mathbf{u}$ we have

$$\mathbf{u} \in \mathbf{J}_{d(n)}, \quad n = 1, 2, \ldots.$$

Suppose we had $\mathbf{u} \in \mathbf{J}_m$ for some $m \geq 1$ not in the sequence $\{d(n)\}$. Then we would have

$$n^* \leq d(n^*) < m < d(n^*+1)$$

for some $n^* \geq 0$, on writing $d(0)=0$. As $\mathbf{u} \in \mathbf{J}_m$ and $\mathbf{u} \in \mathbf{J}_{d(n)}$ for all n, and as the order of $\mathbf{J}_{d(n)}$ tends to infinity as $n \to \infty$, we can choose $n \geq n^*+1$, so that

$$\mathbf{J}_{d(n)} \subset \mathbf{J}_m.$$

Hence
$$E_n^{(x)} \subset \mathbf{J}_{d(n)} \subset \mathbf{J}_m, \quad E_n^{(x)} \subset H_{n^*+1}^{(x)},$$

so that
$$H_{n^*+1}^{(x)} \cap \mathbf{J}_m \neq \emptyset,$$

contrary to (30) as
$$n^*+1 \leq m < d(n^*+1).$$

Thus we have $\mathbf{u} \in \mathbf{J}_m$ if, and only if m belongs to the sequence $\{d(n)\}$.

We now study the order relations between the rational numbers $q_{d(n)}$, $n=1, 2, \ldots$ and between the ordinals $\gamma_{d(n)} = \gamma_{d(n)}(x)$, $n=1, 2, \ldots$. Suppose n, m are integers with $1 \leq n < m$. Choose a point $\mathbf{i}$ in $E_{d(m)}^{(x)}$. Then, using (29),

$$\mathbf{i} \in E_{d(m)}^{(x)} \subset E_m^{(x)} \subset \mathbf{J}_{d(m)}.$$

Similarly $\mathbf{i} \in \mathbf{J}_{d(n)}$. Also $\mathbf{i} \in E_0^{(x)}$. Thus, using (25) and (26)

$$\tau C^{(x \times \mathbf{i})} = \gamma_0(x),$$

$$\tau C_{d(n)}^{(x \times \mathbf{i})} = \gamma_{d(n)}(x), \quad \mathbf{i} \in \mathbf{J}_{d(n)},$$

$$\tau C_{d(m)}^{(x \times \mathbf{i})} = \gamma_{d(m)}(x), \quad \mathbf{i} \in \mathbf{J}_{d(m)}.$$

Comparing the definitions of the sieves C, $C_{d(n)}$, $C_{d(m)}$, see (22) and (23), it follows that

$$\gamma_{d(n)} \leq \gamma_0,$$

and that $\gamma_{d(n)} < \gamma_{d(m)}$, $\gamma_{d(n)} = \gamma_{d(m)}$ or $\gamma_{d(n)} > \gamma_{d(m)}$ according as $q_{d(n)} < q_{d(m)}$, $q_{d(n)} = q_{d(m)}$ or $q_{d(n)} > q_{d(m)}$.

Hence
$$\sup_n \gamma_{d(n)} \leq \gamma_0.$$

Further, the order equivalence, implies that the set of all rationals $q_{d(n)}$, $n=1, 2, \ldots$ is well ordered. As this set coincides with $C^{(x \times \mathbf{u})}$ it follows that $x \times \mathbf{u}$ belongs to E.

Also, for each $n \geq 1$, the ordinal

$$\tau C_{d(n)}^{(x \times \mathbf{u})}$$

is the ordinal of the set of rationals, of the form $q_{d(m)}$ with $q_{d(m)} < q_{d(n)}$, a set which is order-isomorphic to the set of ordinals of the form $\gamma_{d(m)}$ with $\gamma_{d(m)} < \gamma_{d(n)}$. Hence

$$\tau C_{d(n)}^{(x \times \mathbf{u})} \leqslant \gamma_{d(n)}, \quad \text{for} \quad n = 1, 2, \ldots. \tag{33}$$

Similarly

$$\tau C^{(x \times \mathbf{u})}$$

is the ordinal of the set of all rationals of the form $q_{d(n)}$, so that

$$\tau C^{(x \times \mathbf{u})} \leqslant \sup_n \gamma_{d(n)} \leqslant \gamma_0, \tag{34}$$

by (32).

We now prove inductively that $\mathbf{u}$ belongs to each set of the sequence

$$E_0^{(x)}, H_1^{(x)}, E_1^{(x)}, H_2^{(x)}, \ldots.$$

Combining (25) and (34) we have

$$\tau C_0^{(x \times \mathbf{u})} = \gamma_0(x) < \Omega,$$

so that $\mathbf{u} \in E_0^{(x)}$ by (26). If $\mathbf{u} \in E_{n-1}^{(x)}$ for some $n \geqslant 1$, then by the definition (27) of $d_n(x)$ we have

$$d_n(x) \leqslant \tau D_n^{(x \times \mathbf{u})},$$

and, by the construction of D_n and the result $\mathbf{u} \in \mathbf{J}_{d(n)}$, we have

$$\tau D_n^{(x \times \mathbf{u})} \leqslant d_n(x),$$

so that

$$\tau D_n^{(x \times \mathbf{u})} = d_n(x)$$

and

$$\mathbf{u} \in H_n^{(x)}.$$

Now, if $\mathbf{u} \in H_n^{(x)}$ for some $n \geqslant 1$, by the definition (25) of $\gamma_n(x)$,

$$\gamma_n(x) \leqslant \tau C_n^{(x \times \mathbf{u})}.$$

If n is not a member of the sequence $\{d(m)\}$ then $\mathbf{u} \notin \mathbf{J}_n$ and so $\gamma_n(x) = \tau C_n^{(x \times \mathbf{u})} = 0$. In this case $\mathbf{u} \in E_n^{(x)}$ by (26). But if $n = d(m)$, for some $m \geqslant 1$, then $\mathbf{u} \in \mathbf{J}_{d(m)}$ so that

$$\tau C_{d(m)}^{(x \times \mathbf{u})} \leqslant \gamma_{d(m)}(x)$$

by (33). Since $\mathbf{u} \in H_n^{(x)}$, it follows from the definition (25) of $\gamma_n(x)$, that

$$\gamma_n(x) \leqslant \tau C_n^{(x \times \mathbf{u})}.$$

Hence

$$\tau C_n^{(x \times \mathbf{u})} = \gamma_n(x)$$

and $\mathbf{u} \in E_n^{(x)}$ by (26). Thus it follows by induction that $\mathbf{u}$ lies in each set of the sequence

$$H_0^{(x)}, E_0^{(x)}, H_1^{(x)}, E_1^{(x)}, H_2^{(x)}, \dots,$$

and so to $U^{(x)}$. This completes the proof.

10. Uniformization in X×Y, when Y is a complete separable metric space

Our aim in this section is to use the mapping technique explained in the Introduction to extend our uniformization Theorem 17 to a corresponding theorem that applies in a space $X \times Y$ where Y is a complete separable metric space. The proof depends on the well-known result that a complete separable metric space is a continuous one-one image of a closed subset of $\mathbf{I}$, see, for example, C. Kuratowski [18] page 443. As it is no more difficult, we state and prove the result for any Hausdorff space Y that is a one-one continuous image of a closed subset of $\mathbf{I}$.

THEOREM 18. *Suppose that each open set in a topological space X is a Souslin set. Let Y be a Hausdorff space that is a continuous one-one image of some closed subset of $\mathbf{I}$ (for example, any complete separable metric space). Let E be the complement of a Souslin set in $X \times Y$. Then there is a set U, that is a complement of a Souslin set in $X \times Y$, and that satisfies:*

(a) $U \subset E$;

(b) $\pi_X U = \pi_X E$; *and*

(c) *for each x in $\pi_X E$ the set $(\{x\} \times Y) \cap U$ consists of a single point.*

Proof. Let f be a function, defined on a closed set $\mathbf{H}$ of $\mathbf{I}$ that maps $\mathbf{H}$ one-one onto Y. Define a function φ on $X \times \mathbf{H}$ by taking

$$\varphi(x \times \mathbf{i}) = x \times f(\mathbf{i}),$$

for all $\mathbf{i}$ in $\mathbf{H}$. Then φ maps $X \times \mathbf{H}$ one-one onto $X \times Y$. As the inverse image of any open rectangle in $X \times Y$ is an open rectangle in $X \times \mathbf{H}$, it follows that φ is continuous from $X \times \mathbf{H}$ to $X \times Y$. Thus the inverse image of any closed set in $X \times Y$ is closed in $X \times \mathbf{H}$ and so closed in $X \times \mathbf{I}$. Hence the set $A^* = \varphi^{-1}(A)$, where A is the Souslin set

$$A = (X \times Y)\backslash E,$$

is a Souslin set in $X \times \mathbf{I}$. Write

$$E^* = (X \times \mathbf{H})\backslash A^*.$$

As $\mathbf{I}\backslash\mathbf{H}$ is a Souslin set in $\mathbf{I}$

$$E^* = (X \times \mathbf{I})\backslash[(X \times \{\mathbf{I}\backslash\mathbf{H}\}) \cup A^*]$$

is a complement of a Souslin set in $X \times \mathbf{I}$.

Now, by Theorem 17, we can choose a complement U^* of a Souslin set in $X \times \mathbf{I}$ that uniformizes E^* in $X \times \mathbf{I}$. As

$$U^* \subset E^* \subset X \times \mathbf{H},$$

and φ is a one-one map from $X \times \mathbf{H}$ to $X \times Y$ it follows immediately that the set

$$U = \varphi(U^*)$$

uniformizes E in $X \times Y$. It remains to prove that U is the complement of a Souslin set in $X \times Y$.

We introduce some notation to facilitate an application of Theorem 8. We note that Y has the descriptive Borel representation $Y = K(\mathbf{I})$ where K is the semi-continuous function from $\mathbf{J}$ to $\mathcal{K}(Y)$ defined by:

$$K(\mathbf{j}) = \{f(\mathbf{j})\}, \quad \text{if} \quad \mathbf{j} \in \mathbf{H};$$

$$K(\mathbf{j}) = \emptyset, \quad \text{if} \quad \mathbf{j} \notin \mathbf{H}.$$

Further the sets $K(\mathbf{j})$, $\mathbf{j} \in \mathbf{J}$ are disjoint. Similarly the space $X \times Y$ has the disjoint Souslin representation

$$X \times Y = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}|n),$$

where we take

$$F^*(\mathbf{i}) = X \times K(\mathbf{i})$$

for each $\mathbf{i}$ in $\mathbf{I}$ and define $F(\mathbf{i}|n)$ by

$$F(\mathbf{i}|n) = \text{cl}\, F^*(\mathbf{I}_{\mathbf{i}|n}).$$

By the Corollary to Theorem 2, this yields the formula

$$F(\mathbf{i}) = F^*(\mathbf{i}) = X \times K(\mathbf{i}).$$

Hence for each $\mathbf{i}$ in $\mathbf{I}$ and each integer $m \geq 1$ we have

$$F(\mathbf{I}_{\mathbf{i}|m}) \subset X \times K(\mathbf{J}_{\mathbf{i}|m}).$$

Further the map ω: $X \times Y \to X \times \mathbf{I}$ that takes $x \times y$ in $X \times Y$ to the point

$$\omega(x \times y) = x \times \mathbf{i},$$

where $\mathbf{i} = \mathbf{i}(x \times y)$ is the unique $\mathbf{i}$ in $\mathbf{I}$ with $x \times y \in F(\mathbf{i})$ clearly coincides with the map φ^{-1}. Now all the preliminary conditions of Theorem 8 are satisfied. Further

$$(X \times \mathbf{I}) \backslash U^*$$

is a Souslin set in $X \times \mathbf{I}$. So, by the last assertion of Theorem 8,

$$\omega^{-1}[(X \times \mathbf{I})\backslash U^*] = (\varphi^{-1})^{-1}[(X \times \mathbf{I})\backslash U^*] = \varphi[(X \times \mathbf{H})\backslash U^*] = (X \times Y)\backslash U$$

is a Souslin set in $X \times Y$. Hence U is the complement of a Souslin set in $X \times Y$ as required.

11. Partial uniformization in X×Y, when Y is a descriptive Borel space

In this section we use the mapping Theorem 8 together with the uniformization Theorem 17 to prove the partial uniformization Theorem 19 stated in the Introduction.

Proof of Theorem 19. Let the complement A of E in $X \times Y$ have the Souslin representation

$$A = A(\mathbf{I}), \quad A(\mathbf{i}) = \bigcap_{n=1}^{\infty} A(\mathbf{i}\,|\,n),$$

the sets $A(\mathbf{i}\,|\,n)$ being closed.

As each open set in $X \times Y$ has a disjoint Souslin representation, it follows from Theorem 7 that $X \times Y$ has a disjoint Souslin representation

$$X \times Y = F_1(\mathbf{I}), \quad F_1(\mathbf{i}) = \bigcap_{n=1}^{\infty} F_1(\mathbf{i}\,|\,n),$$

with the sets $F_1(\mathbf{i}\,|\,n)$ all closed, such that each set $A(\mathbf{j}\,|\,m)$ is the union of those of the fragments $F_1(\mathbf{i})$ that it meets.

As Y has the representation $Y = K(\mathbf{I})$ as a descriptive Borel set it has the repesentation

$$Y = K(\mathbf{I}), \quad K(\mathbf{i}) = \bigcap_{n=1}^{\infty} Y(\mathbf{i}\,|\,n),$$

where

$$Y(\mathbf{i}\,|\,n) = \text{cl } K(\mathbf{I}_{\mathbf{i}|n}),$$

by the corollary to Theorem 2. Hence $X \times Y$ has the disjoint Souslin representation

$$X \times Y = X \times K(\mathbf{I}), \quad X \times K(\mathbf{i}) = \bigcap_{n=1}^{\infty} [X \times Y(\mathbf{i}\,|\,n)].$$

Applying Theorem 3 to these two disjoint Souslin representations of $X \times Y$ we obtain a disjoint Souslin representation of $X \times Y$ as

$$X \times Y = F(\mathbf{I}), \quad F(\mathbf{i}) = \bigcap_{n=1}^{\infty} F(\mathbf{i}\,|\,n),$$

the sets $F(\mathbf{i}\,|\,n)$ being closed, so that each set $F_1(\mathbf{j})$ and each set $X \times K(\mathbf{k})$ is the union of those fragments $F(\mathbf{i})$ that it meets. This ensures that each set $A(\mathbf{j}\,|\,n)$ is the union of those sets

$F(\mathbf{i})$ that it meets. Further, by the corollary to Theorem 3, we can ensure that, for each $\mathbf{i}$ in $\mathbf{I}$ there is a $\mathbf{j}$ in $\mathbf{J}$ such that for each integer $m \geq 1$, there is an integer n with

$$F(\mathbf{I}_{\mathbf{i}|n}) \subset X \times K(\mathbf{J}_{\mathbf{j}|m}). \tag{35}$$

We now define a map $\omega: X \times Y \to X \times \mathbf{I}$ by taking

$$\omega(x \times y) = x \times \mathbf{i},$$

where $\mathbf{i} = \mathbf{i}(x \times y)$ is the unique $\mathbf{i}$ in $\mathbf{I}$ with

$$x \times y \in F(\mathbf{i}).$$

We note that all the conditions of Theorem 8 are satisfied by the sets and mappings that we have introduced in this proof. It follows from Theorem 8 that ωA is a Souslin set in $X \times \mathbf{I}$, and that $\omega(X \times Y)$ is closed in $X \times \mathbf{I}$.

Now, given an open set G in X, the set $G \times Y$ is open in $X \times Y$ and has a disjoint Souslin representation, and is in particular a Souslin set in $X \times Y$. It follows by [25] that

$$G = \pi_X\{G \times Y\}$$

is a Souslin set in X, so that $G \times \mathbf{I}$ is a Souslin set in $X \times \mathbf{I}$. As $\mathbf{I}$ has a countable basis for its open sets it follows that each open set in $X \times \mathbf{I}$ is a Souslin set. Hence

$$[\omega(X \times Y)] \backslash [\omega A]$$

is the complement in $X \times \mathbf{I}$ of the Souslin set

$$[\omega A] \cup \{(X \times \mathbf{I}) \backslash \omega(X \times Y)\}.$$

Thus the conditions of Theorem 17 are satisfied and there is a set W, that is the complement of a Souslin set B in $X \times \mathbf{I}$ and that satisfies:

(a) $W \subset [\omega(X \times Y)] \backslash [\omega A]$;
(b) $\pi_X W = \pi_X[\omega(X \times Y)] \backslash [\omega A]$;
(c) for each x in $\pi_X W$ the set $(\{x\} \times \mathbf{I}) \cap W$ consists of a single point.

By Theorem 8 the set $\omega^{-1}B$ is a Souslin set in $X \times Y$. Then

$$U = [X \times Y] \backslash \omega^{-1} B = \omega^{-1} W$$

is a complement of a Souslin set in $X \times Y$.

By (a) we have

$$B=[X\times\mathbf{I}]\backslash W\supset \omega A$$

so that $\omega^{-1}B\supset A$ and $U\subset E$. This implies that

$$\pi_X U\subset \pi_X E.$$

Now, if $x\in\pi_X E$, then

$$x\in\pi_X\omega E=\pi_X\{[\omega(X\times Y)]\backslash[\omega A]\}.$$

Hence, by (*b*) we have $x\in\pi_X W$. So there is a point $\mathbf{w}$ in $\mathbf{I}$ with $x\times\mathbf{w}$ in W, and so in $\omega(X\times Y)$ but not in ωA. Hence there is a point y of Y with $x\times y$ in $\omega^{-1}W=U$. Thus

$$\pi_X E\subset\pi_X U,$$

and

$$\pi_X E=\pi_X U.$$

Now, for each x in $\pi_X E$, we have $x\in\pi_X W$ from the last paragraph, and so by (*c*), the set

$$(\{x\}\times\mathbf{I})\cap W$$

consists of a single point, $\mathbf{w}(x)$ say, and

$$(\{x\}\times Y)\cap U=[\{x\}\times Y]\cap F(\mathbf{w}(x)).$$

So, by (35), there is $\mathbf{j}$ in $\mathbf{J}$ with

$$(\{x\}\times Y)\cap U\subset\{x\}\times K(\mathbf{j}).$$

Thus

$$\pi_Y\{\pi_X^{-1}(x)\cap U\}=\pi_Y[(\{x\}\times K(\mathbf{j}))\cap F(\mathbf{w}(x))],$$

and so is a compact subset of $K(\mathbf{j})$. This completes the proof.

12. Uniformizing functions

In this section we use some of the arguments of Sion [28] in conjunction with a transfinite application of Theorem 19 to prove Theorem 20, stated in the introduction.

Proof of Theorem 20. Let $\{G_\alpha\}_{\alpha<\Omega}$ be a base for the open sets of Y. For each ordinal α with $\alpha<\Omega$, the sets G_α, $Y\backslash G_\alpha$ are $\mathfrak{F}_\sigma$-sets.

By Lemma 2 of [24], it follows, from the suppositions that Y is descriptive Borel and that each open set of Y is an $\mathfrak{F}_\sigma$-set, that each $\mathfrak{F}_\sigma$-set of Y is descriptive Borel. Hence G_α and $Y\backslash G_\alpha$ are disjoint descriptive Borel sets and Y has a representation

$$Y=K_\alpha(\mathbf{I}),$$

where K_α is a semi-continuous map from $\mathbf{I}$ to $\mathcal{K}(Y)$, with

$$K_\alpha(\mathbf{i}) \cap K_\alpha(\mathbf{j}) = \emptyset$$

whenever $\mathbf{i}$, $\mathbf{j}$ are distinct points of $\mathbf{I}$, and both G_α and $Y \backslash G_\alpha$ are unions of the fragments $K_\alpha(\mathbf{i})$, with $\mathbf{i} \in \mathbf{I}$, that they meet.

Write $U_1 = E$. For each ordinal α, with $1 \leqslant \alpha < \Omega$, let $U_{\alpha+1}$ be the set obtained from U_α by appliction of Theorem 19 with the descriptive Borel representation

$$Y = K_\alpha(\mathbf{I}).$$

For each limit ordinal α with $\alpha \leqslant \Omega$, let

$$U_\alpha = \bigcap_{\beta < \alpha} U_\beta.$$

This provides an inductive definition of U_α for $1 \leqslant \alpha \leqslant \Omega$.

It follows inductively that the sequence U_α, $1 \leqslant \alpha \leqslant \Omega$ is a decreasing sequence and that the sets U_α, $1 \leqslant \alpha < \Omega$ are complements of Souslin sets in $X \times Y$.

Now for each x in $\pi_X E$, Theorem 19 ensures that $U_2^{(x)}$ is compact and non-empty. Similarly, given that $U_\alpha^{(x)}$ is compact and non-empty, Theorem 19 ensures that $U_{\alpha+1}^{(x)}$ is compact and non-empty. Further if α is a limit ordinal with $\alpha \leqslant \Omega$, knowledge that the sets $U_\beta^{(x)}$, $\beta < \alpha$, are compact and non-empty, ensures that $U_\alpha^{(x)}$ is compact and nonempty. It follows, by transfinite induction, that $U_\alpha^{(x)}$ is compact and non-empty for all α with $2 \leqslant \alpha \leqslant \Omega$ and for all x in $\pi_X E$.

We now prove that for each x in $\pi_X E$ the set $U_\Omega^{(x)}$ consists of a single point. Suppose we had

$$x \times y \in U_\Omega, \quad x \times z \in U_\Omega$$

with $y \neq z$. Then we can choose ordinals α, β with

$$y \in G_\alpha, \quad z \in G_\beta, \quad G_\alpha \cap G_\beta = \emptyset.$$

By Theorem 19,
$$U_{\alpha+1}^{(x)} \subset K_\alpha(\mathbf{i})$$

for some $\mathbf{i}$ in $\mathbf{I}$. So
$$y \in U_\Omega^{(x)} \subset U_{\alpha+1}^{(x)} \subset K_\alpha(\mathbf{i}), \quad y \in G_\alpha.$$

Thus, $G_\alpha \cap K_\alpha(\mathbf{i}) \neq \emptyset$ and, by the choice of K_α we have

$$K_\alpha(\mathbf{i}) \subset G_\alpha.$$

Hence
$$U_\Omega^{(x)} \subset U_{\alpha+1}^{(x)} \subset G_\alpha.$$

Similarly
$$U_\Omega^{(x)} \subset U_{\beta+1}^{(x)} \subset G_\beta,$$

contrary to the results $U_\Omega^{(x)} \neq \emptyset$, $G_\alpha \cap G_\beta = \emptyset$. This shows that $U_\Omega^{(x)}$ consists of a single point for each x of $\pi_X E$ and enables us to define a function f on $\pi_X E$ by the requirement

$$\{f(x)\} = U_\Omega^{(x)}$$

for all x in $\pi_X E$.

Clearly f is a uniformizing function from $\pi_X E$ to Y with the property $x \times f(x) \in E$ for all x in $\pi_X E$.

Let V be any open set in Y. By hypothesis, V is the union of a countable sequence, say

$$G_{\alpha(1)},\ G_{\alpha(2)}, \ldots$$

of elements of the basis for Y. So

$$f^{-1}[V] = \pi_X[(X \times V) \cap U_\Omega] = \bigcup_{n=1}^{\infty} \pi_X[(X \times G_{\alpha(n)}) \cap U_\Omega].$$

Hence to prove that $f^{-1}[V]$ is the projection on X of the complement of a Souslin set in $X \times Y$ it suffices to prove that each set

$$\pi_X[(X \times G_\alpha) \cap U_\Omega], \quad 1 \leqslant \alpha < \Omega,$$

is of this form.

But, given an ordinal α with $1 \leqslant \alpha < \Omega$ it is clear that

$$\pi_X[(X \times G_\alpha) \cap U_\Omega] \subset \pi_X[(X \times G_\alpha) \cap U_{\alpha+1}].$$

On the other hand, if
$$x \in \pi_X[(X \times G_\alpha) \cap U_{\alpha+1}]$$
we have
$$G_\alpha \cap U_{\alpha+1}^{(x)} \neq \emptyset.$$
As before this implies
$$U_\Omega^{(x)} \subset U_{\alpha+1}^{(x)} \subset G_\alpha$$
so that
$$x \in \pi_X[(X \times G_\alpha) \cap U_\Omega].$$
Thus
$$\pi_X[(X \times G_\alpha) \cap U_\Omega] = \pi_X[(X \times G_\alpha) \cap U_{\alpha+1}]$$

and so is the projection on X of the complement of a Souslin set in $X \times Y$. This completes the proof.

References

[1]. BRAUN, S., Sur l'uniformisation des ensembles fermés. *Fund. Math.*, 28 (1937), 214–218.

[2]. BRESSLER, D. W. & SION, M., The current theory of analytic sets. *Canad. J. Math.*, 16 (1964), 207–230.

[3]. CHOQUET, G., Ensembles boreliens et analytiques dans les espaces topologiques. *C.R. Acad. Sci. Paris*, 232 (1951), 2174–2176.

[4]. ——, Theory of capacities. *Ann. Inst. Fourier (Grenoble)*, 5 (1953–54), 131–295.

[5]. ——, Ensembles K-analytiques et K-sousliniens. *Ann. Inst. Fourier (Grenoble)*, 9 (1959), 75–89.

[6]. ——, Convergences. *Ann. Univ. Grenoble, Sci. Math. Phys.*, 23 (1947), 57–112.

[7]. FROLÍK, Z., On analytic spaces. *Bull. Acad. Polon.*, 9 (1961), 721–726.

[8]. ——, On borelian and bianalytic spaces. *Czechoslovak Math. J.*, 11 (86), (1961), 629–631.

[9]. ——, A contribution to the descriptive theory of sets and spaces. *General topology and its relations to modern analysis and algebra.* (Proc. Sym. Prague, Sept. 1961), Adademic Press, New York, 1962, pp. 157–173.

[10]. ——, On the descriptive theory of sets. *Czechoslovak Math. J.*, 13 (88), (1963), 335–359.

[11]. ——, On bianalytic spaces. *Czechoslovak Math. J.*, 13 (88), (1963), 561–573.

[12]. ——, On coanalytic and bianalytic spaces. *Bull. Acad. Polon.*, 12 (1964), 527–530.

[13]. ——, Baire sets that are Borelian subspaces. *Proc. Roy. Soc. Ser. A*, 299 (1967), 287–290.

[14]. KNOWLES, J. D. & ROGERS, C. A., Descriptive sets. *Proc. Roy. Soc. Ser. A*, 291 (1965), 353–367.

[15]. KONDÔ, M., Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe. *Japan J. Math.*, 15 (1939), 197–230.

[16]. ——, Quelques principes dans la théorie descriptive des ensembles. *J. Math. Soc. Japan*, 3 (1951), 91–98.

[17]. KUNUGUI, K., La théorie des ensembles analytiques et les espaces abstraits. *J. Fac. Sci. Hokkaido Imp. Univ.*, Ser. 1, 4 (1935), 1–40.

[18]. KURATOWSKI, K., *Topology, Vol. 1.* Academic Press, New York, 1966.

[19]. LUSIN, N., *Leçons sur les ensembles analytiques et leurs applications.* Paris, 1930.

[20]. ——, Sur le problème de M. J. Hadamard d'uniformisation des ensembles. *Mathematica*, 4 (1930), 54–66.

[21]. LUSIN, N. & NOVIKOFF, P., Choix effectif d'un point dans un complémentaire analytique arbitraire, donné par un crible. *Fund. Math.*, 25 (1935), 559–560.

[22]. MICHAEL, E., Continuous selections I. *Ann. of Math.* (*2*), 63 (1956), 361–382.

[23]. ROGERS, C. A., Analytic sets in Hausdorff spaces. *Mathematika*, 11 (1964), 1–8.

[24]. ——, Descriptive Borel sets. *Proc. Roy. Soc. Ser. A*, 286 (1965), 455–478.

[25]. ROGERS, C. A. & WILLMOTT, R. C., On the projection of Souslin sets. *Mathematika*, 13 (1966), 147–150.

[26]. SAMPEI, Y., On the uniformization of the complement of an analytic set. *Comment. Math. Univ. St. Paul.*, 10 (1960), 57–62.

[27]. SIERPINSKI, W., Sur l'uniformisation des ensembles mesurables (B). *Fund. Math.*, 16 (1930), 136–139.

[28]. SION, M., On uniformization of sets in topological spaces. *Trans. Amer. Math. Soc.*, 96 (1960), 237–245.

[29]. ——, On analytic sets in topological spaces. *Trans. Amer. Math. Soc.*, 96 (1960), 341–354.

[30]. ——, Topological and measure theoretic properties of analytic sets. *Proc. Amer. Math. Soc.*, 11 (1960), 769–776.

[31]. ——, Continuous images of Borel sets. *Proc. Amer. Math. Soc.*, 12 (1961), 385–391.

[32]. ——, On capacitability and measurability. *Ann. Inst. Fourier* (*Grenoble*), 13 (1963), 83–99.

[33]. ŠNEIDER, V. E., Continuous images of Souslin and Borel sets: Metrization theorems (in Russian). *Dokl. Akad. Nauk SSSR*, 50 (1945), 77–79.

[34]. ——, Descriptive theory of sets in topological spaces (in Russian). *Dokl. Akad. Nauk SSSR*, 50 (1945), 81–83.

[35]. ——, Descriptive theory of sets in topological spaces (in Russian). *Učenye Zapiski Moskov. Gos. Univ.*, 135, II (1948), 37–85.

[36]. STONE, M. H. & VON NEUMANN, J., The determination of representative elements in the residual classes of a Boolean algebra. *Fund. Math.*, 25 (1935), 353–378.

[37]. SUZUKI, Y., On the uniformization principle. *Proc. Symp. on the Foundations of Math., Katada* (*Japan*) 1962, 137–144.

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