HARMONIC ANALYSIS AND THETA-FUNCTIONS

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We know that theta-functions have appeared in mathematics in two different ways, one in the theory of polarized abelian varieties over $\mathbf{C}$ and another in the analytic theory of numbers. We shall give a definition of theta-functions which is general enough to cover both cases. This has been made possible by a recent work of A. Weil that has appeared in two papers [11], [12]. We shall also discuss a supplement to his work by proving certain continuity theorems. The following is a more detailed explanation of this paper.

If G is a locally compact abelian group, the regular representation of G and the Fourier transform of the regular representation of its dual satisfy a well-known commutation relation. Therefore, the images of G and its dual by these representations generate a locally compact, two-step nilpotent subgroup $\mathbf{A}(G)$ of the full unitary group $\mathrm{Aut}(L^2(G))$ with the strong operator topology. Consider the normalizer $\mathbf{B}(G)$ of $\mathbf{A}(G)$ in $\mathrm{Aut}(L^2(G))$. Then, if $\mathcal{S}(G)$ is the Schwarz–Bruhat space of G, there exists a mapping $\mathbf{B}(G)\times\mathcal{S}(G)\rightarrow\mathcal{S}(G)$ defined by $(\mathrm{s}, \Phi)\rightarrow \mathrm{s}\Phi$. On the other hand, if Γ is a closed subgroup of G, every Φ in $\mathcal{S}(G)$ gives rise to a function F_Φ on $\mathbf{B}(G)$ by the following integral

$$F_\Phi(\mathrm{s}) = \int_\Gamma (\mathrm{s}\Phi)(\xi)\,d\xi$$

taken with respect to the Haar measure $d\xi$ on Γ. Weil has shown that $\mathrm{s}\Phi$ depends continuously on Φ and that F_Φ has an invariance property with respect to a certain subgroup of $\mathbf{B}(G)$ determined by Γ. Then he has specialized to the arithmetic case, i.e., to the case when G is the localization or the adelization of a finite dimensional vector space over a field, which is either a number field or a function field of one variable with a finite constant field. In this case, he has introduced the metaplectic group $\mathrm{Mp}(G)$ and proved the continuity of $(\mathrm{s}, \Phi)\rightarrow \mathrm{s}\Phi$ and $\mathrm{s}\rightarrow F_\Phi(\mathrm{s})$ restricting s to $\mathrm{Mp}(G)$. We shall discuss these continuity properties in general, i.e., in the case when G is a locally compact abelian group satisfying

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no further assumption. The mapping $\mathbf{B}(G)\times \mathcal{S}(G)\to\mathcal{S}(G)$ is separately continuous but *not* continuous in general. However, if Σ is a locally compact subset of $\mathbf{B}(G)$, the induced mapping $\Sigma\times\mathcal{S}(G)\to\mathcal{S}(G)$ is continuous. Moreover, if $\mathfrak{G}\to\mathrm{Sp}\,(G)$ is a continuous homomorphism of a locally compact group $\mathfrak{G}$ to the symplectic group $\mathrm{Sp}\,(G)$ of G, the fiber-product

$$\mathbf{B}(G)_{\mathfrak{G}}=\mathbf{B}(G)\times_{\mathrm{Sp}(G)}\mathfrak{G}$$

is always locally compact. This will be shown by carefully analyzing the homomorphism $\mathbf{B}(G)\to\mathrm{Sp}\,(G)$. It will be shown also that the function F_Φ is always continuous on $\mathbf{B}(G)$. It appears that these results settle basic continuity problems concerning the group $\mathbf{B}(G)$. This is the outline of Part I. In Part II, we shall introduce a dense subspace $\mathcal{G}(G)$ of $\mathcal{S}(G)$ which is $\mathbf{B}(G)$-stable. We then define theta-functions as functions of the form F_Φ with Φ in $\mathcal{G}(G)$. Among other things, it will be shown that there exists essentially but one theta-function and that every automorphic function can be uniformly approximated on a given compact set by a theta-function. At the end of Part II, we shall discuss the case when $G=\mathbf{R}^n$ and Γ is a lattice in $\mathbf{R}^n$ as an example.

I. The group $\mathbf{B}(G)$ and a continuity theorem

1. We shall start by recalling some of the definitions and terminology in Weil's work cited in the introduction. We shall use the asterisk to denote the autodualization in the category of all locally compact abelian groups. If G is a locally compact abelian group, therefore, its dual is denoted by G^*. We shall denote by T the multiplicative group of complex numbers t satisfying $t\bar{t}=1$ and by $(x, x^*)\to\langle x, x^*\rangle=\langle x^*, x\rangle$ the bicharacter of $G\times G^*$ which puts G and G^* into duality. We shall denote by F the bicharacter of $(G\times G^*)\times(G\times G^*)$ defined by

$$F(w_1, w_2)=F((u_1, u_1^*), (u_2, u_2^*))=\langle u_1, u_2^*\rangle.$$

Also, we shall denote by $A(G)$ the product space $(G\times G^*)\times T$ with the law of composition

$$(w_1, t_1)(w_2, t_2)=(w_1+w_2, F(w_1, w_2)t_1t_2).$$

Then $A(G)$ is a locally compact, two-step nilpotent group with T as its center (if we identify t with $(0, t)$). We shall denote by $B(G)$ the group of topological automorphisms of $A(G)$ inducing the identity on T. We recall that the group of topological automorphisms of an arbitrary locally compact group $\mathfrak{G}$ forms a topological group. If C is a (non-empty) compact subset of $\mathfrak{G}$ and V an open neighborhood of the identity in $\mathfrak{G}$, the set $W(C, V)$ consisting of all topological automorphisms σ of $\mathfrak{G}$ such that the images by $g\to(g\sigma)g^{-1}$, $(g\sigma^{-1})g^{-1}$

of C are both contained in V is a typical open neighborhood of the identity in Aut $(\mathfrak{G})$. Because of this fact, Aut $(A(G))$ forms a topological group and $B(G)$ is an open subgroup of Aut $(A(G))$ of index two. If s is an element of $B(G)$ and if (w, t) is an element of $A(G)$, we have $(w, t)s=(w\sigma, f(w)t)$ with a topological automorphism σ of $G\times G^*$ and a continuous mapping $f: G\times G^* \to T$. The necessary and sufficient condition for such pair (σ, f) to define an element of $B(G)$ as above is that

$$f(w_1+w_2)f(w_1)^{-1}f(w_2)^{-1} = F(w_1\sigma, w_2\sigma)\,F(w_1, w_2)^{-1}$$

for every w_1, w_2 in $G\times G^*$. It follows from this fact that σ is an element of the symplectic group Sp (G) of G and f, an element of the group of second degree characters $X_2(G\times G^*)$ of $G\times G^*$. We recall that Sp (G) is the subgroup of Aut $(G\times G^*)$ consisting of those σ which keep $(w_1, w_2)\to F(w_1, w_2)\,F(w_2, w_1)^{-1}$ invariant. It is clear that Sp (G) is a closed subgroup of Aut $(G\times G^*)$. We shall denote s by (σ, f).

Now, we shall use Haar measures on G and G^* which are dual in the sense that the Fourier transformation gives a (norm-preserving) isomorphism of $L^2(G)$ to $L^2(G^*)$. If Φ is an element of $L^2(G)$ and if we put

$$(U(w, t)\Phi)(x) = t\cdot\Phi(x+u)\langle x, u^*\rangle$$

for $w=(u, u^*)$, we get a unitary representation U of $A(G)$ in $L^2(G)$. That is to say, U is a continuous homomorphism of $A(G)$ to the full unitary group Aut$(L^2(G))$ considered as a topological group by the strong operator topology.

LEMMA 1. *The unitary representation U gives rise to a topological isomorphism of $A(G)$ to its image group* $\mathbf{A}(G)$.

For the sake of completeness, we shall give a proof for this easy lemma. We have only to show that, if $U(w,t)$ is close to the identity in Aut $(L^2(G))$, (w,t) is close to the identity in $A(G)$. We take a complex-valued continuous function Φ on G with compact support C of norm one. Then the condition that $\|U(w, t)\Phi-\Phi\|$ is less than $2^{\frac{1}{2}}$ implies that u is a difference of two points of C. Therefore, if C is small, u is close to 0 in G. If we pass to the equivalent representation of U in $L^2(G^*)$ by the Fourier transformation, the roles of u and u^* interchange. Therefore u^* is close to 0 in G^*. Consequently $U(w, 1)$ is close to the identity in Aut $(L^2(G))$, hence t in $U(w, t)=t\cdot U(w, 1)$ is close to 1 in T. This proves the lemma.

We shall denote by $\mathbf{B}(G)$ the normalizer of $\mathbf{A}(G)$ in Aut $(L^2(G))$. We note that both $\mathbf{A}(G)$ and $\mathbf{B}(G)$ are closed subgroups of Aut $(L^2(G))$. In fact, every locally compact subgroup of a topological group is closed, and also the normalizer of a closed subgroup of a topological group is closed. If $\mathbf{s}$ is an element of $\mathbf{B}(G)$, there exists an element s of $B(G)$ satisfying

$$\mathbf{s}^{-1}U(w, t)\mathbf{s} = U((w, t)s)$$

for every (w, t) in $A(G)$. If we denote by $\mathbf{T}$ the image of T in $\mathbf{A}(G)$, Theorem 1 in [11] asserts that the centralizer of $\mathbf{A}(G)$ in Aut $(L^2(G))$ is T (i.e., the unitary representation U is irreducible) and that the homomorphism $\pi:\mathbf{B}(G)\to B(G)$ defined by $\mathbf{s}\to s$ is surjective with $\mathbf{T}$ as its kernel. We shall examine the homomorphism π closely.

LEMMA 2. *Let $\mathfrak{G}$ denote an arbitrary topological group, H a Hilbert space and U a projective unitary representation of $\mathfrak{G}$ in H. Then the corresponding homomorphism $\mathfrak{G}\to\mathrm{Aut}(H)/\mathbf{T}$, in which $\mathbf{T}$ is the compact group of scalar multiplications in H by elements of T, is continuous.*

Again, for the sake of completeness, we shall give a proof for this lemma. We recall that a projective unitary representation U of $\mathfrak{G}$ in H is a mapping $U:\mathfrak{G}\to\mathrm{Aut}\,(H)$ satisfying

$$U(g)\,U(g') = \lambda(g, g')\,U(g)\,U(g')$$

with $\lambda(g, g')$ in T for every g, g' in $\mathfrak{G}$ such that, if K is a "Hilbert–Schmidt operator" on H and x is in H, the correspondence $g\to U(g)^{-1}KU(g)x$ defines a continuous mapping $\mathfrak{G}\to H$. By the first condition, we get a well-defined homomorphism $\mathfrak{G}\to\mathrm{Aut}\,(H)/\mathbf{T}$. Now, if x is an arbitrary unit vector in H, we can modify $U(g)$ by an element of T so that we get $(U(g)x, x)\geqslant 0$. Then, taking the projection of H to the subspace $\mathbf{C}x$ as K, we get the following inequality

$$\|U(g)^{-1}KU(g)x - x\|^2 = 1-(U(g)x, x)^2 \geqslant 1-(U(g)x, x) = (\tfrac{1}{2})\|U(g)x-x\|^2.$$

This certainly implies the continuity of $\mathfrak{G}\to\mathrm{Aut}\,(H)/\mathbf{T}$.

Now, we shall prove the following result:

PROPOSITION 1. *The isomorphism $\mathbf{B}(G)/\mathbf{T}\to B(G)$ is topological.*

Proof. First of all, if a locally compact group A is contained in a topological group B as a normal subgroup, the inner automorphisms of B give rise to topological automorphisms of A. In this way, we get a homomorphism $B\to\mathrm{Aut}\,(A)$. We can verify easily that this homomorphism is continuous. Therefore, by taking $A=\mathbf{A}(G)$ and $B=\mathbf{B}(G)$, we get a continuous homomorphism $\mathbf{B}(G)\to\mathrm{Aut}\,(\mathbf{A}(G))$. Since $B(G)$ is the image group by the product of this homomorphism and the topological isomorphism Aut $(\mathbf{A}(G)) \rightarrow$ Aut $(A(G))$, we see that π is continuous. Therefore $\mathbf{B}(G)/\mathbf{T} \rightarrow B(G)$ is a continuous isomorphism. We shall show that its inverse is also continuous. We pick $\mathbf{s}$ arbitrarily from $\pi^{-1}(s)$ for each s in $B(G)$. Then, by Lemma 2 we have only to show that the correspondence $s \rightarrow \mathbf{s}$ defines a projective unitary representation of $B(G)$ in $L^2(G)$. This has been proved by Segal [7] (in a slightly different case under the assumption that $x \rightarrow 2x$ defines a topological automorphism of G). We shall briefly recall the key points of the proof: (i) If $\mathfrak{G}$ is an arbitrary locally compact group, ψ a complex-valued continuous function on $\mathfrak{G}$ with a compact support, and ε a positive real number, the set $W(\psi, \varepsilon)$ of those σ in Aut $(\mathfrak{G})$ satisfying

$$\|\psi^{\sigma} - \psi\|_1, \ \|\psi^{\sigma^{-1}} - \psi\|_1 \leqslant \varepsilon,$$

in which ψ^{σ} is defined by $\psi^{\sigma}(g) = \psi(g\sigma^{-1})$, contains $W(C, V)$ for some C, V. (ii) If φ is a complex-valued continuous function on $G \times G^*$ with a compact support and

$$U(\varphi) = \int U(w, 1)\varphi(w)\,dw,$$

in which $dw = du\,du^*$ is the Haar measure on $G \times G^*$, we get a Hilbert–Schmidt operator $U(\varphi)$ on $L^2(G)$. Moreover, the set of such $U(\varphi)$ is dense in the Hilbert space of all Hilbert–Schmidt operators on $L^2(G)$. (iii) If φ is as in (ii) and if we put $\varphi^{\#}(w, t) = \varphi(w)\bar{t}$. we have $(\varphi^{\#})^s = (\varphi^s)^{\#}$ with φ^s defined by

$$\varphi^s(w) = f(w\sigma^{-1})\varphi(w\sigma^{-1})$$

for every $s = (\sigma, f)$ in $B(G)$. Moreover, we have $(\varphi^s)^{s'} = \varphi^{ss'}$ and $\|\varphi^s\|_1 = \|\varphi\|_1$ for s, s' in $B(G)$. (iv) If $\mathbf{s}_1, \mathbf{s}_2$ are elements of $\mathbf{B}(G)$ and s_1, s_2 are their π-images, we have

$$\|\mathbf{s}_1^{-1} U(\varphi)\mathbf{s}_1 \cdot \Phi - \mathbf{s}_2^{-1} U(\varphi)\mathbf{s}_2 \cdot \Phi\| \leqslant \|\Phi\| \, \|\varphi^{s_1} - \varphi^{s_2}\|_1.$$

If we combine (i)–(iv), we see that the correspondence $s \rightarrow \mathbf{s}$ does define a projective unitary representation of $B(G)$ in $L^2(G)$. This completes the proof.

Now, if G, H are locally compact abelian groups, the group Mor (G, H) of continuous mappings from G to H forms a topological abelian group by the compact open topology, and it contains Hom (G, H) as a closed subgroup. Moreover, the asterisk gives a topological isomorphism Hom $(G, H) \rightarrow$ Hom (H^*, G^*). This follows from the definition (and from the Pontrjagin duality). In the case when $H = G^*$, therefore, we get a topological automorphism of Hom (G, G^*). We shall denote by $\Sigma(G)$ the closed subgroup of Hom (G, G^*) of invariant or, more suggestively, symmetric elements of Hom (G, G^*). We observe that an element f of $X_2(G)$ gives rise to an element $\varrho = \varrho(f)$ of $\Sigma(G)$ by

$$f(x+y)f(x)^{-1}f(y)^{-1} = \langle x, y\varrho \rangle.$$

Also $X_2(G)$ is a closed subgroup of Mor (G, T), hence it is a topological abelian group (containing $X_1(G)=G^*$ as a closed subgroup). The point is that the homomorphism $X_2(G)\to\Sigma(G)$ defined by $f\to\varrho(f)$ is continuous. On the other hand, if we consider the product space Aut $(G\times G^*)\times X_2(G\times G^*)$ with the law of composition

$$(\sigma, f)(\sigma', f')=(\sigma\sigma', f''),$$

in which f'' is defined by $f''(w)=f(w)f'(w\sigma)$, we get a topological group $B^{\#}(G)$ which contains $B(G)$ as a closed subgroup. All these follow quite easily from the definitions.

Now, the correspondence $s=(\sigma, f)\to\sigma$ defines a homomorphism $B(G)\to$ Aut $(G\times G^*)$ such that the kernel is $X_1(G\times G^*)$ (if we identify $1\times X_1(G\times G^*)$ with $X_1(G\times G^*)$) and the image is contained in Sp (G). Moreover, since this is the product of the inclusion map $B(G)\to B^{\#}(G)$ and the projection $B^{\#}(G)\to$ Aut $(G\times G^*)$, it is continuous. Therefore the product of π and this homomorphism gives a continuous homomorphism $\mathbf{B}(G)\to$ Sp (G). On the other hand, we have

$$(a, 1)^{-1}(w, t)(a, 1)=(w, F(w, a)F(a, w)^{-1}t)$$

for every a in $G\times G^*$. This implies that the kernel of $\mathbf{B}(G)\to$ Sp (G) is $\mathbf{A}(G)$. In this way, we get the following situation

$$\mathbf{B}(G)/\mathbf{A}(G)\to B(G)/X_1(G\times G^*)\to \mathrm{Sp}\ (G),$$

in which the first arrow is a topological isomorphism by Proposition 1 and the second arrow, a continuous monomorphism. We shall see later that it is also a topological isomorphism.

2. If G is a locally compact abelian group, we shall denote by $\mathcal{S}(G)$ the Schwarz–Bruhat space of G. Since otherwise our later proofs may become unintelligible, we shall briefly recall the definition and basic properties of $\mathcal{S}(G)$ at the same time fixing some notations. We refer to [1], [6], [11] for the details.

First of all, we shall recall structure theorems of locally compact abelian groups (cf. [5], [10]). The group G is the union of its "compactly generated" subgroups. A compactly generated abelian group H contains a unique maximal compact subgroup K such that H/K is topologically isomorphic to $\mathbf{R}^n\times\mathbf{Z}^p$ and that the extension splits. Moreover, the compact abelian group K is the inverse limit of K/H' topologically isomorphic to $(\mathbf{R}/\mathbf{Z})^q\times F$, in which F denotes a finite abelian group. In this way, the locally compact abelian group G is represented as a limit of elementary abelian groups $E=H/H'$. A topological abelian group is called "elementary" if it is topologically isomorphic to $\mathbf{R}^n\times\mathbf{Z}^p\times(\mathbf{R}/\mathbf{Z})^q\times F$. We observe that a locally compact abelian group E is elementary if and only if both E and its dual E^* are compactly generated. The class of compactly generated abelian groups and the class of elementary abelian groups have the property that, for a topological abelian group G and for its closed subgroup H, "H and G/H are in the class" implies "G is in the class", and conversely.

Now, if E is an elementary abelian group, since the quotient of E by its maximal compact subgroup is topologically isomorphic to $\mathbf{R}^n \times \mathbf{Z}^p$, we can consider the coimage of a polynomial in $n+p$ letters with complex coefficients by the product of $E \to \mathbf{R}^n \times \mathbf{Z}^p \to \mathbf{R}^{n+p}$. In this way, we get a function P on E which we call a polynomial function on E. In the same way, the euclidean distance from the origin in $\mathbf{R}^{n+p}$ gives rise to a distance function r on E. All non-negative even powers of r are polynomial functions on E. The concept of polynomial functions is intrinsic. On the other hand, since E is an abelian Lie group, we know what we mean by an invariant differential operator D on E. We shall denote by $\mathcal{S}(E)$ the vector space of complex-valued differentiable functions Φ on E such that

$$\| P \cdot D\Phi \|_\infty = \sup_{x \in E} | P(x) (D\Phi)(x) |$$

is finite for every (P, D). The set of seminorms $\|P \cdot D\Phi\|_\infty$ converts $\mathcal{S}(E)$ into a topological vector space over $\mathbf{C}$. Since the set of seminorms $\|P \cdot D\Phi\|_\infty$, in which P, D are "monomials", is countable and defines the same topology in $\mathcal{S}(E)$, the space is metrizable. We note also that the set of seminorms $\|r^m \cdot D\Phi\|_\infty$ for $m = 0, 1, \ldots$ defines the same topology in $\mathcal{S}(E)$.

If G is a general locally compact abelian group, we take a compactly generated subgroup H and its compact subgroup H' such that $E = H/H'$ is elementary. For the sake of simplicity, we say that such pairs (H, H') are "admissible". If we consider the coimage of Φ in $\mathcal{S}(E)$ by the canonical homomorphism $H \to E$, we get a complex-valued function on G with its support contained in H. We shall denote by $\mathcal{S}(H, H')$ the set of all such functions and we introduce in $\mathcal{S}(H, H')$ the structure of a topological vector space by the $\mathbf{C}$-linear bijection $\mathcal{S}(H, H') \to \mathcal{S}(E)$. Then we take the union of all $\mathcal{S}(H, H')$ and introduce the inductive topology. In this way, we get a locally convex topological vector space $\mathcal{S}(G)$ over $\mathbf{C}$, and this is the Schwarz–Bruhat space of G.

If we denote by dh' the Haar measure on H' of total measure 1, we get a continuous $\mathbf{C}$-linear mapping $\pi_{H,H'}: \mathcal{S}(G) \to \mathcal{S}(H, H')$ by

$$\pi_{H, H'}(\Phi)(h) = \int_{H'} \Phi(h + h') dh'.$$

Since the restriction of $\pi_{H,H'}$ to $\mathcal{S}(H, H')$ is the identity, it is a projection of $\mathcal{S}(G)$ to $\mathcal{S}(H, H')$. Therefore $\mathcal{S}(H, H')$ is a direct factor of $\mathcal{S}(G)$ and, in particular, it is a closed subspace of

$\mathcal{S}(G)$. Consequently, the $\mathcal{S}(E)$ defined explicitly for the elementary abelian group E is the Schwarz–Bruhat space of E. A less obvious property of the Schwarz–Bruhat space is that any bounded subset, hence a fortiori any compact subset, of $\mathcal{S}(G)$ is contained in some $\mathcal{S}(H, H')$. For the sake of completeness, we shall prove the following lemma:

LEMMA 3. *If Γ is a closed subgroup of G, the restriction to Γ gives a $\mathbf{C}$-linear continuous mapping* $\mathrm{res}_\Gamma : \mathcal{S}(G) \to \mathcal{S}(\Gamma)$.

Proof. Suppose first that G is elementary and denote it by E. If K is its maximal compact subgroup, $K' = K \cap \Gamma$ is the maximal compact subgroup of Γ. Therefore, the restriction to Γ gives an epimorphism of the ring of polynomial functions on E to that of Γ. On the other hand, there exists a unique monomorphism of the ring of invariant differential operators on Γ to that of E (extending the Lie algebra monomorphism associated with $\Gamma \to E$). Suppose that P' is a polynomial function and D' an invariant differential operator on Γ. Let P denote a polynomial function and D an invariant differential operator on E such that $P \to P'$ and $D' \to D$. Then, for every Φ in $\mathcal{S}(E)$, we have

$$P' \cdot D'(\mathrm{res}_\Gamma \Phi) = \mathrm{res}_\Gamma (P \cdot D\Phi).$$

This implies the continuity of $\mathrm{res}_\Gamma : \mathcal{S}(E) \to \mathcal{S}(\Gamma)$.

We shall consider the general case. We first observe that, if (H, H') is an admissible pair for G, $(H \cap \Gamma, H' \cap \Gamma)$ is one for Γ. The point is that, for any admissible pair (H_Γ, H'_Γ) for Γ, there exists an admissible pair (H, H') for G satisfying $H \cap \Gamma \supset H_\Gamma$ and $H' \cap \Gamma \subset H'_\Gamma$. We can certainly find a compactly generated subgroup H which contains H_Γ. Then $H \cap \Gamma$ is compactly generated, and hence $(H \cap \Gamma)/H_\Gamma$ is finitely generated. Therefore $(H \cap \Gamma)/H'$ is elementary. Consider an admissible pair (H, H') for this H. Then $((H' \cap \Gamma) + H'_\Gamma)/H'_\Gamma$ is a compact subgroup of the elementary abelian group $(H \cap \Gamma)/H'_\Gamma$. Since H' can be taken arbitrarily small and since no elementary abelian group has a small subgroup, we can find H' satisfying $H' \cap \Gamma \subset H'_\Gamma$. The rest follows from the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}(G) & \longrightarrow & \mathcal{S}(\Gamma) \\ \uparrow & & \uparrow \\ \mathcal{S}(H, H') & \to & \mathcal{S}(H \cap \Gamma, H' \cap \Gamma') \end{array}$$

and from the continuity of the bottom arrow. This follows from the continuity of the two arrows in $\mathcal{S}(H/H') \to \mathcal{S}(((H \cap \Gamma) + H')/H') \to \mathcal{S}((H \cap \Gamma)/(H' \cap \Gamma))$.

By definition, every element Φ of $\mathcal{S}(G)$ is in $L^p(G)$ for all p. In particular, if $\mathbf{s}$ is an element of the group $\mathbf{B}(G)$, $\mathbf{s}\Phi$ is defined as an element of $L^2(G)$. The fact is that it has a representative in $\mathcal{S}(G)$. If we denote this unique representative also by $\mathbf{s}\Phi$, the correspondence $\Phi \to \mathrm{s}\Phi$ defines a continuous automorphism of $\mathcal{S}(G)$. This is proved by Weil in [11]. On the other hand, since the group $\mathbf{B}(G)$ is a topological group, we can inquire on the continuity of $\mathrm{s} \to \mathrm{s}\Phi$. We shall first establish the following basic lemma:

LEMMA 4. *If E is an elementary abelian group, $\mathbf{B}(E)$ is a Lie group. Moreover, the mapping $\mathbf{B}(E) \times \mathcal{S}(E) \to \mathcal{S}(E)$ defined by $(\mathrm{s}, \Phi) \to \mathrm{s}\Phi$ is continuous.*

Proof. We shall show that $\mathbf{B}(E)$ is a Lie group. Since it is an extension of $B(E)$ by $\mathbf{T}$, we have only to show that $B(E)$ is a Lie group. Since $B(E)$ is a closed subgroup of $B^{\#}(E)$ introduced before, which is an extension of Aut $(E \times E^*)$ by $X_2(E \times E^*)$, and since $E \times E^*$ is elementary, the problem is reduced to showing that Aut (E) and $X_2(E)$ are Lie groups. We may identify E with $\mathbf{R}^n \times \mathbf{Z}^p \times (\mathbf{R}/\mathbf{Z})^q \times F$. Then an element σ of Aut (E) can be expressed by the following matrix

$$\begin{bmatrix} \sigma_1 & 0 & \sigma_{13} & 0 \\ \sigma_{21} & \sigma_2 & \sigma_{23} & \sigma_{24} \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_{43} & \sigma_4 \end{bmatrix}.$$

If σ is in a small neighborhood of the identity in Aut (E), we have $\sigma_2 = 1$, $\sigma_{24} = 0$, $\sigma_3 = 1$, $\sigma_{43} = 0$, $\sigma_4 = 1$. The remaining submatrices σ_1, σ_{13}, σ_{21}, σ_{23} are in $GL_n(\mathbf{R})$, $M_{nq}(\mathbf{R})$, $M_{pn}(\mathbf{R})$, $M_{pq}(\mathbf{R}/\mathbf{Z})$ respectively. If we take the neighborhood small enough, σ_{23} can be represented by an element of $M_{pq}(\mathbf{R})$ with coefficients less than $\frac{1}{2}$ in absolute values. Restricting σ to this neighborhood, we can easily convince ourselves that σ is close to the identity in Aut (E) if and only if σ_1 is close to 1 in $GL_n(\mathbf{R})$ and σ_{13}, σ_{21}, σ_{23} are close to 0 in $M_{nq}(\mathbf{R})$, $M_{pn}(\mathbf{R})$, $M_{pq}(\mathbf{R})$ with respect to their standard topology. Therefore Aut (E) is locally isomorphic to a closed subgroup of $GL_{n+p+q}(\mathbf{R})$ of dimension $(n+p)(n+q)$, hence it is a Lie group. We shall next examine $X_2(E)$. More generally, suppose that H is a compactly generated abelian group and K, its maximal compact subgroup. Then H/K is topologically isomorphic to $\mathbf{R}^n \times \mathbf{Z}^p$, and hence H^* has no small subgroup. On the other hand, $\varrho = \varrho(f)$ depends continuously on f. Therefore, if we take f from a small neighborhood of the identity in $X_2(H)$, we get $\varrho = 0$ on K. Furthermore, since T has no small subgroup, if we take the neighborhood small enough, f will become the coimage of an element of $X_2(H/K)$ by the canonical homomorphism $H \to H/K$. We observe that the mapping $X_2(H/K) \to X_2(H)$ identifies $X_2(H/K)$ with a closed subgroup of $X_2(H)$. Therefore, we may assume from the beginning that $H = \mathbf{R}^n \times \mathbf{Z}^p$. Then we get

$$f(x) = e(\tfrac{1}{2}xh^tx + x^ta)$$

with a symmetric matrix h in $M_{n+p}(\mathbf{R})$ and with an element a of $\mathbf{R}^{n+p}$. Furthermore, if f is close to the identity in $X_2(H)$, by replacing some of the coefficients of h and a mod 1, we see that they all become close to 0 with respect to the standard topology. Since the converse is clear, we see that $X_2(H)$ is locally isomorphic to $\mathbf{R}^N$ with $N=\frac{1}{2}(n+p)(n+p+3)$, and hence it is a Lie group. We have thus shown that $\mathbf{B}(E)$ is a Lie group.

We shall show that the mapping $\mathbf{B}(E)\times \mathcal{S}(E)\to \mathcal{S}(E)$ defined by $(\mathrm{s}, \Phi)\to \mathrm{s}\Phi$ is continuous. As we have said, Weil has shown (in the general case) that $\mathrm{s}\Phi$ depends continuously on Φ. The proof is based on what might be called a "five-step decomposition" of $\Phi\to \mathrm{s}\Phi$. In the general case, this consists of (a) a mapping of the form $\Phi\to\Phi\otimes\overline{\Phi}_0$ with a fixed Φ_0 in $\mathcal{S}(G)$; (b) $\mathcal{S}(G\times G)\to\mathcal{S}(G\times G^*)$ given as a product of an automorphism of $\mathcal{S}(G\times G)$, coming from a topological automorphism of $G\times G$, and a partial Fourier transformation; (c) $\mathcal{S}(G\times G^*)\to\mathcal{S}(G\times G^*)$ given by $\varphi\to\varphi^s$ for $s=\pi(\mathrm{s})$; (d) the inverse of (b); (e) a mapping of the form $\mathrm{s}\Phi\otimes\overline{\mathrm{s}\Phi_0}\to \mathrm{s}\Phi$, $\mathrm{s}\Phi\otimes\overline{\mathrm{s}\Phi_0}$ being the image of Φ by (a)–(d). Since the continuity of other steps are known, we have only to make certain that (c′) $(s, \varphi)\to\varphi^s$ and (e) are continuous. Since $\varphi\to\varphi^s$ can be decomposed into $\varphi\to f\varphi\to(f\varphi)^\sigma$ and since $E\times E^*$ is elementary, the continuity of (c′) is reduced to showing that the correspondence $(\sigma, \Phi)\to\Phi^\sigma$ defines a continuous mapping $\mathrm{Aut}\,(E)\times\mathcal{S}(E)\to\mathcal{S}(E)$, and also that the correspondence $(f, \Phi)\to f\Phi$ defines a continuous mapping $X_2(E)\times\mathcal{S}(E)\to\mathcal{S}(E)$. We shall treat them separately.

We shall use the same notations as before. For instance, we write $E=\mathbf{R}^n\times\mathbf{Z}^p\times(\mathbf{R}/\mathbf{Z})^q\times F$. Also, we shall denote by r and D a distance function and an invariant differential operator on E. We shall show that Φ^σ depends continuously on (σ, Φ). If we write

$$\Phi^{\sigma_0\sigma}-\Phi_0^{\sigma_0}=((\Phi^{\sigma_0})^\sigma-\Phi^{\sigma_0})+(\Phi-\Phi_0)^{\sigma_0},$$

then, because of the continuity of $\Phi\to\Phi^{\sigma_0}$, we have only to show that $\|r^m\cdot D(\Phi^\sigma-\Phi)\|_\infty$ becomes uniformly small provided that Φ satisfies a certain boundedness condition with respect to a finite number of seminorms depending on D and on the non-negative integer m (in which upper bounds are arbitrary) and σ is in a small neighborhood of the identity in $\mathrm{Aut}\,(E)$ the size of which depends on the said data. We may replace σ by σ^{-1}. We have seen that, if we restrict σ to a small neighborhood of the identity in $\mathrm{Aut}\,(E)$, we get

$$\sigma=\begin{bmatrix}\sigma_1 & 0 & \sigma_{13} & 0\\ \sigma_{21} & 1 & \sigma_{23} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Consequently, we have

$$\begin{aligned}D(\Phi^{\sigma^{-1}}-\Phi)&=\sum_\alpha(Q_\alpha(\sigma)(D_\alpha\Phi)^{\sigma^{-1}}-Q_\alpha(1)(D_\alpha\Phi))\\&=\sum_\alpha R_\alpha(\sigma)(D_\alpha\Phi)^{\sigma^{-1}}+\sum_\alpha Q_\alpha(1)((D_\alpha\Phi)^{\sigma^{-1}}-D_\alpha\Phi)\end{aligned}$$

with $R_\alpha(\sigma)=Q_\alpha(\sigma)-Q_\alpha(1)$. We are denoting by $Q_\alpha(\sigma)$ certain polynomials in the coefficients of σ and by D_α certain invariant differential operators on E. If we restrict σ to a small compact neighborhood C of 1, there exist two real numbers c and c' satisfying $0<c<1<c'$ such that we have

$$c\cdot r\leqslant r^{\sigma^{-1}},\ r^{\sigma}\leqslant c'\cdot r$$

for every σ in C, We may, in fact, take c and c' as close as possible to 1 by making C smaller. At any rate, we then get

$$\|r^m\cdot R_\alpha(\sigma)\cdot(D_\alpha\Phi)^{\sigma^{-1}}\|_\infty=|R_\alpha(\sigma)|\cdot\|(r^\sigma)^m\cdot(D_\alpha\Phi)\|_\infty\leqslant(c')^m\cdot|R_\alpha(\sigma)|\cdot\|r^m\cdot(D_\alpha\Phi)\|_\infty,$$

and this approaches 0 uniformly in Φ when σ approaches 1 provided that Φ is restricted by $\|r^m\cdot(D_\alpha\Phi)\|_\infty\leqslant M$ for any positive real number M. Restrict Φ further by $\|r^{m+1}\cdot(D_\alpha\Phi)\|_\infty\leqslant M'$ for any positive real number M' and put $D_\alpha\Phi=\Psi$. Then, for any positive real number ε, we have $|r(x)^m\Psi(x)|\leqslant\varepsilon/3$ provided that $r(x)\geqslant r_0=3M'/\varepsilon$. We shall also restrict Φ by the condition that all "first partial derivatives" of Ψ are not greater than a fixed positive real number in absolute values. Then the family of Ψ is equicontinuous, hence, by making C smaller if necessary, we will have

$$|(\Psi^{\sigma^{-1}}-\Psi)(x)|\leqslant\varepsilon/(2r_0)^m$$

for every σ in C and for every x in E satisfying $r(x)\leqslant 2r_0$. Consequently, we get $|r(x)^m(\Psi^{\sigma^{-1}}-\Psi)(x)|\leqslant\varepsilon$ for the same σ and x. Now, suppose that $r(x)\geqslant 2r_0$. Then, at any rate, we have

$$|r(x)^m(\Psi^{\sigma^{-1}}-\Psi)(x)|\leqslant|r(x)^m\Psi(x\sigma)|+|r(x)^m\Psi(x)|.$$

We know that $|r(x)^m\Psi(x)|$ is at most equal to $\varepsilon/3$ for $r(x)\geqslant r_0$, hence a fortiori for $r(x)\geqslant 2r_0$. Since we have

$$|r(x)^m\Psi(x\sigma)|=(r(x)/r(x\sigma))^m\cdot|r(x\sigma)^m\Psi(x\sigma)|,$$

this will be at most equal to $\varepsilon/2$ for $r(x)\geqslant 2r_0$ provided that $r(x\sigma)\geqslant r_0$ for $r(x)\geqslant 2r_0$ and $(r(x)/r(x\sigma))^m\leqslant 3/2$. These conditions are satisfied if we have $c\geqslant(2/3)^{1/m}$. As we have remarked, this can be achieved by making C smaller. Therefore, putting them together, we get

$$\|r^m\cdot((D_\alpha\Phi)^{\sigma^{-1}}-D_\alpha\Phi)\|_\infty\leqslant\varepsilon$$

for every Φ in $\mathcal{S}(E)$ satisfying a certain boundedness condition that we have made explicit and for every σ in the compact neighborhood C of the identity in Aut (E). We have thus shown that the mapping Aut $(E)\times\mathcal{S}(E)\to\mathcal{S}(E)$ defined by $(\sigma,\Phi)\to\Phi^\sigma$ is continuous.

We shall show that $f\Phi$ depends continuously on (f, Φ). If we write

$$ff_0\Phi - f_0\Phi_0 = (f(f_0\Phi) - f_0\Phi) + f_0(\Phi - \Phi_0),$$

then, because of the continuity of $\Phi \to f_0\Phi$, we have only to show that $\|r^m \cdot D(f\Phi - \Phi)\|_\infty$ becomes uniformly small provided that Φ satisfies a certain boundedness condition and f is in a small neighborhood of the identity in $X_2(E)$. We have seen that, if we restrict f to a small neighborhood of the identity in $X_2(E)$, it depends only on the $(\mathbf{R}^n \times \mathbf{Z}^p)$-coordinate x of a point u of E. Moreover, we have

$$f(u) = e(\tfrac{1}{2}xh^tx + x^ta)$$

with a symmetric matrix h in $M_{n+p}(\mathbf{R})$ and with an element a of $\mathbf{R}^{n+p}$. Consequently, we have

$$D(f\Phi - \Phi) = \Sigma_\alpha P_\alpha \cdot Q_\alpha(f) f(D_\alpha \Phi) + (f-1)\, D\Phi,$$

in which P_α are polynomial functions on E and $Q_\alpha(f)$ are certain polynomials in the coefficients of h, a satisfying $Q_\alpha(1) = 0$, i.e., $Q_\alpha(f) = 0$ for $h = 0$, $a = 0$. Moreover, D_α are invariant differential operators on E (many of which may be identical). We then get

$$\|r^m \cdot P_\alpha \cdot Q_\alpha(f) f(D_\alpha \Phi)\|_\infty = |Q_\alpha(f)| \cdot \|r^m P_\alpha \cdot (D_\alpha \Phi)\|_\infty,$$

and this approaches 0 uniformly in Φ when f approaches 1 provided that Φ is restricted by $\|r^m P_\alpha \cdot (D_\alpha \Phi)\|_\infty \leqslant M$ for any positive real number M. Restrict Φ further by $\|r^k \cdot (D\Phi)\|_\infty \leqslant M'$ for $k = m, m+1$, in which M' is any positive real number, and put $D\Phi = \Psi$. Then, for any positive real number ε, we have $|r(u)^m \Psi(u)| \leqslant \varepsilon/3$ provided that $r(u) \geqslant r_0 = 3M'/\varepsilon$. This implies $|r(u)^m (f(u) - 1)\Psi(u)| \leqslant 2\varepsilon/3$ for $r(u) \geqslant r_0$. If we take the neighborhood small enough, we have $|f(u) - 1| \leqslant \varepsilon/M'$ for $r(u) \leqslant r_0$ and for every f in this neighborhood. Then we get $|r(u)^m (f(u) - 1)\Psi(u)| \leqslant \varepsilon$ for $r(u) \leqslant r_0$. Therefore, putting them together, we get

$$\|r^m \cdot (f-1) \cdot (D\Phi)\|_\infty \leqslant \varepsilon$$

for every Φ in $\mathcal{S}(E)$ satisfying a certain boundedness condition and for every f in the said neighborhood of the identity in $X_2(E)$. We have thus shown that the mapping $X_2(E) \times \mathcal{S}(E) \to \mathcal{S}(E)$ defined by $(f, \Phi) \to f\Phi$ is continuous.

By what we have shown, if Φ_0 is a fixed element of $\mathcal{S}(E)$, the mapping $\mathbf{B}(E) \times \mathcal{S}(E) \to \mathcal{S}(E \times E)$ defined by $(\mathrm{s}, \Phi) \to \mathrm{s}\Phi \otimes \overline{\mathrm{s}\Phi_0}$ is continuous. Let s_0 denote a fixed element of

$\mathbf{B}(E)$. Then, the mapping $\mathbf{B}(E)\times\mathcal{S}(E)\to\mathcal{S}(E)$ defined by $(\mathrm{s}, \Phi)\to(\mathrm{s}_0\Phi_0, \mathrm{s}\Phi_0)\mathrm{s}\Phi$ is continuous. In fact, this can be decomposed into (i) $(\mathrm{s}, \Phi)\to\mathrm{s}\Phi\otimes\overline{\mathrm{s}\Phi_0}$, (ii) multiplication by $1\otimes\mathrm{s}_0\Phi_0$, (iii) partial Fourier transformation, (iv) restriction to $E\times 0$, and each step is continuous. On the other hand, by the definition of the topology in $\mathbf{B}(E)$, the scalar factor $(\mathrm{s}_0\Phi_0, \mathrm{s}\Phi_0)$ depends continuously on s. Therefore, its inverse depends continuously on s in some neighborhood of s_0 (provided that $\Phi_0\neq 0$). This implies that $\mathrm{s}\Phi$ depends continuously on (s, Φ) when s is in the said neighborhood of s_0, hence in general. We have thus completed the proof.

In the course of the proof of Lemma 4, we have obtained some further information, e.g. about Aut (E), which permits us to prove the following result:

LEMMA 5. *If E is an elementary abelian group, the continuous homomorphism $B(E)\to\mathrm{Sp}\,(E)$ has a continuous local cross-section in some neighborhood of the identity in* $\mathrm{Sp}\,(E)$.

Proof. We recall that $\mathrm{Sp}\,(E)$ is a closed subgroup of Aut $(E\times E^*)$. Since we are interested only in a small neighborhood of the identity in $\mathrm{Sp}\,(E)$, for an obvious reason, we may assume that $E=\mathbf{R}^n\times\mathbf{Z}^p\times(\mathbf{R}/\mathbf{Z})^q$. If we write $E\times E^*$ in the form $(\mathbf{R}^n\times(\mathbf{R}/\mathbf{Z})^q\times\mathbf{Z}^p)\times(\mathbf{R}^n\times\mathbf{Z}^q\times(\mathbf{R}/\mathbf{Z})^p)$ so that we have

$$\langle(x\ y\ z),\ (x^*\ y^*\ z^*)\rangle = e(x^tx^* + y^ty^* + z^tz^*),$$

we see that $\mathrm{Sp}(E)$ is locally isomorphic to the closed subgroup of $Sp_{2(n+p+q)}(\mathbf{R})$ consisting of matrices of the following form

$$\sigma=\begin{bmatrix}\alpha_1 & \alpha_{12} & 0 & \beta_1 & 0 & \beta_{13}\\ 0 & 1 & 0 & 0 & 0 & 0\\ \alpha_{31} & \alpha_{32} & 1 & \beta_{31} & 0 & \beta_3\\ \gamma_1 & \gamma_{12} & 0 & \delta_1 & 0 & \delta_{13}\\ \gamma_{21} & \gamma_2 & 0 & \delta_{21} & 1 & \delta_{23}\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

Now, we recall that, if G is a locally compact abelian group for which $x\to 2x$ defines a topological automorphism, the continuous homomorphism $B(G)\to\mathrm{Sp}\,(G)$ has an explicitly defined continuous cross-section $\sigma\to(\sigma, f)$. In fact, if α, β, γ, δ are the submatrices of σ, we have only to take

$$f(u, u^*) = \langle u\alpha, 2^{-1}u\beta\rangle\langle u\beta, u^*\gamma\rangle\langle u^*\gamma, 2^{-1}u^*\delta\rangle.$$

Therefore, taking $G=\mathbf{R}^{n+p+q}$, we get a continuous cross-section for $B(\mathbf{R}^{n+p+q})\rightarrow Sp_{2(n+p+q)}(\mathbf{R})$. The point is that, if we restrict this cross-section to the subgroup of matrices σ of the above form, $f(u, u^*)$ for $u=(x\ y\ z)$ and $u^*=(x^*\ y^*\ z^*)$ does not contain the ambiguous coordinates y and z^*. Therefore, the cross-section for $B(\mathbf{R}^{n+p+q})\rightarrow Sp_{2(n+p+q)}(\mathbf{R})$ gives a local cross-section for $B(E)\rightarrow \mathrm{Sp}\ (E)$, which is certainly continuous. This proves the lemma.

3. We shall consider the general case. We shall start by examining Aut (G) and $X_2(G)$, in which G is an arbitrary locally compact abelian group. Suppose that (H, H') is an admissible pair for G. We shall denote by Aut $(H, H')=$ Aut $(G; H, H')$ the subset of Aut (G) consisting of those σ with the property that $H\sigma=H$, $H'\sigma=H'$. Then Aut (H, H') is an open subgroup of Aut (G). We have only to show that it is open. Let C denote a compact neighborhood and V an open neighborhood of 0 in H. We take C large enough that it generates H. Then $W(C, V)$ is an open neighborhood of the identity in Aut (G). If we take σ from $W(C, V)$, we have $C\sigma\subset V+C\subset H$, hence $H\sigma\subset H$, and also $H\sigma^{-1}\subset H$. Therefore, we have $H\sigma=H$. On the other hand, if H_*, $(H')_*$ denote the annihilators of H, H' in G^*, we know that $((H')_*, H_*)$ is an admissible pair for G^*. We define $W(C', V')$ for this admissible pair in the same way as we have defined $W(C, V)$ for (H, H'). It is well known (and easy to show) that the asterisk gives a topological anti-isomorphism Aut $(G)\rightarrow$ Aut (G^*). Consequently $W(C', V')^*$ is an open neighborhood of the identity in Aut (G). If we take σ from $W(C', V')^*$, we have $(H')_*\sigma^*=(H')_*$, and hence $H'\sigma=H'$. Therefore, the intersection of $W(C, V)$ and $W(C', V')^*$ is contained in Aut (H, H'), and this shows that Aut (H, H') is open. On the other hand, it is clear that every element of Aut (H, H') gives rise to a topological automorphism of H/H'. In this way, we get a homomorphism Aut $(H, H')\rightarrow$ Aut (H/H'). This homomorphism is continuous. We leave it as an exercise to show that the kernel of this homomorphism becomes arbitrarily small in Aut (G) by making H larger and H' smaller.

We shall examine $X_2(G)$. We take an admissible pair (H, H') and denote by $X_2(H, H')=X_2(G; H, H')$ the subset of $X_2(G)$ consisting of those f with the property that $f(h+h')=f(h)$ for every h, h' in H, H' respectively. Then $X_2(H, H')$ is an open subgroup of $X_2(G)$. As we have seen in the proof of Lemma 4, this follows from the fact that H^* and T have no small subgroups. On the other hand, every element of $X_2(H, H')$ gives rise to a second degree character of H/H'. In this way, we get a homomorphism $X_2(H, H')\rightarrow X_2(H/H')$, and this is continuous. Moreover, a remark similar to the one before applies to its kernel.

Now, we know that there exists a mapping $\mathbf{B}(G)\times \mathcal{S}(G)\rightarrow \mathcal{S}(G)$ defined by $(\mathbf{s}, \Phi)\rightarrow \mathbf{s}\Phi$ and that there exists a $\mathbf{C}$-linear topological isomorphism $\mathcal{S}(H, H')\rightarrow \mathcal{S}(H/H')$. After recalling this, we shall prove the following "continuity theorem":

THEOREM 1. *If G is a locally compact abelian group and (H, H') an admissible pair, there exists an open subgroup $\mathbf{B}(H, H') = \mathbf{B}(G; H, H')$ of $\mathbf{B}(G)$ with elements $\mathbf{s}$ satisfying $\mathbf{s}\mathcal{S}(H, H') = \mathcal{S}(H, H')$. The corresponding mapping $\mathbf{B}(H, H') \to \mathrm{Aut}\,(L^2(H/H'))$ gives a continuous homomorphism $\mathbf{B}(H, H') \to \mathbf{B}(H/H')$ such that the following diagram*

$$\begin{array}{ccc} \mathbf{B}(H, H') \times \mathcal{S}(H, H') & \to & \mathcal{S}(H, H') \\ \downarrow & & \downarrow \\ \mathbf{B}(H/H') \times \mathcal{S}(H/H') & \to & \mathcal{S}(H/H') \end{array}$$

is commutative. Moreover, the bottom arrow is continuous.

Proof. We first observe that $(H \times (H')_*, H' \times H_*)$ is an admissible pair for $G \times G^*$. For the sake of simplicity, we put $\mathrm{Aut} = \mathrm{Aut}\,(H \times (H')_*, H' \times H_*)$ and $X_2 = X_2(H \times (H')_*, H' \times H_*)$. Then, as we have seen, Aut and X_2 are open subgroups of $\mathrm{Aut}\,(G \times G^*)$ and $X_2(G \times G^*)$. Moreover, $\mathrm{Aut} \times X_2$ is an open subgroup of $B^{\#}(G)$, hence

$$\mathbf{B}(H, H') = \pi^{-1}(B(G) \cap (\mathrm{Aut} \times X_2))$$

is an open subgroup of $\mathbf{B}(G)$. On the other hand, the five-step decomposition of $\Phi \to \mathbf{s}\Phi$ shows that a sufficient condition for an element $\mathbf{s}$ of $\mathbf{B}(G)$ to keep $\mathcal{S}(H, H')$ stable is that the step (c), i.e., the correspondence $\varphi \to \varphi^s$ defined by

$$\varphi^s(w) = f(w\sigma^{-1})\varphi(w\sigma^{-1})$$

for $\pi(\mathbf{s}) = s = (\sigma, f)$, keep $\mathcal{S}(H \times (H')_*, H' \times H_*)$ stable. We shall show that every element of $\mathbf{B}(H, H')$ has this property. Since $\mathbf{B}(H, H')$ is a group, this will imply $\mathbf{s}\mathcal{S}(H, H') = \mathcal{S}(H, H')$ for every $\mathbf{s}$ in $\mathbf{B}(H, H')$. We observe that the dual of $E = H/H'$ can be identified with $(H')_*/H_*$. Moreover, if σ and f are in Aut and X_2, respectively, they determine unique elements σ_E and f_E of $\mathrm{Aut}\,(E \times E^*)$ and $X_2(E \times E^*)$. Furthermore, the pair $s_E = (\sigma_E, f_E)$ is an element of $B(E)$. Let φ_E denote the element of $\mathcal{S}(E \times E^*)$ which corresponds to an element φ of $\mathcal{S}(H \times (H')_*, H' \times H_*)$. Then, it is clear that φ^s and $(\varphi_E)^{s_E}$ are the corresponding elements of $\mathcal{S}(H \times (H')_*, H' \times H_*)$ and $\mathcal{S}(E \times E^*)$. This proves the assertion, hence the first part of the theorem.

As for the second part, suppose that Φ and Φ_E are the corresponding elements of $\mathcal{S}(H, H')$ and $\mathcal{S}(E)$. Then, for every $\mathbf{s}$ in $\mathbf{B}(H, H')$, $\mathbf{s}\Phi$ is in $\mathcal{S}(H, H')$, hence it determines a unique element $\mathbf{s}_E\Phi_E$ of $\mathcal{S}(E)$. Moreover, if we take the restriction to H of the Haar measure dx on G as dh and decompose it into the Haar measures dh_E and dh' on E and H' respectively such that the total measure of H' is 1, we have $\|\Phi\| = \|\Phi_E\|$, and hence $\|\mathbf{s}_E\Phi_E\| = \|\Phi_E\|$. Since $\mathcal{S}(E)$ is dense in $L^2(E)$, this determines $\mathbf{s}_E$ uniquely as an element of $\mathrm{Aut}\,(L^2(E))$. On the other hand, the five-step decompositions of $\Phi \to \mathbf{s}\Phi$ and $\Phi_E \to \mathbf{s}_E\Phi_E$ are compatible.

Namely, each step commutes with the $\mathbf{C}$-linear bijection of the form "$S(H, H') \to S(H/H')$". In fact, we see that $\mathfrak{s}_E$ is an element of $\mathbf{B}(E)$ satisfying $\pi_E(\mathfrak{s}_E) = s_E = (\sigma_E, f_E)$. Therefore, the correspondence $\mathfrak{s} \to \mathfrak{s}_E$ gives a homomorphism $\mathbf{B}(H, H') \to \mathbf{B}(E)$. Furthermore, because of $\|\Phi\| = \|\Phi_E\|$, this homomorphism is continuous. The said compatibility implies, finally, that the diagram in question is commutative. The continuity of the bottom arrow has been proved in Lemma 4.

We observe that the kernel of the homomorphism $\mathbf{B}(H, H') \to \mathbf{B}(H/H')$ becomes arbitrarily small in $\mathbf{B}(G)$ by making H larger and H' smaller. In fact, we have only to use the fact that any finite subset of $S(G)$ is contained in some $S(H, H')$ (and the fact that $S(G)$ is dense in $L^2(G)$) for the proof of this supplement.

Before we start deriving corollaries of Theorem 1, which are suitable for applications, we shall analyze the homomorphism $\mathbf{B}(G) \to \mathrm{Sp}\,(G)$. We need the following lemma:

LEMMA 6. *Suppose that H is a closed subgroup of G and ϱ an element of $\Sigma(G)$. Then every element f_0 of $X_2(H)$ satisfying $f_0(h+h') = f_0(h)f_0(h')\langle h, h'\varrho\rangle$ for every h, h' in H can be extended to an element f of $X_2(G)$ satisfying $\varrho(f) = \varrho$. Moreover, if f is one of them, all possible extensions are of the form $x \to \langle x, x^*\rangle f(x)$ with x^* in the annihilator H_* of H in G^*.*

Proof. Since the second part of the lemma is obvious, we shall prove only the first part. The proof consists of three steps: We shall first consider the case when H is an open subgroup of G. In this case, if we can extend f_0 to a mapping $f: G \to T$ satisfying $f(x+y) = f(x)f(y)\langle x, y\varrho\rangle$ for every x, y in G, the continuity of f_0 implies the continuity of f, and hence f will be an element of $X_2(G)$ satisfying $\varrho(f) = \varrho$. Moreover, a standard application of Zorn's lemma reduces the problem to the extendability of f_0 to $\mathbf{Z}z + H$ for any given element z of G. Now, if the image of z in G/H is free, we put $m = 0$. Otherwise, we shall denote by m the order of the image of z in G/H. Then we choose t arbitrarily from T subject to the condition

$$t^m = f_0(mz)\left\langle z, -\binom{m}{2} z\varrho\right\rangle$$

and we put
$$f(nz+h) = t^n f_0(h) \cdot \left\langle z, \left(\binom{n}{2} z + nh\right) \varrho\right\rangle$$

for every n in $\mathbf{Z}$ and h in H. It is easy to verify that f is well defined on $\mathbf{Z}z + H$ and satisfies $f(x+y) = f(x)f(y)\langle x, y\varrho\rangle$ for every x, y in $\mathbf{Z}z + H$. This settles the case when H is open.

We shall next show that the continuous homomorphism $X_2(G) \to \Sigma(G)$ is surjective. Suppose that ϱ is an element of $\Sigma(G)$. We take a compactly generated subgroup H of G and then a compactly generated subgroup of G^* which contains $H\varrho + H_*$. Let H' denote its annihilator in G. Then (H, H') is an admissible pair for G and we have $\langle h\varrho, h'\rangle = 1$ for every h in H and h' in H'. Consequently, we have $H\varrho \subset (H')_*$ and $H'\varrho \subset H_*$. Put $E = H/H'$. Then we can identify E^* with $(H')_*/H_*$ in an obvious manner. By what we have said, ϱ determines an element ϱ_E of $\Sigma(E)$. Suppose that the surjectivity in question is proved for an elementary abelian group. Then, there exists an element f_E of $X_2(E)$ satisfying $\varrho(f_E) = \varrho_E$. Let f_0 denote the coimage of f_E by the canonical homomorphism $H \to E$. Then f_0 is an element of $X_2(H)$ satisfying $f_0(x+y) = f_0(x) f_0(y) \langle x, y\varrho\rangle$ for every x, y in H. Since H is an open subgroup of G, we can extend f_0 to an element f of $X_2(G)$ satisfying $\varrho(f) = \varrho$. We shall now consider the case when G is elementary. We shall denote it by E. As we have seen, we can replace E by any one of its open subgroups, and hence by its connected component. Therefore, we may assume that $E = \mathbf{R}^n \times (\mathbf{R}/\mathbf{Z})^q$. Then we have $E^* = \mathbf{R}^n \times \mathbf{Z}^q$ with $\langle (x\, y), (x^*\, y^*)\rangle = e(x{}^t x^* + y{}^t y^*)$, and ϱ has the following form:

$$\varrho = \begin{pmatrix} \varrho_1 & 0 \\ 0 & 0 \end{pmatrix},$$

in which ϱ_1 is a symmetric matrix in $M_n(\mathbf{R})$. Hence, if we put $f(x\, y) = e(2^{-1} x \varrho_1 {}^t x)$, we get an element f of $X_2(E)$ satisfying $\varrho(f) = \varrho$.

We shall complete the proof of Lemma 6. We take an element f_1 of $X_2(G)$ satisfying $\varrho(f_1) = \varrho$. Then $h \to f_0(h) f_1(h)^{-1}$ defines an element of $X_1(H)$. Since H is a closed subgroup of G, it can be extended to an element, say χ, of $X_1(G)$. Then $f = \chi f_1$ is an extension of f_0 to an element of $X_2(G)$ satisfying $\varrho(f) = \varrho$.

PROPOSITION 2. *If G is a locally compact abelian group, the continuous homomorphism $B(G) \to \mathrm{Sp}\,(G)$ is surjective. Moreover, $B(G)$ contains a local subgroup $U^{\#}$, which we can take arbitrarily small, such that its image, say U, by the homomorphism $B(G) \to \mathrm{Sp}\,(G)$ is open and that the induced mapping $p: U^{\#} \to U$ is open and proper. Therefore, the homomorphism $B(G) \to \mathrm{Sp}\,(G)$ is open and* $\mathbf{B}(G)/\mathbf{A}(G)$ *is topologically isomorphic to* $\mathrm{Sp}\,(G)$.

Proof. We first observe that the surjectivity of $B(G) \to \mathrm{Sp}\,(G)$ is a simple consequence of the surjectivity of $X_2(G \times G^*) \to \Sigma(G \times G^*)$. Therefore, we have only to prove the second statement in Proposition 2.

Let W denote an open neighborhood of the identity in $\mathrm{Aut}\,(G \times G^*)$; C, a compact subset of $G \times G^*$; and V, an open neighborhood of 1 in T. Let $M(C, V)$ denote the set of those f in $X_2(G \times G^*)$ which map C to V. Then $(W \times M(C, V)) \cap B(G)$ is an open neighborhood of the identity in $B(G)$, and it becomes arbitrarily small by making W, V smaller and C larger. We choose an admissible pair (H, H') for G such that $H \times (H')_*$ contains C. We have only to take, as H, a compactly generated subgroup of G containing the projection of C to G and take, as H', the annihilator in G of a compactly generated subgroup of G^* containing H_* and the projection of C to G^*. (We can further show that the set of admissible pairs of the form $(H\times(H')_*, H'\times H_*)$ is "cofinal" in the set of all admissible pairs for $G\times G^*$.) As before, we put $\mathrm{Aut}=\mathrm{Aut}\,(H\times(H')_*, H'\times H_*)$ and $E=H/H'$. Also, we shall denote by C_E the image of C under the canonical homomorphism $H\times(H')_*\to E\times E^*$. Then C_E is a compact subset of $E\times E^*$. We know, on the other hand, that $\mathrm{Sp}\,(G)\cap\mathrm{Aut}$ is an open subgroup of $\mathrm{Sp}\,(G)$ and that there exists a continuous homomorphism $\mathrm{Sp}\,(G)\cap\mathrm{Aut}\to\mathrm{Sp}\,(E)$. We take an element σ of this open subgroup and put

$$\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\qquad \varrho=\begin{pmatrix}\alpha\beta^* & \beta\gamma^*\\ \gamma\beta^* & \gamma\delta^*\end{pmatrix}.$$

Then ϱ is an element of $\Sigma(G\times G^*)$. Let U denote an open neighborhood of the identity in $\mathrm{Sp}\,(G)$ contained in $\mathrm{Aut}\cap W$. If we choose U sufficiently small and take σ from U, its image σ_E in $\mathrm{Sp}\,(E)$ can be lifted to an element (σ_E, f_E) of $B(E)$ by Lemma 5. Furthermore, since $\sigma_E\to(\sigma_E, f_E)$ defines a continuous cross-section over some neighborhood of the identity in $\mathrm{Sp}\,(E)$, we may assume that U has been taken so small that the image of C_E by f_E is contained in V for every σ in U. Let f_0 denote the coimage of f_E by the canonical homomorphism $H\times(H')_*\to E\times E^*$. Then f_0 is an element of $X_2(H\times(H')_*)$ satisfying $f_0(w_1+w_2)=f_0(w_1)f_0(w_2)\langle w_1, w_2\varrho\rangle$ for every w_1, w_2 in $H\times(H')_*$. Therefore, we can extend f_0 to an element f of $X_2(G\times G^*)$ satisfying $\varrho(f)=\varrho$ by Lemma 6. Then $s=(\sigma, f)$ is an element of $B(G)$. Moreover, we have $f(C)=f_0(C)=f_E(C_E)\subset V$, and hence s is contained in $W\times M(C, V)$. We observe that f is obtained from σ by the following four steps:

$$\sigma\to\sigma_E\to f_E\to f_0\to f,$$

in which the first three steps are unique (and continuous) when we fix (H, H'). We shall consider all possible f for every σ in U and denote by $U^{\#}$ the set of all such $s=(\sigma, f)$. Then $U^{\#}$ is contained in $(W\times M(C, V))\cap B(G)$ and the homomorphism $B(G)\to\mathrm{Sp}\,(G)$ induces a continuous surjection $p: U^{\#}\to U$. Moreover, if $s=(\sigma, f)$, $s'=(\sigma', f')$ are in $U^{\#}$ and $\sigma\sigma'$ is in U, then ss' is in $U^{\#}$. Similarly, if $s=(\sigma, f)$ is in $U^{\#}$ and σ^{-1} is in U, then s^{-1} is in $U^{\#}$. It follows from these facts that $U^{\#}$ forms a local subgroup of $B(G)$ and that p is a homomorphism of the local groups. Furthermore, the kernel of p is topologically isomorphic to the compact group $H_*\times H'$ by Lemma 6. We shall show that p is an open mapping. We have only to show that the image of every neighborhood of the identity in $U^{\#}$ contains a neighborhood of the identity in U.

We take new data W_1, C_1, V_1 similar to W, C, V. Also, we shall denote by U_1 an open neighborhood of the identity in U. Then it will be sufficient to show that, for some choice of U_1, the intersection $(W_1 \times M(C_1, V_1)) \cap p^{-1}(\sigma)$ is not empty for every σ in U_1. At any rate, we have to assume that U_1 is contained in W_1. We put $N = H \times (H')_*$ and take a compactly generated subgroup N_1 of $G \times G^*$ containing both N and C_1. Then N_1/N is a finitely generated abelian group, and hence it is a product of a finite number, say t, of cyclic groups. We shall denote by $w_1, \ldots, w_t$ representatives in N_1 of the generators of these cyclic groups. Since N is open and C_1 is compact, we can find a finite set, say I, of linear combinations of $w_1, \ldots, w_t$ with integer coefficients such that the union of $N + w$ for w in I contains C_1. We shall denote by C_2 the union of $(C_1 - w) \cap N$ for w in I. Since N is closed, C_2 is a compact subset of N. Moreover, the union of $C_2 + w$ for w in I contains C_1. Now, we choose an open neighborhood V_2 of 1 in T satisfying $(V_2)^3 \subset V_1$. Then, by taking U_1 sufficiently small, we can cause $f_0(w_0)$ and $\langle w_0, w\varrho \rangle$ to be contained in V_2 for every w_0 in C_2 and w in I provided that σ is in U_1. On the other hand, we observe that $f(w)$ for w in I can be expressed as a monomial in $f(w_i)^{\pm 1}$ and $\langle w_i, \pm w_j \varrho \rangle$ for $i, j = 1, \ldots, t$. Therefore, if we choose an open neighborhood V_3 of 1 in T sufficiently small and, at the same time, make U_1 smaller, $f(w)$ will be in V_2 when $f(w_i)$ are in V_3 and σ is in U_1 for every w in I. We observe also that, if m is an integer, the mapping $U_1 \to T$ defined by

$$\sigma \to f_0(mw_i) \left\langle w_i, -\binom{m}{2} w_i \varrho \right\rangle$$

is continuous (and takes the value 1 at the identity). We take $m = 0$ if the image of w_i in N_1/N is free. Otherwise, we take as m the order of the image of w_i in N_1/N. Then, by making U_1 still smaller, we may assume that the image of U_1 in T is contained in V_3 for $i = 1, \ldots, t$. We recall that, to extend f_0 to f, we may choose $f(w_i)$ arbitrarily subject to the condition

$$f(w_i)^m = f_0(mw_i) \left\langle w_i, -\binom{m}{2} w_i \varrho \right\rangle.$$

Therefore, if V_3 is connected, we can find an extension f of f_0 such that $f(w_i)$ are in V_3. Then $f(w_0 + w) = f_0(w_0) f(w) \langle w_0, w\varrho \rangle$ is in $(V_2)^3 \subset V_1$ for every w_0 in C_2 and w in I provided that σ is in U_1. This implies $f(C_1) \subset V_1$, and hence $(W_1 \times M(C_1, V_1)) \cap p^{-1}(\sigma)$ is not empty for every σ in U_1.

We have thus shown that $p: U^{\#} \to U$ is an open surjection of the local groups with the following properties: (i) if s, s' are in $U^{\#}$ and $p(s)p(s')$ is defined, ss' is defined and $p(ss') = p(s)p(s')$; (ii) if s is in $U^{\#}$ and $p(s)^{-1}$ is defined, s^{-1} is defined and $p(s^{-1}) = p(s)^{-1}$; (iii) the kernel of p is compact. This implies that p is proper. In fact, the proof is entirely similar to the one in the case when U is a group (cf. [10], pp. 16–18). This completes the proof.

COROLLARY 1. *The mapping* $\mathbf{B}(G) \times S(G) \to S(G)$ *is separately continuous. Moreover, if* Σ *is an arbitrary locally compact subset of* $\mathbf{B}(G)$, *the induced mapping* $\Sigma \times S(G) \to S(G)$ *is continuous.*

The first part follows immediately from Theorem 1. As for the second part, since the continuity is a local property, we may assume that Σ is compact. Let (H, H') denote an arbitrary admissible pair. Then we can find a finite number of elements $(\mathrm{s}_i)_{i\in I}$ of $\mathbf{B}(G)$ satisfying

$$\Sigma \subset \bigcup_{i\in I} \mathrm{s}_i \mathbf{B}(H, H').$$

Theorem 1 implies that the restriction of $\mathbf{B}(G)\times S(G)\to S(G)$ to each $\mathrm{s}_i\mathbf{B}(H, H')\times S(H, H')$ is continuous. Therefore, its restriction to $\Sigma\times S(H, H')$ is continuous. Now, we recall that a subset X of $S(G)$ is open in $S(G)$ if and only if it is locally convex and if $X\cap S(H, H')$ is open in $S(H, H')$ for every $S(H, H')$. Let (s_0, Φ_0) denote an arbitrary element of $\Sigma\times S(G)$. Also, let X denote an open convex neighborhood of $\mathrm{s}_0\Phi_0$ in $S(G)$. Since $\mathrm{s}\Phi_0$ depends continuously on s, we can find a compact neighborhood Σ_0 of s_0 in Σ such that $\Sigma_0\Phi_0$ is contained in X. Consider the set Y of those elements Φ in $S(G)$ such that $\Sigma_0\Phi$ is contained in X. Then Y is a convex subset of $S(G)$ containing Φ_0. We shall show that $Y\cap S(H, H')$ is open in $S(H, H')$ for every $S(H, H')$. Suppose that Φ_1 is an element of $Y\cap S(H, H')$. Then Φ_1 is in $S(H, H')$ and $\Sigma_0\Phi_1$ is contained in X. Since Σ_0 is compact and since we have seen that the mapping $\Sigma\times S(H, H')\to S(G)$ is continuous, there exists a neighborhood of Φ_1 in $S(H, H')$ which is contained in Y. This proves the assertion. Therefore Y is open in $S(G)$, and hence the mapping $\Sigma\times S(G)\to S(G)$ is continuous.

There is an important process by which we can construct various groups out of $\mathbf{B}(G)$. Suppose that $\mathfrak{G}$ is a topological group and $\mathfrak{G}\to\mathrm{Sp}\,(G)$, a continuous homomorphism. We shall consider the fiber-product

$$\mathbf{B}(G)_{\mathfrak{G}} = \mathbf{B}(G)\times_{\mathrm{Sp}(G)}\mathfrak{G}.$$

This is the closed subgroup of the product $\mathbf{B}(G)\times\mathfrak{G}$ consisting of those pairs (s, g) which project down to the same elements of $\mathrm{Sp}\,(G)$.

COROLLARY 2. *If $\mathfrak{G}$ is a locally compact group, $\mathbf{B}(G)_{\mathfrak{G}}$ is also a locally compact group. Moreover, the mapping $\mathbf{B}(G)_{\mathfrak{G}}\times S(G)\to S(G)$ defined by $((\mathrm{s}, g), \Phi)\to \mathrm{s}\Phi$ is continuous.*

Let W denote a neighborhood of the identity in $B(G)$ such that $X_1(G\times G^*)\cap W$ has a compact closure in $X_1(G\times G^*)=1\times X_1(G\times G^*)$. Choose a smaller neighborhood W' of the identity in $B(G)$ such that $W'(W')^{-1}$ is contained in W. On the other hand, let $\mathfrak{K}$ denote a compact neighborhood of the identity in $\mathfrak{G}$ and denote its image in $\mathrm{Sp}\,(G)$ by K. If we take $\mathfrak{K}$ sufficiently small, we can find a compact subset F of $B(G)$ contained in W' such that the homomorphism $B(G)\to\mathrm{Sp}\,(G)$ induces a surjection $F\to K$. The existence of F follows from Proposition 2. Then $X_1(G\times G^*)F\cap W'$ has a compact closure in $X_1(G\times G^*)F$, and hence $X_1(G \times G^*) F$ is locally compact. We observe that this is the inverse image of K under the homomorphism $B(G) \to \mathrm{Sp}\,(G)$. On the other hand, the kernel of the homomorphism $\pi: \mathbf{B}(G) \to B(G)$ is compact. Therefore $\pi^{-1}(X_1(G \times G^*) F)$ is a locally compact subset of $\mathbf{B}(G)$. We have thus shown that the set of those s in $\mathbf{B}(G)$ such that (s, g) is in $\mathbf{B}(G)_{\mathfrak{G}}$ for some g in the small compact neighborhood $\mathfrak{K}$ of the identity in $\mathfrak{G}$ is locally compact. This implies the local compactness of $\mathbf{B}(G)_{\mathfrak{G}}$ and, by Corollary 1, the continuity of the mapping $\mathbf{B}(G)_{\mathfrak{G}} \times \mathcal{S}(G) \to \mathcal{S}(G)$ defined by $((\mathrm{s}, g), \Phi) \to \mathrm{s}\Phi$.

We note that, if we have a continuous homomorphism $\mathfrak{G} \to B(G)$, we can consider the fiber-product of $\mathbf{B}(G)$ and $\mathfrak{G}$ over $B(G)$. This fiber-product is a closed subgroup of the fiber-product $\mathbf{B}(G)_{\mathfrak{G}}$ defined by the product homomorphism $\mathfrak{G} \to B(G) \to \mathrm{Sp}\,(G)$. Therefore, if $\mathfrak{G}$ is locally compact, the fiber-product over $B(G)$ will have the same properties as $\mathbf{B}(G)_{\mathfrak{G}}$. We can prove these properties directly, i.e., without going through $\mathbf{B}(G)_{\mathfrak{G}}$, and the proof is simpler. At any rate, if we take G to be the localization or the adelization of a finite dimensional vector space over an algebraic number field or a function field of one variable with a finite constant field, and if we take $\mathfrak{G}$ to be the localization or the adelization of the corresponding pseudosymplectic group, the fiber-product over $B(G)$ will become the metaplectic group in the terminology of Weil [11]. In other words, the weaker version of Corollary 2 implies the continuity property for the metaplectic group.

COROLLARY 3. *If G is a locally compact abelian group and Γ is a closed subgroup with the Haar measure $d\xi$, every element Φ of $\mathcal{S}(G)$ gives rise to a continuous function F_Φ on $\mathbf{B}(G)$ as*

$$F_\Phi(\mathrm{s}) = \int_\Gamma (\mathrm{s}\Phi)(\xi)\, d\xi.$$

We have shown, in Lemma 3, that the restriction to Γ gives a continuous mapping $\mathcal{S}(G) \to \mathcal{S}(\Gamma)$. Therefore, the integral for $F_\Phi(\mathrm{s})$ is absolutely convergent. We next observe that, because of the obvious formula $F_{\mathrm{s}_0\Phi}(\mathrm{s}) = F_\Phi(\mathrm{s}\mathrm{s}_0)$, we have only to prove the continuity of F_Φ at the identity of $\mathbf{B}(G)$. We take an admissible pair (H, H') such that Φ is contained in $\mathcal{S}(H, H')$. We shall use the same notations as in the proof of Theorem 1. We identify $\Gamma_E = ((\Gamma \cap H) + H')/H'$ with $(\Gamma \cap H)/(\Gamma \cap H')$ and denote the image in Γ_E of an element ξ of $\Gamma \cap H$ by ξ_E. Also, we decompose the restriction of $d\xi$ to $\Gamma \cap H$ into the Haar measures $d\xi_E$ and $d\xi'$ on Γ_E and $\Gamma \cap H'$ respectively such that the total measure of $\Gamma \cap H'$ is 1. Then, for every s in $\mathbf{B}(H, H')$ we have

$$F_\Phi(\mathrm{s}) = \int_{\Gamma_E} (\mathrm{s}_E \Phi_E)(\xi_E)\, d\xi_E = F_{\Phi_E}(\mathrm{s}_E).$$

This reduces the problem to the case when $G=E$ and $\Gamma=\Gamma_E$. We take a distance function r on E such that its restriction to Γ gives its distance function. This is possible. On the other hand, we choose a positive integer m satisfying $m \geqslant \dim(\Gamma^*)+1$. Then, for any positive real number ε, we can find a neighborhood of the identity in $\mathbf{B}(E)$ such that we have $\|r^k\cdot(\mathrm{s}\Phi-\Phi)\|_\infty \leqslant \varepsilon$ for $k=0$, m and for every s in this neighborhood. This implies

$$|F_\Phi(\mathrm{s})-F_\Phi(\mathbf{1})| \leqslant c\cdot\varepsilon$$

for the same s, in which $\mathbf{1}$ is the identity element of $\mathbf{B}(E)$ and c is a constant depending only on Γ (and on the choice of r). This completes the proof.

We note that Corollary 3 is a generalization of Theorem 6 in [11]. What seems to be quite remarkable is the fact that, in the general case, *the continuous function F_Φ on $\mathbf{B}(G)$ can be considered locally everywhere as a coimage of continuous functions on Lie groups.*

Finally, we shall discuss the example of a locally compact abelian group mentioned in the introduction. Let $\mathbf{F}_q$ denote a finite field with q elements and consider the quotient field of the ring of formal power-series in one variable, say t, with coefficients in $\mathbf{F}_q$. We shall denote by H_m the subset of this field consisting of elements of the form $a_m t^m+\ldots$ for every m in $\mathbf{Z}$. Then the additive group of this field with $(H_n)_{n\geqslant 0}$ as a fundamental system of neighborhoods of 0 forms a locally compact abelian group. We shall show that this will give an example of G for which the mapping $\mathbf{B}(G)\times\mathcal{S}(G)\to\mathcal{S}(G)$ is not continuous.

We observe that (H_{-n}, H_n) forms an admissible pair for G and that the sequence $((H_{-n}, H_n))_{n\geqslant 0}$ is cofinal in the set of all admissible pairs. Therefore $\mathcal{S}(G)$ is the inductive limit of $\mathcal{S}(H_{-n}, H_n)$ and this is topologically isomorphic to $\mathbf{C}^{2qn}$ in an obvious manner. After this remark, we take a null sequence $(N_n)_{n\geqslant 0}$ of positive real numbers. We then consider the subset X of $\mathcal{S}(G)$ consisting of those Φ with the property that $|\Phi(x)|<N_n$ for every x in $H_{-n}-H_{-n+1}$. Then X is an open neighborhood of 0 in $\mathcal{S}(G)$. We shall show that, if Σ is a neighborhood of the identity in $\mathbf{B}(G)$ and Y is a neighborhood of 0 in $\mathcal{S}(G)$, the image ΣY of $\Sigma\times Y$ by the mapping $\mathbf{B}(G)\times\mathcal{S}(G)\to\mathcal{S}(G)$ is never contained in X. If we take a positive integer k sufficiently large, the element of $\mathbf{B}(G)$ defined by $\Phi(x)\to|\alpha|^{\frac{1}{2}}\Phi(x\alpha)$ is contained in Σ for every α in $W(H_{-k}, H_k)$. We then take a positive real number δ sufficiently small such that $\Phi_\delta=\delta$-times the characteristic function of H_{-k-1} is contained in Y. Since we have $N_n\to 0$ for $n\to\infty$, we can find an integer m larger than $k+1$ for which $N_m\leqslant\delta$. We put

$$\begin{cases} t^n\alpha=t^n & n\neq -k-1,\ -m\\ t^{-k-1}\alpha=t^{-m}, & t^{-m}\alpha=t^{-k-1}. \end{cases}$$

Then α can be extended uniquely to an element of Aut (G). If we denote this extension also by α, it is in $W(H_{-k}, H_k)$ and we have $|\alpha|=1$. Therefore, the function $x\to\Phi_\delta(x\alpha)$ is in ΣY. However, because of

$$\Phi_\delta(t^{-m}\alpha)=\Phi_\delta(t^{-k-1})=\delta\geqslant N_m,$$

it is not in X. This proves the assertion. We observe that $\mathbf{B}(G)$ is not locally bounded and this is responsible for the discontinuity of the mapping $\mathbf{B}(G)\times S(G)\to S(G)$.

II. Theta-functions on the group $\mathbf{B}(G)$

4. We shall denote by Γ a closed subgroup of a locally compact abelian group G. Then, by Corollary 3 of Theorem 1, every element Φ of $S(G)$ gives rise to a continuous function F_Φ on $\mathbf{B}(G)$. On the other hand, let $B(G, \Gamma)$ denote the set of those elements $s=(\sigma, f)$ of $B(G)$ satisfying $(\Gamma\times\Gamma_*)\sigma=\Gamma\times\Gamma_*$ and $f=1$ on $\Gamma\times\Gamma_*$. Then $B(G,\Gamma)$ forms a closed subgroup of $B(G)$. The point is that the homomorphism $\pi:\mathbf{B}(G)\to B(G)$ has a continuous cross-section over $B(G,\Gamma)$ and that the function F_Φ possesses a remarkable semi-invariance property with respect to the image group. We shall briefly recall this part of the Weil theory.

The said cross-section over $B(G,\Gamma)$ is not constructed directly but in terms of another realization of $\mathbf{B}(G)$. We take dual measures $d\xi$ and $d\dot{x}^*$ on Γ and G^*/Γ_* and dual measures $d\dot{x}$ and $d\xi^*$ on G/Γ and Γ_*. Then $dx=d\dot{x}d\xi$ and $dx^*=d\dot{x}^*d\xi^*$ are dual measures on G and G^*. We put $Q=(G\times G^*)/(\Gamma\times\Gamma_*)$. Let $L(G,\Gamma)$ denote the vector space of those complex-valued, continuous functions Θ on $G\times G^*$ satisfying the functional equation

$$\Theta(x-\xi,\, x^*-\xi^*)=\langle\xi, x^*\rangle\cdot\Theta(x, x^*)$$

for every ξ, ξ^* in Γ, Γ_* and with the property that the functions on Q well defined by $(\dot{x},\dot{x}^*)\to|\Theta(x,x^*)|$ have compact supports. Then $L(G,\Gamma)$ forms a pre-Hilbert space with

$$(\Theta,\Theta')_Q=\iint_Q\Theta(x,x^*)\overline{\Theta'(x,x^*)}\,d\dot{x}\,d\dot{x}^*$$

as its scalar product. Let $H(G,\Gamma)$ denote its completion. Then, there exists a (norm-preserving) isomorphism Z of $L^2(G)$ to $H(G,\Gamma)$. If we take Φ from $S(G)$, its image is represented by a continuous function. If we denote this unique representative also by $Z\Phi$, we have

$$(Z\Phi)(x,x^*)=\int_\Gamma\Phi(x+\xi)\langle\xi,x^*\rangle\,d\xi.$$

On the other hand, if $s=(\sigma,f)$ is an element of $B(G,\Gamma)$ and if we put

$$(\mathbf{r}_\Gamma(s)\Theta)(z)=|\sigma|_Q^{\frac{1}{2}}\,\Theta(z\sigma)f(z)$$

with $|\sigma|_Q$ denoting the module of the topological automorphism of Q determined by σ, we get a homomorphism $\mathbf{r}_\Gamma : B(G, \Gamma) \to \mathrm{Aut}\,(H(G, \Gamma))$. Moreover, it is easy to show that $\mathbf{r}_\Gamma$ is continuous. If we consider $Z^{-1}\mathbf{r}_\Gamma Z$, which we shall denote also by $\mathbf{r}_\Gamma$ as in [11], we get a continuous cross-section for $\pi : \mathbf{B}(G) \to B(G)$ with the property

$$F_\Phi(\mathbf{r}_\Gamma(s)\mathbf{s}) = |\sigma|_Q^{\frac{1}{2}} F_\Phi(\mathbf{s})$$

for every s in $B(G, \Gamma)$ and $\mathbf{s}$ in $\mathbf{B}(G)$. This is (the corollary of) Theorem 4 in [11]. The module $|\sigma|_Q$ is not, in general, equal to 1. However, in the important special cases when Q is either compact or discrete, it is certainly equal to 1, hence we have the invariance instead of the semi-invariance. At any rate, we may call F_Φ an *automorphic function* on $\mathbf{B}(G)$ belonging to $\mathbf{r}_\Gamma$ $(B(G, \Gamma))$ or simply to Γ. We shall introduce a theta-function on $\mathbf{B}(G)$.

The idea is simply to introduce a suitable subspace of $\mathcal{S}(G)$ and to consider the corresponding vector subspace of the space of all automorphic functions on $\mathbf{B}(G)$. If E is an elementary abelian group, we consider a complex-valued function with its support contained in the union of a finite number of cosets of the connected component $E_0 = \mathbf{R}^n \times (\mathbf{R}/\mathbf{Z})^q$ of $E = \mathbf{R}^n \times \mathbf{Z}^p \times (\mathbf{R}/\mathbf{Z})^q \times F$. If x, y denote the coordinates on $\mathbf{R}^n$, $(\mathbf{R}/\mathbf{Z})^q$, we assume that, on each one of these cosets, it is a finite linear combination (with complex coefficients) of functions of the following form

$$e(\tfrac{1}{2}x\tau{}^t x + x{}^t a + y{}^t b),$$

in which τ is a symmetric matrix in $M_n(\mathbf{C})$ with a positive non-degenerate imaginary part and a, b are in $\mathbf{C}^n$, $\mathbf{Z}^q$ respectively. We shall denote the complex vector space of all such functions by $\mathcal{G}(E)$. We see immediately that a topological isomorphism $E \to E'$ gives a $\mathbf{C}$-linear bijection $\mathcal{G}(E') \to \mathcal{G}(E)$. In particular, $\mathcal{G}(E)$ is intrinsically defined and, clearly, it is contained in $\mathcal{S}(E)$. Now, if G is a general locally compact abelian group and (H, H'), an admissible pair, we define $\mathcal{G}(H, H')$ as the image of $\mathcal{G}(H/H')$ under the $\mathbf{C}$-linear bijection $\mathcal{S}(H/H') \to \mathcal{S}(H, H')$. Then we define $\mathcal{G}(G)$ as the union of all $\mathcal{G}(H, H')$. It is clear that $\mathcal{G}(G)$ forms a complex vector space (contained in $\mathcal{S}(G)$). We shall prove the following lemma:

LEMMA 7. *If G' is a compact subgroup of G, the coimage of $\mathcal{G}(G/G')$ under the canonical homomorphism $G \to G/G'$ is contained in $\mathcal{G}(G)$; if G' is an open subgroup of G, the image of $\mathcal{G}(G')$ under the "extension by 0" is contained in $\mathcal{G}(G)$; $\mathcal{G}(G_1 \times G_2)$ contains $\mathcal{G}(G_1) \otimes \mathcal{G}(G_2)$; the Fourier transformation gives a $\mathbf{C}$-linear bijection $\mathcal{G}(G) \to \mathcal{G}(G^*)$; if f is in $X_2(G)$ and Φ is in $\mathcal{G}(G)$, $f\Phi$ is in $\mathcal{G}(G)$; if G is compact, $\mathcal{G}(G)$ contains all finite linear combinations of elements of $X_1(G)$; if x denotes the coordinate on $\mathbf{R}$, the function $x \to \exp(-\pi x^2)$ is in $\mathcal{G}(\mathbf{R})$. Moreover, if $\mathfrak{J}(G)$ is an intrinsically defined complex vector space of functions on G with these seven properties, for every G, $\mathfrak{J}(G)$ contains $\mathcal{G}(G)$.*

Proof. Among the seven properties, only the fourth and the fifth need proofs. Instead of the fourth property, we shall prove a stronger statement to the effect that, if G decomposes into $G_1 \times G_2$, the partial Fourier transformation, say on the second factor, gives a $\mathbf{C}$-linear bijection $\mathcal{G}(G_1 \times G_2) \to \mathcal{G}(G_1 \times G_2^*)$. We have only to show that it maps $\mathcal{G}(G_1 \times G_2)$ to $\mathcal{G}(G_1 \times G_2^*)$. The verification of this fact is reduced immediately to the case when G_1 and G_2 are elementary. We shall denote them by E_1 and E_2. Since the Fourier transformation decomposes into a product when E_2 decomposes into a product, we may assume that E_2 is one of the four types of "simple" factors. Then, only the case when $E_2 = \mathbf{R}^{n_2}$ is not obvious. In this case, because of the particular form of elements of $\mathcal{G}(E_1 \times E_2)$, we may assume that $E_1 = \mathbf{R}^{n_1}$. We may also assume, by a change of coordinates in E_2^*, that we have $\langle x_2, x_2^* \rangle = e(x_2{}^t x_2^*)$. Then the partial Fourier transform Φ^* of an element Φ of $\mathcal{G}(E_1 \times E_2)$ defined by

$$\Phi(x_1, x_2) = e(\tfrac{1}{2}(x_1 x_2)\tau^t(x_1 x_2) + (x_1 x_2)^t(a_1 a_2))$$

with

$$\tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{21} & \tau_2 \end{pmatrix}$$

is given by

$$\Phi^*(x_1, x_2^*) = \det(i^{-1}\tau_2)^{-\frac{1}{2}} e(-\tfrac{1}{2} a_2 \tau_2^{-1}\, {}^t a_2) e(\tfrac{1}{2}(x_1 x_2^*)\tau^{*t}(x_1 x_2^*) + (x_1 x_2^*)^t(a_1^* a_2^*))$$

with

$$\tau^* = \begin{pmatrix} \tau_1 - \tau_{12}\tau_2^{-1}\tau_{21} & -\tau_{12}\tau_2^{-1} \\ -\tau_2^{-1}\tau_{21} & -\tau_2^{-1} \end{pmatrix}$$

$$a_1^* = a_1 - a_2\tau_2^{-1}\tau_{21}, \qquad a_2^* = -a_2\tau_2^{-1}.$$

We observe that τ^* is also a symmetric matrix with a positive non-degenerate imaginary part. Therefore Φ^* is an element of $\mathcal{G}(E_1 \times E_2^*)$. We shall prove the fifth property. We recall that, for any f in $X_2(G)$ and for any compactly generated subgroup H of G, there exists an admissible pair (H, H'') such that $f(h + h'') = f(h)$ for every h, h'' in H, H'' respectively. Therefore, if Φ is in $\mathcal{G}(H, H')$, by taking a smaller H' if necessary and putting $E = H/H'$, the problem is reduced to showing that, if f is in $X_2(E)$ and Φ is in $\mathcal{G}(E)$, then $f\Phi$ is also in $\mathcal{G}(E)$. We recall that, if $E = \mathbf{R}^n \times \mathbf{Z}^p \times (\mathbf{R}/\mathbf{Z})^q \times F$ and if (x, y) denote coordinates on $E_0 = \mathbf{R}^n \times (\mathbf{R}/\mathbf{Z})^q$, the restriction of f to each coset of E_0 is given by $t \cdot e(\frac{1}{2}xh^t x)\chi(x,y)$ with t in T and χ in $X_1(E_0)$, depending on the coset, and with a symmetric matrix h in $M_n(\mathbf{R})$. Therefore $f\Phi$ is certainly an element of $\mathcal{G}(E)$. We leave the verification of the additional remark as an exercise.

THEOREM 2. *If G is a locally compact abelian group, the vector subspace $\mathcal{G}(G)$ of $\mathcal{S}(G)$ is dense in $\mathcal{S}(G)$. Moreover, every element of $\mathbf{B}(G)$ gives an automorphism of $\mathcal{G}(G)$.*

Proof. We shall prove the first part. We may assume that G is elementary. Let Φ denote an arbitrary element of $\mathcal{S}(E)$ for $E=\mathbf{R}^n\times\mathbf{Z}^p\times(\mathbf{R}/\mathbf{Z})^q\times F$. We shall show that Φ can be approximated by an element of $\mathcal{G}(E)$. It is well known that Φ can be approximated by an element of $\mathcal{S}(E)$ with a compact support (cf. [6], [1]). Therefore, we may assume that Φ has a compact support. Then we have only to approximate Φ by an element of $\mathcal{G}(E)$ on each one of the finite number of cosets of $E_0=\mathbf{R}^n\times(\mathbf{R}/\mathbf{Z})^q$ where Φ is different from the constant 0. This reduces the problem to the case when $E=E_0$. In this case, if Φ^* is the Fourier transform of Φ, we have only to approximate Φ^* by an element of $\mathcal{G}(E^*)$. Since we have $E^*=\mathbf{R}^n\times\mathbf{Z}^q$, by what we have said, we may assume that $q=0$. In this way, the problem is finally reduced to the case when $E=\mathbf{R}^n$. This is, perhaps, a classically known case. However, for the sake of completeness, we shall give a proof. The function Φ in $\mathcal{S}(\mathbf{R}^n)$ is assumed to have a compact support, say C. We shall introduce a function Φ_λ depending on a positive real number λ by the following integral

$$\Phi_\lambda(x)=\lambda^n\int_{\mathbf{R}^n}\Phi(y)\exp(-\pi\lambda^2r(x-y)^2)\,dy,$$

in which r is the distance function on $\mathbf{R}^n$. Then, for any invariant differential operator D on $\mathbf{R}^n$, we have

$$(D\Phi_\lambda)(x)=\lambda^n\int_{\mathbf{R}^n}\Phi(y)\,(D_x\exp(-\pi\lambda^2r(x-y)^2))\,dy$$

$$=\lambda^n\int_{\mathbf{R}^n}(D\Phi)(y)\exp(-\pi\lambda^2r(x-y)^2)\,dy.$$

If m is a non-negative integer, the function $x\to r(x)^m\exp(-\pi\lambda^2r(x-y)^2)$ is uniformly bounded when y is restricted to C. Therefore $\|r^m\cdot(D\Phi_\lambda)\|_\infty$ is finite for every λ. We shall show that we have

$$\lim_{\lambda\to\infty}\Phi_\lambda=\Phi.$$

We take a large sphere of radius r_0 which contains C. Then we have

$$|r(x)^m(D\Phi_\lambda-D\Phi)(x)|\leqslant\|D\Phi\|_\infty(2r_0)^n\cdot\lambda^{-m-1}r(\lambda x)^{m+n+1}\exp(-\tfrac{1}{4}\pi r(\lambda x)^2)\leqslant\text{const}\cdot\lambda^{-1},$$

provided that $r(x)\geqslant 2r_0\geqslant 1$ and $\lambda\geqslant 1$. On the other hand, if $r(x)\leqslant 2r_0$, for any positive real number ε, we choose another positive real number δ so that $r(x-y)\leqslant\delta$ implies $|(D\Phi)(x)-(D\Phi)(y)|\leqslant\varepsilon$. Then, integrating over the sphere of radius δ centered at x and over its exterior, we get

$$|r(x)^m(D\Phi_\lambda-D\Phi)(x)|\leqslant\text{const}\cdot\varepsilon+\text{const}\cdot\lambda^{-1}.$$

The constants are certain positive real numbers that are independent of x and λ. Therefore Φ_λ converges to Φ for $\lambda \to \infty$ in $\mathcal{S}(\mathbf{R}^n)$. Now, we take a large hypercube with vertices in $\mathbf{Z}^n$ which contains C and we subdivide it into hypercubes δ_i with vertices in $(1/k)\mathbf{Z}^n$ for some positive integer k. For each i, we shall denote by y_i the smallest vertex of δ_i with respect to the lexicographic ordering, say, in $\mathbf{R}^n$ and introduce a function S_k by

$$S_k(x) = (\lambda/k)^n \sum_i \Phi(y_i) \exp(-\pi\lambda^2 r(x-y_i)^2).$$

Then $S_k(x)$ is a Riemann sum for the integral defining $\Phi_\lambda(x)$ and $(DS_k)(x)$ is one for the first integral for $(D\Phi_\lambda)(x)$. We take a large sphere of radius r_1 which contains the hypercube. We observe that, if we write

$$D_x \exp(-\pi\lambda^2 r(x-y)^2) = P(x, y, \lambda) \exp(-\pi\lambda^2 r(x-y)^2),$$

we have $|P(x, y, \lambda)| \leqslant c\cdot(\lambda r(x))^{m'}$ for some positive integer m' and for some positive real number c depending only on D, when $r(x) \geqslant 2r_1$, y in C and $\lambda \geqslant 1$. Therefore, we get

$$|r(x)^m(DS_k - D\Phi_\lambda)(x)| \leqslant 2c\|\Phi\|_\infty (2r_1)^n \cdot \lambda^{-m-1} r(\lambda x)^{m+m'+n+1} \exp(-\tfrac{1}{4}\pi r(\lambda x)^2) \leqslant \text{const}\cdot\lambda^{-1},$$

provided that $r(x) \geqslant 2r_1 \geqslant 1$ and $\lambda \geqslant 1$. On the other hand, if λ is fixed, the family of functions $y \to \lambda^n \Phi(y)(D_x \exp(-\pi\lambda^2 r(x-y)^2))$ indexed by x satisfying $r(x) \leqslant 2r_1$ is certainly equicontinuous. Therefore, for any positive real number ε, we have

$$|r(x)^m(DS_k - D\Phi_\lambda)(x)| \leqslant \varepsilon,$$

provided that k is sufficiently large. Consequently, if we take λ and then k sufficiently large, the Riemann sum S_k approximates Φ as closely as possible in $\mathcal{S}(\mathbf{R}^n)$. We have only to observe, finally, that the said Riemann sum is an element of $\mathcal{G}(\mathbf{R}^n)$. This completes the proof of the first part. As for the second part, we have only to show that every element $\mathbf{s}$ of $\mathbf{B}(G)$ keeps $\mathcal{G}(G)$ stable. We shall use the five-step decomposition of $\Phi \to \mathbf{s}\Phi$. Since $\mathcal{G}(G)$ is kept stable by the complex conjugation as well as by any topological automorphism of G, the third, the fourth (generalized to a partial Fourier transformation) and the fifth properties of $\mathcal{G}(G)$ in Lemma 7 imply that, for Φ and Φ_0 in $\mathcal{G}(G)$ and $\mathbf{s}$ in $\mathbf{B}(G)$, $\mathbf{s}\Phi \otimes \overline{\mathbf{s}\Phi_0}$ is in $\mathcal{G}(G \times G)$. Therefore, we have only to remark that, if Φ_i are elements of $\mathcal{S}(G_i)$ for $i = 1, 2$ such that $\Phi_1 \otimes \Phi_2$ is in $\mathcal{G}(G_1 \times G_2)$, Φ_i are necessarily in $\mathcal{G}(G_i)$ (provided that $\Phi_i \neq 0$). This completes the proof.

We shall prove a supplement to Theorem 2 as its first corollary. We recall that, if G is a locally compact abelian group, it is topologically isomorphic to a product $\mathbf{R}^n \times G_0$, in which G_0 is a closed subgroup of G with compact open subgroups. In fact, we have only to take an open subgroup H topologically isomorphic to $\mathbf{R}^n \times K$ with K compact, extend the projection $H \to \mathbf{R}^n$ to a homomorphism $G \to \mathbf{R}^n$, and take its kernel as G_0. We shall identify G with $\mathbf{R}^n \times G_0$. Although this decomposition is not intrinsic (except when G_0 is the union of totally disconnected compact open subgroups), the dimension n is unique and G_0 contains all compact subgroups of G.

COROLLARY 1. *Let $G = \mathbf{R}^n \times G_0$ denote a decomposition of a locally compact abelian group G such that G_0 has compact open subgroups. We consider a function $\Phi = \Phi_\infty \otimes \Phi_0$ on G defined by*

$$\Phi_\infty(x) = e((\tfrac{1}{2})\, i x^t x) = \exp(-\pi x^t x)$$

$$\Phi_0 = \textit{the characteristic function of a compact open subgroup of } G_0.$$

Then $\mathcal{S}(G)$ is the complex vector space of finite linear combinations of elements of $\mathbf{B}(\mathbf{R}^n)\Phi_\infty \otimes \mathbf{A}(G_0)\Phi_0$.

We observe that, for the said decomposition $G = \mathbf{R}^n \times G_0$, $\mathcal{S}(G)$ coincides with the tensor product $\mathcal{S}(\mathbf{R}^n) \otimes \mathcal{S}(G_0)$. Therefore, we have only to show that $\mathbf{B}(\mathbf{R}^n)\Phi_\infty$ and $\mathbf{A}(G_0)\Phi$ respectively generate $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}(G_0)$. We shall examine the two cases when $G = \mathbf{R}^n$ and $G = G_0$. In the case when $G = \mathbf{R}^n$ and $\langle x, x^* \rangle = e(x^t x^*)$, for any Φ in $\mathcal{S}(G)$, $\mathbf{B}(G)$ contains $\Phi(x) \to |\det(\alpha)|^{\frac{1}{2}} \Phi(x\alpha) e(\frac{1}{2} x h^t x + x^t a)$ and the Fourier transformation $\Phi \to \Phi^*$, in which α is in $GL_n(\mathbf{R})$, h is a symmetric matrix in $M_n(\mathbf{R})$ and a is in $\mathbf{R}^n$. (We shall review this property of $\mathbf{B}(G)$ more generally in the next section.) Therefore, applying the two elements of $\mathbf{B}(G)$ in that order to Φ_∞ and then applying $\Phi(x) \to \Phi(x) e(x^t b)$ with b in $\mathbf{R}^n$, we will get $\Phi(x) = e(\frac{1}{2} x \tau^t x + x^t a)$, in which τ is a symmetric matrix in $M_n(\mathbf{C})$ with a positive non-degenerate imaginary part and a is in $\mathbf{C}^n$. This settles the first case. We shall consider the case when $G = G_0$. Suppose that K, K' are compact open subgroups of G and φ, φ' are their characteristic functions. We take a set of representatives $u_1, u_2, \ldots$ of the finite group $(K+K')/K$ and a set of representatives $u_1^*, u_2^*, \ldots$ of the finite group $(K')_*/(K+K')_*$. Then we have

$$\varphi' = [K+K' : K']^{-1} \cdot \sum_{i,j} U((u_i, u_j^*), 1)\varphi.$$

We observe that the right-hand side is a finite linear combination of elements of $\mathbf{A}(G)\varphi$. On the other hand, we know that $\mathcal{S}(G)$ consists of finite linear combinations of functions of the form $U((u, u^*), 1)\varphi$, in which φ is the characteristic function of a compact open subgroup of G. Putting them together, we see that every element of $\mathcal{S}(G)$ is a finite linear combination of elements of $\mathbf{A}(G)\Phi_0$.

We shall derive a significant consequence from what we have shown. Let $\mathbf{R}_+^\times$ denote the multiplicative group of positive real numbers and consider the product space $(\mathbf{R}_+^\times) \times \mathbf{R}^n$ with the law of composition

$$(\lambda_1, a_1)(\lambda_2, a_2) = (\lambda_1 \lambda_2, a_1 \lambda_2 + a_2).$$

Then we get a connected solvable Lie group. If $G = \mathbf{R}^n \times G_0$ is the decomposition in Corollary 1, we shall denote by $\Lambda(G)$ the product of this solvable group and $A(G_0)$. We observe that

$$\Phi(x) \to \lambda^{\frac{1}{2}n} \Phi(x\lambda + a)$$

defines a unitary representation of $(\mathbf{R}_+^\times) \times \mathbf{R}^n$ (with the above defined law of composition) in $L^2(\mathbf{R}^n)$ and that the image group is contained in $\mathbf{B}(\mathbf{R}^n)$. In this way, we get a continuous homomorphism $\Lambda(G) \to \mathbf{B}(\mathbf{R}^n) \times \mathbf{A}(G_0)$. If we combine this homomorphism with the obvious homomorphism $\mathbf{B}(\mathbf{R}^n) \times \mathbf{B}(G_0) \to \mathbf{B}(G)$, we get a continuous homomorphism $\Lambda(G) \to \mathbf{B}(G)$.

COROLLARY 2. *If I is a tempered distribution on $G = \mathbf{R}^n \times G_0$ and Φ is an element of $\mathcal{S}(G)$, we get a continuous function I_Φ on the locally compact solvable group $\Lambda(G)$. Moreover, if Φ is the particular function introduced in Corollary 1, $I_\Phi = 0$ implies $I = 0$.*

The continuous function I_Φ on $\Lambda(G)$ is defined as the product of $\Lambda(G) \to \mathbf{B}(G)$, the mapping $\mathbf{B}(G) \to \mathcal{S}(G)$ defined by $\mathbf{s} \to \mathbf{s}\Phi$, and the $\mathbf{C}$-linear mapping $I: \mathcal{S}(G) \to \mathbf{C}$. The first mapping is continuous by what we have said; the second mapping is continuous by Corollary 1 of Theorem 1; the third mapping is continuous by definition. Now, if $\Phi = \Phi_\infty \otimes \Phi_0$ is the function in Corollary 1, the proof of Theorem 2 shows that functions of the form $x \to \Phi_\infty(x\lambda + a)$ for (λ, a) in $(\mathbf{R}_+^\times) \times \mathbf{R}^n$ generate a dense subspace of $\mathcal{S}(\mathbf{R}^n)$. Therefore, if we have $I_\Phi = 0$, Corollary 1 shows that we have $I = 0$.

We fix a closed subgroup Γ of G and define a *theta-function* as an automorphic function F_Φ on $\mathbf{B}(G)$ belonging to Γ in which Φ is an element of $\mathcal{G}(G)$. We note that there is "essentially" but one theta-function. In fact, every theta-function is a finite linear combination of $F_{\mathbf{s}\Phi}$, in which Φ is the function in Corollary 1 and $\mathbf{s}$ is in the image group of $\mathbf{B}(\mathbf{R}^n) \times \mathbf{A}(G_0)$ by the homomorphism $\mathbf{B}(\mathbf{R}^n) \times \mathbf{B}(G_0) \to \mathbf{B}(G)$. The set of theta-functions forms a vector subspace of the complex vector space of all automorphic functions. This subspace is fairly large. In fact, we have the following result:

COROLLARY 3. *On any compact subset of $\mathbf{B}(G)$, every automorphic function can be uniformly approximated by a theta-function.*

Suppose that Φ is an arbitrary element of $\mathcal{S}(G)$. Then Φ is in $\mathcal{S}(H, H')$ for some admissible pair (H, H'). Since $\mathcal{S}(H, H')$ is metrizable and $\mathcal{G}(H, H')$ is dense in $\mathcal{S}(H, H')$, we can find a sequence of elements $\Phi_1, \Phi_2, \ldots$ of $\mathcal{G}(H, H')$ converging to Φ in $\mathcal{S}(H, H')$. Let Σ denote a compact subset of $\mathbf{B}(G)$. Then the union of $\Sigma\Phi_1, \Sigma\Phi_2, \ldots, \Sigma\Phi$ is a compact subset of $\mathcal{S}(G)$ by Corollary 1 of Theorem 1. Hence, by making H larger and H' smaller, we may assume that it is contained in $\mathcal{S}(H, H')$. We shall show that the convergence of $\mathbf{s}\Phi_1, \mathbf{s}\Phi_2, \ldots$ to $\mathbf{s}\Phi$ in $\mathcal{S}(H, H')$ is uniform with respect to $\mathbf{s}$ in Σ. We note that this implies the corollary. In fact, we have only to proceed as in the proof of Corollary 3 of Theorem 1. Now, since Σ can be covered by a finite number of cosets $\mathbf{B}(H, H')\mathbf{s}_i$ with $\mathbf{s}_i$ in Σ, it is enough to show that, for each i, the convergence of $\mathbf{s}\mathbf{s}_i\Phi_1, \mathbf{s}\mathbf{s}_i\Phi_2, \ldots$ to $\mathbf{s}\mathbf{s}_i\Phi$ in $\mathcal{S}(H, H')$ is uniform with respect to $\mathbf{s}$ in $\Sigma_i = \mathbf{B}(H, H') \cap \Sigma\mathbf{s}_i^{-1}$. Since $\mathbf{B}(H, H')$ is an open subgroup of $\mathbf{B}(G)$, it is closed, and hence Σ_i is a compact subset of $\mathbf{B}(H, H')$. If we replace the sequence $\mathbf{s}_i\Phi_1, \mathbf{s}_i\Phi_2, \ldots$ and $\mathbf{s}_i\Phi$ by $\Phi_1, \Phi_2, \ldots$ and Φ, and Σ_i by Σ, we are in the case where Σ is contained in $\mathbf{B}(H, H')$. Put $E = H/H'$. Then, by Theorem 1, the problem is finally reduced to showing that, if a sequence $\Phi_1, \Phi_2, \ldots$ converges to Φ in the metrizable space $\mathcal{S}(E)$, for any compact subset Σ of $\mathbf{B}(E)$, the convergence of $\mathbf{s}\Phi_1, \mathbf{s}\Phi_2, \ldots$ to $\mathbf{s}\Phi$ in $\mathcal{S}(E)$ is uniform with respect to $\mathbf{s}$ in Σ. However, since $\mathbf{B}(E)$ is a Lie group, Σ is sequentially compact. Therefore, this is certainly the case.

We note that Corollary 2 of Theorem 1 permits us to consider the restrictions of automorphic and theta-functions on $\mathbf{B}(G)$ to

$$\mathbf{B}(G)_{\mathfrak{G}} = \mathbf{B}(G) \times_{\mathrm{Sp}(G)} \mathfrak{G}$$

for any locally compact group $\mathfrak{G}$ and continuous homomorphism $\mathfrak{G} \to \mathrm{Sp}(G)$. We shall call them *automorphic* and *theta-functions* on $\mathbf{B}(G)_{\mathfrak{G}}$. It seems that all complex-valued theta-functions can be obtained by this process. In this connection, we observe that *the Siegel formula* (for the orthogonal group) *as formulated and proved by Weil in* [11], [12] *and the classical Siegel formula by Siegel involving theta-series and Eisenstein series* (cf. [8], [9]) *are equivalent* (once both sides of the Siegel formula are shown to be tempered distributions). This follows from Corollary 2.

5. The locally compact abelian group G has been arbitrary. We shall now discuss that part of the Weil theory where further assumptions are necessary. We consider the group $B(G, \Gamma)$ for $\Gamma = 0$. It is clear that this group consists of elements

$$s = (\sigma, f) = \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, f\right)$$

of $B(G)$, in which $\gamma = 0$ and f is in $X_2(G)$. Moreover, if we put $\mathbf{r}_\Gamma(s) = \mathbf{p}(s)$ for $\Gamma = 0$, we have

$$(\mathbf{p}(s)\Phi)(x) = |\alpha|^{\frac{1}{2}} f(x)\Phi(x\alpha),$$

in which $|\alpha|$ is the module of the topological automorphism α of G. According to the general property of $\mathbf{r}_\Gamma$, the correspondence $s \to \mathbf{p}(s)$ defines a continuous cross-section for $\pi: \mathbf{B}(G) \to B(G)$ over the subgroup $P(G)$ of $B(G)$ defined by $\gamma = 0$ and f in $X_2(G)$.

On the other hand, we consider the subset $\Omega(G)$ of $B(G)$ defined by the condition that γ is in Is (G^*, G), i.e., in the subset of Hom (G^*, G) consisting of topological isomorphisms. We note that $\Omega(G)$ is empty unless G is autodual, i.e., topologically isomorphic to G^*. Although $\Omega(G)$ is not a subgroup, we have $\Omega(G)^{-1}=\Omega(G)$ and $P(G)\Omega(G)=\Omega(G)P(G)=\Omega(G)$. More precisely, if we put

$$d'(\gamma)=\left(\begin{pmatrix}0 & -\gamma^{*-1}\\ \gamma & 0\end{pmatrix}, \langle x, -x^*\rangle\right)$$

for any γ in Is (G^*, G), we have $\Omega(G)=P(G)d'(\gamma)P(G)$. In fact, if s is an arbitrary element of $\Omega(G)$, we can write it in the form $s=s_1d'(\gamma)s_2$ with s_1, s_2 in $P(G)$ such that "$\alpha=1$" in one of them. Now, if $s=(\sigma, f)$ is in $\Omega(G)$, we put

$$(\mathbf{r}(s)\Phi)(x)=|\gamma|^{\frac{1}{2}}\int\Phi(x\alpha+x^*\gamma)f(x, x^*)\,dx^*$$

for every Φ in $S(G)$. Then, by a completely formal verification, we get $\mathbf{r}(s)^{-1}U(w, t)\mathbf{r}(s)=U((w, t)s)$, $(\mathbf{r}(s)\Phi, \Phi')=(\Phi, \mathbf{r}(s^{-1})\Phi')$, and $\mathbf{p}(s)\mathbf{r}(s')=\mathbf{r}(ss')$, $\mathbf{r}(s')\mathbf{p}(s)=\mathbf{r}(s's)$. Moreover, if Φ^* denotes the Fourier transform of Φ, we have

$$(\mathbf{r}(d'(\gamma))\Phi)(x)=|\gamma|^{-\frac{1}{2}}\Phi^*(-x\gamma^{*-1}).$$

Therefore $\mathbf{r}$ defines a continuous cross-section for $\pi:\mathbf{B}(G)\to B(G)$ over $\Omega(G)$ satisfying $\mathbf{r}(s^{-1})=\mathbf{r}(s)^{-1}$.

Now, if s is an element of $\Omega(G)$ and if f is an element of $X_2(G)$ which is non-degenerate in the sense that $\varrho=\varrho(f)$ is in Is (G, G^*), the element s' of $B(G)$ defined by

$$s^{-1}\left(\begin{pmatrix}1 & \varrho\\ 0 & 1\end{pmatrix}, f\right)s=s'$$

is also in $\Omega(G)$. Therefore, we get a relation of the form

$$\mathbf{r}(s)^{-1}\mathbf{p}\left(\begin{pmatrix}1 & \varrho\\ 0 & 1\end{pmatrix}, f\right)\mathbf{r}(s)=\gamma(f)\mathbf{r}(s')$$

in $\mathbf{B}(G)$ with some $\gamma(f)$ in T. We choose an arbitrary γ from Is (G^*, G) and write s in the form $s_1d'(\gamma)s_2$ with s_1, s_2 in $P(G)$ and "$\alpha=1$" in s_1. Then we will get a similar relation in which s is replaced by $d'(\gamma)$ and $\gamma(f)$ remains the same. Consequently $\gamma(f)$ is independent of the element s of $\Omega(G)$. Moreover, if we consider f as a tempered distribution on G, its Fourier transform f^* is given by

$$f^*(x^*)=\gamma(f)|\varrho|^{-\frac{1}{2}}f(x^*\varrho^{-1})^{-1}.$$

This is Theorem 2 in [11]. As a consequence, if we replace f by x^*f, i.e., by $x\to\langle x, x^*\rangle f(x)$, we get $\gamma(x^*f)=f(x^*\varrho^{-1})^{-1}\gamma(f)$. We also remark that, if $s=(\sigma, f)$, $s'=(\sigma', f')$, $ss'=s''=(\sigma'', f'')$ are in $\Omega(G)$ and if we put

$$f_0(x)=f(0, x\gamma^{-1})f'(x, -x\alpha'\gamma'^{-1}),$$

we have $\mathbf{r}(s)\mathbf{r}(s')=\gamma(f_0)\mathbf{r}(s'')$. This is Theorem 3 in [11]. We shall show that, under a certain general condition, the constant $\gamma(f)$ can be expressed by an integral over a compact group which is, in a certain sense, a generalization of Gaussian sums.

Suppose the Γ is a closed subgroup of G. We say that the pair (G, Γ) is autodual if there exists a topological isomorphism from G to G^* mapping Γ isomorphically to its annihilator Γ_* in G^*. Also a complex number $z\neq 0$ can be decomposed uniquely into a product of a positive real number and an element t of T. We call t the T-part of z. We shall first prove the following lemma:

LEMMA 8. *Suppose that $s=(\sigma, f)$ is an element of $\Omega(G)\cap B(G, \Gamma)$ with the property that $\Gamma/\Gamma_*\gamma$ is compact. Then we have $\mathbf{r}_\Gamma(s)=\lambda(s)\mathbf{r}(s)$ with $\lambda(s)$ given by*

$$\lambda(s)=T\text{-part of }\int_{\Gamma/\Gamma_*\gamma} f(0, \xi\gamma^{-1})^{-1}d\dot{\xi},$$

in which $d\dot{\xi}$ is the Haar measure on $\Gamma/\Gamma_*\gamma$.

Proof. We observe that $\Gamma_*\gamma$ is contained in Γ by the condition that s is in $B(G, \Gamma)$. By the definition of $\mathbf{r}_\Gamma$, if Φ is an element of $S(G)$, we have

$$(\mathbf{r}_\Gamma(s)\Phi)(x)=|\sigma|_\Omega^{\frac{1}{2}}\int\left(\int\Phi(x\alpha+x^*\gamma+\xi)\langle\xi, x\beta+x^*\delta\rangle d\xi\right)f(x, x^*)d\dot{x}^*$$

provided that the second integral over G^*/Γ_* is absolutely convergent. We are denoting by $d\xi$ and $d\dot{x}^*$ the Haar measures on Γ and G^*/Γ_* respectively. We observe that, if we put $g(x)=f(0, x\gamma^{-1})^{-1}$, we have

$$\langle\xi, x\beta+x^*\delta\rangle f(x, x^*)=f(x, x^*+\xi\gamma^{-1})g(\xi).$$

We observe also that we have $g(\xi+\xi^*\gamma)=g(\xi)$ for every ξ in Γ and ξ^* in Γ_*. Now, the integral over Γ can be decomposed into an integral over $\Gamma_*\gamma$ and an integral over $\Gamma/\Gamma_*\gamma$ with respect to their Haar measures. Since $\Gamma/\Gamma_*\gamma$ is compact, the triple integral is absolutely convergent. Therefore, the expression for $\mathbf{r}_\Gamma(s)\Phi$ is legitimate for every Φ in $S(G)$. Moreover, we can change the order of the second integration over $\Gamma/\Gamma_*\gamma$ and the third integration over G^*/Γ_*. In this way, we get

$$(\mathbf{r}_\Gamma(s)\Phi)(x) = |\sigma|_Q^{\frac{1}{2}} \int \left(\int \Phi(x\alpha + x^*\gamma) f(x, x^*) dx^* \right) g(\xi) d\xi$$

$$= c \cdot (\mathbf{r}(s)\Phi)(x) \cdot \int g(\xi) d\xi,$$

in which $c = |\sigma|_Q^{\frac{1}{2}} |\gamma|^{-\frac{1}{2}}$. Therefore, we have

$$\lambda(s) = c \cdot \int_{\Gamma/\Gamma_{*\gamma}} f(0, \xi\gamma^{-1})^{-1} d\xi.$$

This completes the proof of Lemma 8.

THEOREM 3. *Suppose that* (G, Γ) *is autodual. Then, for every non-degenerate element* f *of* $X_2(G)$ *such that* $f=1$ *on* $\Gamma_*\varrho^{-1}$ *and* $\Gamma/\Gamma_*\varrho^{-1}$ *is compact, we have*

$$\gamma(f) = T\text{-part of } \int_{\Gamma/\Gamma_*\varrho^{-1}} f(\xi) d\xi$$

in which $\varrho = \varrho(f)$ *and the integral is taken with respect to the Haar measure* $d\xi$ *on* $\Gamma/\Gamma_*\varrho^{-1}$. *Actually, the group* $\Gamma/\Gamma_*\varrho^{-1}$ *is finite.*

Proof. We observe that $\Gamma_*\varrho^{-1}$ is contained in Γ because of the condition that $f=1$ on $\Gamma_*\varrho^{-1}$. Moreover, if we introduce the following element

$$s = \left(\begin{pmatrix} 1 & 0 \\ -\varrho^{-1} & 1 \end{pmatrix}, f(x^*\varrho^{-1})^{-1} \right)$$

of $\Omega(G)$, this condition is equivalent to the condition that s is in $B(G, \Gamma)$. In this case, both $\mathbf{r}(s)$ and $\mathbf{r}_\Gamma(s)$ are defined. We shall show that they are related as $\mathbf{r}_\Gamma(s) = \gamma(f)\mathbf{r}(s)$. In order to save some printing space, we put

$$t(f) = \left(\begin{pmatrix} 1 & \varrho \\ 0 & 1 \end{pmatrix}, f \right).$$

Then, by definition, we have

$$\mathbf{r}(d'(\varrho^{-1}))^{-1} \mathbf{p}(t(f)) \mathbf{r}(d'(\varrho^{-1})) = \gamma(f)\mathbf{r}(s).$$

We shall show that the left-hand side is $\mathbf{r}_\Gamma(s)$. Since (G, Γ) is autodual by assumption, there exists an element γ of Is (G^*, G) mapping Γ_* isomorphically to Γ. We then have

$$\mathbf{r}(d'(\varrho^{-1}))^{-1} \mathbf{p}(t(f)) \mathbf{r}(d'(\varrho^{-1})) = \mathbf{r}(d'(\gamma))^{-1} \mathbf{p}(t(f')) \mathbf{r}(d'(\gamma))$$

with $f'(x) = f(x(\varrho\gamma^*)^{-1})$. The point is that $d'(\gamma)$ and $t(f')$ are in $B(G, \Gamma)$. As a special case of Lemma 8, we have $\mathbf{r}(d'(\gamma)) = \mathbf{r}_\Gamma(d'(\gamma))$. Also the verification of $\mathbf{p}(t(f')) = \mathbf{r}_\Gamma(t(f'))$ is immediate.

Since $\mathbf{r}_\Gamma$ is a cross-section and since the π-image of the left-hand side is s, the right-hand side is $\mathbf{r}_\Gamma(s)$. This proves the assertion. Therefore, by Lemma 8 we get the "T-part representation" for $\gamma(f)$. The fact that the compact group $\Gamma/\Gamma_*\varrho^{-1}$ is actually finite can be proved, e.g. by integrating $(\xi, \eta)\to f(\xi+\eta)f(\eta)^{-1}$ over the product of $\Gamma/\Gamma_*\varrho^{-1}$. This completes the proof.

We note that Theorem 3 implies Theorem 5 in [11]. We leave it as an exercise to calculate $\gamma(f)$ in various cases using Theorem 3. We shall, as an example, prove the following result in [11]:

COROLLARY. *Suppose that we take $\langle x, x^*\rangle = e(x{}^tx^*)$ for x in $\mathbf{R}^n$ and x^* in its dual. Then, for $f(x)=e(\frac{1}{2}xh{}^tx)$ with a symmetric, non-degenerate matrix h in $M_n(\mathbf{R})$ of signature* sgn (h), *we have $\gamma(f)=e((1/8)\operatorname{sgn}(h))$.*

We choose an element u of $GL_n(\mathbf{R})$ so that we have

$$uh{}^tu = \tfrac{1}{2}\begin{pmatrix}1_p & 0\\ 0 & -1_q\end{pmatrix} \qquad (p+q=n).$$

Then we can apply Theorem 3 to $\Gamma=\mathbf{Z}^n u$, and we get

$$\gamma(f) = 2^{-\frac{1}{2}n}\sum_{m \bmod 2} e\left(\tfrac{1}{4}m\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}{}^tm\right) = 2^{-\frac{1}{2}n}(1+i)^p(1-i)^q.$$

This is clearly equal to $e((1/8)(p-q))=e((1/8)\operatorname{sgn}(h))$, and the corollary is proved.

We shall now obtain an explicit local expression for our theta-function in the case when $G=\mathbf{R}^n$ and Γ is a lattice in $\mathbf{R}^n$. If we denote coordinates in $G\times G^*$ by (x, x^*), we have $\langle x, x^*\rangle = e(xh{}^tx^*)$ with a symmetric non-degenerate matrix h in $M_n(\mathbf{R})$. Therefore, by using either xh or x^*h as new coordinates in G or G^*, we may assume that $h=1$. Then Sp (G) becomes $Sp_{2n}(\mathbf{R})$. According to Corollary 1 of Theorem 2, we have only to calculate F_Φ for

$$\Phi(x) = e((\tfrac{1}{2})ix{}^tx) = \exp(-\pi x{}^tx).$$

We take a point $s=(\sigma, f)$ of $\Omega(G)$, a point $(w, 1)$ of $A(G)$ and put $\mathbf{s}=U(w, 1)\mathbf{r}(s)$. If $\alpha, \beta, \gamma, \delta$ are the submatrices of σ, we have

$$f(x, x^*) = \langle x\alpha, 2^{-1}x\beta\rangle\langle x\beta, x^*\gamma\rangle\langle x^*\gamma, 2^{-1}x^*\delta\rangle\langle x, m^*\rangle\langle m, -x^*\rangle$$

with (m, m^*) in $G\times G^*$. Therefore, if we put

$$\theta_{mm^*}(\tau, z) = \sum_{\xi\in\Gamma} e(\tfrac{1}{2}(\xi+m)\tau{}^t(\xi+m) + (\xi+m){}^t(z+m^*)),$$

by using a formula in the proof of Lemma 7, we get

$$F_\Phi(\mathrm{s}) = (\operatorname{sgn}(\det(\gamma))\det(i^{-1}(\gamma i+\delta)))^{-\frac{1}{2}}\, e(\tfrac{1}{2}u\tau{}^tu+u{}^tm^*-u^{*t}m-m{}^tm^*-\tfrac{1}{2}m\alpha\gamma^{-1t}m)\cdot\theta_{mm^*}(\tau,\, u\tau+u^*),$$

in which $\tau=(\alpha i+\beta)(\gamma i+\delta)^{-1}$. We have thus shown that the theta-function F_Φ coincides, except for an elementary factor, with the classical theta-function.

We shall examine the invariance property of F_Φ. Since we shall have to use s for a different meaning, we shall denote the previous s by $s'=(\sigma', f')$ keeping m, m^* and other notations; for instance, we have $\mathrm{s}=U(w,1)\mathrm{r}(s')$ and $\tau=(\alpha' i+\beta')(\gamma' i+\delta')^{-1}$. Now, if $s=(\sigma, f)$ is an element of $B(G,\Gamma)$, we have

$$F_\Phi(\mathrm{r}_\Gamma(s)\mathrm{s}) = F_\Phi(\mathrm{s}).$$

As before, we have

$$f(x, x^*) = \langle x\alpha,\, 2^{-1}x\beta\rangle\langle x\beta,\, x^*\gamma\rangle\langle x^*\gamma,\, 2^{-1}x^*\delta\rangle\langle x,\, a^*\rangle\langle a,\, -x^*\rangle$$

with some (a, a^*) in $G\times G^*$. We shall assume that s is in $\Omega(G)$, i.e. that γ is non-degenerate. Then, changing the variable point s' slightly, we may assume that ss' is also contained in $\Omega(G)$. Then, using some of the results that we have either recalled or proved, we can translate the invariance property of F_Φ into

$$\begin{aligned}&\theta_{(m\, m^*)\sigma^{-1}+(a\, a^*)}((\alpha\tau+\beta)(\gamma\tau+\delta)^{-1}, z(\gamma\tau+\delta)^{-1})\\ &\qquad = \varkappa(\sigma)e(\varphi_{mm^*}(\sigma))\det(\gamma\tau+\delta)^{\frac{1}{2}}e(\tfrac{1}{2}z(\gamma\tau+\delta)^{-1}\gamma{}^tz)\cdot\theta_{mm^*}(\tau, z),\end{aligned}$$

in which $z=u\tau+u^*$. Furthermore, $\varkappa(\sigma)$ and $\varphi_{mm^*}(\sigma)$ are given by

$$\begin{aligned}\varkappa(\sigma) &= (i^{-n}\operatorname{sgn}(\det(\gamma)))^{\frac{1}{2}}e(\tfrac{1}{2}a\alpha\gamma^{-1\,t}a+a{}^ta^*)\\ &\qquad (T\text{-part of} \sum_{\xi \bmod \Gamma_*\gamma} e(\tfrac{1}{2}\xi\gamma^{-1}\delta{}^t\xi-\xi\gamma^{-1\,t}a))\\ \varphi_{mm^*}(\sigma) &= -(\tfrac{1}{2})(m{}^t\beta\delta{}^tm+m^{*t}\alpha\gamma{}^tm^*-2m{}^t\beta\gamma{}^tm^*-2a^{*t}(m{}^t\delta-m^{*t}\gamma)).\end{aligned}$$

This is one of the forms in which we express the functional equation of the classical theta-functions (cf. [3], pp. 180–182). We can, of course, use $\langle x, x^*\rangle=e(xh{}^tx^*)$ without making the normalization $h=1$. Then the formulas have to be modified accordingly. We also note that, if we use the five-step decomposition instead of $\mathbf{r}$, although the calculations become more involved, we get other local expressions for F_Φ especially in the case when $\det(\gamma)=0$.

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