POTENTIAL THEORY OF RANDOM WALKS ON ABELIAN GROUPS

BY

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1. Introduction

Let $\mathfrak{G}$ be a locally compact Abelian group and let μ denote a regular probability measure on $\mathfrak{G}$. If $\{\xi_n, n\geq 1\}$ is a sequence of independent $\mathfrak{G}$ valued random variables each having μ for their probability law, then the *random walk* with initial point S_0 is the Markov chain $S_n = S_0 + \xi_1 + \ldots + \xi_n$. If $\mathfrak{G}_0$ is the minimal closed subgroup of $\mathfrak{G}$ generated by the support $S(\mu)$ of μ, then $P_0(S_n \in \mathfrak{G}_0$ for all $n\geq 1) = 1$, where $P_x(\cdot)$ denotes conditional probability given $S_0 = x$. Henceforth we will assume that $\mathfrak{G}_0 = \mathfrak{G}$. This entails no real loss in generality and is essential for the proper formulation of our results. In addition, throughout the first 13 sections of the paper we always assume that $\mathfrak{G}$ is also noncompact. For a compact $\mathfrak{G}$ the corresponding results (where meaningful) are far easier to establish. We will discuss these in our final § 14.

Basic notation and concepts used throughout the paper are listed in § 2. The reader should refer to this section while reading the introduction as the need arises.

A random walk is said to be *recurrent* if for some compact neighborhood N of 0, $\sum_{n=1}^{\infty} P_0(S_n \in N) = \infty$. Otherwise the walk is called *transient*. It is a known fact (see Loynes [7]) that for a recurrent walk $\sum_{n=1}^{\infty} P_x(S_n \in N) = \infty$ for all x and open sets N, while for a transient walk $\sum_{n=1}^{\infty} P_x(S_n \in K) < \infty$ for all x and compact sets K. Moreover (Loynes [7]) in a recurrent walk, $P_x(V_N < \infty) \equiv 1$ for all open sets $N \neq \emptyset$. A random walk is *nonsingular* if for some $n \geq 1$, $\mu^{(n)}$ has a nonsingular component relative to the Haar measure on $\mathfrak{G}$. For a nonsingular walk the sets N in the above statements may be taken to be Borel sets of positive Haar measure.

Briefly, our main goals in this paper are five-fold. First, to establish the renewal theorem for transient random walks on $\mathfrak{G}$. This will be done in § 4 and will be the only place that transient walks are discussed. The remainder of the paper is devoted to recurrent

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walks. Our second main objective of this paper is to establish the existence and basic limit properties of recurrent potentials and to use these recurrent potentials to establish the asymptotic behavior of the hitting distribution and Green's function for relatively compact sets. This will be carried out in §§ 5–9. Our third goal is to find all solutions of the "Poisson" equation $Pf = f + \varphi$ that are bounded from below. This will be done in § 10. Our fourth goal, carried out in §§ 11 and 12, is to investigate the analytic properties of recurrent potentials. Our fifth and final goal is to investigate the behavior of $P_x(V_B > n)$ for large n. This will be carried out in § 13.

As in the Euclidian case our methods will be a mixture of Fourier-analytic and probabilistic arguments. While for the groups R^d the prerequisite Fourier analytic facts were standard tools of the trade, this is not the case for arbitrary locally compact groups $\mathfrak{G}$ In § 3 we gather together those necessary preliminary facts which are needed to proceed further. Some of these may be of intrinsic interest as, for example, the analogue of the well-known fact that on R^d, $1 - \operatorname{Re} \hat{\mu}(\theta) \geqslant c|\theta|^2$.

For a transient random walk the renewal measure $\nu(A) = \sum_{n=1}^{\infty} P(S_n \in A) < \infty$ for all relatively compact sets. A primary problem in the study of transient random walks on Z^d was the asymptotic behavior of $\nu(A+x)$ as $x \to \infty$. In § 4 we examine this problem for transient walks on $\mathfrak{G}$. A transient walk is said to be *type two* if

(1) $\mathfrak{G} \cong R \oplus H$ or $Z \oplus H$ (where H is a compact group), and

(2) the random walk induced on R or Z has a finite non-zero mean m.

A transient walk is *type one* if it is not type two. The behavior of $\nu(A+x)$ is given by the renewal theorem (Theorem 4.1) which asserts that $\nu(A+x) \to 0$, $x \to \infty$ except for a type two walk. In that case if say $m > 0$ and $A \in \mathcal{A}$, then

$$\lim_{x \to +\infty} \nu(A+x) = m^{-1}|A| \quad \text{and} \quad \lim_{x \to -\infty} \nu(A+x) = 0. \tag{1.1}$$

Let $V_B = \min\{n > 0 : S_n \in B\}$ denote the first hitting time after time 0 of the Borel set B, and let

$$U_B(x, A) = E\left[\sum_{n=1}^{V_B} 1_A(S_n - x)\right] \tag{1.2}$$

be the expected number of visits to A starting from x on or before time V_B. The renewal theorem easily yields the following facts about the behavior of $U_B(x, A)$ as $x \to \infty$ in the type two case. Suppose $m > 0$, $B \in \mathfrak{G}^+$, $|\partial B| = 0$. Then for any $A \in \mathcal{A}$,

$$\lim_{x \to -\infty} U_B(x, A) = m^{-1} \int_A \tilde{P}_x(\tilde{V}_B = \infty)\, dx. \tag{1.3}$$

In particular, if $A \subset B$ we obtain the first hitting distribution of B from $-\infty$. Sharper forms of the renewal theorems and of the behavior of $U_B(x, A)$ are available (see Theorem 4.2) when the walk is non-singular.

The renewal theorem for discrete groups was obtained by Kesten and Spitzer [5] and the present proof is patterned after theirs. When specialized to R^d, $d \geq 2$, the renewal theorem gives the first complete proof that for an arbitrary d-dimensional transient random walk $\lim_{x\to\infty} \nu(x+A)=0$ for all relatively compact sets A. The first results in this direction were obtained by Chung. Complete results in the lattice case were obtained by Spitzer [13]. When specialized to $d=1$, the renewal theorem is, of course, the ordinary renewal theorem proven in its final form in the general case by Feller and Orey [2] and strengthened in the non-singular case by Stone [17].

The remainder of the paper is devoted to recurrent random walks on $\mathfrak{G}$. For recurrent walks a major problem is to find the asymptotic behavior of $U_B(x, A)$ as $x \to \infty$ and, dually, the asymptotic behavior of $U_B(x, A+y)$ as $y \to \infty$, when B is a relatively compact set. A recurrent walk is said to be *type two* if

(1) $\mathfrak{G} \cong R \oplus H$ or $Z \oplus H$, where H is a compact group, and

(2) the random walk induced on R or Z has mean 0 and finite variance σ^2.

Otherwise, the recurrent walk is called *type one*. In Theorems 5.5 and 5.7 we show that if $B \in \mathcal{A}$, $\operatorname{int} B \neq \varnothing$, then for $A \in \mathcal{A}$ there are functions $L_B(x)$, $\tilde{L}_B(x)$ such that

$$\lim_{y\to\infty} U_B(x, A+y) = |A| \, L_B(x) \tag{1.4}$$

$$\lim_{x\to\infty} U_B(x, A) = \int_A \tilde{L}_B(t)\, dt \tag{1.5}$$

for every type one walk. For every type two walk there are functions $L_B^+(x)$, $L_B^-(x)$, $\tilde{L}_B^+(x)$, $\tilde{L}_B^-(x)$ such that

$$\lim_{y\to\pm\infty} U_B(x, A+y) = |A| \, L_B^{\pm}(x) \tag{1.6}$$

$$\lim_{x\to\pm\infty} U_B(x, A) = \int_A \tilde{L}_B^{\pm}(t)\, dt. \tag{1.7}$$

If $A \subset B$ then $U_B(x, A)$ is just the probability that the set B is first entered at some point in A. Thus the result in (1.4) shows that in every type one recurrent walk there is a well defined first hitting distribution at infinity that is absolutely continuous. On the other hand, if $A \subset B^c$ then $U_B(x, A)$ is the expected number of visits to A before hitting B. The results in (1.4) and (1.5) show that for every type one walk there is a well defined meaning to the expected number of visits to A before hitting B starting from infinity and also a well defined meaning to the expected number of visits to infinity before hitting B

starting from x. Of course, similar interpretations hold for the results in (1.6)–(1.7) for type two recurrent walks.

To establish these results we follow roughly the same procedure that we did in the case of vector groups. Let $\mathfrak{F}$ be the class of continuous functions whose Fourier transforms have compact support and satisfy several other technical requirements (see Section 5). Let $g \geqslant 0$, $g \in \mathfrak{F}$, $J(g)=1$. We show in Theorem 5.3 that for all $f \in \mathfrak{F}$ the potential

$$Df(x) = \lim_{\lambda \uparrow 1} E\left\{\sum_{n=1}^{\infty} \lambda^n [J(f)\, g(S_n) - f(S_n - x)]\right\} \tag{1.8}$$

exists and has the asymptotic behavior

$$\lim_{y\to\infty} [Df(x-y) - Df(-y)] = 0 \tag{1.9}$$

for every type one walk while

$$\lim_{y\to\pm\infty} [Df(x-y) - Df(-y)] = \mp \sigma^2 J(f)\, \psi(x) \tag{1.10}$$

for every type two walk.

As in the Euclidian case we show that (1.8)–(1.10) are equivalent to a certain problem in Fourier analysis. In Theorem 5.2 this Fourier analysis problem is solved for a compactly generated $\mathfrak{G}$ by appealing to the basic structure theorem for such groups to reduce the problem to the Euclidian case where the results of Port and Stone [11] apply. We then assume that Theorem (5.2) holds in general to show that (1.8) to (1.10) are universally valid. Having these results we proceed (as in the Euclidian case) to show that the fundamental identity

$$Df(x) - \Pi_B Df(x) = -G_B f(x) + J(f) L_B(x) \tag{1.11}$$

holds for all $B \in \mathcal{B}$ having non-empty interior and all $f \in \mathfrak{F}$. The desired results (1.4)–(1.7) then follow from (1.8)–(1.11) just as in the Euclidian case. The remainder of Section 5 is devoted to establishing various additional properties of the potentials $Df(x)$ and of the behavior of $G_B f(x)$ for $f \in \mathfrak{F}$ which are needed later.

The establishment of all these results then rests upon showing that Theorem 5.2 is indeed valid in general. By using the methods of Stone [15] and some ideas from Kesten and Spitzer [5] this is shown to be the case for all nonsingular walks on an arbitrary $\mathfrak{G}$ in § 6. The singular case then follows at once by use of the approximation procedure spelled out in Theorem 3.2.

A by-product of the investigations §§ 5 and 6 is the establishment of the conjecture of Kesten and Spitzer stating that the random walk generated by μ is recurrent if and only if for a compact neighborhood N of 0

$$\int_N \operatorname{Re} \frac{1}{1-\hat{\mu}(\theta)}\, d\theta = \infty. \tag{1.12}$$

For a discrete group this was done by Kesten and Spitzer in [5]. For the groups R^d this was first done by Ornstein [8] and later by another method by Stone [15].

As was outlined above, potentials of functions in $\mathfrak{F}$ suffice to establish the main probabilistic results (1.4)–(1.7) for sets in $\mathcal{A}$. For a singular walk $\mathcal{A}$ is the most general class of sets for which these results are universally valid. For many purposes (in particular, for the establishment of a rich analytic theory) it is desirable to have potentials of functions f having compact support. Counter-examples show however that for singular walks there is no non-trivial class of these functions for which we can assure that the potential Df exists. Thus we turn to consider nonsingular walks in § 7 where we show that all these desired strengthenings are valid. These may all be summarized by saying that for a nonsingular walk all the results of § 5 are true when the class of set $\mathcal{A}$ is replaced with the class $\mathcal{B}$ and the class of functions $\mathfrak{F}$ is replaced with Φ.

The operator Df was defined in the sense of Abel summability and it is natural to inquire if this mode of convergence may be replaced by ordinary convergence. The procedure for doing this is outlined in § 8.

In § 9 we examine some further asymptotic properties of $Af(x)$ and $L_B(x)$. It is shown that if $J(f)>0$ and $f \in \mathfrak{F}$ (or Φ in the nonsingular case) then $\lim_{x\to\infty} Af(x) = \infty$ except perhaps if $\mathcal{B}$ is isomorphic to $R^1 \oplus H$ or $Z^1 \oplus H$. In the exceptional case there is a unique constant L, $0 \leqslant L < \infty$, such that for $f \in \mathfrak{F}$ (or Φ in the nonsingular case), $J(f)>0$, either

$$\lim_{x\to+\infty} Af(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} Af(x) = LJ(f)$$

or

$$\lim_{x\to+\infty} Af(x) = LJ(f) \quad \text{and} \quad \lim_{x\to-\infty} Af(x) = \infty.$$

Similar results are shown to be valid for $L_B(x)$. Results of this type were first given for discrete groups by Kesten and Spitzer [5] and the method used here was patterned after theirs.

Up until now we have focused our attention on mostly probabilistic matters. In §§ 10–12 we consider some analytical questions connected with potentials for recurrent random walks. We call a locally integrable function $f \# Q_B$ superregular if f is defined on $\mathfrak{G} - B$, bounded from below, and $Q_B f \leqslant f$ a.e. on $\mathfrak{G} - B$. In § 10 (Theorem 10.2) we show that if $B \in \mathcal{A}^*$, $|B|>0$, then when μ generates a recurrent type one walk there is a unique $c \geqslant 0$ such that $f = c\mathcal{L}_B + G_B(f - Q_B f)$ a.e., while if μ generates a recurrent type two walk there are unique constants c_1, $c_2 \geqslant 0$ such that $f = c_1 \mathcal{L}_B^+ + c_2 \mathcal{L}_B^- + G_B(f - Q_B f)$ a.e.

The operator $(P-I)$ is the analogue of the Laplace operator so the equation $(P-I)f=\varphi$ a.e. can be considered a Poisson type equation. For potential theoretic matters the correct recurrent potential operator is $A_b\varphi = D\varphi - \varphi + bJ(\varphi)$ where b is a constant. Using this operator we show that potentials $A_b\varphi$ provide solutions of the Poisson equation for $\varphi \in \mathfrak{F}^+$, (or Φ^+ in the nonsingular case) that are bounded from below. For nonsingular walks we then show that the Poisson equation with $\varphi \in \mathfrak{F}$ or Φ has a solution bounded from below if and only if $J(\varphi) \geq 0$, and that for a type one walk the only such solutions are $f = A\varphi + \beta$ a.e., while for a type two walk the only such solutions are

$$f = A\varphi + \frac{\alpha J(\varphi)\psi}{\sigma^2} + \beta \quad \text{a.e.},$$

where β is an arbitrary constant and α a constant such that $|\alpha| \leq 1$. Results in this direction were found by Spitzer [13] for recurrent random walks on Z^d and by Ornstein [8] for nonsingular walks on R^d.

In the singular case our uniqueness results are not as general as those in the nonsingular case. If we let E be the direct sum space of the bounded measurable functions with the one dimensional space of multiples of Ag (where $g \in \mathfrak{F}$, $g \geq 0$, $J(g)=1$) then for $\varphi \in \mathfrak{F}$, $A\varphi \in E$. In Theorem 10.4 we show that the only other solutions of Poisson's equation which are in E are $f = A\varphi + \beta$ a.e.

Quite a different kind of problem is the Poisson equation with boundary condition. Given a set B, a bounded function φ on B, and a function f having support on B' we seek a function h bounded from below such that $(P-I)h = -f$ a.e. on B' and $h = \varphi$ a.e. on B. We conclude § 10 by showing that for sets $B \in \mathcal{A}^*$ having nonempty interior the only locally integrable such solutions are $h = c\mathfrak{L}_B + H_B\varphi + G_Bf$ a.e. in the type one case and $h = c_1\mathfrak{L}_B^+ + c_2\mathfrak{L}_B^- + H_B\varphi + G_Bf$ a.e. in the type two case.

Our purpose in § 11 is to investigate to what extent the basic principles of logarithmic potentials have analogues for our potentials $A\varphi$. Naturally since many of these have to do with charges having compact support we cannot expect that there are results of this type in the singular case, so in the main, the section is devoted to the nonsingular case where we show that such things as the minimum principle, domination principle, and Balyage have their counterpart. Another basic principle is the equilibrium principle. In general, for sets $B \in \mathcal{A}_g^*$, we define the Robin's constant of B as $k_b(B) = \lim_x [A_b g(x) - \mathfrak{L}_B(x)]$, where $J(g)=1$, $g \in \mathfrak{F}$. This is actually a fairly close analogue of one definition of this constant in the case of logarithmic potentials. So defined, we show in Theorem 11.1 that $k(B)$, as a set function of B, has all the desired properties of such a constant. In the case of a nonsingular walk we show that for $B \in \mathcal{B}$, $k(B)$ is the unique constant such that for some

φ, $J(\varphi)=1$, having support on B, $A\varphi(x)=k(B)$ a.e. on B. It turns out that any such φ must coincide with $l_B(x)$ a.e. We have thus an interesting hierarchy in the definitions of $k(B)$. For very nice symmetric walks (e.g. those having the operator $A\varphi(x)=\int a(y-x)\varphi(y)\,dy$ where $a(x)$ is a continuous function) we can show that $k(B)=\sup_\varphi(\varphi, A\varphi)$, where φ has support on B and $J(\varphi)=1$. For these walks then there are three characterizations of $k(B)$ as in the classical case. For general nonsingular recurrent walks there are two, and for arbitrary recurrent walks just one. This shows that the limit definition of $k(B)$ is, at least from our point of view, most intrinsic.

Potential theoretic facts of the above type were first given by Spitzer [13] for walks on Z^d. The arguments used here are more or less patterned after those of Port [10] for the stable processes.

Let $C_0(\mathfrak{G})$ be the usual space of continuous functions vanishing at ∞ if the random walk is of type one and the closed subspace of the *two* point compactification of $\mathfrak{G}$ that is the kernel of the linear functional, $f\to f(+\infty)+f(-\infty)$, in the case of a type two walk. Further, set $\chi=C_0(\mathfrak{G})\oplus\{a(-x)\}$. Then for a nonsingular walk $A\varphi\in\chi$ if $\varphi\in C_c(\mathfrak{G})$. The main result of § 12 is to show that χ is the correct range space of A on $C_c(\mathfrak{G})$. More precisely, we show that $A[C_c(\mathfrak{G})]$ is dense in χ and that $A[C_c(\mathfrak{G})\cap N]$ is dense in $C_0(\mathfrak{G})$ where $N=\{\varphi: J(\varphi)=0\}$. Analogous facts were shown by Port [10] to be valid for potentials associated with recurrent stable processe.

Assume $|B|>0$ and $k_b(B)\neq 0$. Let $\Phi(B)$ be the Banach space of bounded measurable functions on B with ess sup norm. Then another result of some interest in § 12 is that the restriction of A to $\Phi(B)$ is a topological isomorphism of $\Phi(B)$ onto $\Phi(B)$. This fact is an extension to arbitrary $\mathfrak{G}$ for a nonsingular walk of the fact (due to Spitzer [13]) that on Z^d, A restricted to a finite set B with $k(B)\neq 0$ is an isomorphism of R^B onto R^B.

In § 13 we again return to probabilistic problems concerning recurrent random walks. Here we are interested in several questions about the asymptotic behavior of the hitting times. Let $E_B(n)=\int_{\mathfrak{G}}P_x(V_B\leqslant n)\,dx$. (Two different interpretations of $E_B(n)$ are given in the body of the text.) Let $B, A\in\mathcal{A}^*$. Then for any type two walk,

$$\lim_{n\to\infty}\sqrt{n}\,P_x(V_B>n)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sigma L_B(x)$$

uniformly on compacts, and

$$\lim_{n\to\infty}[E_B(n)-E_A(n)]=(2\,\sigma^2)\,[k(B)-k(A)],$$

where $k(\cdot)$ is the Robin's constant of B. For an arbitrary type one walk our results are much more meager. In general, if A, $B\in\mathcal{A}^*$ $|A|$, $|B|>0$, then, $E_B(n)\sim E_A(n)$ and

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} \int_B P_x(V_B > j)\, dx}{\sum_{j=0}^{n} \int_A P_x(V_A > j)\, dx} = 1, \tag{1.13}$$

and

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} \int_A P_x(V_B > j)\, dx}{\sum_{j=0}^{n} \int_B P_x(V_B > j)\, dx} = \int_A L_B(x)\, dx. \tag{1.14}$$

In the general nonsingular case we may improve (1.14) to

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} P_x(V_B > j)}{\sum_{j=0}^{n} \int_B P_x(V_B > j)\, dx} = L_B(x) \tag{1.15}$$

uniformly on compacts, for any $B \in \mathcal{B}$, $|B| > 0$. In addition, in the nonsingular case, for A, $B \in \mathcal{B}$ we can show that $E_B(n) - E_A(n)$ approximates $k(B) - k(A)$ in the following sense. Let $F \in \mathcal{B}$, $|F| = 1$, and set $q_n = \int_F P_x(V_F > n)\,dx$. Then

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} [E_B(n) - E_A(n)]}{\sum_{n=0}^{N} \left[\sum_{j=0}^{n} q_{n-j} q_j\right]} = k(B) - k(A). \tag{1.16}$$

For random walks on Z^d or R^d far sharper results are known to be valid. It was shown by Kesten and Spitzer [6] that in this case ratios of individual terms rather than partial sums exist in (1.13) and (1.15) and by Port [9] that this is true also in (1.16). Ornstein [8] showed that when A, B are intervals, then limits of individual terms in (1.13) and (1.15) also exist for all recurrent walks (singular or not) on R^d. Whether all the strong facts known to be valid on Z^d are true in general remains an open problem. Perhaps the methods of Ornstein could be used to obtain results of this nature on an arbitrary $\mathfrak{G}$, but we have made no attempt in this direction.

2. Notation

In this section we will introduce the notation that will be used throughout the paper.

$\mathfrak{G}$ will be a fixed locally compact and, except in § 14, noncompact Abelian group. The *Borel sets* of $\mathfrak{G}$ are the elements of the minimal σ-field generated by the open sets. *Haar measure* on $\mathfrak{G}$ will be denoted by $|\cdot|$ or dx. The phrase a.e. (almost everywhere) will always be with respect to Haar measure, and the phrase *essentially* will mean except on a set of Haar measure 0.

$\mathcal{B}$ will be the class of all relatively compact Borel sets, and $\mathcal{A}$ will be the subset of $\mathcal{B}$ consisting of those sets B whose boundaries ∂B have 0 Haar measure. Define the class $\mathcal{A}^*$ as $\mathcal{A}$ in the general case as $\mathcal{B}$ in the nonsingular case. The class $\mathfrak{F}$ is the class of all functions f whose Fourier transforms have compact support and satisfy several other properties (see § 3). Φ is the class of bounded measurable functions having compact support and $C_c(\mathfrak{G})$ the continuous functions having compact support. If χ is one of the above class of functions χ^+ will denote the nonnegative elements.

The complement of a set B will be denoted by B' or B^c. The nth power of the transition operator is P^n, where $P^0 = I$ (identity) $Pf = \int_{\mathfrak{G}} f(y+x)\mu(dy)$, and $P^{n+1} = PP^n$. The hitting times T_B and V_B are respectively

$$T_B = \min\{n \geqslant 0 : S_n \in B\} \quad (= \infty \text{ if } S_n \notin B \text{ for all } n \geqslant 0)$$

$$V_B = \min\{n > 0 : S_n \in B\} \quad (= \infty \text{ if no such } n).$$

For a function f set $f_y(x) = f(x-y)$ and set $J(f) = \int_{\mathfrak{G}} f(x)\,dx$. For functions f, g set $(f, g) = \int_{\mathfrak{G}} f(x)g(x)\,dx$.

Let $0 < \lambda < 1$ and define operators on bounded measurable functions or nonnegative measurable functions as follows:

$$U^\lambda = \sum_{n \geqslant 1} \lambda^n P^n$$

$$G^\lambda = I + U^\lambda$$

$$D^\lambda = U^\lambda g(0)\,J - U^\lambda,$$

where $g \geqslant 0$ and $J(g) = 1$, $g \in \mathfrak{F}$.

$$A^\lambda = G^\lambda g(0)\,J - G^\lambda = D^\lambda + g(0)\,J - I$$

$${}_BP^n f(x) = E_x[f(S_n);\ V_B \geqslant n], \quad n \geqslant 1$$

$$U_B^\lambda = \sum_{n \geqslant 1} \lambda^n {}_BP^n$$

$$I_B f(x) = 1_B(x) f(x)$$

where $1_B(x)$ is the indicator function of B.,

$$\Pi_B^\lambda = U_B^\lambda I_B$$

$$H_B^\lambda = I_B + I_{B'}\,\Pi_B^\lambda$$

$$G_B^\lambda = I_{B'}\,(I + U_B^\lambda)\,I_B$$

$$L_B^\lambda(x) = (1-\lambda)\, U^\lambda g(0) \sum_{n=0}^{\infty} P_x(V_B > n)\,\lambda^n$$

$$\mathfrak{L}_B^\lambda(x) = (1-\lambda)\, G^\lambda g(0) \sum_{n=0}^{\infty} P_x(T_B > n)\,\lambda^n.$$

When any of the above quantities have a finite limit as $\lambda \uparrow 1$ we will denote that limit by the same symbol without the λ, e.g. $\lim_{\lambda\uparrow 1} H_B^\lambda f = H_B f$.

Of all the groups $\mathfrak{G}$, two particular compactly generated groups will play a distinguished role. These are, when $\mathfrak{G}$ is isomorphic to either $R^1 \oplus H$ or $Z^1 \oplus H$, where H is a compact group. In this case we will simply identify $\mathfrak{G}$ with either $R^1 \oplus H$ or $Z^1 \oplus H$. The random walk on $R^1(Z^1)$ induced by μ is the random walk generated by the measure μ', where for a Borel set B of $R^1(Z^1)$, $\mu'(B) = \mu(\psi^{-1}(B))$. Here ψ is the natural projection of $R^1 \oplus H\,(Z^1 \oplus H)$ onto $R^1(Z^1)$. The mean and variance of μ' will be denoted by m and σ^2 respectively. We set $\mathfrak{G}^- = \psi^{-1}(-\infty, 0]$ and $\mathfrak{G}^+ = \psi^{-1}(0, \infty)$.

By $\lim_{x\to\infty} f(x) = f(\infty)$ we mean that given any $\varepsilon > 0$ there is a compact set K such that $|f(x) - f(\infty)| < \varepsilon$ for all $x \notin K$. When the group can be identified with either $R^1 \oplus H$ or $Z^1 \oplus H$ we define $\lim_{x\to\pm\infty} f(x) = f(\pm\infty)$ as $\lim_{x\to\infty, x\in\mathfrak{G}^\pm} f(x)$. We introduce the convention that $\mathrm{Lim}_x f(x)$ is $[f(+\infty) + f(-\infty)]/2$ when $\mathfrak{G}$ is one of the distinguished groups. In all other cases $\mathrm{Lim}_x f(x) = \lim_{x\to\infty} f(x)$.

The measure $\tilde{\mu}(dx) = \mu(-dx)$ generates a random walk on $\mathfrak{G}$, recurrent or transient according as the one generated by μ is, which is called the *dual* random walk. Quantities referring to the dual walk are denoted by $\sim$, e.g. $\tilde{H}_B$ is the quantity H_B for the dual walk. It easily follows that for any f, $g \in \Phi$ or any nonnegative measurable f, g $(g, P^n f) = (\tilde{P}^n g, f)$ and $(g, {}_B P^n f) = ({}_B\tilde{P}^n g, f)$. From these, other duality relations follow for the operators defined above.

If γ is a bounded regular measure then the *Fourier transform* $\hat{\gamma}(\theta)$ of γ is $\hat{\gamma}(\theta) = \int_{\mathfrak{G}} \langle\theta, x\rangle \gamma(dx)$, where $\langle\theta, x\rangle$ is a character of $\mathfrak{G}$. For a function $f \in L_1(\mathfrak{G})$, the Fourier transform $\hat{f}(\theta)$ is $\int_{\mathfrak{G}} \langle\theta, x\rangle f(x)\,dx$. Haar measure on $\hat{\mathfrak{G}}$ is chosen so that $f(x) = \int_{\mathfrak{G}} \overline{\langle\theta, x\rangle} \hat{f}(\theta)\,d\theta$ whenever f is continuous and $\hat{f}$ is integrable.

In future sections we will show that various limits exist. For ease in reference we gather these together here;

$$L_B(x) = \lim_{\lambda\uparrow 1} L_B^\lambda(x).$$

It turns out that $L_B(x)$ is also given by

$$|A|\, L_B(x) = \mathrm{Lim}_y\, U_B(x, A + y).$$

The function $L_B^\pm$ are defined for type two walks by

$$L_B^{\pm}(x) = L_B(x) \pm \sigma^{-2}\int_B \Pi_B(x, dz)\,\psi(x-z)$$

and for a type two walk

$$\lim_{y\to\pm\infty} U_B(x, A+y) = |A|\,L_B^{\pm}(x).$$

The functions $\mathcal{L}_B$ and $\mathcal{L}_B^{\pm}$ are slight modifications of the functions L_B and $L_B^{\pm}$.

$$\mathcal{L}_B(x) = L_B(x)\,1_{B'}(x)$$

and for a type two walk

$$\mathcal{L}_B^{\pm}(x) = \mathcal{L}_B(x) \pm \sigma^{-2}\int_B H_B(x, dz)\,\psi(x-z).$$

Let $g \in \mathfrak{F}$, $g \geqslant 0$, $J(g) = 1$. The Robbin's constant $k(B)$ is

$$k(B) = \lim_x\,[Ag(x) - \mathcal{L}_B(x)].$$

For a nonsingular recurrent walk the potential Df can be written as

$$Df(x) = \int a(y-x)\,f(y)\,dy - U_2 f(x),$$

where $a(x)$ is a continuous function and $U_2(x, dy) = U_2(0, dy-x)$ is a bounded measure.

3. Some Fourier Analysis

In this section we extend some of the basic properties of characteristic functions of probability measures on Euclidean space to those on locally compact Abelian groups.

THEOREM 3.1. *Let μ be a probability measure on $\mathfrak{G}$ whose support S generates $\mathfrak{G}$. Then*

$$\frac{1 - \Re\langle x, \theta\rangle}{1 - \Re\hat{\mu}(\theta)}$$

is bounded for x and θ in compacts and $\theta \neq 0$.

We begin the proof with

LEMMA 3.1. *Let η and ω be complex numbers such that $|\eta| = |\omega| = 1$. Then*

$$(1 - \Re\eta) + (1 - \Re\omega\eta) \geqslant (1 - \Re\omega)/6. \tag{3.1}$$

Proof. Note first that

$$(\Im\eta)^2 = 1-(\Re\eta)^2 = (1+\Re\eta)(1-\Re\eta) \leqslant 2(1-\Re\eta)$$

and similarly $(\Im\omega)^2 \leqslant 2(1-\Re\omega)$. Thus

$$|\Im\eta\,\Im\omega| \leqslant (\Im\eta)^2 + \tfrac{1}{4}(\Im\omega)^2 \leqslant 2(1-\Re\eta)+\tfrac{1}{2}(1-\Re\omega).$$

Observe next that

$$\begin{aligned}(1-\Re\eta)+(1-\Re\omega\eta) &= 2(1-\Re\eta)+(1-\Re\omega)+\Im\eta\,\Im\omega-(1-\Re\eta)(1-\Re\omega)\\ &\geqslant (1-\Re\omega)\left(\tfrac{1}{2}-(1-\Re\eta)\right).\end{aligned}$$

If $1-\Re\eta \leqslant \frac{1}{3}$, the result now follows immediately. If $1-\Re\eta \geqslant \frac{1}{3}$, the conclusion of the lemma is trivial.

COROLLARY 3.1. *Let U be a subset of $\mathfrak{G}$. Then for $x, y \in \mathfrak{G}$ and $\theta \in \hat{\mathfrak{G}}$*

$$\inf_{y+U}(1-\Re\langle z,\theta\rangle) + \inf_{x+y+U}(1-\Re\langle z,\theta\rangle) \geqslant \tfrac{1}{6}\inf_{x+U-U}(1-\Re\langle z,\theta\rangle). \tag{3.2}$$

LEMMA 3.2. *Let μ be as in Theorem 3.1. Then for every $x \in \mathfrak{G}$ and open neighborhood U of the origin of $\mathfrak{G}$, there is a $c>0$ such that*

$$1-\Re\hat{\mu}(\theta) \geqslant c\inf_{x+U}(1-\Re\langle z,\theta\rangle), \quad \theta\in\hat{\mathfrak{G}}. \tag{3.3}$$

Proof. Choose $x\in\mathfrak{G}$ and an open neighborhood U of the origin of $\mathfrak{G}$. Let U_1 be a neighborhood of the origin of $\mathfrak{G}$ such that $U_1 - U_1 \subseteq U$. Then there exist positive integers m and n and a $y\in\mathfrak{G}$ such that

$$\mu^{(n)}(y+U_1)>0 \quad\text{and}\quad \mu^{(m)}(x+y+U_1)>0.$$

Thus there is a constant $c>0$ such that for $\theta\in\hat{\mathfrak{G}}$

$$1-\Re\hat{\mu}^n(\theta) \geqslant 12\,cn\inf_{y+U_1}(1-\Re\langle z,\theta\rangle)$$

and

$$1-\Re\hat{\mu}^m(\theta) \geqslant 12\,cm\inf_{x+y+U_1}(1-\Re\langle z,\theta\rangle).$$

Consequently

$$|1-\hat{\mu}(\theta)| \geqslant |1-\hat{\mu}^n(\theta)|/n \geqslant 12\,c\inf_{y+U_1}(1-\Re\langle z,\theta\rangle)$$

and similarly

$$|1-\hat{\mu}(\theta)| \geqslant 12\,c\inf_{x+y+U_1}(1-\Re\langle z,\theta\rangle).$$

Therefore, by Corollary 3.1,

$$|1-\hat{\mu}(\theta)| \geqslant c\inf_{x+U}(1-\Re\langle z,\theta\rangle).$$

This proves the lemma if μ is symmetric or, equivalently, if $\hat{\mu}$ is real. The general result follows by looking at $(\hat{\mu}+\bar{\hat{\mu}})/2=\Re\hat{\mu}$.

LEMMA 3.3. *Let V be a vector group and L a lattice group. Let C_1 be a compact subset of $V\oplus L$ and C_2 a compact subset of $\widehat{V\oplus L}$. Then there is a constant $0<c<\infty$ and an open neighborhood U_1 of the origin of $V\oplus L$ and there are n points $x_1, \ldots, x_n$ in $V\oplus L$ such that*

$$\sum_{j=1}^{n} \inf_{x_j+U_1} (1-\Re\langle z,\theta\rangle) \geqslant c(1-\Re\langle y,\theta\rangle), \quad y\in C_1 \quad \textit{and} \quad \theta\in C_2. \tag{3.4}$$

Proof. The proof follows easily from the form of $V\oplus L$ and is left to the reader.

Proof of Theorem 3.1. Let C be a compact subset of $\mathfrak{G}$. Then there is an open compactly generated subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ containing C. We can write $\mathfrak{G}_1=V\oplus L\oplus H$, where V is a vector group, L is a lattice group, and H is compact (see Hewitt and Ross [4] p. 90). Let C_1 be the projection of C into $V\oplus L$. Then C_1 is a compact subset of $V\oplus L$.

Let Λ denote the subgroup of $\hat{\mathfrak{G}}$ which annihilates H. Then Λ is open and closed (for $\hat{\mathfrak{G}}/\Lambda\cong\hat{H}$ is discrete). Let C_3 be a compact subset of Λ.

Let $M:\hat{\mathfrak{G}}\to\widehat{V\oplus L}$ be the map which takes $\theta\in\hat{\mathfrak{G}}$ into its restriction to $V\oplus L$. Then M is a continuous map ([4], p. 377) and, in particular, the image C_2 of C_3 under M is compact. Also if $\theta\in\Lambda$ and $x=y+h\in\mathfrak{G}_1$, where $y\in V\oplus L$ and $h\in H$, then $\langle x,\theta\rangle=\langle y, M(\theta)\rangle$.

With V, L, C_1, and C_2 as just defined, let c, U_1, and $x_1, \ldots, x_n$ be as in Lemma 3.3. Set $U=U_1+H$.

By Lemma 3.2 there is a constant $c_1>0$ such that for $\theta\in\Lambda$ and $1\leqslant j\leqslant n$

$$1-\Re\hat{\mu}(\theta)\geqslant c_1c^{-1}n \inf_{x_j+U_1}(1-\Re\langle z,\theta\rangle)=c_1c^{-1}n\inf_{x_j+U_1}(1-\Re\langle z,M(\theta)\rangle)$$

and hence for $\theta\in\Lambda$

$$1-\Re\hat{\mu}(\theta)\geqslant c_1c^{-1}\sum_{j=1}^{n}\inf_{x_j+U_1}(1-\Re\langle z,M(\theta)\rangle).$$

Thus by equation (3.4) (since $\theta\in C_3$ if and only if $M(\theta)\in C_2$)

$$1-\Re\hat{\mu}(\theta)\geqslant c_1(1-\Re\langle y, M(\theta)\rangle), \quad \theta\in C_3 \text{ and } y\in C_1.$$

Consequently for $\theta\in C_3$ and $x=y+h\in C$ ($y\in V\oplus L$ and $h\in H$),

$$1-\Re\hat{\mu}(\theta)\geqslant c_1(1-\Re\langle y, M(\theta)\rangle)=c_1(1-\Re\langle x,\theta\rangle).$$

Now Λ is open and hence C_3 can be assumed to contain a neighborhood of $\theta=0$. Since $1-\Re\hat{\mu}(\theta)$ is continuous and vanishes only at $\theta=0$, the proof of the theorem is now complete.

THEOREM 3.2. *Let μ be a probability measure on $\mathfrak{G}$ which defines a recurrent random walk on $\mathfrak{G}$. Then there is a nonsingular probability measure ν on $\mathfrak{G}$ which defines a recurrent random walk on $\mathfrak{G}$ and is such that for some compact subset C of $\mathfrak{G}$, some open neighborhood P of the origin of $\hat{\mathfrak{G}}$ and some $0 < c < \infty$*

$$|\hat{\mu}(\theta) - \hat{\nu}(\theta)| \leqslant c[\max_{x \in C} (1 - \Re \langle x, \theta \rangle)]^2, \;\; \theta \in P. \tag{3.5}$$

If $V \cong R^{d_1}$ is a vector group and $L \cong Z^{d_2}$ is a lattice group, the dimension of $V \oplus L$ is defined to be $d_1 + d_2$. Note that every closed subgroup of $V \oplus L$ is of the same form and hence has a well defined dimension.

We begin the proof of Theorem 3.2 with

LEMMA 3.4. *Let $\mathfrak{G}_1$ be a compactly generated open subgroup of $\mathfrak{G}$ written as $\mathfrak{G}_1 = V \oplus L \oplus H$, where $V \cong R^{d_1}$, $L \cong Z^{d_2}$, $d_1 + d_2 = d$, and H is compact. Let $\alpha \in \mathfrak{G}_1$ and let $\mathfrak{G}_2$ be a closed subgroup of $\mathfrak{G}$ such that $\bigcup_{\infty}^{1} (n\alpha + \mathfrak{G}_2)$ is dense in $\mathfrak{G}$. Then the projection of $\mathfrak{G}_2 \cap \mathfrak{G}_1$ into $V \oplus L$ is a d-dimensional closed subgroup of $V \oplus L$.*

Proof. Note first that $\mathfrak{G}_3 = \mathfrak{G}_2 \cap \mathfrak{G}_1$ is a closed subgroup of $\mathfrak{G}_1$ and that $\bigcup_1^{\infty} (n\alpha + \mathfrak{G}_3)$ is dense in $\mathfrak{G}_1$. Let β denote the projection of α on $V \oplus L$ and let $\mathfrak{G}_4$ denote the projection of $\mathfrak{G}_3$ on $V \oplus L$. Then $\mathfrak{G}_4$ is a closed subgroup of $V \oplus L$ and $\bigcup_1^{\infty} (n\beta + \mathfrak{G}_4)$ is dense in $V \oplus L$. This clearly implies that $\mathfrak{G}_4$ is of the same dimension as $V \oplus L$, as desired.

Let $t_1, t_2, \ldots, t_k$ be elements of $\mathfrak{G}$. Their *span* $S(t_1, \ldots, t_k)$ is defined to consist of all $t \in \mathfrak{G}$ for which there exists a compact subset C of $\mathfrak{G}$ and integer valued functions $m_1(n)$, $\ldots$, $m_k(n)$, $-\infty < n < \infty$, such that

$$nt - m_1(n) t_1 - \ldots - m_k(n) t_k \in C, \quad -\infty < n < \infty.$$

LEMMA 3.5. *$S(t_1, \ldots, t_k)$ is a closed subgroup of $\mathfrak{G}$.*

Proof. The result clearly holds if $\mathfrak{G}$ is the direct sum of a vector group and a lattice group.

It is obvious, in general, that $\mathfrak{G}$ is a group. To prove it is closed, let s be in the closure of $S(t_1, \ldots, t_k)$. Let $\mathfrak{G}_1$ be the group generated by $t_1, \ldots, t_k$ and a compact neighborhood of s. Then $\mathfrak{G}_1 = V \oplus L \oplus H$, as usual. Let W be the subgroup of $V \oplus L$ spanned by the projection of $t_1, \ldots, t_k$ into $V \oplus L$. Then by the first part of this proof, W is a closed subgroup of $V \oplus L$. The points in $\mathfrak{G}_1 \cap S(t_1, \ldots, t_k)$ are those which, when projected into $V \oplus L$, are mapped into W. Thus $\mathfrak{G}_1 \cap S(t_1, \ldots, t_k)$ is a closed subset of $\mathfrak{G}_1$ and hence $s \in S(t_1, \ldots, t_k)$. Thus $S(t_1, \ldots, t_k)$ is closed, as desired.

LEMMA 3.6. *Let $\mathfrak{G}$ be such that there is a finite bound to the dimension of lattice subgroups of $\mathfrak{G}$. Let T denote a closed subset of $\mathfrak{G}$ and $\mathfrak{G}_2$ the subgroup of $\mathfrak{G}$ generated by T. Let $\alpha \in \mathfrak{G}$ and suppose that $\bigcup_1^\infty (n\alpha + \mathfrak{G}_2)$ is dense in $\mathfrak{G}$. Then there is a compactly generated open subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ of the form $\mathfrak{G}_1 = V \oplus L \oplus H$, where $V \cong R^{d_1}$, $L \cong Z^{d-d_1}$, and H is compact, and such that the projection of $T \cap \mathfrak{G}_1$ into $V \oplus L$ generates a d-dimensional subgroup of $V \oplus L$.*

Proof. By the assumption on $\mathfrak{G}$, there are nonnegative integers d_1 and $d \geqslant d_1$ and there is a compactly generated subgroup $\mathfrak{G}_3$ of $\mathfrak{G}$ such that if $\mathfrak{G}_1$ is a compactly generated subgroup of $\mathfrak{G}$ containing $\mathfrak{G}_3$, then $\mathfrak{G}_1 = V \oplus L \oplus H$, where $V \cong R^{d_1}$, $L \cong Z^{d-d_1}$, and H is compact.

Suppose that for all such $\mathfrak{G}_1$, the projection of $T \cap \mathfrak{G}_1$ into $V \oplus L$ generates at most a k-dimensional subgroup, where $k < d$. Then we can find k elements $t_1, \ldots, t_k \in T$ such that $S(t_1, \ldots, t_k) \supseteq \mathfrak{G}_2$ by Lemma 3.5. Let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$ containing α, $\mathfrak{G}_3$, $t_1, \ldots, t_k$ and decomposed as $\mathfrak{G}_1 = V \oplus L \oplus H$, as above. Then $S(t_1, \ldots, t_k) \supseteq \mathfrak{G}_1 \cap \mathfrak{G}_2$ and hence the projection of $\mathfrak{G}_1 \cap \mathfrak{G}_2$ on $V \oplus L$ is at most $k < d$-dimensional, which contradicts Lemma 3.4.

LEMMA 3.7. *Let ϱ be a finite measure on $R^{d_1} \oplus Z^{d_2}$, where $0 \leqslant d_1 + d_2 = d \leqslant 2$, with finite third moment and nonsingular covariance matrix. Then there is a finite measure χ on $R^{d_1} \oplus Z^{d_2}$ which is nonsingular with respect to Haar measure on $R^{d_1} \oplus Z^{d_2}$ and whose moments of orders zero through three agree with those of ϱ.*

Proof. The result is rather straightforward and depends on the version of the Schwarz inequality involving strict inequality. The details will be omitted.

Proof of Theorem 3.2. Since μ defines a recurrent random walk on $\mathfrak{G}$, it follows that if $V \oplus L$ is a closed subgroup of $\mathfrak{G}$, where $V \cong R^{d_1}$ and $L \cong Z^{d_2}$, then $0 \leqslant d_1 + d_2 \leqslant 2$.

Let S denote the support of μ, choose $\alpha \in S$ and set $T = S - \alpha$. Let $\mathfrak{G}_2$ denote the subgroup of $\mathfrak{G}$ generated by T. Then by the definition of recurrence, $\bigcup_1^\infty (n\alpha + \mathfrak{G}_2)$ is dense in $\mathfrak{G}$.

Lemma 3.6 is now applicable. There is a compactly generated open subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ of the form $\mathfrak{G}_1 = V \oplus L \oplus H$, where $V \cong R^{d_1}$, $L \cong Z^{d-d_1}$, H is compact, and the projection of $T \cap \mathfrak{G}_1$ into $V \oplus L$ generates a d-dimensional subgroup of $V \oplus L$. Note that $0 \leqslant d \leqslant 2$.

There is a compact subset C of $\mathfrak{G}_1$ such that the projection of $T \cap (\operatorname{int} C)$ into $V \oplus L$ generates a d-dimensional closed subgroup of $V \oplus L$. Let $2\varrho_1$ denote the restriction of μ to C, and let ϱ denote the measure induced on $V \oplus L$ by the projection of ϱ_1. Then ϱ satisfies the assumptions of Lemma 3.7.

Let χ be as in the conclusion of Lemma 3.7. Let χ_1 be the measure on $\mathfrak{G}_1$ which is the direct product of χ on $V \oplus L$ and Haar measure on H (normalized to be a probability measure on H). Set $\nu = \mu + \chi_1 - \varrho_1$. Then ν is a nonsingular probability measure on $\mathfrak{G}$ whose support contains that of μ and hence generates all of $\mathfrak{G}$. Also $\hat{\nu} - \hat{\mu} = \hat{\chi}_1 - \hat{\varrho}_1$.

Let Λ denote the annihilator of H. Then Λ is an open closed subgroup of $\hat{\mathfrak{G}}$. Let $M : \hat{\mathfrak{G}} \to \widehat{V \oplus L}$ be the map which takes an element in $\hat{\mathfrak{G}}$ into its restriction to $V \oplus L$. If $\theta \in \Lambda$ and $x = y + h \in \mathfrak{G}_1$, where $y \in V \oplus L$ and $h \in H$, then $\langle \theta, x \rangle = \langle M(\theta), y \rangle$. Consequently

$$\hat{\nu}(\theta) - \hat{\mu}(\theta) = \hat{\chi}(M(\theta)) - \hat{\varrho}(M(\theta)),$$

where $\hat{\chi}$ and $\hat{\varrho}$ are functions on $\widehat{V \oplus L}$.

Let $P \subseteq \Lambda$ denote an open relatively compact neighborhood of the origin of $\hat{\mathfrak{G}}$. Since the moments of χ and ϱ of orders zero through three agree, and since they both have finite fourth moments, there is a constant c, $0 < c < \infty$, and a compact subset C_1 of $V \oplus L$ such that

$$|\hat{\chi}(M(\theta)) - \hat{\varrho}(M(\theta))| \leqslant c\,[\max_{y \in C_1} (1 - \Re \langle y, M(\theta) \rangle)]^2, \ \ \theta \in P.$$

Let C_2 denote the compact subset of $\mathfrak{G}_1$ which is projected onto C_1. Then

$$|\hat{\nu}(\theta) - \hat{\mu}(\theta)| \leqslant c[\max_{x \in C} (1 - \Re \langle x, \theta \rangle)]^2, \ \ \theta \in P.$$

It remains only to show that ν defines a recurrent random walk. Note first that for $0 \leqslant \lambda \leqslant 1$

$$|1 - \lambda \hat{\mu}(\theta)| \geqslant \lambda |1 - \hat{\mu}(\theta)|, \ \ \theta \in \hat{\mathfrak{G}}, \tag{3.6}$$

and the same result holds with $\hat{\mu}$ replaced by $\hat{\nu}$. Since

$$\left| \frac{1}{1 - \lambda \hat{\mu}(\theta)} - \frac{1}{1 - \lambda \hat{\nu}(\theta)} \right| \leqslant \frac{|\hat{\nu}(\theta) - \hat{\mu}(\theta)|}{|1 - \hat{\mu}(\theta)|\,|1 - \hat{\nu}(\theta)|}$$

is uniformly bounded for $\frac{1}{2} \leqslant \lambda \leqslant 1$ and θ in some open neighborhood of the origin of $\hat{\mathfrak{G}}$, the recurrence of ν follows from the recurrence of μ and the Chung–Fuchs criterion (Loynes [7, p. 453]). This completes the proof of Theorem 3.2.

Let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$. Then $\mathfrak{G}_1 \cong V \oplus L \oplus H$. where $V \cong R^{d_1}$, $L = Z^{d_2}$, and H is compact. The numbers d_1 and d_2 and hence also $d = d_1 + d_2$ depend only on $\mathfrak{G}_1$, not on the choice of V or L. This justifies calling $\mathfrak{G}_1$ d-dimensional.

The next result is obvious but will be useful in a number of places later on.

THEOREM 3.3. *Let $\mathfrak{G}_1$ be a d-dimensional compactly generated subgroup of $\mathfrak{G}$, let H be the group of compact elements of $\mathfrak{G}_1$ and let Λ denote the annihilator of H. Then there are functions $\psi : \mathfrak{G}_1 \to R^d$ and $\varphi : \hat{\mathfrak{G}} \to R^d$ such that:* (i) *ψ is a continuous homomorphism which maps $\mathfrak{G}_1$ onto a closed d-dimensional subgroup of R^d;* (ii) *φ is a Borel function which is con-*

tinuous near the origin, maps compact sets into relatively compact sets, and maps every neighborhood of the origin of $\hat{\mathfrak{G}}$ onto a neighborhood of the origin of R^d; and (iii) *for $x \in \mathfrak{G}_1$ and $\theta \in \Lambda$*

$$\langle x, \theta \rangle = e^{i\psi(x)\cdot\varphi(\theta)}.$$

Let $\mathfrak{F}$ denote the collection of functions $f(x)$, $x \in \mathfrak{G}$, such that

(i) f is a continuous, nonnegative, and integrable function with integral

$$J(f) = \int f(x)\,dx;$$

(ii) f is supported by a compactly generated subgroup of $\mathfrak{G}$;

(iii) $\hat{f}$ has compact support; and

(iv) there is a compact subset C of $\mathfrak{G}$, a constant c such that $0 < c < \infty$, and an open neighborhood P of the origin of $\hat{\mathfrak{G}}$ such that

$$J(f) - \Re\hat{f}(\theta) \leqslant c \max_{x \in C} (1 - \Re\langle x, \theta\rangle), \quad \theta \in P. \tag{3.7}$$

Let $\mathfrak{F}_s$ denote the collection of symmetric functions in $\mathfrak{F}$.

THEOREM 3.4. *Given $\varepsilon > 0$ and an open neighborhood U of the origin of $\mathfrak{G}$, there is an $f \in \mathfrak{F}_s$ such that*

$$\int_U f(x)\,dx \geqslant 1 - \varepsilon. \tag{3.8}$$

We begin the proof of Theorem 3.4 with

LEMMA 3.8. *Given $\varepsilon > 0$ and an open neighborhood U of the origin of $\mathfrak{G}$, there is a continuous symmetric probability density function f on $\mathfrak{G}$ such that $\hat{f}$ has compact support and (3.8) holds.*

Proof. Let $T \subseteq U$ be a relatively compact open neighborhood of the origin of $\mathfrak{G}$ and set $g = 1_T/|T|$. Then $\|g\|_1 = \|g^{\frac{1}{2}}\|_2 = 1$. Let δ, $0 < \delta < 1$, be a number to be chosen later. Choose (by Plancherel's theorem) $h \in \mathfrak{L}_2$ such that h is continuous, $\hat{h}$ has compact support and $\|h - g^{\frac{1}{2}}\|_2 \leqslant \delta$ Then $\|h\|_2 \leqslant 1 + \delta < 2$. Also $|h|^2$ is continuous and in $\mathfrak{L}_1$ and its Fourier transform has compact support. Moreover,

$$\|g - |h|^2\|_1 = \|g^{\frac{1}{2}}(g^{\frac{1}{2}} - h) + h(g^{\frac{1}{2}} - \bar{h})\|_1 \leqslant 3\delta.$$

Set $f_1 = |h|^2/\| |h|^2\|_1$. Then f_1 is a continuous probability density, $\hat{f}_1$ has compact support and

$$\int_T f_1(x)\,dx \geqslant (1-3\delta)/(1+3\delta) \geqslant 1-\varepsilon$$

if δ is sufficiently small. Define f by $f(x)=(f_1(x)+f_1(-x))/2$. Then f is the desired function.

LEMMA 3.9. *Let $\mathfrak{G}_1$ be an open closed subgroup of $\mathfrak{G}$ and let $f\in \mathfrak{L}_1(\mathfrak{G})$ have support on $\mathfrak{G}_1$. Let g denote the Fourier transform of f as a function on $\hat{\mathfrak{G}}_1$. If g has compact support, then $\hat{f}$ has compact support.*

Proof. Let $M:\hat{\mathfrak{G}}\to\hat{\mathfrak{G}}_1$ map $\theta\in\hat{\mathfrak{G}}$ into the restriction of θ to $\mathfrak{G}_1$. Then M is a continuous open map onto $\hat{\mathfrak{G}}_1$ ([4, p. 377]). Furthermore $g(M(\theta))=\hat{f}(\theta)$, $\theta\in\hat{\mathfrak{G}}$. Let C_1 be a compact set supporting g. Since M is open, there is a compact subset C of $\hat{\mathfrak{G}}$ such that $M(C)=C_1$. Let Λ denote the annihilator of $\mathfrak{G}_1$. Then $\Lambda\cong\widehat{\mathfrak{G}/\mathfrak{G}_1}$ is compact, since $\mathfrak{G}_1$ is open. Choose $\theta\in\hat{\mathfrak{G}}$ such that $\hat{f}(\theta)\neq 0$. Then $M(\theta)\in C_1$. Thus there is a $\theta_1\in C$ such that $M(\theta_1)=M(\theta)$ and hence $\theta-\theta_1\in\Lambda$ or $\theta\in C+\Lambda$. Thus $\hat{f}$ is supported by the compact set $C+\Lambda$, as desired.

We next define a collection $\mathfrak{F}_2$ of functions f on $\mathfrak{G}$ as all functions which can be constructed in a particular way.

Let $\mathfrak{G}_1$ be a compactly generated open subgroup of $\mathfrak{G}$ which can be written as $\mathfrak{G}_1= V\oplus H$, where V is a vector group and H is compact. Let dv and dh be Haar measures on V and H respectively such that, on $\mathfrak{G}_1$, $dg=dv\cdot dh$. Let f_1 be a continuous symmetric probability density function on V having finite second moment and whose Fourier transform (as a function on $\hat{V}$) has compact support. Let f_2 be a continuous symmetric probability density function on H whose Fourier transform (as a function on $\hat{H}$) has compact support. Let $f(g)$, $g\in\mathfrak{G}$, be defined by $f(g)=f_1(v)f_2(h)$ for $g=v+h\in\mathfrak{G}_1$ with $v\in V$ and $h\in H$, and $f(g)=0$ for $g\notin\mathfrak{G}_1$. Finally let $\mathfrak{F}_2$ denote the collection of all functions f that can be constructed in this manner.

If $f\in\mathfrak{F}_2$, then f is a continuous symmetric probability density function. Also its Fourier transform, as a function on $\mathfrak{G}_1$, has compact support. Therefore, by Lemma 3.9, $\hat{f}(\theta)$, $\theta\in\hat{\mathfrak{G}}$, has compact support.

LEMMA 3.10. *Let $\varepsilon>0$ and U an open neighborhood of the origin of $\mathfrak{G}$. Then there is an $f\in\mathfrak{F}_2$ such that* (3.8) *holds.*

Proof. Let W be an open subset of $\mathfrak{G}$ containing the origin and such that $W+W\subseteq U$. Consider the construction used in the definition of $\mathfrak{F}_2$. The function f_1 can be chosen so that

$$\int_{V\cap W} f_1(v)\,dv \geqslant 1-\frac{\varepsilon}{2}$$

and the function f_2 can be chosen so that

$$\int_{H\cap W} f_2(h)\,dh \geqslant 1-\frac{\varepsilon}{2}.$$

Then for the corresponding f

$$\int_U f(x)\,dx \geqslant \int_{W+W} f(x)\,dx \geqslant 1-\varepsilon,$$

as desired

In order to complete the proof of Theorem 3.4 we need only prove

LEMMA 3.11. *For $f\in\mathfrak{F}_2$ there is a compact subset C of $\mathfrak{G}$, a constant c such that $0<c<\infty$, and an open neighborhood P of the origin of $\hat{\mathfrak{G}}$ such that*

$$1-\Re\hat{f}(\theta)\leqslant c\max_{x\in C}(1-\Re\langle x,\theta\rangle),\quad \theta\in P.$$

Proof. Let $\mathfrak{G}_1$, V, H, etc. be as in the construction used in defining $\mathfrak{F}_2$. Let Λ denote the annihilator of H. Then $\hat{\mathfrak{G}}/\Lambda$ is isomorphic to the dual of H and hence discrete. Thus Λ is open and hence is an open closed subgroup of $\hat{\mathfrak{G}}$. Clearly $\langle v+h,\theta\rangle=\langle v,\theta\rangle$ for $v\in V$, $h\in H$, and $\theta\in\Lambda$.

We can endow V with a dot product. Then for every $\theta\in\hat{\mathfrak{G}}$ there is a unique $M(\theta)\in\hat{V}$ such that

$$\langle v,\theta\rangle=e^{iv\cdot M(\theta)},\quad v\in V.$$

Thus for $\theta\in\Lambda$

$$1-\Re\hat{f}(\theta)=\int(1-\cos v\cdot M(\theta))\,f_1(v)\,dv\leqslant\tfrac{1}{2}|M(\theta)|^2\int|v|^2f_1(v)\,dv=c_1|M(\theta)|^2,$$

where $0<c_1<\infty$. Also $1-\Re\hat{f}(\theta)\leqslant 2$ and hence

$$1-\Re\hat{f}(\theta)\leqslant\min(2,c_1|M(\theta)|^2). \tag{3.9}$$

There is a constant c_2, $0<c_2<\infty$, such that

$$\max_{|v|\leqslant 1}(1-\cos v\cdot M(\theta))\geqslant c_2|M(\theta)|^2,\quad |M(\theta)|\leqslant\frac{\pi}{2}.$$

Then for $\theta\in\Lambda$

$$\max_{|v|\leqslant 1}(1-\Re\langle v,\theta\rangle)=\max_{|v|\leqslant 1}(1-\cos v\cdot M(\theta))\geqslant c_2|M(\theta)|^2,\quad |M(\theta)|\leqslant\frac{\pi}{2},$$

and

$$\max_{|v|\leqslant 1}(1-\Re\langle v,\theta\rangle)\geqslant 1,\quad |M(\theta)|>\frac{\pi}{2}.$$

In particular

$$\max_{|v|\leqslant 1}(1-\Re\langle v,\theta\rangle)\geqslant\min(1,c_2|M(\theta)|^2). \tag{3.10}$$

It follows from equations (3.9) and (3.10) that for some constant c, $0 < c < \infty$,

$$1 - \Re\hat{f}(\theta) \leqslant c \max_{|v| \leqslant 1} (1 - \Re\langle v, \theta\rangle), \quad \theta \in \Lambda.$$

Since Λ is an open neighborhood of the origin of $\hat{\mathfrak{G}}$, and since $\{v, |v| \leqslant 1\}$ is a compact subset of $\mathfrak{G}$, this completes the proof of the lemma and hence of Theorem 3.4.

Let $f \in \mathfrak{F}$ and let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$ containing the support of f. Let φ, ψ, and Λ be as in Theorem 3.3. Then

$$1 - \Re\hat{f}(\theta) = \int_{\mathfrak{G}_1} (1 - \cos \psi(x) \cdot \varphi(\theta)) f(x)\, dx, \quad \theta \in \Lambda.$$

From this we can easily get

THEOREM 3.5. *Let $f \in \mathfrak{F}$ and let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$ containing the support of f. Then if ψ is as in Theorem 3.3,*

$$\int_{\mathfrak{G}_1} |\psi(x)|^2 f(x)\, dx < \infty. \tag{3.11}$$

It is also easy to obtain from Theorems 3.3 and 3.5 and the fact that Λ is open

THEOREM 3.6. *Let $f \in \mathfrak{F}$, let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$ containing the support of f, let ψ and φ be as in Theorem 3.3, and set*

$$K(f) = \int \psi(x) f(x)\, dx.$$

Then there is an open neighborhood P of the origin of $\hat{\mathfrak{G}}$, a compact subset C of $\mathfrak{G}_1$ and a $0 < c < \infty$ such that

$$|\Im\hat{f}(\theta) - \varphi(\theta) \cdot K(f)| \leqslant c \max_{y \in C} (1 - \Re\langle y, \theta\rangle), \quad \theta \in P. \tag{3.12}$$

4. Renewal theory for transient random walks

In this section we consider probability measures μ which define a transient random walk, so that the renewal measure

$$\nu = \sum_0^\infty \mu^{(n)}$$

assigns finite measure to compact sets.

We say that the transient random walk is of Type I if the renewal measure vanishes at infinity, i.e., if

$$\lim_{x\to\infty} \nu(x+C) = 0 \tag{4.1}$$

for all compact subsets C of $\mathfrak{G}$. Otherwise the transient random walk is referred to as of Type II.

This nomenclature is justified by Theorem 4.1 below. In this theorem $\mathfrak{G}$ will be of the form $R\oplus H$ or $Z\oplus H$, where H is compact. The Haar measures are assumed to be of the form $dg = dx\,dh$, where dh assigns unit probability to H and dx is either Lebesgue measure on R or counting measure on Z. The function $\psi:\mathfrak{G}\to R$ or $\mathfrak{G}\to Z$ is defined by $\psi(x+h)=x$.

We set $$\mathfrak{G}^+ = \{x\in\mathfrak{G} \mid \psi(x)\geq 0\}$$
and $$\mathfrak{G}^- = \{x\in\mathfrak{G} \mid \psi(x)< 0\}.$$

By "$x\to+\infty$" or "$x\to-\infty$" we mean that $x\to\infty$ and $x\in\mathfrak{G}^+$ or $x\in\mathfrak{G}^-$ respectively.

THEOREM 4.1. *Let μ define a Type II transient random walk. Then $\mathfrak{G}\cong R\oplus H$ or $\mathfrak{G}\cong Z\oplus H$, where H is compact. Suppose $\mathfrak{G}=R\oplus H$ or $\mathfrak{G}=Z\oplus H$, the Haar measure being chosen as indicated above. Then*

$$m = \int \psi(x)\,\mu(dx)$$

is finite and nonzero. Let $\pm m>0$. Then for $A\in\mathcal{A}$

$$\lim_{x\to\pm\infty} \nu(x+A) = |m|^{-1}|A| \quad \textit{and} \quad \lim_{x\to\mp\infty} \nu(x+A)=0 \tag{4.2}$$

If some iterate of μ is nonsingular, then stronger results are possible.

THEOREM 4.2. *Let μ define a nonsingular transient random walk. Then its renewal measure ν can be written as $\nu=\nu'+\nu''$, ν' being a finite measure and ν'' being an absolutely continuous measure having a continuous density p such that for any $A\in\mathcal{B}$*

$$\lim_{x\to\infty} (\nu(x+A) - p(x)\,|A|) = 0. \tag{4.3}$$

In particular, in Theorem 4.1, $\mathcal{A}$ can be replaced by $\mathcal{B}$.

COROLLARY 4.1. *Let μ and p be as in Theorem 4.2. If the random walk is of Type I, then*

$$\lim_{x\to\infty} p(x) = 0. \tag{4.4}$$

If the random walk is of Type II and m is as in Theorem 4.1, then for $\pm m>0$

$$\lim_{x\to\pm\infty} p(x) = |m|^{-1} \quad \textit{and} \quad \lim_{x\to\mp\infty} p(x) = 0. \tag{4.5}$$

Theorem 4.1 can be used to derive results concerning the asymptotic hitting distributions in the Type II case. Let $\mathcal{D}$ denote the collection of Borel subsets B of $\mathfrak{G}^+$ such that $|\partial B| = 0$ or some iterate of μ is nonsingular. For $B \in \mathcal{D}$ set

$$\tilde{L}_B(x) = \tilde{P}_x(S_n \notin B \text{ for all } n \geq 1),$$

where $\tilde{P}_x$ refers to the random walk with transition distribution $\tilde{\mu}(dy) = \mu(-dy)$. The proof of the next result will be omitted since the basic ideas of the proof can be found in Theorem 4 of Stone [16].

THEOREM 4.3. *Le μ define a transient Type II random walk and suppose that $m > 0$. Then for $A \in \mathcal{A}^*$ and $B \in \mathcal{D}$*

$$\lim_{x \to -\infty} U_B(x, A) = m^{-1} \int_A \tilde{L}_B(x)\, dx.$$

We now will prove the results of this section. In doing so we will assume without further mention that μ defines a transient random walk on $\mathfrak{G}$. We begin the proof of Theorem 4.1 with

LEMMA 4.1. *There is a constant L, $0 \leq L < \infty$ such that if P is a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$, then*

$$\lim_{r \uparrow 1} \int_P \Re\left(\frac{1}{1 - r\hat{\mu}(\theta)}\right) d\theta = \frac{L}{2} + \int_P \Re\left(\frac{1}{1 - \hat{\mu}(\theta)}\right) d\theta. \tag{4.6}$$

Also

$$\int \Re\left(\frac{1}{1 - \hat{\mu}(\theta)}\right) d\theta < \infty. \tag{4.7}$$

Proof. Equation (4.7) and also the fact that if P is relatively compact,

$$\lim_{r \uparrow 1} \sup \int \Re\left(\frac{1}{1 - r\hat{\mu}(\theta)}\right) d\theta < \infty \tag{4.8}$$

are part of the Chung–Fuchs criterion (see Loynes [7, p. 453]). Choose $f \in \mathfrak{F}_s$. Then for $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n \int f(z)\, \mu^{(n)}(dz) = \int \hat{f}(\theta)\, \Re\left(\frac{1}{1 - r\hat{\mu}(\theta)}\right) d\theta. \tag{4.9}$$

It follows from (4.8), Theorem 3.1, and the nonnegativity of f that both sides of (4.9) have finite limits as $r \uparrow 1$. From this (4.6) follows as desired

LEMMA 4.2. *For $f \in \mathfrak{F}_s$*

$$\sup_{x \in \mathfrak{G}} \int f(x+z)\, \nu(dz) < \infty, \tag{4.10}$$

$$\lim_{x\to\infty} \int (f(x+z) + f(-x+z))\, \nu(dz) = L, \tag{4.11}$$

and

$$\lim_{x\to\infty} \int (f(x+2y+z) - 2f(x+y+z) + f(x+z))\, \nu(dz) = 0 \tag{4.12}$$

uniformly for y in compacts.

Proof. It follows easily from Lemma 4.1 and Theorem 3.1 that for $f \in \mathfrak{F}_s$ and $x \in \mathfrak{G}$

$$\frac{1}{2}\int (f(x+z) + f(-x+z))\, \nu(dz) = \frac{L}{2} + \int \Re\langle x, \theta\rangle \hat{f}(\theta)\, \Re\left(\frac{1}{1-\hat{\mu}(\theta)}\right) d\theta \tag{4.13}$$

and the Riemann–Lebesgue Lemma yields (4.10) and (4.11). We also have that for $f \in \mathfrak{F}_s$, $x \in \mathfrak{G}$, and $y \in \mathfrak{G}$

$$\int f(x+2y+z) - 2f(x+y+z) + f(x+z))\, \nu(dz) = \int \langle x, \theta\rangle\, (\langle y, 0\rangle - 1)^2 \hat{f}(0)\, \frac{1}{1-\hat{\mu}(0)}\, d\theta$$

and (4.12) now follows by Theorem 3.1 and the Riemann–Lebesgue lemma.

Lemma 4.3. *For $f \in \mathfrak{F}_s$*

$$\lim_{x\to\infty} \int (f(x+y+z) - f(x+z))\, \nu(dz) = 0 \tag{4.14}$$

uniformly for y in compacts.

Proof. Otherwise, by (4.12), for some $\varepsilon > 0$ and compact set C and any positive integer N, we can find $x \in \mathfrak{G}$ and $y \in C$ such that for $1 \leqslant n \leqslant N$

$$\int (f(x+ny+z) - f(x+(n-1)\,y+z))\, \nu(dz) \geqslant \varepsilon$$

and hence

$$\int (f(x+Ny+z) - f(x+z))\, \nu(dz) \geqslant N\varepsilon,$$

which contradicts (4.10).

Lemma 4.4. *For $A \in \mathcal{A}$ such that $A = -A$ and $|A| > 0$*

$$\sup_{x \in \mathfrak{G}} \nu(x+A) < \infty, \tag{4.15}$$

$$\lim_{x\to\infty} (\nu(x+A)+\nu(-x+A)) = L|A|, \tag{4.16}$$

and

$$\lim_{x\to\infty} (\nu(x+y+A)-\nu(x+A)) = 0 \tag{4.17}$$

uniformly for y in compacts.

Proof. Equation (4.15) follows easily from the fact that μ defines a transient random walk. Choose $k \in \mathfrak{F}_s$ and define $f \in \mathfrak{F}_s$ by

$$f(x) = |A|^{-1}\int 1_A(x-y)\,k(y)\,dy.$$

Then

$$\int f(x+z)\,\nu(dz) = \int \nu(z-x+A)\,k(z)\,dz.$$

Thus by (4.11) we have that

$$\lim_{x\to\infty} \int (\nu(x+z+A)+\nu(-x+z+A))\,k(z)\,dz = L|A|. \tag{4.18}$$

The proof of (4.16) now follows along the lines of the proof of Theorem 2.1 of Stone [14] by using (4.15), (4.18), and Theorem 3.4.

From (4.16) we obtain easily that

$$\limsup_{x\to\infty} \nu(x+A) \leqslant L|A|, \quad A\in\mathcal{A} \text{ and } A=-A. \tag{4.19}$$

From (4.14) we see that for symmetric $A\in\mathcal{A}$ and $k\in\mathfrak{F}_s$

$$\lim_{x\to\infty} \int (\nu(x+y+z+A)-\nu(x+z+A))\,k(z)\,dz = 0 \tag{4.20}$$

uniformly for y in compacts. Choose $\varepsilon>0$. There is a symmetric open neighborhood P of the origin of $\mathfrak{G}$ such that if

$$B = [A^c \cap (A+P)] \cup [A \cap (A^c+P)],$$

then B is symmetric and $|\mathrm{cl}\, B| \leqslant \varepsilon/L$. It follows easily from (4.19) that

$$\limsup_{x\to\infty} \nu(x+B) \leqslant \varepsilon. \tag{4.21}$$

Consequently

$$\limsup_{x\to\infty} \sup_{w\in P} |\nu(x+w+A)-\nu(x+A)| \leqslant \varepsilon. \tag{4.22}$$

Equation (4.17) now follows from (4.15), (4.20), (4.22), and Theorem 3.4.

LEMMA 4.5. *If C is compact, then*

$$\lim_{x\to\infty} \nu(x+C)\,\nu(-x+C)=0. \tag{4.23}$$

Proof. This follows by an easy probabilistic argument which asserts that if D is compact, then the probability that the random walk defined by μ will hit $x+C$ and thereafter hit D approaches zero as $x\to\infty$.

LEMMA 4.6. *Suppose $\mathfrak{G}$ has a compactly generated open subgroup $\mathfrak{G}_1$ such that $\mathfrak{G}_1$ is noncompact and $\mathfrak{G}/\mathfrak{G}_1$ is infinite. Then μ defines a Type I random walk.*

Proof. Let $A\in\mathcal{A}$ be symmetric and such that $|A|>0$. If $z_n+\mathfrak{G}_1$ are disjoint for $n\geqslant 1$, then $z_n\to\infty$ as $n\to\infty$. The z_n's can be chosen so that

$$\lim_{n\to\infty} \nu(z_n+A)=L\,|A|.$$

This follows from Lemmas 4.4 and 4.5.

Since $\mathfrak{G}_1$ is compactly generated, but not compact, we can choose $x\in\mathfrak{G}_1$ such that $nx\to\infty$ as $n\to\infty$. By Lemmas 4.4 and 4.5 we can assume that x is also such that

$$\lim_{n\to\infty} \nu(nx+A)=0.$$

Now $z_n+kx\to\infty$ as $n\to\infty$ uniformly in k. Suppose $L>0$. Then there is an $n_0>0$ such that for all $n\geqslant n_0$ there is a $k_n>0$ such that

$$\nu(z_n+k_nx+A)>L|A|/2$$

and
$$\nu(z_n+(k_n+1)x+A)\leqslant L|A|/2.$$

By Lemma 4.4.
$$\lim_{n\to\infty} \nu(z_n+k_nx+A)=L\,|A|/2,$$

which contradicts Lemmas 4.4 and 4.5. Thus $L=0$ and (4.16) yields (4.1) as desired.

LEMMA 4.7. *Suppose $\mathfrak{G}$ has a closed subgroup $\mathfrak{G}_1\cong R^{d_1}\oplus Z^{d_2}$, where $d_1+d_2>1$. Then μ defines a Type I random walk.*

Proof. We can choose elements x and y in $\mathfrak{G}_1$ such that $nx+ky\to\infty$ as $n+k\to\infty$ and then use the proof of Lemma 4.6.

LEMMA 4.8. *Let H be a compact subgroup of $\mathfrak{G}$. Then*

$$\nu(x+H)\leqslant\nu(H),\quad x\in\mathfrak{G} \tag{4.24}$$

and
$$\nu(x+H)\nu(y+H)\leqslant\nu(x+y+H)\nu(H),\quad x,y\in\mathfrak{G}. \tag{4.25}$$

Proof. This result follows by an easy probabilistic argument based on the first passage into the set $x+H$ and the fact that $H=H+H=H-H$.

LEMMA 4.9. *Suppose that every element of $\mathfrak{G}$ is a compact element. Then μ defines a Type I random walk.*

Proof. We suppose that $L>0$ in Lemma 4.4 and will arrive at a contradiction. Let H be an open compactly generated subgroup of $\mathfrak{G}$. Then H is compact. Without loss of generality we can assume that $|H|=1$. The remainder of the proof is exactly the same as that of the corresponding result in the discrete case (Kesten and Spitzer [5, pp. 259–260]) upon replacing their $g(x)$ by our $\nu(x+H)$ and using Lemma 4.8 (note that since $\mathfrak{G}/H$ is infinite there exist $z_n \in \mathfrak{G}$ with $z_n \to \infty$).

LEMMA 4.10. *Suppose that $\mathfrak{G}=R\oplus H$ or $\mathfrak{G}=Z\oplus H$, where H is compact, and that*

$$\int \psi(x)\,\mu(dx)$$

is either infinite or undefined. Then μ defines a Type I random walk.

Proof. This result reduces immediately to the corresponding one-dimensional result of Feller and Orey [2].

Proof of Theorem 4.1. Let μ define a Type II transient random walk. Then, by Lemmas 4.6, 4.7, and 4.9, $\mathfrak{G}\cong R\oplus H$ or $\mathfrak{G}\cong Z\oplus H$, where H is compact. Suppose $\mathfrak{G}=R\oplus H$ or $\mathfrak{G}=Z\oplus H$, the Haar measures being chosen as indicated just prior to the statement of Theorem 4.1. Then

$$m=\int_{-\infty}^{\infty} \psi(x)\,\mu(dx)$$

is finite and nonzero. This follows by Lemma 4.10 and the fact that if the integral vanished, the random walk would be recurrent.

It now follows by Blackwell's renewal theorem that if $A=A_1+H$, where A_1 is a relatively compact subset of R or Z whose boundary has measure zero (in R or Z), then $A\in\mathcal{A}$ and equation (4.2) holds for this A. From this it follows that for $A\in\mathcal{A}$

$$\lim_{x\to\pm\infty} \nu(x+A)=0. \tag{4.26}$$

In order to complete the proof of (4.2) it suffices to show that for $A\in\mathcal{A}$

$$\lim_{x\to\infty} (\nu(x+A)+\nu(-x+A))=L\,|A|. \tag{4.27}$$

For this it suffices to show that for $f \in \mathfrak{F}$

$$\lim_{x\to\infty} \int (f(x+z)+f(-x+z))\,\nu(dz)=L. \tag{4.28}$$

But we have the formula

$$\int (f(x+z)+f(-x+z))\,\nu(dz)=L+2\int (\Re\langle x,\theta\rangle)\,\hat{f}(\theta)\,\frac{1}{1-\hat{\mu}(\theta)}\,d\theta. \tag{4.29}$$

Since $\Re(1/(1-\hat{\mu}(\theta)))$ is integrable on compacts the Riemann–Lebesgue lemma implies that

$$\lim_{x\to\infty}\int \Re\langle x,\theta\rangle\,\Re\hat{f}(\theta)\,\Re\left(\frac{1}{1-\hat{\mu}(\theta)}\right)d\theta=0.$$

In order to complete the proof of (4.28) it suffices to show that

$$\Im\hat{f}(\theta)\,\Im\left(\frac{1}{1-\hat{\mu}(\theta)}\right)$$

is absolutely integrable. Write $\mathfrak{G}=\Theta_1\oplus\hat{H}$ where $\Theta_1=\hat{R}$ or $\hat{Z}$. Since $\hat{H}$ is discrete, it suffices to show that the above term is integrable on Θ_1. This is true because

$$\Im\hat{f}(\theta_1)=O(|\theta_1|)$$

and

$$\Im\left(\frac{1}{1-\hat{\mu}(\theta_1)}\right)=O(|\theta_1|^{-1}).$$

This completes the proof of Theorem 4.1.

Proof of Theorem 4.2 and Corollary 4.1. The proof of these results follows from Theorem 4.1 and the arguments found in Stone [17, pp. 271–272]. Note that for present purposes the assumption in [17] that "φ has a twice continuously differentiable density" can be replaced by the assumption that φ has a continuous density.

5. The general recurrent case

Throughout this section it will be assumed that μ defines a recurrent random walk on $\mathfrak{G}$. We will state in this section those results which hold in general. The proofs will not be given completely in this section, however. In particular Theorem 5.1, which is not needed for any other results of this section follows from Theorem 3.1, Theorem 3.7 and Theorem 6.1 of the following section (the proof of Theorem 6.1 is independent of the results of

Section 5). Theorem 5.2 will be proven in this section only in the compactly generated case. The remainder of the results of this section, through Corollary 5.3, will be proven in general, given that Theorem 5.2 holds. Theorem 5.2 itself follows in general immediately from Theorem 3.1, Theorem 3.2 and Theorem 6.3. Since Theorem 6.3 depends on Theorem 5.2 and its consequences only in the compactly generated case, there is no danger of circular reasoning.

Theorem 5.13 and the results which follow depend on Section 6. This is permissible, however, since Section 6 depends on Section 5 only through Corollary 5.1.

Suppose $\mathfrak{G}$ is compactly generated. Then we can assume that $\mathfrak{G}=\mathfrak{G}_1\oplus H$, where H is compact and $\mathfrak{G}_1=R$, Z, $R\oplus R$, $R\oplus Z$, or $Z\oplus Z$. In this case Haar measure on H is normalized to be a probability measure, while the Haar measures on R and Z are Lebesgue measure and counting measure respectively. The function ψ denotes the projection of $\mathfrak{G}$ onto $\mathfrak{G}_1$ and μ_1 denotes the probability measure induced on $\mathfrak{G}_1$ by μ and ψ. If $\mathfrak{G}_1$ is one-dimensional, then $x\to\pm\infty$ is taken to mean that $\psi(x)\to\pm\infty$. If $\mathfrak{G}_1$ is one-dimensional, then σ^2 denotes the variance of μ_1. The random walk is said to be of Type II if $\mathfrak{G}_1$ is one-dimensional and $\sigma^2<\infty$. Otherwise the random walk is of Type I.

If $\mathfrak{G}$ is not compactly generated, the random walk is always considered to be of Type I.

THEOREM 5.1. *Let P be an open neighborhood of the origin on $\hat{\mathfrak{G}}$. Then*

$$\int_P \Re\left(\frac{1}{1-\hat{\mu}(\theta)}\right) d\theta=\infty. \tag{5.1}$$

Proof. This result follows as indicated above.

THEOREM 5.2. *Let P be a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then*

$$\lim_{\lambda\uparrow 1}\int_P \frac{\langle x,\theta\rangle-1}{1-\lambda\hat{\mu}(\theta)}\, d\theta \tag{5.2}$$

exists and is finite. In the Type I case

$$\lim_{y\to\infty}\lim_{\lambda\uparrow 1}\int_P \frac{\langle y,\theta\rangle\,(\langle x,\theta\rangle-1)}{1-\lambda\hat{\mu}(\theta)}\, d\theta=0. \tag{5.3}$$

In the Type II case

$$\lim_{y\to\pm\infty}\lim_{\lambda\uparrow 1}\int_P \frac{\langle y,\theta\rangle\,(x,\theta\rangle-1)}{1-\lambda\hat{\mu}(\theta)}\, d\theta=\mp\sigma^2\psi(x), \tag{5.4}$$

where
$$\sigma^2=\int\psi^2\, d\mu.$$

The convergence in these limits is uniform for x in compacts.

Proof in the compactly generated case. We can assume that P is a subset of $\hat{\mathfrak{G}}_1$. Suppose first that $\mathfrak{G}_1$ is one-dimensional. Then

$$\int_P \frac{\langle x, \theta\rangle - 1}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \int_P \frac{e^{i\theta_1\psi(x)}-1}{1-\lambda\hat{\mu}_1(\theta_1)}\, d\theta_1.$$

In order to prove (5.2) it suffices to show that

$$\lim_{\lambda\uparrow y} \int_P \frac{i\theta_1}{1-\lambda\hat{\mu}_1(\theta_1)}\, d\theta_1$$

exists and is finite. But this is Theorem 3.1″ of Port and Stone [11]. The proof there depended on results of Ornstein [8]. For a selfcontained proof see Theorem 1 of Stone [15]. Similarly

$$\int_P \frac{\langle y, \theta\rangle\,(\langle x, \theta\rangle - 1)}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \int_P \frac{e^{i\theta_1\psi(y)}\,(e^{i\theta_1\psi(x)}-1)}{1-\lambda\hat{\mu}_1(\theta_1)}\, d\theta_1.$$

Thus to complete the proof of Theorem 5.2 in this case it suffices to show that in the Type I case

$$\lim_{y\to\infty}\lim_{\lambda\uparrow 1} \int_P \frac{e^{i\theta_1\psi(y)}\, i\theta_1}{1-\lambda\hat{\mu}_1(\theta_1)}\, d\theta_1 = 0, \tag{5.5}$$

while in the Type II case

$$\lim_{y\to\pm\infty}\lim_{\lambda\uparrow 1} \int_P \frac{e^{i\theta_1\psi(y)}\, i\theta_1}{1-\lambda\hat{\mu}_1(\theta_1)}\, d\theta_1 = \mp\sigma^{-2}. \tag{5.6}$$

Again for a proof see either Theorem 3.1″ of [11] or Theorem 1 of [15].

If $\mathfrak{G}_1$ is two-dimensional then $|\theta|/|1-\hat{\mu}(\theta_1)|$ is integrable on compact subsets of $\hat{\mathfrak{G}}_1$ and the theorem follows trivially from the Riemann–Lebesgue lemma.

Choose $g\in\mathfrak{F}$ such that g is symmetric and nonnegative and $J(g)=1$. Define D^λ for $f\in\mathfrak{F}$ and $x\in\mathfrak{G}$ by

$$\mathrm{D}^\lambda f(x) = J(f)\, U^\lambda g(0) - U^\lambda f(x).$$

THEOREM 5.3. *For $f\in\mathfrak{F}$ and $x\in\mathfrak{G}$*

$$\lim_{\lambda\uparrow 1} D^\lambda f(x) = Df(x) \tag{5.7}$$

exists and is finite. In the Type I case

$$\lim_{y\to\infty} (Df_y(x) - Df_y(0)) = 0 \tag{5.8}$$

and in the Type II case

$$\lim_{y\to\pm\infty} (Df_y(x) - Df_y(0)) = \mp\psi(x)\,\sigma^{-2} J(f). \tag{5.9}$$

Proof. Observe that

$$D^{\lambda} f(x) = \lambda \int_{\hat{\mathfrak{G}}} \frac{\hat{g}(-\theta) J(f) - \langle x, \theta \rangle \hat{f}(-\theta)}{1 - \lambda \hat{\mu}(\theta)} \hat{\mu}(\theta)\, d\theta \tag{5.10}$$

and
$$D^{\lambda} f_y(x) - D^{\lambda} f_y(0) = \lambda \int_{\hat{\mathfrak{G}}} \frac{\overline{\langle x, \theta \rangle}\, (1 - \langle x, \theta \rangle) \hat{f}(-\theta)}{1 - \lambda \hat{\mu}(\theta)} \hat{\mu}(\theta)\, d\theta.$$

Theorem 5.3 now follows immediately from Theorem 5.2 and Theorem 3.1.

PROPOSITION 5.1. *Let* $B \in \mathfrak{B}$ *with* $\operatorname{int} B \neq \emptyset$. *Let* $f(x)$, $x \in \mathfrak{G}$, *be a nonnegative Borel-function such that for some open neighborhood* Q *of the origin of* $\mathfrak{G}$ *the function* $f_Q(x)$, $x \in \mathfrak{G}$, *defined by*

$$f_Q(x) = \sup_{y \in Q} f_y(x)$$

is integrable on $\mathfrak{G}$. *Then*

$$\lim_{\lambda \uparrow 1} U_B^{\lambda} f(x) = U_B f(x), \quad x \in \mathfrak{G} \tag{5.11}$$

exists and is finite and the convergence is uniform for x *in compacts.*

Proof. Since f is nonnegative the limit in (5.11) clearly exists. We next prove it is finite. Let A be a nonempty open subset of $\mathfrak{G}$ and R a relatively compact symmetric open neighborhood of the origin of $\mathfrak{G}$ such that $R + R \subseteq Q$ and $A + R \subseteq B$. For $x_0 \in \mathfrak{G}$ and $x \in x_0 + R$

$$|R| U_B f(x) = |R| \int U_B(x, dz) f(z) = |R| \int U_{y+B}(x + y, dz) f_y(z)$$

$$= \int_R dy\, U_{y+B} f_y(x + y) \leqslant \int_R dy\, U_A f_R(x + y) \leqslant \int_{x_0 + R + R} dy U_A f_R(y)$$

$$= \int_{\mathfrak{G}} f_R(y)\, dy\, \tilde{U}_A(y, x_0 + R + R) < \infty$$

since $x_0 + R + R$ is relatively compact and consequently, $\tilde{U}_A(y, x_0 + R + R)$ is bounded in y. This proves that the limit in (5.11) is finite and, in fact, bounded uniformly on compacts.

In particular for f as in the statement of the proposition and A as in the proof. $U_A f_R(z) < \infty$, $z \in \mathfrak{G}$, and hence for $x_0 \in \mathfrak{G}$

$$\lim_{n \to \infty} \int_{A^c} P_A^n(x_0, dz) U_A f_R(z) = 0.$$

For $x \in x_0 + R$

$$\int_{B^c} {}_B P^n(x, dz) U_B f(z) \leqslant \int_{B^c + x_0 - x} {}_A P^n(x_0, dz) U_B f(z + x - x_0)$$

$$\leqslant \int_A {}_A P^n(x_0, dz) U_A f_R(z) \to 0 \quad \text{as} \quad n \to \infty.$$

From this it follows that

$$\lim_{n\to\infty} \int_{B^c} {}_B P^n(x, dz) U_B f(z) = 0$$

uniformly for x in compacts and hence that (5.11) holds uniformly for x in compacts. This completes the proof of Proposition 5.1.

Let $B \in \mathcal{B}$ and int $B \neq \emptyset$. Then $P_x(V_B < \infty) = 1$, $x \in \mathfrak{G}$. As in [11] we have the fundamental identity

$$D^\lambda f(x) - \Pi_B^\lambda D^\lambda f(x) = -U_B^\lambda f(x) + L_B^\lambda(x) J(f) \tag{5.12}$$

for $0 < \lambda < 1$, $f \in \mathfrak{F}$, and $x \in \mathfrak{G}$. It is easy to find an $f \in \mathfrak{F}$ satisfying the conditions of Proposition 5.1 and such that $J(f) > 0$. As a consequence of Theorem 5.3 and Proposition 5.1 we now have

THEOREM 5.4. *Let $B \in \mathcal{B}$ and* int $B \neq \emptyset$ *and let $f \in \mathfrak{F}$. Then*

$$\lim_{\lambda \uparrow 1} L_B^\lambda(x) = L_B(x) \tag{5.13}$$

and

$$\lim_{\lambda \uparrow 1} U_B^\lambda f(x) = U_B f(x) \tag{5.14}$$

exist and are finite and the convergence is uniform for x in compact subsets of $\mathfrak{G}$. Also

$$Df(x) - \Pi_B Df(x) = -U_B f(x) + L_B(x) J(f). \tag{5.15}$$

From Theorems 5.1 and 5.2 we see that in the Type I case

$$\lim_{y\to\infty} U_B f_y(x) = L_B(x) J(f) \tag{5.16}$$

and in the Type II case

$$\lim_{y\to\pm\infty} U_B f_y(x) = L_B^{\pm}(x) J(f), \tag{5.17}$$

where

$$L_B^{\pm}(x) = L_B(x) \pm \sigma^{-2}(\psi(x) - \Pi_B \psi(x)).$$

From these results we will obtain

THEOREM 5.5. *Let $B \in \mathcal{B}$ and* int $B \neq \emptyset$ *and let $A \in \mathcal{A}$. In the Type I case*

$$\lim_{y\to\infty} U_B(x, y + A) = L_B(x) |A|, \tag{5.18}$$

and in the Type II case

$$\lim_{y\to\pm\infty} U_B(x, y+A) = L_B^{\pm}(x)\,|A|. \tag{5.19}$$

The convergence in these limits is uniform for x in compact subsets of $\mathfrak{G}$.

Proof. This result uses Theorem 3.4 and is otherwise essentially that of Theorem 3.3 of [11].

Using Theorem 5.5 and duality, we have

THEOREM 5.6. *Let $B \in \mathcal{B}$ and* int $B \neq \emptyset$. *Let $C \in \mathcal{A}$ and let $A \in \mathcal{B}$. In the Type I case*

$$\lim_{x\to\infty} \int_C U_B(x+z, A)\,dz = |C| \int_A \tilde{L}_B(x)\,dx \tag{5.20}$$

and in the Type II case

$$\lim_{x\to\pm\infty} \int_C U_B(x+z, A)\,dz = |C| \int_A \tilde{L}_B^{\pm}(x)\,dx. \tag{5.21}$$

By following the proof of Theorem 3.5 of [11] we obtain

THEOREM 5.7. *Let $A \in \mathcal{A}$, $B \in \mathcal{A}$, and* int $B \neq \emptyset$. *In the Type I case*

$$\lim_{x\to\infty} U_B(x, A) = \int_A \tilde{L}_B(x)\,dx. \tag{5.22}$$

and in the Type II case

$$\lim_{x\to\pm\infty} U_B(x, A) = \int_A \tilde{L}_B^{\pm}(x)\,dx. \tag{5.23}$$

We rephrase this result in a form more useful for application in Section 6.

COROLLARY 5.1. *Let $B \in \mathcal{A}$ and* int $B \neq \emptyset$. *Let the random walk start at x. In the Type I case as $x \to \infty$ the hitting distribution of B has a limit which is absolutely continuous with density $\tilde{L}_B(x)1_B(x)$. In the Type II case as $x \to \pm\infty$ the hitting distribution of B has a limit which is absolutely continuous with density $\tilde{L}_B^{\pm}(x)\,1_B(x)$.*

In the Type II case let

$$K(f) = \int \psi(x)\,f(x)\,dx.$$

THEOREM 5.8. *Let $f \in \mathfrak{F}$ with $J(f) = 0$. Then*

$$\lim_{\lambda\uparrow 1} U^{\lambda} f(x) = Uf(x), \quad x \in \mathfrak{G}, \tag{5.24}$$

exists and is finite and the convergence is uniform for x in compacts. In the Type I case

$$\lim_{x\to\infty} Uf(x)=0 \tag{5.25}$$

and in the Type II case

$$\lim_{x\to\pm\infty} Uf(x)=\pm\sigma^{-2}K(f). \tag{5.26}$$

COROLLARY 5.2. *Let $f, f_1 \in \mathfrak{F}$. In the Type I case*

$$\lim_{x\to\infty} J(f_1)\,Df(x)-J(f)\,Df_1(x)=0 \tag{5.27}$$

and in the Type II case

$$\lim_{x\to\pm\infty} (J(f_1)\,Df(x)-J(f)\,Df_1(x))=\mp\sigma^{-2}(J(f_1)\,K(f)-K(f_1)\,J(f)). \tag{5.28}$$

Proof of Theorem 5.8. Let $\mathfrak{G}_1$ be an open compactly generated subgroup of $\mathfrak{G}$ containing the support of f. Let H, ψ, φ, and Λ be as in Theorem 3.3. We have easily

LEMMA 5.1. *There is a relatively compact open neighborhood P of the origin of $\hat{\mathfrak{G}}$, a compact subset C of $\mathfrak{G}$ and a $0<c<\infty$ such that for $x\in\mathfrak{G}_1$ and $\theta\in P$*

$$|\mathfrak{J}\langle x,\theta\rangle-\psi(x)\cdot\varphi(\theta)|\leqslant c\,|\psi(x)|^2\max_{y\in C}(1-\mathfrak{R}\langle y,\theta\rangle).$$

From Lemma 5.1 and Theorem 5.2 we have

LEMMA 5.2. *Let P be a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then*

$$\lim_{\lambda\uparrow 1}\int_P \frac{i\varphi(\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta \tag{5.29}$$

exists and is finite. In the Type I case

$$\lim_{x\to\infty}\lim_{\lambda\uparrow 1}\int_P \frac{\langle x,\theta\rangle\, i\varphi(\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta=0 \tag{5.30}$$

and in the Type II case

$$\lim_{x\to\pm\infty}\lim_{\lambda\uparrow 1}\int_P \frac{\langle x,\theta\rangle\, i\varphi(\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta=\mp\sigma^{-2}. \tag{5.31}$$

Theorem 5.8 follows from Lemma 5.2 and Theorems 3.1 and 3.6.

THEOREM 5.9. *Let $\mathfrak{G}_1$ be a compactly generated open subgroup of $\mathfrak{G}$ of the form $\mathfrak{G}_1 = V\oplus L\oplus H$ where $V\cong R^{d_1}$, $L\cong Z^{d_2}$, $d=d_1+d_2$, and H is compact. Let ψ be a continuous homomorphism of $\mathfrak{G}_1$ onto a closed d-dimensional subgroup of R^d. Let $B\in\mathcal{B}$ have a nonempty interior and let $A\in\mathcal{A}$. In the Type I case*

$$\lim_{x\to\infty,\, x\in\mathfrak{G}_1} U_B(x, x+A)/\psi(x) = 0 \tag{5.32}$$

and in the Type II case (setting $\mathfrak{G}_1 = \mathfrak{G}$)

$$\lim_{x\to\infty} U_B(x, x+A)/|\psi(x)| = 2\sigma^{-2}|A|. \tag{5.33}$$

Proof. Choose $f\in\mathfrak{F}$. Then from 5.15

$$U_B f_x(x) = Df(x) + Df(-x) - Df(0) + (\Pi_B Df_x(x) - Df_x(0)) - \Pi_B Df(x) + U_B f(x).$$

Now
$$-Df(0) + (\Pi_B Df_x(x) - Df_x(0)) - \Pi_B Df(x) + U_B f(x)$$

is bounded in x since the individual terms are bounded. It now follows from Theorem 5.3 that in the Type I case

$$\lim_{x\to\infty,\, x\in\mathfrak{G}_1} U_B f_x(x)/\psi(x) = 0 \tag{5.34}$$

and in the Type II case

$$\lim_{x\to\infty} U_B f_x(x)/|\psi(x)| = \sigma^{-2}J(f). \tag{5.35}$$

It is easy to go from (5.34) to (5.32). To go from (5.35) to (5.33) one need only use an "unsmoothing" argument, noting firstly that (5.35) implies that

$$U_B(x, x+A) = O(|\psi(x)|) \quad \text{as} \quad x\to\infty.$$

The main role of the functions in $\mathfrak{F}$ is to act as smoothing functions. For some results the class $\mathfrak{F}$ is too large to be easily useful. One possible out is to further restrict $\mathfrak{F}$. Another is provided by

PROPOSITION 5.2. *Let* $k_n\in\mathfrak{F}$ *be supported on an open compactly generated subgroup* $\mathfrak{G}_1$ *of* $\mathfrak{G}$, *uniformly bounded, let*

$$\lim_{n\to\infty} k_n(x) = 1, \quad x\in\mathfrak{G}_1,$$

the convergence being uniform on compact subsets of $\mathfrak{G}_1$, *and let* $\hat{k}_n$ *be nonnegative and supported on a compact subset* C *of* $\hat{\mathfrak{G}}$. *For* $f\in\mathfrak{F}$ *supported by* $\mathfrak{G}_1$ *define* f_n *by* $f_n(x) = f(x)k_n(x)$, $x\in\mathfrak{G}$. *Then* $f_n\in\mathfrak{F}$ *and*

$$\lim_{n\to\infty} Df_n(x) = Df(x), \quad x\in\mathfrak{G} \tag{5.36}$$

uniformly for x *in compacts.*

Proof. Note first that $J(f_n)\to J(f)$. Also

$$\hat{f}_n(\theta) = \int \hat{f}(\theta-\tau)\,\hat{k}_n(\tau)\,d\tau$$

and hence $\hat{f}_n$, $\hat{f}$ are supported by a compact set $C_1 \subseteq \hat{\mathfrak{G}}$, where C_1 is independent of n. Moreover

$$J(f_n) = \hat{f}_n(0) = \int \hat{f}(-\tau)\,\hat{k}_n(\tau)\,d\tau$$

and $\hat{f}_n(\theta) \to \hat{f}(\theta)$ uniformly on C_1.

Let ψ, φ, and Λ be as in Theorem 3.3. It follows that there is an $0 < M < \infty$ and a compact subset C_2 of $\mathfrak{G}$ such that

$$|\varphi(\theta)|^2 \leqslant M \max_{x \in C_2} (1 - \Re\langle x, \theta\rangle), \quad \theta \in C_1 \cap \Lambda.$$

Note that
$$\int \hat{k}_n(\tau)\,d\tau \int i\varphi(\theta)\cdot\psi(x)\,\langle -\tau, x\rangle\, f(x)\,dx = i\varphi(\theta)\cdot K(f_n)$$

and that $K(f_n) \to K(f)$ as $n \to \infty$.

For $\theta \in C_1 \cap \Lambda$

$$\hat{f}_n(\theta) - J(f_n) - i\varphi(\theta)\cdot K(f_n) = \int \hat{k}_n(\tau)\,d\tau \int (e^{i\psi(x)\cdot\varphi(\theta)} - 1 - i\varphi(\theta)\cdot\psi(x))\,\langle -\tau, x\rangle\, f(x)\,dx.$$

In particular $|\Re\hat{f}_n(\theta) - J(f_n)| = O(\max_{x \in C_2}(1 - \Re\langle x, \theta\rangle))$ as $\theta \to 0$ and hence $f_n \in \mathfrak{F}$. The conclusion of the proposition now follows easily from the formula

$$Df_n(x) = \int \frac{\hat{g}(-\theta)\,J(f_n) - \langle x, \theta\rangle\,\hat{f}_n(-\theta)}{1 - \lambda\hat{\mu}(\theta)}\,d\theta.$$

PROPOSITION 5.3. *Let $\mathfrak{G}_1$ and ψ be as in Theorem 5.9. Let $B \in \mathcal{A}$ with* int $B \neq \emptyset$. *Let $f \in \mathfrak{F}$ be supported by $\mathfrak{G}_1$ and suppose that $f(x) = O(|\psi(x)|^{-4})$ as $x \to \infty$ in $\mathfrak{G}_1$. In the Type I case*

$$\lim_{x \to \infty} U_B f(x) = \int \bar{L}_B(x)\,f(x)\,dx \tag{5.37}$$

and in the Type II case

$$\lim_{x \to \pm\infty} U_B f(x) = \int \bar{L}_B^{\pm}(x)\,f(x)\,dx. \tag{5.38}$$

The right sides of (5.37) and (5.38) are finite.

Proof. Write
$$U_B f(x) = \int U_B(x, dy)\,f(y).$$

The desired result now follows easily from Theorem 5.7, Theorem 5.9 and the fact that $d \leqslant 2$.

THEOREM 5.10. *Let $B \in \mathcal{A}$ with* int $B \neq \varnothing$. *In the Type I case there is a finite constant $k(B)$ such that for $f \in \mathfrak{F}$*

$$\lim_{x\to\infty} (Df(x) - L_B(x)\,J(f)) = J(f)\,k(B). \tag{5.39}$$

In the Type II case there exist finite constants $k^{\pm}(B)$ such that for $f \in \mathfrak{F}$

$$\lim_{x\to\pm\infty} (Df(x) - L_B(x)\,J(f)) = J(f)\,k^{\pm}(B) \mp \sigma^{-2}K(f). \tag{5.40}$$

Proof. One can easily find an $f_1 \in \mathfrak{F}$ of the form of Proposition 5.3 and with $J(f_1) = 1$. It can also be assumed that in the Type II case $K(f_1) = 0$. Recall the identity

$$Df(x) - \Pi_B Df(x) = -U_B f(x) + L_B(x) J(f). \tag{5.41}$$

In the Type I case it follows from Theorem 5.7 and Proposition 5.3 that for some finite constant $k(B)$

$$\lim_{x\to\infty} (Df_1(x) - L_B(x)) = k(B),$$

and (5.39) now follows from Corollary 5.2.

In the Type II case it follows from Theorem 5.7 and Proposition 5.3 that for some finite constants $k^{\pm}(B)$

$$\lim_{x\to\pm\infty} (Df_1(x) - L_B(x)) = k^{\pm}(B),$$

and (5.40) now follows from Corollary 5.2.

From Theorem 5.7, Theorem 5.10, and equation (5.41) we have

THEOREM 5.11. *Let $B \in \mathcal{A}$ with* int $B \neq \varnothing$ *and let $f \in \mathfrak{F}$. In the Type I case*

$$\lim_{x\to\infty} U_B f(x) = -k(B)\,J(f) + \int_B \tilde{L}_B(x)\,Df(x)\,dx \tag{5.42}$$

and in the Type II case

$$\lim_{x\to\pm\infty} U_B f(x) = -k^{\pm}(B)\,J(f) \pm \sigma^{-2}K(f) + \int_B \tilde{L}_B^{\pm}(x)\,Df(x)\,dx. \tag{5.43}$$

THEOREM 5.12. *Let $B \in \mathcal{A}$ with* int $B \neq \varnothing$ *and let $f \in \mathfrak{F}$. In the Type I case*

$$\int \tilde{L}_B(x)\,f(x)\,dx = -k(B)\,J(f) + \int_B \tilde{L}_B(x)\,Df(x)\,dx \tag{5.44}$$

and in the Type II case

$$\int \tilde{L}_B^{\pm}(x)\,f(x)\,dx = -k^{\pm}(B)\,J(f) \pm \sigma^{-2}K(f) + \int_B \tilde{L}_B^{\pm}(x)\,Df(x)\,dx. \tag{5.45}$$

Proof. If f satisfies the conditions of Proposition 5.3 the result follows from Proposition 5.3 and Theorem 5.11. The general case can be reduced to this special case by an application of Proposition 5.2, since it is always possible to choose k_n as in Proposition 5.2 and such that each $k_n(x) = O(|\psi(x)|^{-4})$ as $x \to \infty$.

From Theorem 5.11 and Theorem 5.12 we obtain immediately

COROLLARY 5.3. *Let $B \in \mathcal{A}$ with* int $B \neq \emptyset$ *and let $f \in \mathfrak{F}$. In the Type I case*

$$\lim_{x\to\infty} U_B f(x) = \int \tilde{L}_B(x) f(x)\, dx \tag{5.46}$$

and in the Type II case

$$\lim_{x\to\pm\infty} U_B f(x) = \int \tilde{L}_B^{\pm}(x) f(x)\, dx. \tag{5.47}$$

The remainder of the results of this section depend on the results of Section 6. There is no danger of circular reasoning since the results of Section 6 depend on Section 5 only to Corollary 5.1.

THEOREM 5.13. *Let C be a compact subset of $\mathfrak{G}$ and P a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then there is an $0 < M < \infty$ such that*

$$\left| \int_P \frac{\langle y, \theta\rangle (\langle x, \theta\rangle - 1)}{1 - \lambda\hat{\mu}(\theta)}\, d\theta \right| \leqslant M \tag{5.48}$$

for $y \in \mathfrak{G}$, $x \in C$, and $0 < \lambda < 1$.

Proof. Let some $\mu^{(n)}$ be nonsingular. Let $f \in \mathfrak{F}_1$ be nonnegative with $J(f) > 0$ and let C be a compact subset of $\mathfrak{G}$. Then by Corollary 7.1 and the arguments leading up to Lemma 6.4 we see that there is an $0 < M < \infty$ such that for $y \in \mathfrak{G}$, $x \in C$, and $0 \leqslant \lambda < 1$

$$|D^\lambda f_y(x) - D^\lambda f_y(0)| \leqslant M. \tag{5.49}$$

When expressed in terms of Fourier analysis and used with Theorem 3.2, (5.49) yields Theorem 5.13.

From Theorem 5.13 we easily get

COROLLARY 5.4. *Let $f \in \mathfrak{F}$ and C be a compact subset of $\mathfrak{G}$. Then there is an $0 < M < \infty$ such that*

$$|D^\lambda f_y(x) - D^\lambda f_y(0)| \leqslant M \tag{5.50}$$

for $y \in \mathfrak{G}$, $x \in C$, and $0 \leqslant \lambda \leqslant 1$.

From Corollary 5.4 and the identity

$$\Pi_B Df_y(x) - Df_y(0) = \int \Pi_B(x, dz)\,(Df_y(z) - Df_y(0)),$$

we get

COROLLARY 5.5. *Let $B \in \mathcal{A}$ with* int $B \neq \emptyset$ *and let $f \in \mathfrak{F}$. Then*

$$\Pi_B Df_y(x) - Df_y(0)$$

is bounded uniformly for $x, y \in \mathfrak{G}$.

By exchanging the roles of x and y we also get as a consequence of Theorem 5.13

COROLLARY 5.6. *Let C be a compact subset of $\mathfrak{G}$ and P a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then there is an $0 < M < \infty$ such that for $x \in \mathfrak{G}$, $y \in C$, and $0 \leqslant \lambda < 1$*

$$\left| \int_P \frac{\langle x, \theta\rangle \,\mathfrak{J}\langle y, \theta\rangle}{1 - \lambda\hat{\mu}(\theta)}\, d\theta \right| \leqslant M. \tag{5.51}$$

THEOREM 5.14. *Let P be a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then there is an $0 < M < \infty$ such that*

$$\int_P \frac{1 - \langle x, \theta\rangle}{1 - \lambda\hat{\mu}(\theta)}\, d\theta \geqslant -M, \quad x \in \mathfrak{G} \text{ and } 0 \leqslant \lambda < 1. \tag{5.52}$$

Proof. Using Corollary 7.1 and the proof of Lemma 6.6 we can find an $f \in \mathfrak{F}^+_{1s}$ such that $J(f) > 0$ and for some $0 < M < \infty$

$$D^\lambda f(x) \geqslant -M, \quad x \in \mathfrak{G} \text{ and } 0 \leqslant \lambda \leqslant 1. \tag{5.53}$$

When expressed in terms of Fourier analysis and used with Theorem 3.2, (5.53) yields Theorem 5.14.

THEOREM 5.15. *Let $f \in \mathfrak{F}$ with $J(f) \geqslant 0$. Then there is an $0 < M < \infty$ such that*

$$D^\lambda f(x) \geqslant -M, \quad x \in \mathfrak{G} \text{ and } 0 \leqslant \lambda \leqslant 1. \tag{5.54}$$

COROLLARY 5.7. *Let $f \in \mathfrak{F}$ with $J(f) = 0$. Then there is an $0 < M < \infty$ such that*

$$|U^\lambda f(x)| \leqslant M, \quad x \in \mathfrak{G} \text{ and } 0 \leqslant \lambda \leqslant 1. \tag{5.55}$$

Proof of Theorem 5.15. Let $\mathfrak{G}_1$ be an open compactly generated subgroup of $\mathfrak{G}$ containing the support of f. Let ψ, φ, and Λ be as in Theorem 3.3.

From Theorem 3.1, Lemma 5.1 and Corollary 5.6 it follows that there is an $0 < M < \infty$ such that if $P \subseteq \Lambda$ is a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$

$$\left|\int \frac{\langle x, \theta\rangle\, i\varphi(-\theta)}{1-\lambda\hat{\mu}(\theta)}\right| \leqslant M, \quad x \in \mathfrak{G} \text{ and } 0 \leqslant \lambda \leqslant 1. \tag{5.56}$$

From Theorem 3.1, Theorem 3.6, and equation (5.56) it follows that there is an $0 < M < \infty$ such that

$$\left|\int_P \frac{\langle x, \theta\rangle\, \mathfrak{F}f(-\theta)}{1-\lambda\hat{\mu}(\theta)}\, d\theta\right| \leqslant M, \quad x \in \mathfrak{G}. \tag{5.57}$$

Theorem 5.15 follows from Theorem 5.15 and equation (5.57).

6. The nonsingular case

In this section we prove those results in the nonsingular case which are necessary to complete the proof of Theorems 5.1 and 5.2. Further results for the nonsingular case will be given in Section 7.

Throughout this section it will be assumed that μ defines a recurrent random walk on $\mathfrak{G}$ and that some iterate of μ is nonsingular.

Let $\mathfrak{F}_1$ denote the collection of continuous functions on $\mathfrak{G}$ having compact support and integrable Fourier transform. Let $\mathfrak{F}_{1s}$ denote the symmetric functions in $\mathfrak{F}_1$. Let $\mathfrak{F}_1^+$ and $\mathfrak{F}_{1s}^+$ denote respectively the nonnegative functions in $\mathfrak{F}_1$ and $\mathfrak{F}_{1s}$.

Let g be an element of F_{1s}^+ with $J(g)=1$ and let D^λ be defined for $f \in \mathfrak{F}_1$ and $x \in \mathfrak{G}$ by

$$D^\lambda f(x) = J(f)\, U^\lambda g(0) - U^\lambda f(x).$$

We have first

PROPOSITION 6.1. *For* $f \in \mathfrak{F}_{1s}$

$$\lim_{\lambda \uparrow 1} (D^\lambda f(y) + D^\lambda f(-y)) \tag{6.1}$$

exists and is finite and the convergence is uniform for y *in compacts.*

Proof. For $0 < \lambda < 1$

$$D^\lambda f(y) + D^\lambda f(-y) = 2\lambda \int \frac{\hat{g}(-\theta)\, J(f) - \Re\langle y, \theta\rangle \hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)}\, \hat{\mu}(\theta)\, d\theta.$$

By Theorem 3.1 we need only prove that if P is a relatively compact neighborhood of the origin of $\hat{\mathfrak{G}}$, then

$$\lim_{\lambda \uparrow 1} \int_P \frac{1 - \Re\langle y, \theta\rangle}{1-\lambda\hat{\mu}(\theta)}\, d\theta$$

exists and is finite and the convergence is uniform for y in compacts. But this result also follows easily from Theorem 3.1, as desired.

Let $\mathfrak{G}_1$ be an open compactly generated subgroup of $\mathfrak{G}$. Let ψ, φ, and Λ be as in Theorem 3.3. Let $C \subseteq \Lambda$ be a compact neighborhood of the origin of $\hat{\mathfrak{G}}$.

Set
$$d^\lambda = \lambda \int_C \frac{i\varphi(\theta)}{1-\lambda\hat{\mu}(\theta)}\,\hat{\mu}(\theta)\,d\theta, \quad 0<\lambda<1.$$

Also for $0<\lambda<1$ and $f \in \mathfrak{F}$, define the function $D_0^\lambda f(x)$, $x \in \mathfrak{G}_1$, by

$$D_0^\lambda f(x) = D^\lambda f(x) + (\psi(x)\cdot d^\lambda)\,J(f).$$

PROPOSITION 6.2. *For $f \in \mathfrak{F}_{1s}$*

$$\lim_{\lambda \uparrow 1} D_0^\lambda f(x) = D_0 f(x), \quad x \in \mathfrak{G}_1, \tag{6.2}$$

exists and is finite and the convergence is uniform for x in compact subsets of $\mathfrak{G}_1$.

Proof. We have that for $x \in \mathfrak{G}_1$

$$D_0^\lambda f(x) = \lambda \int_{\theta \notin C} \left(\frac{\hat{g}(-\theta)\,J(f) - \langle x,\theta\rangle \hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)}\right)\hat{\mu}(\theta)\,d\theta$$
$$+\lambda \int_C \left(\frac{\hat{g}(-\theta)\,J(f) - e^{i\psi(x)\cdot\varphi(\theta)}\hat{f}(-\theta) + i\psi(x)\cdot\varphi(\theta)\,J(f)}{1-\lambda\hat{\mu}(\theta)}\right)\hat{\mu}(\theta)\,d\theta.$$

The first term causes no problems. To study the second term it suffices to study

$$\int_C \left(\frac{1+i\psi(x)\cdot\varphi(\theta) - e^{i\psi(x)\cdot\varphi(\theta)}}{1-\lambda\hat{\mu}(\theta)}\right)\hat{\mu}(\theta)d\theta.$$

Let E be a compact subset of $\mathfrak{G}_1$. There is an M, $0<M<\infty$ such that for $x \in F$ and $\theta \in C$

$$\left|e^{i\psi(x)\cdot\varphi(\theta)} - 1 - i\psi(x)\cdot\varphi(\theta)\right| \leqslant M(1-\Re\, e^{i\psi(x)\cdot\varphi(\theta)}) = M(1-\Re\langle x,\theta\rangle).$$

It now follows from Theorem 2.1 that

$$\lim_{\lambda\uparrow 1}\int_C \left(\frac{1+i\psi(x)\cdot\varphi(\theta)\,e^{i\psi(x)\cdot\varphi(\theta)}}{1-\lambda\hat{\mu}(\theta)}\right)\hat{\mu}(\theta)$$

exists and is finite and the convergence is uniform for $x \in E$. This completes the proof of Proposition 6.2.

Let B be relatively compact subset of $\mathfrak{G}_1$ having a nonempty interior. Then (5.12) holds and consequently for $0<\lambda<1$, $f \in \mathfrak{F}_1$, and $x \in \mathfrak{G}_1$

$$D_0^\lambda f(x) - \Pi_B^\lambda D_0^\lambda f(x) = -U_B^\lambda f(x) + J(f)\left(L_B^\lambda(x) + d^\lambda \cdot \int (\psi(x) - \psi(z))\, \Pi_B^\lambda(x, dz)\right).$$

Since we can choose $f \in \mathfrak{F}_1$ such that $J(f) \neq 0$, we get immediately from Proposition 6.2

PROPOSITION 6.3. *For* $x \in \mathfrak{G}_1$

$$\lim_{\lambda \uparrow 1} \left(L_B^\lambda(x) + d^\lambda \cdot \int (\psi(x) - \psi(z))\, \Pi_B^\lambda(x, dz)\right) \tag{6.3}$$

exists and is finite. Also the convergence is uniform for x in compact subsets of $\mathfrak{G}_1$.

Next we will obtain

PROPOSITION 6.4. *d^λ is bounded in λ, $0 \leqslant \lambda < 1$.*

Proof. There is an M, $0 < M < \infty$, such that

$$\left|\int_B \psi(z)\, \Pi_B^\lambda(x, dz)\right| \leqslant M, \quad 0 \leqslant \lambda < 1, x \in \mathfrak{G}_1.$$

Suppose d^λ is unbounded. Then there exist $\lambda_n \uparrow 1$ as $n \to \infty$ such that $|d^{\lambda_n}| \to \infty$ and

$$\frac{d^{\lambda_n}}{|d^{\lambda_n}|} \to d, \quad |d| = 1.$$

We can choose $x \in \mathfrak{G}_1$ such that $\psi(x) \cdot d > M$. Then

$$\lim_{n\to\infty} \frac{1}{|d^{\lambda_n}|}\left(\psi(x) \cdot d^{\lambda_n} - d^{\lambda_n} \cdot \int_B \psi(z)\, \Pi_B^\lambda(y, dz)\right) > 0$$

and hence
$$\lim_{n\to\infty} d^{\lambda_n} \cdot \int (\psi(x) - \psi(z))\, \Pi_B^\lambda(x, dz) = +\infty.$$

Since $L_B^\lambda(x) \geqslant 0$ for $0 \leqslant \lambda < 1$ and $x \in \mathfrak{G}_1$, this contradicts Proposition 6.3.

Choose $\lambda_n \uparrow 1$ and d such that

$$\lim_{n\to\infty} d^{\lambda_n} = d. \tag{6.4}$$

Clearly d is finite. By Proposition 6.3

$$\lim_{n\to\infty} L_B^{\lambda_n}(x) = L_B(x), \quad x \in \mathfrak{G}_1.$$

exists and is finite and the convergence is uniform on compact subsets of $\mathfrak{G}_1$.

Next we will obtain

LEMMA 6.1. *For* $f \in \mathfrak{F}_{1s}$

$$\lim_{n\to\infty} D^{\lambda_n} f_y(x) = Df_y(x), \quad x, y \in \mathfrak{G}, \tag{6.5}$$

exists and is finite and the convergence is uniform for x and y in compacts.

Proof. Since $D^\lambda f_y(x) = D^\lambda f(x-y)$, it suffices to prove the result for $y=0$. This result follows from the definition of D_0^λ, Proposition 6.2, equation (6.4) and the fact that $\mathfrak{G}$ is σ-compact since μ defines a recurrent random walk on $\mathfrak{G}$.

LEMMA 6.2. *For* $x \in \mathfrak{G}$

$$\lim_{n\to\infty} L_B^{\lambda_n}(x) = L_B(x) \tag{6.6}$$

exists and is finite and the convergence is uniform for x in compacts. For $f \in \mathfrak{F}_{1s}$, $x \in \mathfrak{G}$, *and* $y \in \mathfrak{G}$

$$Df_y(x) - \Pi_B Df_y(x) = -U_B f_y(x) + L_B(x) J(f). \tag{6.7}$$

Proof. The first statement follows from equation (5.12) and Lemma 6.1, since $f \in \mathfrak{F}_{1s}$ can be chosen so that $J(f) \neq 0$. The remainder of the lemma now follows immediately.

Let $f \in \mathfrak{F}_1$ be supported by a compact set C having a nonempty interior. Then for $0 \leq \lambda < 1$

$$U^\lambda f(y) = E_y \lambda^{V_C}(f(S_{V_C}) + U^\lambda f(S_{V_C})). \tag{6.8}$$

LEMMA 6.3. *Let* $f \in \mathfrak{F}_{1s}$, C_1 *a compact subset of* $\mathfrak{G}$ *and* C *a compact subset of* $\mathfrak{G}$ *having a nonempty interior and containing the support of* $f(-y)$, $y \in \mathfrak{G}$, *and of* $f(x-y)$, $y \in C_1$. *Then*

$$Df_y(x) - Df_y(0) = E_y(f(S_{V_C}) - f(x+S_{V_C})) + E_y(Df(x+S_{V_C}) - Df(S_{V_C})). \tag{6.9}$$

Proof. By (6.8) for $x \in C_1$

$$\begin{aligned} D^\lambda f_y(x) - D^\lambda f_y(0) &= U^\lambda f(-y) - U^\lambda f(x-y) \\ &= E_{-y}\lambda^{V_C}(f(S_{V_C}) - f(x+S_{V_C})) + E_{-y}\lambda^{V_C}(U^\lambda f(S_{V_C}) - U^\lambda f(x+S_{V_C})) \\ &= E_{-y}\lambda^{V_C}(f(S_{V_C}) - f(x+S_{V_C})) + E_{-y}\lambda^{V_C}(D^\lambda f(x+S_{V_C}) - D^\lambda f(S_{V_C})), \end{aligned}$$

and the desired result now follows from Lemma 6.1.

LEMMA 6.4. *For* $f \in \mathfrak{F}_{1s}$ *and* C_1 *a compact subset of* $\mathfrak{G}$ *there is an* M, $0 < M < \infty$, *such that*

$$|Df_y(x) - Df_y(0)| \leq M, \quad y \in \mathfrak{G} \text{ and } x \in C_1. \tag{6.10}$$

Proof. The result follows immediately from Lemmas 6.1 and 6.3.

LEMMA 6.5. *For* $f \in \mathfrak{F}_{1s}$

$$\Pi_B Df_y(x) - Df_y(0)$$

is bounded uniformly for $x \in \mathfrak{G}$ *and* $y \in \mathfrak{G}$.

Proof. Since

$$\Pi_B Df_y(x) - Df_y(0) = \int \Pi_B(x, dz)\,(Df_y(z) - Df_y(0)),$$

the desired result follows from Lemma 6.4.

From (6.7) we see that for $f \in \mathfrak{F}_{1s}$, $x \in \mathfrak{G}$, and $y \in \mathfrak{G}$

$$Df_y(x) - Df(x) - Df_y(0) = \Pi_B Df_y(x) - Df_y(0) - \Pi_B Df(x) - U_B f_y(x) + U_B f(x). \tag{6.11}$$

We study now the right side of (6.11). It follows from Lemma 6.4 that $\Pi_B Df_y(x) - Df_y(0)$ is bounded uniformly for $x \in \mathfrak{G}$ and $y \in \mathfrak{G}$. Also $\Pi_B Df(x)$ is bounded for $x \in \mathfrak{G}$, since B is relatively compact and Df is bounded on compacts. Clearly $U_B f(x)$ is bounded for $x \in \mathfrak{G}$.

PROPOSITION 6.4.1. *If* $f \in \mathfrak{F}_1^+$ *and* $J(f) > 0$, *then*

$$\lim_{y \to \infty} U_B f_y(y) = \infty \tag{6.12}$$

Proof. Note that

$$U_B f_y(y) = E_y \sum_1^{V_B} f_y(S_n) = E_y \sum_1^{V_B} f(S_n - y) = E_0 \sum_1^{V_{B-y}} f(S_n). \tag{6.13}$$

With probability one

$$\lim_{y \to \infty} V_{B-y} = \infty. \tag{6.14}$$

If $f \in \mathfrak{F}_1^+$ and $J(f) > 0$, then with probability one

$$\sum_1^{\infty} f(S_n) = \infty. \tag{6.15}$$

Proposition 6.4.1 follows from (6.13)—(6.15).

PROPOSITION 6.5. *If* $f \in \mathfrak{F}_{1s}^+$ *and* $J(f) > 0$, *then*

$$\lim_{y \to \infty} \lim_{\lambda \uparrow 1} (D^\lambda f(y) + D^\lambda f(-y)) = \infty. \tag{6.16}$$

Proof. From (6.11) we have that

$$Df(y) + Df(-y) = Df(0) - (\Pi_B Df_y(y) - Df_y(0)) + \Pi_B Df(y) + U_B f_y(y) - U_B f(y).$$

From the discussion following (6.11) and from Proposition 6.4.1 we have that

$$\lim_{y\to\infty} (Df(y) + Df(-y)) = \infty. \tag{6.17}$$

The desired result now follows from Proposition 6.1 and Lemma 6.1.

THEOREM 6.1. *Let P be a neighborhood of the origin of $\mathfrak{G}$. Then*

$$\int_P \Re\left(\frac{1}{1-\hat{\mu}(\theta)}\right) d\theta = \infty. \tag{6.18}$$

Proof. By the proof of Proposition 6.1 it suffices to find $f\in\mathfrak{F}_{1s}$ with $J(f)>0$ such that (6.16) holds. The desired result now follows from Proposition 6.5, since there exist $f\in\mathfrak{F}_{1s}^+$ with $J(f)>0$.

LEMMA 6.6. *If $f\in\mathfrak{F}_{1s}^+$, then $Df(x)$, $x\in\mathfrak{G}$, is bounded from below.*

Proof. Let C be a compact set having a nonempty interior and containing the support of f. Then by Lemma 6.1 there is a finite constant M such that

$$U^{\lambda_n}f(0) - U^{\lambda_n}f(y) \geqslant -M, \quad n\geqslant 1 \text{ and } y\in C. \tag{6.19}$$

We can also assume that

$$f(y) \leqslant M, \quad y\in\mathfrak{G}. \tag{6.20}$$

Then by (6.8), (6.19), and (6.20) for $x\in\mathfrak{G}$

$$\begin{aligned} U^{\lambda_n}f(0) - U^{\lambda_n}f(x) &= U^{\lambda_n}f(0) - E_x\lambda_n^{V_C}f(S_{V_C}) - E_x\lambda_n^{V_C}U^{\lambda_n}f(S_{V_C}) \\ &\geqslant E_x\lambda_n^{V_C}(U^{\lambda_n}f(0) - U^{\lambda_n}f(S_{V_C})) - M \geqslant -2M. \end{aligned}$$

Thus by Lemma 6.1

$$Df(x) - Df(0) \geqslant -2M, \quad x\in\mathfrak{G},$$

from which the desired result follows.

LEMMA 6.7. *Suppose all elements of $\mathfrak{G}$ are compact elements. Then for $f\in\mathfrak{F}_{1s}$*

$$\lim_{y\to\infty} (Df_y(x) - Df_y(0)) = 0, \quad x\in\mathfrak{G}, \tag{6.21}$$

and the convergence is uniform for x in compacts.

Proof. Let C_1 be a compact subset of $\mathfrak{G}$. Let C be an open compactly generated subgroup of $\mathfrak{G}$ containing C_1 and the support of f. Then C is compact and Lemma 6.3 is applicable. Note that for $x\in C_1$ and $y\in\mathfrak{G}$

$$E_y f(x+S_{V_C}) = E_x f(y+S_{V_{C+x-y}}) = E_x f(y+S_{V_{C-y}}).$$

For y large, the random walk starting at x will, with probability close to one, hit any given neighborhood of the origin before hitting $C-y$. For this reason, since f is continuous,

$$\lim_{y\to\infty} E_y(f(x+S_{V_C}) - f(S_{V_C})) = 0$$

uniformly for x in compacts Similarly

$$\lim_{y\to\infty} E_y(Df(x+S_{V_C}) - Df(S_{V_C})) = 0$$

uniformly for x in compacts. The desired result now follows from Lemma 6.3.

PROPOSITION 6.6. *Suppose that $\mathfrak{G}_1$ is a noncompact open compactly generated subgroup of $\mathfrak{G}$ and that C is a compact subset of $\mathfrak{G}_1$. Then*

$$\lim_{y\to\infty} \Pi_{\mathfrak{G}_1}(y, C) = 0. \tag{6.22}$$

Proof. The result says that for y large the hitting distribution of $\mathfrak{G}_1$ for a random walk starting at y should be concentrated near infinity. If $\mathfrak{G}/\mathfrak{G}_1$ is finite the result is obvious (and not needed). Suppose $\mathfrak{G}/\mathfrak{G}_1$ is infinite so that

$$\mathfrak{G} = \bigcup_{n=1}^{\infty} (\alpha_n + \mathfrak{G}_1),$$

where $\alpha_n + \mathfrak{G}_1, n \geq 1$, are disjoint subsets of $\mathfrak{G}$. Let N be a positive integer. If $y\to\infty$ within

$$\bigcup_1^N (\alpha_n + \mathfrak{G}_1),$$

the result is again obvious.

On the other hand, choose $x_1, \ldots, x_k$ in $\mathfrak{G}_1$ and $P \subseteq \mathfrak{G}_1$ an open neighborhood of the origin such that the sets

$$C,\ x_1 + C + P,\ \ldots,\ x_k + C + P$$

are disjoint. Then N can be chosen large enough so that if $y \in \bigcup_{N+1}^{\infty}(\alpha_n + \mathfrak{G}_1)$, then for $1 \leq j \leq k$ the probability that a random walk starting from y will hit $y + x_j + P$ before hitting $\mathfrak{G}_1$ is at least $\frac{1}{2}$. We will then have that for such y for $1 \leq j \leq k$

$$\Pi_{\mathfrak{G}_1}(y, C) \leq 2\, \Pi_{\mathfrak{G}_1}(y, x_j + P + C)$$

so that
$$\Pi_{\mathfrak{G}_1}(y, C) \leq 2/k, \quad y \in \bigcup_{N+1}^{\infty} (\alpha_n + \mathfrak{G}_1).$$

Since k can be made arbitrarily small, this completes the proof of (6.22).

PROPOSITION 6.7. *Suppose $\mathfrak{G}$ has noncompact elements. Then there exist s_1 and s_2 in the support S of μ such that $s_1 - s_2$ is not a compact element.*

Proof. Let x be a noncompact element of $\mathfrak{G}$. It follows easily from the structure theorem that $nx \to \infty$ as $x \to \infty$. Let c_n be compact elements of $\mathfrak{G}$. Then $nx + c_n \to \infty$ as $n \to \infty$. Otherwise we could find $n_j \to \infty$ as $j \to \infty$ and a compact set C such that $n_j x + c_{n_j} \in C$, $n \geqslant 1$. Let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$ containing x and C. Then c_{n_j} and $\mathfrak{G}_1$. Thus the c_{n_j} lie in the compact set H of compact elements of $\mathfrak{G}_1$. Consequently the $n_j x$ lie in the compact set $C - H$, which contradicts the fact that $nx \to \infty$ as $n \to \infty$.

If all elements of S were compact, then S would lie in the proper closed subgroup consisting of the compact elements of $\mathfrak{G}$. Thus S has noncompact elements.

Suppose all elements in S are of the form $x + c$ where c is a compact element and x is a fixed noncompact element. If S_n is the random walk, then with probability one $S_n = nx + c_n$, where c_n is compact and $S_n \to \infty$ as $n \to \infty$. But this contradicts the fact that the random walk is recurrent. Thus we have eliminated all possibilities other than the conclusion of Proposition 6.7.

PROPOSITION 6.8. *Let $\mathfrak{G}_1$ be an open compactly generated subgroup of $\mathfrak{G}$ which is isomorphic to either $R \oplus H$ or $Z \oplus H$, with H compact, and which contains elements s_1 and s_2 in the support S of μ such that $s_1 - s_2$ is not compact. Also let $\mathfrak{G}/\mathfrak{G}_1$ be infinite. Let ξ_n be the embedded random walk on $\mathfrak{G}_1$. Then the one-dimensional random walk $\psi(\xi_n)$ has infinite variance.*

Proof. As observed by Kesten and Spitzer [5], this result depends on the fact that the projection of the random walk on $\mathfrak{G}/\mathfrak{G}_1$ is nullrecurrent, and hence has infinite mean recurrence time. It follows that the number T of jumps having values in $\mathfrak{G}_1$ before the first return to $\mathfrak{G}_1$ has infinite mean. Let σ_1^2 denote the conditional variance of $\psi(S_1)$ given that $S_1 \subseteq \mathfrak{G}_1$ (starting at the origin). Then $0 < \sigma_1^2 \leqslant \infty$.

It is clear that for a random walk starting at the origin

$$\operatorname{Var}[\psi(\xi) \mid T] \geqslant \sigma_1^2 T.$$

Since $ET = \infty$, it follows that $\operatorname{Var} \psi(\xi_1) = \infty$, as desired.

PROPOSITION 6.9. *Let $\mathfrak{G}$ be neither compactly generated nor consisting exclusively of compact elements. Then for an $A \in \mathcal{A}$ having a nonempty interior the hitting distribution of A for a random walk starting at y has a limit as $y \to \infty$.*

Proof. Let $\mathfrak{G}_1$ be an open compactly generated subgroup of $\mathfrak{G}_1$ containing A and containing points s_1 and s_2 in S such that $s_1 - s_2$ is not compact. This is possible by Proposition 6.7. Now $\mathfrak{G}/\mathfrak{G}_1$ is infinite. By Proposition 6.8 the embedded random walk on $\mathfrak{G}_1$ is of Type I and hence for this embedded random walk by Corollary 5.1, the hitting distribution of A has a limit distribution as the starting point y approaches infinity. Proposition 6.9 now follows immediately from Proposition 6.6.

LEMMA 6.8. *Suppose $\mathfrak{G}$ is not compactly generated. Then for $f\in\mathfrak{F}_{1s}$*

$$\lim_{y\to\infty} (Df_y(x)-Df_y(0))=0, \quad x\in\mathfrak{G}, \tag{6.23}$$

and the convergence is uniform for x in compacts.

Proof. We can suppose that $\mathfrak{G}$ has noncompact elements, for otherwise Lemma 6.7 is applicable. From Lemma 6.3 and Proposition 6.9 it follows that for $f\in\mathfrak{F}_{1s}$

$$\lim_{y\to\infty} (Df_y(x)-Df_y(0))=g(x), \quad x\in\mathfrak{G}, \tag{6.24}$$

and the convergence is uniform for x in compacts.

To complete the proof of Lemma 6.8, we will show that $g(x)=0$, $x\in\mathfrak{G}$. Suppose first that x is a compact element. Then there is a compact set C such that $nx\in C$ for $n\geqslant 1$. It follows from (6.24) that

$$\lim_{y\to\infty} (Df_y(nx)-Df_y(0))=ng(x).$$

Lemma 6.4 yields that $g(x)=0$.

Suppose now that x is noncompact. Then $nx\to\infty$ as $|n|\to\infty$ and by (6.4),

$$\lim_{|n|\to\infty} (Df((n+1)x)-Df(nx))=c(x).$$

Lemma 6.6 now yields that $c(x)=0$.

LEMMA 6.9. *Suppose $\mathfrak{G}$ is not compactly generated. Then for $f\in\mathfrak{F}_{1s}$*

$$\lim_{y\to\infty} U_B f_y(x)=L_B(x)J(f), \quad x\in\mathfrak{G}, \tag{6.25}$$

and the convergence is uniform for x in compacts.

Proof. From Lemma 6.2 we have that

$$Df_y(x)-Df_y(0)-\Pi_B(Df_y-Df_y(0))(x) = -U_Bf_y(x)+L_B(x)J(f)$$

and the desired result now follows from Lemma 6.8.

It follows from Lemma 6.9 that the value of $L_B(x)$ is independent of the choice of λ_n. Thus from Proposition 6.3 the value of d is independent of the choice of λ_n, and hence by

Proposition 6.2 for $f \in \mathfrak{F}_{1s}$ the value of $Df(x)$, $x \in \mathfrak{G}$, is independent of the choice of λ_n. In other words we have

THEOREM 6.2. *Suppose $\mathfrak{G}$ is not compactly generated. Then*

$$\lim_{\lambda \uparrow 1} d^{\lambda} = d \tag{6.26}$$

exists and is finite. Also

$$\lim_{\lambda \uparrow 1} L_B^{\lambda}(x) = L_B(x), \quad x \in \mathfrak{G} \tag{6.27}$$

exists and is finite. If $f \in \mathfrak{F}_{1s}$, then

$$\lim_{\lambda \uparrow 1} D^{\lambda} f_y(x) = Df_y(x), \quad x, y \in \mathfrak{G} \tag{6.28}$$

exists and is finite and

$$\lim_{y \to \infty} (Df_y(x) - Df_y(0)) = 0, \quad x \in \mathfrak{G}. \tag{6.29}$$

The limits in (6.27)—(6.29) *hold uniformly on compacts.*

We finish this section with the main result.

THEOREM 6.3. *Suppose $\mathfrak{G}$ is not compactly generated. Let C be a relatively compact neighborhood of the origin of $\hat{\mathfrak{G}}$. Then for $x \in \mathfrak{G}$*

$$\lim_{\lambda \uparrow 1} \int_C \frac{\langle x, \theta \rangle - 1}{1 - \lambda \hat{\mu}(\theta)} \, d\theta \tag{6.30}$$

exists and is finite and

$$\lim_{y \to \infty} \lim_{\lambda \uparrow 1} \int_C \frac{\langle y, \theta \rangle (\langle x, \theta \rangle - 1)}{1 - \lambda \hat{\mu}(\theta)} \, d\theta = 0. \tag{6.31}$$

The convergence in both limits is uniform for x in compacts.

Proof. This theorem follows from Theorem 3.1, Theorem 6.2 and the formulas for $D^{\lambda} f(x)$ and $D^{\lambda} f_y(x) - D^{\lambda} f_y(0)$ given in the proof of Theorem 5.1.

7. Further results in the nonsingular case

We continue the assumptions of § 6, namely that μ defines a recurrent random walk on $\mathfrak{G}$ and that some iterate of μ is nonsingular. The operator D^{λ} can be written as

$$D^{\lambda} f(x) = \int D^{\lambda}(x, dy) f(y)$$

in the obvious way.

THEOREM 7.1. *The measures* $D^\lambda(x, dy)$ *can be written as*

$$D^\lambda(x, dy) = a^\lambda(y-x)\,dy - U_2^\lambda(x, dy), \tag{7.1}$$

where $U_2^\lambda(x, dy) = \mu_2^\lambda(dy - x)$, μ_2^λ *being a finite positive measure that increases as* $\lambda \uparrow 1$ *to a finite positive measure* μ_2,

$$\lim_{\lambda \uparrow 1} a^\lambda(x) = a(x) \tag{7.2}$$

exists and is finite, in the Type I case

$$\lim_{y \to \infty} (a(y-x) - a(y)) = 0, \tag{7.3}$$

and in the Type II case

$$\lim_{y \to \pm\infty} (a(y-x) - a(y)) = \mp \sigma^{-2} \psi(x). \tag{7.4}$$

The convergence in (7.2)–(7.4) *is uniform for x in compacts.*

Proof. We can find an n_0 and probability measures φ and ψ such that $\mu^{(n_0)} = (\varphi + \psi)/2$ and φ has compact support, absolutely integrable characteristic function and continuous density. The remainder of the proof is, with obvious modifications, that of Theorem 4.1 of [11].

Recall that Φ denotes the collection of bounded Borel functions. We have immediately from the above theorem (in the nonsingular case)

COROLLARY 7.1. *Theorem 5.3, Theorem 5.8, and Corollary 5.2 hold when* $\mathfrak{F}$ *is replaced by* Φ.

PROPOSITION 7.1. *Let* $B \in \mathfrak{B}$ *have positive measure. Then*

$$\lim_{n \to \infty} P_x(V_B \geqslant n) = 0 \tag{7.5}$$

uniformly for x in compacts.

Proof. It suffices to find a nonempty open set P, a positive integer n and a $c > 0$ such that

$$\mu^{(n)}(B - x) \geqslant c, \quad x \in P. \tag{7.6}$$

To this end, we can find a positive integer n, an $x_0 \in \mathfrak{G}$, a nonempty open neighborhood R of the origin of $\mathfrak{G}$, and an $\alpha > 0$ such that

$$\mu^{(n-1)}(dy) > \alpha dy, \quad y \in x_0 + R.$$

Let Q be a nonempty open set such that $Q - Q \subseteq R$. Since

$$\int_{\mathfrak{G}} \mu(B-y)\,dy = |B| > 0$$

there is an $x_1 \in \mathfrak{G}$ such that

$$\int_{x_1+Q} \mu(B-y)\,dy = \beta > 0.$$

Set $P = x_1 - x_0 + Q$ and $c = \alpha\beta$. Then for $x \in P$

$$\mu^{(n)}(B-x) \geqslant \int_{x_1+Q} \mu^{(n-1)}(dy-x)\,\mu(B-y) \geqslant \alpha \int_{x_1+Q} \mu(B-y)\,dy = \alpha\beta = c$$

and (7.6) holds, as desired.

COROLLARY 7.2. *Let $B \in \mathcal{B}$ have positive measure. Then for $A \in \mathcal{B}$*

$$\lim_{\lambda \uparrow 1} U_B^\lambda(x, A) = U_B(x, A) \tag{7.7}$$

exists and is finite and the convergence is uniform for x in compacts.

Let $B \in \mathcal{B}$ have positive measure. Then as before we have the identity

$$D^\lambda(x, dy) - \int \Pi_B^\lambda(x, dz)\,D^\lambda(z, dy) = -U_B^\lambda(x, dy) + L_B^\lambda(x)\,dy. \tag{7.8}$$

From (7.8), Theorem 7.1 and Corollary 7.2 we have

COROLLARY 7.3. *Let $B \in \mathcal{B}$ have positive measure. Then*

$$\lim_{\lambda \uparrow 1} L_B^\lambda(x) = L_B(x), \quad x \in \mathfrak{G}, \tag{7.9}$$

exists and is finite and the convergence is uniform on compacts.

Set
$$u_B^\lambda(x, y) = L_B^\lambda(x) - a^\lambda(y-x) + \int \Pi_B^\lambda(x, dz)\,a^\lambda(y-z) \tag{7.10}$$

and
$${}_2U_B^\lambda(x, dy) = U_2^\lambda(x, dy) - \int \Pi_B^\lambda(x, dz)\,U_2^\lambda(z, dy). \tag{7.11}$$

Then
$$U_B^\lambda(x, dy) = u_B^\lambda(x, y)\,dy + {}_2U_B^\lambda(x, dy). \tag{7.12}$$

Clearly we have that, as $\lambda \uparrow 1$, ${}_2U_B^\lambda(x, dy)$ approaches

$${}_2U_B(x, dy) = U_2(x, dy) - \int \Pi_B(x, dz)\,U_2(z, dy) \tag{7.13}$$

in the sense that for $A \in \mathcal{B}$

$$\lim_{\lambda \uparrow 1} {}_2U_B^\lambda(x, A) = {}_2U_B(x, A) \tag{7.14}$$

and the convergence is uniform for x in compacts. Moreover, from (7.10)

$$\lim_{\lambda \uparrow 1} u_B^\lambda(x, y) = u_B(x, y), \tag{7.15}$$

the convergence being uniform for x and y in compacts, where

$$a(y-x) - \int \Pi_B(x, dz)\, a(y-z) = u_B(x, y) + L_B(x). \tag{7.16}$$

Thus from Theorem 7.1 we get

THEOREM 7.2. *Let $B \in \mathcal{B}$ have positive measure. Then*

$$U_B(x, dy) = u_B(x, y)\, dy + {}_2U_B(x, dy),$$

where for $A \in \mathcal{B}$

$$\lim_{y \to \infty} {}_2U_B(x, y + A) = 0, \tag{7.17}$$

in the Type I case

$$\lim_{y \to \infty} u_B(x, y) = L_B(x), \tag{7.18}$$

and in the Type II case

$$\lim_{y \to \pm\infty} u_B(x, y) = L_B^{\pm}(x). \tag{7.19}$$

The convergence in (7.17) – (7.19) is uniform for x in compacts.

COROLLARY 7.4. *Let $A \in \mathcal{B}$, $B \in \mathcal{B}$ and B have positive measure. Then the conclusion of Theorem 5.5 holds.*

Next we will prove

THEOREM 7.3. *Let $A \in \mathcal{B}$, $B \in \mathcal{B}$ and B have positive measure. Then the conclusion of Theorem 5.7 holds.*

Proof. We give the proof in the Type I case, the proof in the Type II case being essentially the same. Applying duality to Theorem 7.3 we get that if A, B, $C \in \mathcal{B}$ and B has positive measure, then

$$\lim_{x \to \infty} \int_C U_B(x+z, A)\, dz = |C| \int_A \tilde{L}_B(x)\, dx. \tag{7.20}$$

Let $\mathfrak{G}_1$ be the group generated by $S - S$. Since some iterate of μ is nonsingular, it follows that $|\mathfrak{G}_1| > 0$. Let C be a relatively compact subset of $\mathfrak{G}_1$ having positive measure. It is easily seen that for any positive integer n

$$\lim_{x\to\infty}\left(\int \mu^{(n)}(dy-x)\,U_B(y,A) - U_B(x,A)\right) = 0. \tag{7.21}$$

There is a $0 < M < \infty$ such that

$$U_B(y,A) \leqslant M, \quad y \in \mathfrak{G}.$$

Thus

$$\left|\int \mu^{(n)}(dy-x-z)\,U_B(y,A) - \int \mu^{(n)}(dy-x)\,U_B(y,A)\right| \leqslant M \int \left|\mu^{(n)}(dy-z) - \mu^{(n)}(dy)\right|. \tag{7.22}$$

By the methods of Stone [18] it follows that

$$\lim_{n\to\infty} \int |\mu^{(n)}(dy-z) - \mu^{(n)}(dy)| = 0, \quad z \in \mathfrak{G}_1, \tag{7.23}$$

and that the convergence holds uniformly for z in compact subsets of $\mathfrak{G}_1$. It follows from (7.21)–(7.23) that

$$\lim_{x\to\infty} (U_B(x+z,A) - U_B(x,A)) = 0 \tag{7.24}$$

uniformly for z in compact subsets of $\mathfrak{G}_1$ and hence in particular uniformly for $z \in C$. From (7.20) and (7.24) we have that

$$\lim_{x\to\infty} U_B(x,A) = \int_A L_B(x)\,dx$$

as desired.

From Theorem 7.2 and Theorem 7.3 we get

THEOREM 7.4. *Let $B \in \mathcal{B}$ have positive measure and $f \in \Phi$. In the Type I case*

$$\lim_{y\to\infty} U_B f_y(x) = L_B(x)\,J(f) \tag{7.25}$$

and

$$\lim_{x\to\infty} U_B f(x) = \int \tilde{L}_B(x)\,f(x)\,dx. \tag{7.26}$$

In the Type II case

$$\lim_{y\to\pm\infty} U_B f_y(x) = L_B^{\pm}(x)\,J(f) \tag{7.27}$$

and

$$\lim_{x\to\pm\infty} U_B f(x) = \int \tilde{L}_B^{\pm}(x)\,f(x)\,dx. \tag{7.28}$$

The limits in (7.25) and (7.27) hold uniformly for x in compacts.

The proof of Theorem 5.10 now yields

THEOREM 7.5. *Let $B \in \mathfrak{B}$ have positive measure. In the Type I case there is a finite constant $k(B)$ such that for $f \in \Phi$*

$$\lim_{x \to \infty} (Df(x) - L_B(x) J(f)) = J(f)\, k(B). \tag{7.29}$$

In the Type II case there exist finite constants $k_B^{\pm}$ such that for $f \in \Phi$

$$\lim_{x \to \pm\infty} (Df(x) - L_B(x) J(f)) = J(f)\, k^{\pm}(B) \mp \sigma^{-2} K(f). \tag{7.30}$$

Extending Theorems 5.11 and 5.12, we have

THEOREM 7.6 *Let $B \in \mathfrak{B}$ have positive measure and let $f \in \Phi$. In the Type I case*

$$\lim_{x \to \infty} U_B f(x) = \int \tilde{L}_B(x) f(x)\, dx = -k(B) J(f) + \int_B \tilde{L}_B(x) Df(x)\, dx \tag{7.31}$$

and in the Type II case

$$\lim_{x \to \pm\infty} U_B f(x) = \int \tilde{L}_B^{\pm}(x) f(x)\, dx = -k^{\pm}(B) J(f) \pm \sigma^{-2} K(f) + \int_B \tilde{L}_B(x) Df(x)\, dx. \tag{7.32}$$

From Theorem 5.13 and the definition of $a^{\lambda}(x)$ in Port and Stone [11], we easily get

THEOREM 7.7. *Let C be a compact subset of $\mathfrak{G}$. Then there is an $0 < M < \infty$ such that*

$$|a^{\lambda}(y-x) - a^{\lambda}(y)| \leq M, \quad y \in \mathfrak{G},\ x \in C, \text{ and } 0 \leq \lambda \leq 1. \tag{7.33}$$

COROLLARY 7.5. *Let $f \in \Phi$ and let C be a compact subset of $\mathfrak{G}$. Then there is an $0 < M < \infty$ such that*

$$|D^{\lambda} f_y(x) - D^{\lambda} f_y(0)| \leq M \tag{7.34}$$

for $y \in \mathfrak{G}$, $x \in C$, and $0 \leq \lambda \leq 1$.

COROLLARY 7.6. *Let $f \in \Phi$ and let $B \in \mathfrak{B}$ with $|B| > 0$. Then*

$$\Pi_B Df_y(x) - Df_y(0)$$

is bounded uniformly for $x, y \in \mathfrak{G}$.

Using the formula for $a^{\lambda}(x)$ given in [11] together with the arguments that led up to Theorem 5.15 we get

THEOREM 7.8. *There is an $0 < M < \infty$ such that*

$$a^{\lambda}(x) \geq -M, \quad x \in \mathfrak{G} \text{ and } 0 \leq \lambda \leq 1. \tag{7.35}$$

COROLLARY 7.7. *Let $f\in\Phi$ with $J(f)\geqslant 0$. Then there is an $0<M<\infty$ such that*

$$D^\lambda f(x)\geqslant -M,\quad x\in\mathfrak{G} \text{ and } 0\leqslant\lambda\leqslant 1. \tag{7.36}$$

Proof. Note that

$$\int a^\lambda(y-x)\,f(y)\,dy=a^\lambda(-x)\,J(f)+\int(a^\lambda(y-x)-a^\lambda(-x))\,f(y)\,dy.$$

The result now follows from Theorems 7.1, 7.7, and 7.8.

COROLLARY 7.8. *Let $f\in\Phi$ with $J(f)=0$. Then there is an $0<M<\infty$ such that*

$$|U^\lambda f(x)|\leqslant M,\quad x\in\mathfrak{G} \text{ and } 0\leqslant\lambda\leqslant 1. \tag{7.37}$$

8. On replacing Abel summability by ordinary convergence

Set

$$U_n=\sum_{k=1}^{n}P^n$$

and for suitable g set $c_n=U_n g(0)$. Let D_n be defined for suitable f by

$$D_n f=c_n J(f)-U_n f=\sum_{k=1}^{n}(J(f)\,P^k g(0)-P^k f).$$

This should be compared with

$$D^\lambda f=\sum_{k=1}^{\infty}\lambda^k(J(f)\,P^k g(0)-P^k f).$$

Since Abel summability is weaker than ordinary convergence, convergence of $D^\lambda f$ doesn't immediately imply convergence of $D_n f$. Even though this sharpening is not needed in applications it is interesting enough to be worth while.

The key to such results, and what we will confine our attention to, is the following strengthening of the first part of Theorem 5.2.

THEOREM 8.1. *Let μ define a recurrent random walk on $\mathfrak{G}$ and let P be a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then*

$$\lim_{n\to\infty}\int_P\frac{(\langle x,\theta\rangle-1)\,(1-\hat\mu^n(\theta))}{1-\hat\mu(\theta)}\,d\theta \tag{8.1}$$

exists and is finite and the convergence is uniform for x in compacts.

In proving this result we let ν be the probability measure given by Theorem 3.2. We will first prove

LEMMA 8.1. *Let P be a relatively compact neighborhood of the origin of $\hat{\mathfrak{G}}$. Then*

$$\lim_{n\to\infty}\int_P \frac{(\langle x,\theta\rangle-1)\,(\hat{\mu}^n(\theta)-\hat{\nu}^n(\theta))}{1-\hat{\nu}(\theta)}=0 \tag{8.2}$$

uniformly for x in compacts.

Proof. Let $\mathfrak{G}_1$ be a compactly generated subgroup of $\mathfrak{G}$. Then $\mathfrak{G}_1=V\oplus L\oplus H$, where $V\cong R^{d_1}$, $L\cong Z^{d_2}$, and H is compact. Let $d=d_1+d_2$. We can assume that P is contained in the annihilator of H. Then there exist continuous functions ψ from $\mathfrak{G}_1$ to R^d and M from $\hat{\mathfrak{G}}$ to R^d such that

$$\langle x,\theta\rangle=e^{i\psi(x)\cdot M(\theta)},\quad x\in\mathfrak{G}_1 \text{ and } \theta\in P.$$

By Lemma 3.6 we can assume that $\mathfrak{G}_1$ is large enough so that if $S_1\subseteq R^d$ is the image under ψ of the restriction of the support of μ to $\mathfrak{G}_1$, then S_1-S_1 generates a d-dimensional closed subgroup of R^d. It follows that for some $0<c<\infty$

$$|\hat{\mu}(\theta)|\leqslant e^{-c|M(\theta)|^2},\quad \theta\in P. \tag{8.3}$$

Similarly we can assume that

$$|\hat{\nu}(\theta)|\leqslant e^{-c|M(\theta)|^2},\quad \theta\in P. \tag{8.4}$$

Then for $\theta\in P$

$$|\hat{\mu}^{n+1}(\theta)-\hat{\nu}^{n+1}(\theta)|\leqslant(n+1)\,|\hat{\mu}(\theta)-\hat{\nu}(\theta)|\,e^{-cn|M(\theta)|^2}. \tag{8.5}$$

We can also assume that P and $\mathfrak{G}_1$ are such that for some compact subset C_1 of $\mathfrak{G}_1$ and some $0<K_1<\infty$

$$|\hat{\mu}(\theta)-\hat{\nu}(\theta)|\leqslant K_1\,(\max_{x\in C_1}(1-\Re\langle x,\theta\rangle))^2,\quad \theta\in P.$$

Thus we can find a $0<K<\infty$ such that

$$\left|\frac{\hat{\mu}(\theta)-\hat{\nu}(\theta)}{1-\hat{\nu}(\theta)}\right|\leqslant K|M(\theta)|^2,\quad \theta\in P. \tag{8.6}$$

Let C be a compact subset of $\mathfrak{G}_1$. Then we can find an $0<L<\infty$ such that

$$|\langle x,\theta\rangle-1|\leqslant L|M(\theta)|,\quad x\in C \text{ and } \theta\in P. \tag{8.7}$$

Then by (8.5)–(8.7), for $x\in C$ and $\theta\in P$

$$\left|\frac{(\langle x,\theta\rangle-1)\,(\hat{\mu}^{(n+1)}(\theta)-\hat{\nu}^{(n+1)}(\theta))}{1-\hat{\nu}(\theta)}\right|\leqslant(n+1)\,KL\,|M(\theta)|^3\exp\,(-nc\,|M(\theta)|^2). \tag{8.8}$$

It is easily seen from elementary calculus that

$$\lim_{n\to\infty} \sup_{\theta\in P} (n+1)\,|M(\theta)|\, \exp\,(-nc\,|M(\theta)|^2)=0$$

and (8.8) now yields the conclusion of Lemma 8.1.

By Lemma 8.1 in order to prove Theorem 8.1 it suffices to prove

LEMMA 8.2. *Theorem 8.1 holds under the additional hypothesis that μ be nonsingular.*

Proof. The proof of Lemma 8.2 is similar to the methods used in Section 6. We start with the identity

$$P^n=\sum_{k=1}^{n} {}_B\Pi^k P^{n-k}+B^{P^n}$$

and sum on n to get

$$U_n=\sum_{k=1}^{n} {}_B\Pi^k U_{n-k}+{}_BU_n.$$

Then

$$D_n f(x)-\sum_{k=1}^{n} {}_B\Pi^k D_{n-k} f(x)=-{}_BU_n f(x)+L_B^{(n)}(x)\,J(f), \tag{8.9}$$

where

$$L^{(n)}(x)=\sum_{k=1}^{n} c_{n-k}\,{}_B\Pi^k(x,B).$$

The remainder of the proof will be omitted since it follows along the lines of Section 6 with (8.9) being used instead of (5.12).

9. Asymptotic behavior of the recurrent potential operator

In this section we study the asymptotic behavior of $Af(x)$ as $x\to\infty$. This subject was initiated by Kesten and Spitzer in [5]. The results in the recurrent case are analogous to those in the transient case, only less complete.

Throughout this section it will be assumed that μ defines a recurrent random walk on $\mathfrak{G}$. Then the results of §§ 5–7 are applicable.

THEOREM 9.1. *If $\mathfrak{G}\cong R\oplus H$ or $\mathfrak{G}\cong Z\oplus H$, where H is compact, then there is an $0\leqslant L\leqslant\infty$ such that for $f\in\mathfrak{F}$ with $J(f)>0$ either*

$$\lim_{x\to+\infty} Df(x)=LJ(f) \quad \textit{and} \quad \lim_{x\to-\infty} Df(x)=\infty \tag{9.1}$$

or

$$\lim_{x\to+\infty} Df(x)=\infty \quad \textit{and} \quad \lim_{x\to-\infty} Df(x)=LJ(f). \tag{9.2}$$

If $\mathfrak{G}$ is not of the above type then for $f\in\mathfrak{F}$ with $J(f)>0$

$$\lim_{x\to\infty} Df(x)=\infty. \tag{9.3}$$

The next result depends on the decomposition of Theorem 7.1.

THEOREM 9.2. *Let some $\mu^{(n)}$ be nonsingular. If $\mathfrak{G}\cong R\oplus H$ or $\mathfrak{G}\cong Z\oplus H$, where H is compact, there is an $0\leqslant L\leqslant\infty$ such that either*

$$\lim_{x\to-\infty} a(x)=L \quad \textit{and} \quad \lim_{x\to+\infty} a(x)=\infty \tag{9.4}$$

or

$$\lim_{x\to-\infty} a(x)=\infty \quad \textit{and} \quad \lim_{x\to+\infty} a(x)=L. \tag{9.5}$$

If $\mathfrak{G}$ in not of the above type then

$$\lim_{x\to\infty} a(x)=\infty. \tag{9.6}$$

COROLLARY 9.1. *Let some $\mu^{(n)}$ be nonsingular. Then Theorem 9.1 holds with $\mathfrak{F}$ replaced by Φ.*

THEOREM 9.3. *Let $B\in\mathcal{B}$ be such that $|B|>0$ and either $|\partial B|=0$ or some $\mu^{(n)}$ is nonsingular. If $\mathfrak{G}\cong R\oplus H$ or $\mathfrak{G}\cong Z\oplus H$, where H is compact, there is an $0\leqslant L_1\leqslant\infty$ such that either*

$$\lim_{x\to+\infty} L_B(x)=L_1 \quad \textit{and} \quad \lim_{x\to-\infty} L_B(x)=\infty \tag{9.7}$$

or

$$\lim_{x\to+\infty} L_B(x)=\infty \quad \textit{and} \quad \lim_{x\to-\infty} L_B(x)=L_1. \tag{9.8}$$

If $\mathfrak{G}$ is not of the above type then

$$\lim_{x\to\infty} L_B(x)=\infty. \tag{9.9}$$

Remark. Even if $\mathfrak{G}\cong R\oplus H$ or $\mathfrak{G}\cong Z\oplus H$, but provided that the random walk is of Type II, then $L=\infty$ in Theorems 9.1 and 9.2 and Corollary 9.1 and $L_1=\infty$ in Theorem 9.3.

LEMMA 9.1. *Suppose μ defines a Type I recurrent random walk and that some $\mu^{(n)}$ is nonsingular. Let f be continuous, nonnegative and have compact support and $J(f)>0$. Choose b such that $\inf_{y\in\mathfrak{G}} Af(y)>0$ where $Af=Df-f+bJ(f)$. Then for all $0<\varepsilon<\inf_{y\in\mathfrak{G}} Af(y)$, there is a compact set C and an $0<M<\infty$ such that for $x\in\mathfrak{G}$ and $y\in C^c$*

$$\left(\frac{Af(y)-\varepsilon}{Af(x)}\right)(Af(x)+Af(-y)-Af(x-y)-M)\leqslant\left(\frac{Af(y)+\varepsilon}{Af(y)}\right)(Af(y)+Af(-y)+M). \tag{9.10}$$

Proof. We start with the identity (see Theorem 10.1)

$$PAf=Af+f.$$

Let B be a compact set including the support of f. Let ${}_1Q_B$ be the subtransition probability operator on B^c given by

$$ {}_1Q_B(x, dy) = \frac{P(x, dy)\, Af(y)}{Af(x)}, \quad x, y \in B^c. $$

Note that
$$ {}_1Q_B(x, B^c) \leqslant \frac{PAf(x)}{Af(x)} = 1, \quad x \in B^c. $$

Let ${}_1Q_B^n$ denote the nth power of ${}_1Q_B$. It is easily seen that

$$ {}_1Q_B^n(x, dy) = \frac{{}_BP^n(x, dy)\, Af(y)}{Af(x)}, \quad x, y \in B^c. $$

Set ${}_1G_B = \sum_0^\infty {}_1Q_B^n$. Then
$$ {}_1G_B(x, dy) = \frac{G_B(x, dy)\, Af(y)}{Af(x)}, \quad x, y \in B^c. $$

Note that for $y \in \mathfrak{G}$, f_y is supported by $y + B$. It is probabilistically evident that

$$ {}_1G_B f_y(x) \leqslant \sup_{z \in B^c(y+B)} {}_1G_B f_y(z), \quad x \in B^c, y \in \mathfrak{G}. \tag{9.11} $$

For $y \in \mathfrak{G}$ and $z \in B^c$

$$ {}_1G_B f_y(z) = \frac{1}{Af(z)} \int_{B^c(y+C)} G_B(z, du)\, Af(u)\, f_y(u). $$

By Theorem 5.3 and Corollary 7.1

$$ Af(u) = Af(y) + o_y(1), \quad u \in y + C, $$

and hence
$$ {}_1G_B f_y(z) = \left(\frac{Af(y) + o_y(1)}{Af(z)} \right) G_B(1_{B^c} f_y)(z). \tag{9.12} $$

It follows from (6.11) that

$$ G_B(1_{B^c} f_y(x)) = Af(x) + Af(-y) - Af(x-y) + O(1) \tag{9.13} $$

for $x \in \mathfrak{G}$ and $y \in \mathfrak{G}$. From (9.11)—(9.13) we get that for $x \in B^c$, $y \in \mathfrak{G}$

$$ \frac{Af(y) + o_y(1)}{Af(x)} [Af(x) + Af(-y) - Af(x-y) + O(1)] $$
$$ \leqslant \left(\frac{Af(y) + o_y(1)}{Af(y)} \right) [Af(y) + Af(-y) + O(1)]. \tag{9.14} $$

It is obvious that (9.14) holds also for $x \in B$, since B is compact. The conclusion of the lemma follows easily from (9.14).

LEMMA 9.2. *Let μ and f be as in Lemma 9.1. Then either*

$$\lim_{x\to\infty} Af(x) = \infty \tag{9.15}$$

or there is an $0 \leqslant L < \infty$ having the property that for all $N > 0$ there is a compact set C such that for $x \notin C$ either

$$|Af(x) - L| \leqslant N^{-1} \quad or \quad Af(x) \geqslant N. \tag{9.16}$$

Proof. Suppose (9.15) doesn't hold. Let

$$L = \liminf_{x\to\infty} Af(x). \tag{9.17}$$

Then $0 \leqslant L < \infty$. Choose $y \in \mathfrak{G}$. We can find a sequence x_n such that, as $n \to \infty$, $Af(x_n) \to L$, $Af(-x_n) \to \infty$, and $Af(x_n) - Af(x_n - y) \to 0$. Substituting $x = x_n$ in (9.10) and letting $n \to \infty$, wet get that for $y \in C^c$

$$\frac{Af(y)-\varepsilon}{L}(Af(-y)-M) \leqslant \left(\frac{Af(y)+\varepsilon}{Af(y)}\right)(Af(y)+Af(-y)+M). \tag{9.18}$$

Choose $0 < N < \infty$. There is a compact set $C_1 \supseteq C$ such that if $y \notin C_1$ and $Af(y) < N$, then $Af(-y) - M > 0$ and

$$\frac{Af(y)+Af(-y)+M}{Af(-y)-M} \leqslant 1+\varepsilon$$

and hence

$$(Af(y)-\varepsilon) \leqslant L\frac{(1+\varepsilon)(Af(y)+\varepsilon)}{Af(y)}. \tag{9.19}$$

Since ε can be made arbitrarily small, Lemma 9.2 follows from (9.17)–(9.19).

LEMMA 9.3. *Let μ and f be as in Lemma 9.1, suppose that (9.15) doesn't hold, and let L be as in Lemma 9.2. If If $\mathfrak{G} \cong R \oplus H$ or $\mathfrak{G} \cong Z \oplus H$, where H is compact, then either*

$$\lim_{x\to+\infty} Af(x) = L \quad and \quad \lim_{x\to-\infty} Af(x) = \infty \tag{9.20}$$

or

$$\lim_{x\to+\infty} Af(x) = \infty \quad and \quad \lim_{x\to-\infty} Af(x) = L. \tag{9.21}$$

Proof. This result follows easily from Lemma 9.2 and the facts that

$$\lim_{x\to\infty} (Af(x) + Af(-x)) = \infty, \tag{9.22}$$

and

$$\lim_{x\to\infty} (Af(x-y) - Af(x)) = 0, \quad y \in \mathfrak{G}, \tag{9.23}$$

uniformly for y in compacts.

LEMMA 9.4. *Let μ and f be as in Lemma 9.1 and suppose that $\mathfrak{G}$ has an open noncompact compactly generated subgroup $\mathfrak{G}_1$ such that $\mathfrak{G}/\mathfrak{G}_1$ is infinite. Then (9.15) holds.*

Proof. Suppose that (9.15) doesn't hold and let L be as in Lemma 9.2. Let $z_n+\mathfrak{G}_1$ be disjoint for $n \geq 1$. Then $z_n \to \infty$ as $n \to \infty$. It follows from (9.18) and Lemma 9.2 that the z_n's can be chosen so that

$$\lim_{n\to\infty} Af(z_n) = L.$$

Since $\mathfrak{G}_1$ is compactly generated but not compact, we can choose $x \in \mathfrak{G}_1$ such that $nx \to \infty$ as $n \to \infty$. By (9.18), (9.22) and (9.23), we can suppose x is such that

$$\lim_{n\to\infty} Af(nx) = \infty.$$

Now $z_n+kx \to \infty$ as $n \to \infty$ uniformly in k. There is an $n_0>0$ such that for all $n \geq n_0$ there is a $k_n>0$ such that

$$Af(z_n+k_nx) < L+1$$

and

$$Af(z_n+(k_n+1)x) \geq L+1.$$

By (9.23)

$$\lim_{n\to\infty} Af(z+k_nx) = L+1,$$

which contradicts Lemma 9.2.

LEMMA 9.5. *Let μ and f be as in Lemma 9.1 and suppose that $\mathfrak{G}$ has a closed subgroup $\mathfrak{G}_1 \cong R^{d_1} \oplus Z^{d_2}$ where $d_1+d_2=2$. Then (9.15) holds.*

Proof. We can choose elements x and y in $\mathfrak{G}_1$ such that $nx+ky \to \infty$ as $n+k \to \infty$ and use the proof of Lemma 9.4.

LEMMA 9.6. *Let μ and f be as in Lemma 9.1 and suppose that every element of $\mathfrak{G}$ is a compact element. Then (9.15) holds.*

Proof. Suppose (9.15) doesn't hold. Let L be as in Lemma 9.2. Let C be an open compact subgroup of $\mathfrak{G}$ which contains the support of f. Then $\mathfrak{G}/C$ is infinite.

Let $z_n \in \mathfrak{G} \cap C^c$ be such that z_n+C are disjoint and $Af(z_n) \to L$. Necessarily $z_n \to \infty$ and $Af(-z_n) \to \infty$. Let

$$h_n = \min\{m \geq 1 \mid mz_n \in H\}.$$

Then $h_n>1$. Also $(h_n-1)z_n \in -z_n+H$ and hence $Af((h_n-1)z_n) \to \infty$ as $n \to \infty$. Thus $h_n>2$ for n sufficiently large.

For n sufficiently large we can choose k_n to be the largest positive integer less than h_n-1 such that $Af(k_nz_n) \leq 2L+2$. Then for n sufficiently large

$$Af(k_n z_n) \leqslant 2L+2 \leqslant Af(k_n z_n + z_n).$$

We have the identity (see equation (10.1))

$$Af(x) - H_C Af(X) = -G_C f(x) + \mathfrak{L}_C(x) J(f).$$

It follows that

$$\begin{aligned} Af(x+y) - Af(x) - Af(y) &= Af_{-y}(x) - Af(x) - Af_{-y}(0) \\ &= H_C A f_{-y}(x) - Af_{-y}(0) - H_C Af(x) - G_C f_{-y}(x) + G_C f(x). \end{aligned}$$

Since f is supported by C, $G_C f(x) \equiv 0$. Since f_{-y} and Af are nonnegative, $H_C Af(x) \geqslant 0$ and $G_C f_{-y}(x) \geqslant 0$. Since μ defines a Type I recurrent random walk

$$H_C Af_{-y}(x) - Af_{-y}(0) = o_y(1) \text{ as } y \to \infty$$

uniformly in x. Consequently

$$Af(x+y) - Af(x) - Af(y) \leqslant o_y(1).$$

Letting $y = z_n$ and $x = k_n z_n$ we get that

$$Af((k_n+1)z_n) - Af(k_n z_n) \leqslant Af(z_n) + o_n(1) = L + o_n(1).$$

Thus for n sufficiently large

$$L+1 \leqslant Af(k_n z_n) \leqslant 2L+2 \leqslant Af((k_n+1)z_n) \leqslant 3L+3.$$

Since either $k_n z_n$ or $(k_n+1)z_n$ has a subsequence which converges to infinity we have a contradiction to Lemma 9.1.

Proof of Corollary 9.1. In the Type II case the result follows from Corollary 7.1 (the extension of Theorem 5.3). In the Type I case the result follows from Corollary 7.1 (the extension of Corollary 5.2) and Lemmas 9.2 to 9.5.

From Theorem 3.2 and Corollary 9.1 we get immediately

THEOREM 9.4. *Let P be a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Either*

$$\lim_{x\to\infty} \lim_{\lambda \uparrow 1} \int_P \frac{1-\langle x,\theta\rangle}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \infty \tag{9.24}$$

or $\mathfrak{G} \cong R \oplus H$ or $Z \oplus H$, where H is compact and there is an $0 \leqslant L < \infty$ such that either

$$\lim_{\lambda\uparrow 1} \int_P \frac{1-\langle x,\theta\rangle}{1-\lambda\hat{\mu}(\theta)}\, d\theta \to \begin{cases} L & \text{as } x \to +\infty, \\ \infty & \text{as } x \to -\infty, \end{cases} \tag{9.25}$$

or

$$\lim_{\lambda\uparrow 1}\int_P \frac{1-\langle x,\theta\rangle}{1-\lambda\hat{\mu}(\theta)}\,d\theta \to \begin{cases}\infty & \text{as } x\to+\infty\\ L & \text{as } x\to-\infty.\end{cases} \tag{9.26}$$

Proof of Theorem 9.1. Write

$$D^\lambda f(x) = \int \frac{J(f)\hat{g}(-\theta)-\langle x,\theta\rangle \hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta.$$

We need only consider the Type I case. By Theorem 9.4 we need only prove that

$$\lim_{x\to\infty}\lim_{\lambda\uparrow 1}\int \frac{\langle x,\theta\rangle \mathfrak{J}\hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta = 0.$$

But this follows from the arguments used in proving Theorem 5.8.

Proof of Theorem 9.2. The proof of this result is similar to that of Theorem 9.1, except that the inversion formula for $a^\lambda(x)$ (as in [11]) is used.

Proof of Theorem 9.3. This result follows immediately from Theorem 5.10 and Theorem 7.5.

10. Poisson's equation

Throughout this section we will assume that μ generates a recurrent random walk on $\mathfrak{G}$. Our main purpose is to investigate the Poisson equation $(P-I)f=\varphi$ a.e., and some closely allied facts. In the investigation of potential theoretic matters the operator Df is not the correct potential operator since $D\varphi$ fails to yield a solution of the Poisson equation. However a simple modification of Df is correct. Define $A^\lambda f$, $0<\lambda<1$ as

$$A^\lambda f = G^\lambda g(0)J(f) - G^\lambda f = D^\lambda f + g(0)J(f) - f$$

and let

$$Af = \lim_{\lambda\uparrow 1} A^\lambda f = Df + g(0)J(f) - f.$$

This operator possesses all of the properties that the operator Df was shown to possess in § 5. A simple computation shows that the basic identity (5.15) translated into terms of Af becomes the following relation:

$$Af - H_B Af = -G_B f + J(f)\,\mathfrak{L}_B, \tag{10.1}$$

where

$$H_B(x,A) = 1_{A\cap B}(x) + 1_{B'}(x)\Pi_B(x,A),$$

$$G_B(x,A) = 1_{B'}(x)U_B(x,A\cap B') + 1_{B'\cap A}(x),$$

$$\mathfrak{L}_B(x) = 1_{B'}(x)L_B(x).$$

All of the limiting relations established in § 5 are immediately translatable into limit relations for the above quantities. In the sequel we will just refer to the original relations in § 5 and leave their trivial translation to the present quantities to the reader.

Our first task is to show that potentials are solutions of Poisson's equation.

THEOREM 10.1. *Take $f \in \mathfrak{F}$, or if the walk is nonsingular, then we may also take $f \in \Phi$. Then*

$$PAf = Af + f.$$

Proof. By replacing f with $-f$ if necessary we may assume that $J(f) \geq 0$. It then follows from (7.36) if $f \in \Phi$ in the nonsingular case or from (5.54) if $f \in \mathfrak{F}$ in the general case, that there is an M, $0 \leq M < \infty$ such that $A^\lambda f(x) \geq -M$. Now

$$PA^\lambda f = A^\lambda f + \sum_{n=0}^{\infty} \lambda^n (P^n f - P^{n+1} f),$$

and for $f \in \mathfrak{F}$ (or Φ) $P^n f \to 0, n \to \infty$, so

$$\lim_{\lambda \uparrow 1} PA^\lambda f = Af + f,$$

Fatou's lemma then yields that $PAf = Af + f - g$, where

$$g(x) = \lim_{\lambda \uparrow 1} PA^\lambda (x) - PAf(x) \geq 0.$$

Then $$g(x-y) = g_y(x) = \lim_{\lambda \uparrow 1} PA^\lambda f(x-y) - PAf(x-y) = \lim_{\lambda \uparrow 1} PA^\lambda f_y(x) - PAf_y(x).$$

Hence $$g_y(x) - g(x) = \lim_{\lambda \uparrow 1} PA^\lambda (f_y - f)(x) - PA(f_y - f)(x).$$

But by (7.34) and (5.50), for fixed y, $|A^\lambda (f_y - f)| \leq K < \infty$ and thus bounded convergence yields

$$g_y(x) - g(x) = 0.$$

Hence $g(x) \equiv g_0 \geq 0$. However as $Af \geq -M$,

$$P^{n+1}[Af - M] = Af - M + \sum_{j=0}^{n} P^j f - n g_0,$$

so $$0 \leq \lim_n \frac{P^{n+1}(Af - M)}{n} = -g_0,$$

thus $g_0 \leq 0$. Hence $g_0 = 0$ and the theorem is proved.

Let $B \in \mathcal{B}$, $|B| > 0$, $|\partial B| = 0$, or in the nonsingular case just $|B| > 0$, $B \in \mathcal{B}$. Take $Q_B(x, dy)$ to be the transition function of the random walk killed on B, i.e., $Q_B(x, dy) = \mu(dy - x)$ for $x, y \in \mathfrak{G} - B$ and $Q_B(x, dy) = 0$ elsewhere.

THEOREM 10.2. *For any type one random walk the only locally integrable nonnegative solutions of the equation $Q_B f = f$ a.e. are multiples of $\mathcal{L}_B(x)$. For any type two random walk the only such solutions are appropriate linear combinations of $\mathcal{L}_B^+(x)$ and $\mathcal{L}_B^-(x)$, where*

$$\mathcal{L}_B^+(x) = \mathcal{L}_B(x) + \frac{1}{\sigma^2}\int_B H_B(x, dz)\,\psi(x-z) \tag{10.2}$$

and

$$\mathcal{L}_B^-(x) = \mathcal{L}_B(x) - \frac{1}{\sigma^2}\int_B H_B(x, dz)\,\psi(x-z). \tag{10.3}$$

We shall divide the proof into several lemmas.

LEMMA 10.1. *For any type one random walk $\mathcal{L}_B(x)$ is a locally integrable nonnegative solution of $Q_B f = f$.*

Proof. Let $\varphi \in \mathfrak{F}^+$, $J(\varphi) > 0$. The basic identity (10.1) and the fact that $G_B\varphi < \infty$ shows that

$$A\varphi - PA\varphi = PH_B A\varphi - H_B A\varphi + J(\varphi)[\mathcal{L}_B - P\mathcal{L}_B] - [G_B\varphi - PG_B\varphi].$$

But if $x \notin B$ then $PH_B = H_B$, and $PG_B\varphi = G_B\varphi - \varphi$. By Theorem 10.1, $A\varphi - PA\varphi = -\varphi$ so we see that for $x \notin B$ $\mathcal{L}_B = P\mathcal{L}_B$. Since $\mathcal{L}_B(x) = 0$ on B it follows that $Q_B\mathcal{L}_B = \mathcal{L}_B$. The nonnegativity is clear and the local integrability follows from the fact that $\mathcal{L}_B(x)$ is bounded on compacts.

LEMMA 10.2. *For any type two random walk $\mathcal{L}_B^+(x)$ and $\mathcal{L}_B^-(x)$ are nonnegative, locally integrable solutions of $Q_B f = f$.*

Proof. As with $\mathcal{L}_B(x)$ all that needs to be verified is the fact that these are solutions of the equation. The same argument as used in the preceeding lemma shows that $\mathcal{L}_B(x)$ is a solution. The function

$$\delta(x) = \int_B H_B(x, dz)\,\psi(x-z) = \psi(x) - \int_B H_B(x, dz)\,\psi(z)$$

vanishes for $x \in B$ and for $x \notin B$,

$$P\delta(x) = \int P(x, dy)\,\psi(y) - \int_B PH_B(x, dz)\,\psi(z)$$

$$= -\int_B H_B(x, dz)\,\psi(z) + \int \mu(dz)\,\psi(z+x) = -\int_B H_B(x, dz)\,\psi(z) + \psi(x) = \delta(x).$$

Thus $\mathcal{L}_B^+$ and $\mathcal{L}_B^-$ are solutions.

LEMMA 10.3. *Let h be a locally integrable, essentially nonnegative solution of $Q_B h = h$ a.e. Then for a type one random walk, $h = c\mathfrak{L}_B$, a.e. where $c \geq 0$.*

Proof. The potential kernel of the Markov process Q_B is just

$$\sum_{n=0}^{n} Q_B^n f = G_B f,$$

where $Q_B^0 f(x) = f(x) 1_{B'}(x)$. Let $\{E_n\}$, int $E_n \neq \emptyset$ be an increasing family of compacts with union $\mathfrak{G}$, and define functions $h_n(x)$ as follows:

$$h_n(x) = \min\{h(x),\ nG_B(x, E_n)\}.$$

Then $h_n(x) \uparrow h(x)$, and

$$h_n(x) \leq nG_B(x, E_n). \tag{10.4}$$

A simple computation shows that if $g \geq 0$ a.e. then $Q_B g \geq 0$ a.e. Then, a.e. $Q_B h_n \leq \min\{Q_B h, Q_B(nG_B 1_{E_n})\} \leq h_n$, and thus setting $\delta_n = h_n - Q_B h_n$ we see that $\delta_n \geq 0$ a.e. But

$$\sum_{j=0}^{m} Q_B^j \delta_n = \sum_{j=0}^{m} Q_B^j (h_n - Q_B h_n) = h_n - Q_B^{m+1} h_n, \quad \text{a.e.,} \tag{10.5}$$

and then by (10.4)

$$Q_B^{m+1} h_n \leq n \sum_{j=m+1}^{\infty} Q_B^j(x, E_n) \to 0 \quad \text{a.e. as } m \to \infty.$$

Hence $Q_B^{m+1} h_n \downarrow 0$ a.e. Thus (10.5) shows that

$$h_n(x) = G_B \delta_n(x), \text{ a.e.}$$

Let f be any function in C_c^+. Since h is locally integrable $(h_n, f) \leq (h, f) < \infty$. But

$$(h_n, f) = (G_B \delta_n, f) = (\delta_n, \tilde{G}_B f).$$

Since by (5.22) $\tilde{G}_B \varphi(x) \to (\mathfrak{L}_B, \varphi)$ as $x \to \infty$, and by Theorem 9.3 $\mathfrak{L}_B(x) + \mathfrak{L}_B(-x) \to \infty$ we see that there is a $\varphi \in C_c^+$ and a compact set D such that $(\mathfrak{L}_B, \varphi) > 0$ and $\tilde{G}_B \varphi(x) > 0$, $x \notin D$. Thus

$$(h_n, f) = \int_D \tilde{G}_B f(x)\, \delta_n(x)\, dx + \int_{D'} \left[\frac{\tilde{G}_B f(x)}{\tilde{G}_B \varphi(x)} - a\right] \gamma_n(dx) + a\gamma_n(D'), \tag{10.6}$$

where the measures $\gamma_n(dx)$ are defined by

$$\gamma_n(dx) = \tilde{G}_B \varphi(x)\, \delta_n(x)\, dx,$$

and

$$a = \frac{(\mathfrak{L}_B, f)}{(\mathfrak{L}_B, \varphi)}.$$

Now observe that $\gamma_n(\mathfrak{G}) = (\tilde{G}_B\varphi, \delta_n) = (h_n, \varphi) \leqslant (h, \varphi) < \infty$. Moreover, since $h_n(x)$ is locally integrable, we see that for any compact K,

$$\int_K \delta_n(x)\,dx = \int_K h_n(x)\,dx - \int_K Q_B h_n(x)\,dx.$$

Since $h_n \uparrow h$ a.e. and $Q_B h = h$ a.e. we see that

$$\lim_n \int_K \delta_n(x)\,dx = 0. \tag{10.7}$$

Hence
$$\overline{\lim_n}\ \gamma_n(K) \leqslant \left[\sup_{x\in K} \tilde{G}_B\varphi(x)\right] \lim_n \int_K \delta_n(x)\,dx = 0. \tag{10.8}$$

We will now show that for some constant γ, $0 \leqslant \gamma < \infty$,

$$(h, f) = \lim_n\ (h_n, f) = a\gamma. \tag{10.9}$$

Indeed, given $\varepsilon > 0$, choose the compact set K such that

$$\left|\frac{\tilde{G}_B f(x)}{\tilde{G}_B\varphi(x)} - a\right| < \varepsilon$$

for $x \notin K$. Then the second integral in (10.6) may be written as

$$\int_{D'\cap K} + \int_{D'\cap K'},$$

and
$$\left|\int_{D'\cap K'}\right| \leqslant \varepsilon\gamma_n(D' \cap K') \leqslant \varepsilon(h, \varphi),$$

while
$$\left|\int_{D'\cap K}\right| = O\left(\int_K \delta_n(x)\,dx\right).$$

Since the first integral on the right in (10.6) is $O(\int_D \delta_n(x)dx)$, and $(h_n, f) \uparrow (h, f)$ we see that $\gamma_n(D')$ converges to some number γ and that (10.9) holds. Now (10.9) asserts for any $f \in C_c^+$

$$\int_{\mathfrak{G}} h(x)\,f(x)\,dx = c\int_{\mathfrak{G}} \mathfrak{L}_B(x)\,f(x)\,dx,$$

where $c = \gamma/(\mathfrak{L}_B, \varphi)$, and thus $h(x) = c\mathfrak{L}_B(x)$ a.e. This completes the proof.

LEMMA 10.4. *Let $h \geqslant 0$ a.e. be a locally integrable solution of $Q_B h = h$, a.e. Then for any type two random walk*

$$h(x) = c_1 \mathcal{L}_B^+(x) + c_2 \mathcal{L}_B^-(x),$$

where $c_1 \geqslant 0$, $c_2 \geqslant 0$ *are constants, and* $\mathcal{L}_B^+(x)$, $\mathcal{L}_B^-(x)$ *are given by* (10.2) *and* (10.3) *respectively.*

Proof. We know that $\mathcal{L}_B^+$ and $\mathcal{L}_B^-$ are $\geqslant 0$. Suppose $\mathcal{L}_B^+(x) = 0$ a.e. Then

$$\mathcal{L}_B(x) = \sigma^{-2} \int_B H_B(x, dz)\, \psi(z-x) \quad \text{a.e.,}$$

and thus
$$\mathcal{L}_B(x) + \mathcal{L}_B(-x) = \sigma^{-2} \int_B [H_B(x, dz)\, \psi(z) + H_B(-x, dz)\, \psi(z)], \quad \text{a.e.}$$

However, as the left-hand side tends to ∞ as $x \to \infty$ while the right-hand side is bounded, we see that this is impossible. Thus $\mathcal{L}_B^+(x) > 0$ on a set of positive measure. A similar argument shows that $\mathcal{L}_B^-(x) > 0$ on a set of positive measure. Using Urysohn's lemma we may then find a $\varphi \in C_c^+$ such that $(\mathcal{L}_B^\pm, \varphi) > 0$. Since

$$\lim_{x \to \pm\infty} \tilde{G}_B \varphi(x) = (\mathcal{L}_B^\pm, \varphi),$$

we see that there is a compact set D such that $\tilde{G}_B \varphi(x) > 0$ for $x \notin D$.

Let h_n, δ_n, and γ_n be as in the proof of Lemma 10.3. Then we may write

$$(h_n, f) = \int_D \tilde{G}_B f(x)\, \delta_n(x)\, dx + \int_{D' \cap \mathfrak{G}^-} \left[\frac{\tilde{G}_B f(x)}{\tilde{G}_B \varphi(x)} - c_1\right] \gamma_n(dx)$$
$$+ \int_{D' \cap \mathfrak{G}^+} \left[\frac{\tilde{G}_B f(x)}{\tilde{G}_B \varphi(x)} - c_2\right] \gamma_n(dx) + c_1 \gamma_n(D' \cap \mathfrak{G}^-) + c_2 \gamma_n(D' \cap \mathfrak{G}^+).$$

where
$$c_1 = \frac{\tilde{G}_B f(-\infty)}{\tilde{G}_B \varphi(-\infty)} = \frac{(\mathcal{L}_B^-, f)}{(\mathcal{L}_B^-, \varphi)}$$

$$c_2 = \frac{\tilde{G}_B f(+\infty)}{\tilde{G}_B \varphi(+\infty)} = \frac{(\mathcal{L}_B^+, f)}{(\mathcal{L}_B^+, \varphi)}.$$

Arguing as in the preceeding proof we may conclude that

$$(h, f) = \lim_{n \to \infty} [c_1 \gamma_n(D' \cap \mathfrak{G}^-) + c_2 \gamma_n(D' \cap \mathfrak{G}^+]. \tag{10.10}$$

There is a subsequence n' such that $\gamma_{n'}(D' \cap \mathfrak{G}^-) \to \gamma_1'$, $\gamma_{n'}(D' \cap \mathfrak{G}^+) \to \gamma_2'$ and thus $(h, f) = \gamma_1' c_1 + \gamma_2' c_2$. As $f \in C_c^+$ was arbitrary, it follows that $h = \gamma_1 \mathcal{L}_B^+ + \gamma_2 \mathcal{L}_B^-$ a.e. This completes the proof.

Theorem 10.2 now follows from the preceeding lemmas. We will now establish an extension of this theorem which will prove useful later. First we introduce a definition.

Definition. Let h be a locally integrable function defined on B' that is bounded from below. Let $B \in \mathcal{A}^*$, $|B| > 0$. Then h is called Q_B *superregular* provided $Q_B h \leqslant h$ a.e. on B'. It is called Q_B *regular* if $Q_B h = h$ a.e. on B'.

COROLLARY 10.1. *Let h be a Q_B superregular function, and let $\delta = h - Q_B h$ a.e. Then in a type one walk there is a unique constant $c \geqslant 0$ such that*

$$h(x) = c\mathcal{L}_B(x) + G_B\delta \text{ a.e.}$$

while in a type two walk there are unique constants $c_1, c_2 \geqslant 0$ such that

$$h(x) = c_1 \mathcal{L}_B^+(x) + c_2 \mathcal{L}_B^-(x) + G_B\delta \quad \text{a.e.}$$

Moreover, the only Q_B regular functions are the above with $\delta = 0$, a.e.

Proof. Clearly,
$$Q_B^{n+1} h = h - \sum_{j=0}^{n} Q_B^j (h - Q_B h).$$

By definition there is an M, $0 \leqslant M < \infty$ such that

$$-M \leqslant Q_B^{n+1} h \leqslant Q_B^n h \leqslant \dots \leqslant h \quad \text{a.e.}$$

Thus
$$Q_B^\infty h = \lim_n Q_B^n h$$

exists a.e. and thus so does $\lim_n \sum_{j=0}^n Q_B^j (h - Q_B h)$. Moreover,

$$Q_B^\infty h = h - G_B(h - Q_B h) \quad \text{a.e.}$$

Also (by monotone convergence) $Q_B Q_B^\infty h = Q_B^\infty h$ a.e. and $Q_B^\infty h \geqslant -M$. However a.e.,

$$Q_B^\infty h(x) = \lim_n Q_B^n (Q_B^\infty h + M)(x) - M \lim_n P_x(T_B > n) = \lim_n Q_B^n (Q_B^\infty h + M)(x) \geqslant 0.$$

Thus by Theorem 10.2 we see that for a type one walk $Q_B^\infty h(x) = c\mathcal{L}_B(x)$ a.e. while for a type two walk $Q_B^\infty h(x) = c_1 \mathcal{L}_B^+(x) + c_2 \mathcal{L}_B^-(x)$ a.e. Since the functions $\mathcal{L}_B^+$, $\mathcal{L}_B^-$ are linearly independent the constants c_1 and c_2 are unique. This completes the proof.

Consider now a nonsingular walk. We already know that potentials of functions $\varphi \in \Phi$, $J(\varphi) \geqslant 0$ provide solutions of Poisson's equation that are bounded from below. We will now exhibit all such solutions.

THEOREM 10.3. *Let μ generate a nonsingular recurrent random walk, and let $\varphi \in \Phi$. Consider the equation $Pf = f + \varphi$ a.e. In order that there be a solution which is bounded from* *below it is necessary that $J(\varphi) \geqslant 0$. In that case, for a type one walk, the only such solutions are $f = A\varphi + \beta$ a.e., β a constant. For a type two walk the only such solutions are*

$$f = A\varphi + \frac{\alpha J(\varphi)}{\sigma^2}\,\psi(x) + \beta$$

a.e., where β is an arbitrary constant and $|\alpha| \leqslant 1$.

We shall divide the proof into several lemmas. Our first task is to show that an f satisfying $Pf = f + \varphi$ for a nonsingular walk must be locally integrable.

LEMMA 10.5. *If $Pf = f + \varphi$ a.e. and $f \geqslant 0$ then f is locally integrable.*

Proof. It follows from the equation that for any $n \geqslant 1$, $P^n f < \infty$, a.e. But since the walk is nonsingular there is an n such that $\mu^{(n)}$ has a component with a density $k(x) \in L_2(\mathfrak{G})$, and thus $k * k$ must be bounded away from 0 on some relatively compact open set I. Thus there is a measure μ_1 and a constant $\gamma > 0$ such that

$$P^{n_0+1}f(x) = \int \mu_1(dy)\, f(y+x) + \int k * k(y)\, f(y+x)\, dy \geqslant \gamma \int_I f(y+x)\, dy.$$

Since $P^{n_0+1} f < \infty$ a.e. it follows that f is locally integrable.

Next we establish a simple fact which will be needed during the uniqueness proof.

LEMMA 10.6. *For any measurable function f that is bounded on B,*

$$\lim_n P^{n+1} H_B f(x) = (\bar{l}_B, f). \tag{10.11}$$

Proof. Consider a type one random walk. Then $\lim_{y\to\infty} H_B f(y) = (\bar{l}_B, f)$. Now

$$P^{n+1} H_B f = \int_{\mathfrak{G}} P^{n+1}(x, dz)\, [H_B f(z) - (\bar{l}_B, f)] + (\bar{l}_B, f).$$

Equation (10.11) now follows by using the fact that $|H_B f(z) - (\bar{l}_B, f)| < \varepsilon$ if z is outside of some compact set K and $P^n 1_K(x) \to 0$ as $n \to \infty$. The proof for a type two random walk is similar. We write

$$P^{n+1} H_B f = \int_{\mathfrak{G}^+} P^n(x, dz)\, [H_B f(z) - H_B f(+\infty)] + \int_{\mathfrak{G}^-} P^n(x, dz)\, [H_B f(z) - H_B f(-\infty)]$$
$$+ P^n(x, \mathfrak{G}^+)\, H_B f(+\infty) + P^n(x, \mathfrak{G}^-)\, H_B f(-\infty).$$

The central limit theorem for the induced random walk on R^1 or Z^1 implies that

$\lim_{n\to\infty} P^n(x, \mathfrak{G}^+) = \lim_{n\to\infty} P^n(x, \mathfrak{G}^-) = \frac{1}{2}$, and the desired result follows.

We may now show that the theorem is true.

LEMMA 10.7. *The assertions of Theorem 10.3 hold for any nonsingular type one walk.*

Proof. Set $\delta(x) = \int_B P(x, dy) f(y)$. Since $f + c$ satisfies the Poisson equation with charge φ if f does and f is bounded from below we may assume that $f \geqslant 0$. Let B contain the support of φ, $|B| > 0$, $B \in \mathcal{B}$. Then for $x \notin B$, $Q_B f(x) + \delta(x) = f(x)$ a.e. Since $f \geqslant 0$ we see that $\delta \geqslant 0$, and thus f restricted to B' is a Q_B superregular function. It follows from Corollary 10.1 then that there is a $c \geqslant 0$ such that

$$f(x) = c\mathfrak{L}_B(x) + G_B \delta(x) \quad \text{a.e. } x \notin B.$$

However, for $x \notin B$, $G_B \delta(x) = H_B f(x)$ so

$$f(x) = c\mathfrak{L}_B(x) + H_B f(x) \quad \text{a.e. } x \notin B. \tag{10.12}$$

Suppose $J(\varphi) \neq 0$. Then it follows from (10.12) and (10.1) applied to $\varphi(G_B\varphi = 0)$ that

$$f = H_B f + \frac{c}{J(\varphi)} [A\varphi - H_B A\varphi] \quad \text{a.e.}$$

and thus
$$P^{n+1} f = P^{n+1} H_B f + \frac{c}{J(\varphi)} [P^{n+1} A\varphi - P^{n+1} H_B A\varphi].$$

By Theorem 10.1 however,
$$P^{n+1} A\varphi = A\varphi + \sum_{j=0}^{n} P^j \varphi$$

while
$$P^{n+1} f = f + \sum_{j=1}^{n} P^j \varphi \quad \text{a.e.}$$

Thus
$$f + \sum_{j=0}^{n} P^j \varphi = \frac{c}{J(\varphi)} \left[A\varphi + \sum_{j=0}^{n} P^j \varphi - P^{n+1} H_B A\varphi\right] + P^{n+1} H_B f, \quad \text{a.e.}$$

or
$$f + \left(1 - \frac{c}{J(\varphi)}\right) \sum_{j=0}^{n} P^j \varphi = \frac{c}{J(\varphi)} A\varphi + P^{n+1}\left[H_B f - \frac{c}{J(\varphi)} H_B A\varphi\right], \quad \text{a.e.} \tag{10.13}$$

Now as $n \to \infty$ the right-hand side above has a finite limit, and thus so must the left-hand side. An appeal to the Chacon–Ornstein ergodic theorem shows that for any set K of positive measure,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} P^j \varphi}{\sum_{j=1}^{n} P^j 1_K} = \frac{J(\varphi)}{|K|}, \quad \text{a.e.}$$

Thus unless $J(\varphi)=c$ the left-hand side in (10.13) is infinite a.e. Consequently $J(\varphi)=c$ and by letting $n\to\infty$ in (10.13) and using Lemma 10.6 we see that $f=A\varphi+\alpha$, a.e. We have thus shown that if $J(\varphi)\neq 0$ and $Pf=f+\varphi$, a.e., f bounded from below, then it must be that $J(\varphi)>0$ and $f=A\varphi+\alpha$ a.e. Consider now the case when $J(\varphi)=0$. Let $\chi\in\Phi^+$ be such that $J(\chi)>0$. Then Theorem 10.1 shows that $f+A\chi$ is a solution of $Pg=g+\chi+\varphi$ that is bounded from below. Since $J(\chi+\varphi)=J(\chi)>0$ it follows from what has already been shown that $f+A\chi=A(\chi+\varphi)+\alpha$ a.e. Thus $f=A\varphi+\alpha$ a.e. This establishes the lemma.

LEMMA 10.8. *In order that $Pf=f+\varphi$, a.e. have a solution f bounded from below for a type two random walk it is necessary that $J(\varphi)\geqslant 0$. In that case $f=A\varphi+a\psi(x)+b$ a.e. for suitable constants a and b.*

Proof. As before we need only consider solutions $f\geqslant 0$. Arguing as in the proof of the preceeding lemma we find that if $Pf=f+\varphi$, a.e. $f\geqslant 0$ then

$$f=c_2\mathcal{L}_B^+ + c_1\mathcal{L}_B^- + H_B f \quad \text{a.e.,}$$

where $\mathcal{L}_B^+$, $\mathcal{L}_B^-$ are given by (10.2) and (10.3) respectively. Thus

$$f(x)=(c_1+c_2)\,\mathcal{L}_B(x)+\frac{(c_1-c_2)}{\sigma^2}\int_B H(x,dz)\,\psi(z)-\frac{(c_1-c_2)}{\sigma^2}\,\psi(x)+H_Bf.$$

Consequently if $J(\varphi)\neq 0$, then a.e.

$$P^{n+1}f(x)=\frac{c_1+c_2}{J(\varphi)}\,P^{n+1}[A\varphi-H_BA\varphi]+\frac{c_1-c_2}{\sigma^2}\,P^{n+1}H_B\psi(x)-\frac{c_1-c_2}{\sigma^2}\,\psi(x)+P^{n+1}H_Bf$$

or

$$f+\left(1-\frac{c_1+c_2}{J(\varphi)}\right)\sum_{j=1}^{n}P^j\varphi=\frac{c_1+c_2}{J(\varphi)}\,A\varphi-\frac{c_1+c_2}{J(\varphi)}\,P^{n+1}H_BA\varphi+P^{n+1}H_Bf$$

$$+\frac{c_1-c_2}{\sigma^2}\,P^{n+1}H_B\psi(x)-\frac{c_1-c_2}{\sigma^2}\,\psi(x), \quad \text{a.e.}$$

Since the left-hand side has a finite limit as $n\to\infty$ we must have $c_1+c_2=J(\varphi)$ and thus it is necessary that $J(\varphi)>0$ (since c_1, c_2 are $\geqslant 0$) and

$$f(x)=A\varphi(x)+a\psi(x)+b. \tag{10.14}$$

If $J(\varphi)=0$ the same argument used in the preceeding proof shows that f must again be of this form. This establishes the lemma.

Our final lemma is to establish the values of a and b.

LEMMA 10.9. *In order that*

$$A\varphi(x)+a\psi(x)+b$$

be bounded from below it is necessary and sufficient that

$$a=\frac{\alpha J(\varphi)}{\sigma^2} \quad where \quad |\alpha|\leqslant 1.$$

Proof. Suppose there is a positive $M<\infty$ such that

$$A\varphi(x)+\frac{\alpha J(\varphi)}{\sigma^2}\,\psi(x)+b\geqslant -M.$$

It follows from (5.9) that

$$\lim_{x\to\pm\infty}\frac{A\varphi(x)}{\psi(x)}=\pm\frac{J(\varphi)}{\sigma^2}.$$

Consequently,

$$\frac{J(\varphi)}{\sigma^2}+\operatorname{sgn}\psi(x)\,\frac{\alpha J(\varphi)}{\sigma^2}\geqslant 0,$$

so $|\alpha|\leqslant 1$ is necessary. On the other hand,

$$\frac{\alpha J(\varphi)}{\sigma^2}\,\psi(x)+b+A\varphi(x)=\int_B H_B(x,dz)\left[A\varphi(z)+\frac{\alpha J(\varphi)}{\sigma^2}\,\psi(z)+b\right]$$

$$+J(\varphi)\left[\mathcal{L}_B(x)+\frac{\alpha}{\sigma^2}\int_B H_B(x,dz)\,\psi(x-z)\right].$$

The first term on the right is bounded in x. Since

$$\mathcal{L}_B(x)\pm\frac{1}{\sigma^2}\int_B H_B(x,dz)\,\psi(x-z)\geqslant 0$$

we see that $|\alpha|\leqslant 1$ is sufficient. (The proof is that of Spitzer [13].)

Theorem 10.3 now follows from the preceeding three lemmas.

There are some immediate consequences of Theorem 10.3 which are of interest.

COROLLARY 10.2. *Let μ generate a nonsingular recurrent random walk. Then the only superregular functions for P (i.e. functions f, bounded from below, such that $Pf\leqslant f$ a.e.) are $f(x)=\alpha$ a.e. for α a constant.*

Proof. By choosing $\varphi=0$ we see that the only solutions of $Pf=f$ a.e., f bounded from below, are $f(x)=\alpha$ a.e. But if $Pf\leqslant f$ a.e. and $f(x)\geqslant -M$, $0\leqslant M<\infty$ then as $\sum_{j=0}^{n}P^j(f-Pf)=f-P^{n+1}f$ a.e. we see that $\lim_{n\to\infty}\sum_{j=0}^{n}P^j(f-Pf)(x)<\infty$. But by the Chacon–Ornstein ergodic theorem this can only be the case if $J(f-Pf)=0$. Hence $f=Pf$ a.e. and the assertion is proved.

Remark. Suppose $Pf(x)=f(x)$ for all x and $f(x)\geqslant 0$. It follows from the above that if $\mu(dx) << dx$, then $f(x)=\alpha$ for all x. It might be suspected that this should hold in the general nonsingular case. Simple counter-examples however show that this is false, so that in the general nonsingular case the most we can conclude from $Pf(x)=f(x)$ all x, $f(x)\geqslant 0$ is that $f(x)=\alpha$ a.e. In turn, this implies that for $\varphi\in\Phi$, $J(\varphi)\geqslant 0$ even if we assume $Pf(x)= f(x)+\varphi(x)$ for all x we still can only conclude in the general nonsingular case that solutions permissible by Theorem 10.3 hold a.e. However, for bounded solutions things are different.

COROLLARY 10.3. *Let μ generate a nonsingular recurrent random walk. In order that the equation $Pf(x)=f(x)+\varphi(x)$ have a bounded solution it is necessary that $J(\varphi)=0$. In that case the only bounded solutions are $f=A\varphi+\beta$ for some constant β.*

Proof. It follows from Theorem 10.3 that $J(\varphi)=0$ is necessary, and it is clear that $A\varphi$ is a bounded solution of the equation. Since the difference of two bounded solutions is bounded the desired conclusion will follow provided we can show that the only bounded solution of $Pf(x)=f(x)$, $x\in\mathfrak{G}$ is $f(x)\equiv f(0)$. To establish this fact we may proceed as follows. We already know (from Corollary 10.2) that $f(x)=\beta$ a.e. $x\in\mathfrak{G}$, for some β. Let $E= \{x:f(x)=\beta\}$. Then as $|E|>0$, $P_x(V_E<\infty)\equiv 1$. But then for any $n>0$,

$$f(x)=E_xf(S_n)=\sum_{r=0}^{n}\int_E P_x(V_E=r, S_r\in dz)\,P^{n-r}f(z)+\int_{\mathfrak{G}} P_x(V_E>n, S_n\in dz)\,f(z).$$

The second term on the right is bounded in absolute value by $\|f\|_\infty P_x(V_E>n)\downarrow 0$ as $n\to\infty$ while the first term is just $P_x(V_E\leqslant n)\beta\uparrow\beta$ as $n\to\infty$. Thus $f(x)\equiv\beta$, as desired.

For functions $\varphi\in\mathfrak{F}$ there is a similar uniqueness result in the nonsingular case to that for functions $\varphi\in\Phi$.

COROLLARY 10.4. *Let μ generate a nonsingular recurrent random walk, and let $\varphi\in\mathfrak{F}$. In order that $Pf=f+\varphi$ a.e. have a solution bounded from below it is necessary that $J(\varphi)\geqslant 0$. In that case for a type one walk $f=A\varphi+\beta$ a.e. while for a type two walk*

$$f=A\varphi+\frac{J(\varphi)\,\alpha}{\sigma^2}\,\psi(x)+\beta\ \ a.e.,\ |\alpha|\leqslant 1.$$

Proof. Let g have compact support B, $J(g)=1$. Then $A\varphi(x)-J(\varphi)Ag(x)$ is bounded, and by Theorem 10.1, $h(x)=f(x)+J(\varphi)Ag-A\varphi$ is a solution of $Ph=h+J(\varphi)g$ a.e. which by Theorem 10.3 is bounded from below if and only if $J(\varphi)\geqslant 0$. It follows that f is bounded from below if and only if $J(\varphi)\geqslant 0$. In that case Theorem 10.3 shows that $Ph=h+J(\varphi)g$

a.e., h bounded from below, has only the solutions $h=J(\varphi)Ag+\beta$ a.e. in the type one case and

$$h=J(\varphi)\,Ag+\frac{J(\varphi)\,\alpha\psi(x)}{\sigma^2}+\beta$$

a.e., in the type two case. This establishes the result.

We now turn our attention to the question of uniqueness in the case of a singular recurrent walk. Here we have only been able to establish uniqueness in a smaller class of functions. Let $g\geqslant 0$, $J(g)=1$ and $g\in\mathfrak{J}$ be fixed. For any $\varphi\in\mathfrak{J}$, $A\varphi-J(\varphi)Ag$ is a bounded continuous function. Let E be the direct sum space of the Banach space of bounded measurable functions with the one dimensional space generated by multiples of Ag, let $E'=\{f\in E:(P-I)f=\varphi$ a.e. for some $\varphi\in\mathfrak{J}\}$, and let E_c be the subspace of elements of E such that for some α, $f-\alpha Ag$ is a bounded continuous function.

THEOREM 10.4. *Let μ generate a recurrent random walk, and let $\varphi\in\mathfrak{J}$. Then the equation $Pf=f+\varphi$ a.e., $f\in E$ has only the solutions $f=A\varphi+\beta$ a.e. Moreover, if $f\in E_c$ then $f=A\varphi+\beta$ for all $x\in\mathfrak{G}$.*

Proof. Suppose $f=f_0+\alpha Ag\in E'$ and let $\pi_1(f)=\alpha$ and $\pi_2(f)=J((P-I)f)$. Both of these are homomorphisms of E' onto the reals. Suppose $f\in\ker\pi_1$, i.e. $\alpha=0$. Then as f is bounded and

$$P^{n+1}f-f=\sum_{j=0}^{n}P^j(P-I)\,f$$

we see that $\sum_{j=0}^{n}P^j(P-I)\,f$ is also bounded. Let $B\in\mathcal{A}$, $|B|>0$. Then $\sum_{j=0}^{n}P^j 1_B(x)\to\infty$ a.e. and the Chacon–Ornstein theorem shows that

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{n}P^j(P-I)\,f(x)}{\sum_{j=0}^{n}P^j 1_B(x)}=\frac{J((P-I)\,f)}{|B|}\quad\text{a.e.}$$

Thus $J[(P-I)f]=0$, so $f\in\ker\pi_2$. It follows that there is a homomorphism π of the reals onto themselves such that $\pi(\pi_1(f))=\pi_2(f)$. Hence for some real γ, $\gamma\alpha=J[(P-I)f]$. By Theorem 10.1, $(P-I)A\varphi=\varphi$ and the α for $A\varphi$ is $J(\varphi)$. Thus $\gamma=1$ and we see that $\alpha=J[(P-I)f]$. Now suppose f is a solution of $(P-I)f=\varphi$ a.e., $f\in E$. Then it must be that $f=f_0+J(\varphi)Ag$. Thus $f-A\varphi$ is bounded and satisfies the equation $Ph=h$ a.e. It follows from a theorem of Choquet–Deny [1] that $h(x)=\beta$ a.e. for some constant β. Finally if f is required to be in E_c, then since both f and $A\varphi+\beta$ are continuous the equality must hold everywhere. This completes the proof.

Remark. We know that $\lim_{x\to\infty}[Ag(x)+Ag(-x)]=\infty$. Theorem 10.4 then shows that a function f satisfying the Poisson equation that grows like a potential must in fact differ from a potential by an additive constant.

If we examine the Poisson equation in the nonsingular case in the class E then we always have $A\varphi+\beta$ as the solutions.

COROLLARY 10.5. *Let μ generate a nonsingular walk and let $\varphi\in\mathfrak{F}$ or Φ. Suppose $f\in E$ and $Pf(x)=f(x)+\varphi(x)$ for all $x\in\mathfrak{G}$. Then $f(x)=A\varphi(x)+\beta$ for some constant β.*

Proof. The same proof as in Theorem 10.4 shows that $f-A\varphi$ is bounded, and $P(f-A\varphi)=f-A\varphi$. Corollary 10.3 then shows that $f-A\varphi\equiv\beta$ for some β.

Our final results in this section concern the Poisson equation with boundary conditions.

THEOREM 10.5. *Assume μ generates a nonsingular recurrent walk. Let $B\in\mathcal{B}$, $|B|>0$ and let $\varphi\in\Phi$ have support on B and $f\in\Phi$ have support on B'. Then the solutions of the equation $(P-I)h=-f$ a.e. on B' subject to the boundary condition $h=\varphi$ a.e. on B that are bounded from below are as follows: In a type one walk*

$$h=H_B\varphi+G_Bf+c\mathfrak{L}_B\quad \text{a.e. } c\geqslant 0. \tag{10.15}$$

In a type two walk

$$h=H_B\varphi+G_Bf+c_1\mathfrak{L}_B^{+}+c_2\mathfrak{L}_B^{-},\ \text{a.e. } c_1\geqslant 0, c_2\geqslant 0. \tag{10.16}$$

Proof. It is clear that $h(x)=\varphi(x)$ a.e. on B and a simple computation plus Lemma's 10.1 and 10.2 shows that right-hand sides of (10.15) and (10.16) satisfy $(P-I)h(x)=-f(x)$, $x\in B'$. Now consider a type one walk and assume that h satisfies the requirements of the theorem. Set $\delta(x)=[(P-I)h(x)]1_B(x)$. Then $(P-I)h(x)=\delta(x)-f(x)$ a.e. $\mathfrak{G}$, $\delta(x)-f(x)\in\Phi$ and h is bounded from below on $\mathfrak{G}$. Thus by Theorem 10.3 $h=A(\delta-f)+\beta$, a.e. and $J(\delta-f)\geqslant 0$. Setting $c=J(\delta-f)$ we see from the basic identity that a.e.

$$h=H_Bh-G_B(\delta-f)+c\mathfrak{L}_B=H_B\varphi+G_Bf+c\mathfrak{L}_B$$

as desired. The proof of (10.16) for the type two walk is similar. This completes the proof.

There is also an analogue of this result valid in the singular case.

THEOREM 10.6. *Let $B\in\mathcal{A}$, $|B|>0$. Assume $\varphi\in\Phi$ has support on B and $f\in\Phi$ has support on B'. Then the only locally integrable solutions of the Poisson equation $(P-I)h=-f$ a.e. on B' subject to the boundary condition $h=\varphi$ a.e. on B that are bounded from below are as follows. In a type one walk*

$$h=G_Bf+H_B\varphi+c\mathfrak{L}_B\quad \textit{a.e. } c\geqslant 0,$$

while in a type two walk

$$h = G_B f + H_B \varphi + c_1 \mathfrak{L}_B^+ + c_2 \mathfrak{L}_B^- \text{ a.e. } c_1, c_2 \geqslant 0.$$

Proof. A simple computation shows that the right-hand sides satisfy the equation with the boundary value φ. Also it is clear that $h_1 = G_B f + H_B \varphi$ is a bounded solution. Suppose that h is a solution that is bounded from below. Then $g = h - h_1$ is locally integrable, bounded from below, and satisfies the equation $Pg = g$ a.e. on B' and $g = 0$ a.e. on B. Since $g = 0$ a.e. on B this equation is the same as $Q_B g = g$ a.e. Then g is Q_B regular and it follows from Corollary 10.1 $g = c\mathfrak{L}_B$ a.e. in the type one case and $g = c_1 \mathfrak{L}_B^+ + c_2 \mathfrak{L}_B^-$ in the type two case. This completes the proof.

By the same type of argument we may establish the following.

THEOREM 10.7. *Let $B \in \mathcal{B}$ have nonempty interior (or in the nonsingular case $|B| > 0$). Then the only bounded solution of $(P - I)h(x) = 0$ for all $x \in B'$ and $h(x) = \varphi(x)$ on B is $H_B \varphi(x)$.*

Proof. Suppose h is a solution. Set $\delta(x) = \int_B P(x, dy) h(y)$. Then for $x \notin B$

$$Q_B^{n+1} h = h - \sum_{j=0}^{n} Q_B^j \delta.$$

But $|Q_B^{n+1} h(x)| \leqslant \sup_x |h(x)| P_x(V_B > n) \downarrow 0$ as $n \to \infty$, and thus $h(x) = \sum_{j=0}^{\infty} Q_B^j \delta = H_B h(x) = H_B \varphi(x)$, as desired.

11. Recurrent potential theory

Throughout this section we will assume that μ generates a recurrent random walk. Our purpose here is to show that analogues of some of the basic principles of classical logarithmic potential theory are valid for our potentials. Naturally one cannot hope for too much in the singular case, but for the nonsingular case most of the familiar principles have their counterpart. A nice treatment of classical potential theory can be found in the lecture notes of Fuchs [3].

The potential operator A was defined a bit arbitrarily. It is clear that all of the results established for A up till now also hold for the operators A_b defined by $A_b f = Af + bJ(f)$, b a fixed constant. In particular we still have the basic identity

$$A_b f(x) - H_B A_b f(x) = -G_B f(x) + J(f) \mathfrak{L}_B(x). \tag{11.1}$$

Let $g \in \mathfrak{F}$, $g \geqslant 0$, $J(g) = 1$ be fixed. By Corollaries 5.2 and 7.1 we then see that for $f \in \mathfrak{F}$ (or $f \in \Phi$ in the nonsingular case)

$$\operatorname*{Lim}_{x} [A_b f - J(f) A_b g(x)] = 0 \tag{11.2}$$

and by (11.1), Theorems 5.10 and 7.6, and Corollary 5.3

$$-(\tilde{l}_B, A_b f) = -(\tilde{\mathfrak{L}}_B, f) + J(f) \lim_x [\mathfrak{L}_B(x) - A_b g(x)] \tag{11.3}$$

where
$$\tilde{l}_B(y)\, dy = \tilde{L}_B(y)\, 1_B(y)\, dy = \lim_x H_B(x, dy).$$

The constant
$$\operatorname*{Lim}_x [A_b g(x) - \mathfrak{L}_B(x)] = \operatorname*{Lim}_x [A g(x) - \mathfrak{L}_B(x)] + b = k_b(B) \tag{11.4}$$

is independent of g since if $g, g' \in \mathfrak{F}$ (or Φ in the nonsingular case) and $J(g) = 1$, then $\lim_x [A_b g(x) - A_b g'(x)] = 0$. The constant $k_b(B)$ is called the *b-Robin's constant of B*. For $b = 0$ the constant $k(B)$ is called the *Robin's* constant of B. If the walk is nonsingular then $(\tilde{l}_B, A_b f) = (\tilde{A}_b \tilde{l}_B, f)$ and we may rewrite (11.3) as

$$(\tilde{A}_b \tilde{l}_B, f) = (\tilde{\mathfrak{L}}_B, f) + J(f)\, k_b(B).$$

It follows that in this case
$$\tilde{A}_b \tilde{l}_B(x) = \mathfrak{L}_B(x) + k_b(B) \quad \text{a.e.} \tag{11.5}$$

Applying these results to the dual walk we see that for $f \in \mathfrak{F}$ (or $f \in \Phi$ in the nonsingular case)

$$\operatorname*{Lim}_x [\tilde{\mathfrak{L}}_B(x) - \tilde{A} g(x)] = -\tilde{k}(B)$$

and in the nonsingular case
$$A_b l_B(x) = \mathfrak{L}_B(x) + \tilde{k}_b(B) \quad \text{a.e.} \tag{11.6}$$

Also in the nonsingular case

$$\tilde{k}_b(B) = (l_B, \tilde{A}_b \tilde{l}_B) = (A_b l_B, \tilde{l}_B) = k_b(B).$$

[In the singular case, for $B \in \mathcal{A}$, $|B| > 0$, it is also true that $k_b(B) = \tilde{k}_b(B)$. See Theorem 11.1(e) below.]

In the nonsingular case we have thus established

PROPOSITION 11.1. *Let $B \in \mathcal{B}$, $|B| > 0$, and let $\mathfrak{M}(B)$ denote the collection of all $\varphi \in \Phi^+$ having support on B such that $J(\varphi) = 1$. Assume the walk is nonsingular. Then $l_B \in \mathfrak{M}(B)$ and $A_b l_B(x) = k_b(B)$ a.e. $x \in B$.*

Before proceeding further we pause to establish some properties of the Robin's constant $k(B)$.

THEOREM 11.1. *Let sets $\{B_i : i \geq 1\} \in \mathcal{A}^*$, $|B_i| > 0$.*

(a) *If $B_1 \subset B_2$ then $k(B_1) \leq k(B_2)$.*

(b) *If $B = \bigcap_{i=1}^n B_i$ then*

$$k(B) \leq \sum_{i=1}^n k(B_i) - \sum_{i \neq j} k(B_i \cup B_j) + \ldots + (-1)^{n+1} k(B_1 \cup \ldots \cup B_n).$$

(c) *If* $|B|>0$, *and* $B_{n+1}\supset B_n$ *and* $|B_n-B|\to 0$, $n\to\infty$ *then* $k(B_n)\to k(B)$.

(d) *If* $|B_n|>0$ *and* $B_n\subset B_{n+1}$ *and* $|B-B_n|\to 0$, $n\to\infty$ *then* $k(B_n)\to k(B)$.

(e) $\tilde{k}(B)=k(B)=k(-B)$.

Remark. Properties (a)–(d) show that $k(B)$ satisfies the axioms of a Choquet capacity on the sets $B\in\mathcal{A}^*$.

Proof. By definition, $k(B)=\lim_x[Ag(x)-L_B(x)]$ and $|A|L_B(x)=\lim_y G_B(x, A+y)$, $A\in\mathcal{A}^*$, $|A|>0$. To see that (a) holds note that if $B_1\subset B_2$ then $G_{B_1}(x,A)\geqslant G_{B_2}(x,A)$ and thus $L_{B_1}(x)\geqslant L_{B_2}(x)$ and thus $k(B_1)\leqslant k(B_2)$. Similarly if $B=\bigcap_{i=1}^n B_i$ then

$$G_B(x,A)\geqslant\sum_{i=1}^n G_{B_i}(x,A)-\sum_{i\neq j}G_{B_i\cup B_j}(x,A)+\ldots+(-1)^{n+1}G_{B_1\cup\ldots\cup B_n}(x,A),$$

because the left-hand side is just the mean number of visits to A before hitting B while the right-hand side is the mean number of visits to A before time $\max(T_{B_i}:1\leqslant i\leqslant n)\leqslant T_B$. To establish (c) note that

$$G_{B_n}(x,A)=G_B(x,A)-\int_{B_n-B}H_{B_n}(x,dz)\,G_B(z,A),$$

and thus
$$\mathcal{L}_{B_n}(x)=\mathcal{L}_B(x)-\int H_{B_n}(x,dz)\,\mathcal{L}_B(z).$$

Hence
$$k(B_n)=k(B)+\int_{B_n-B}\tilde{l}_{B_n}(z)\,\mathcal{L}_B(z)\,dz\leqslant\int_{B_n-B}\tilde{l}_B(z)\mathcal{L}_B(z)\,dz$$

and the result follows. The proof of (d) is similar.

To establish (e) we may proceed as follows. Since the dual walk is generated by $\mu(-dx)$, it is clear that $G_B(x,A)=\tilde{G}_{-B}(-x,-A)$ and thus $L_B(x)=\tilde{L}_{-B}(-x)$. Also it is quite easy to see that $Ag(x)=\tilde{A}\bar{g}(-x)$ where $\bar{g}(x)=g(-x)$. By definition of $k(B)$,

$$k(B)=\lim_x\,[Ag(x)-L_B(x)]=\lim_x\,[\tilde{A}\bar{g}(-x)-\tilde{L}_{-B}(-x)]=\tilde{k}(-B).$$

In the nonsingular case (e) now follows from the fact that $k(B)=\tilde{k}(B)$. To establish this in the singular case requires a different argument. If we knew that $k(B)-k(A)=\tilde{k}(B)-\tilde{k}(A)$ for $A,B\in\mathcal{A}^*$, then choosing say A symmetric (and using $k(B)=\tilde{k}(-B)$) would yield the desired result, That this is so is the content of our next

LEMMA 11.1. *Let* $A,B\in\mathcal{A}^*$, $A\subset B$, $|A|$, $|B|>0$. *Then*

$$\int_{B-A} dx\, \tilde{L}_B(x)\, L_A(x) = k(B) - k(A) \tag{11.7}$$

and for any two sets $A, B \in \mathcal{A}^*$

$$k(B) - k(A) = \tilde{k}(B) - \tilde{k}(A). \tag{11.8}$$

Proof. Let $K \in \mathcal{A}$, and let $A \subset B$.

$$U_B(x, K+y) = U_A(x, K+y) - \int_{B-A} \Pi_B(x, dz)\, U_A(z, K+y) \tag{11.9}$$

so taking $\lim_y$ we obtain

$$L_B(x) = L_A(x) - \int_{B-A} \Pi_B(x, dz)\, L_A(z).$$

Thus
$$k(B) - k(A) = \int_B \tilde{L}_B(z)\, L_A(z)\, dz.$$

On the other hand, taking lim on x in (11.9) yields

$$\int_{K+y} [\tilde{L}_B(z) - \tilde{L}_A(z)]\, dz = -\int_{B-A} \tilde{L}_B(z)\, U_A(z, K+y)\, dz.$$

Now take $\lim_y$ to obtain

$$\tilde{k}(B) - \tilde{k}(A) = \int_{B-A} \tilde{L}_B(z)\, L_A(z)\, dz.$$

Thus $k(B) - k(A) = \tilde{k}(B) - \tilde{k}(A)$ if $A \subset B$. Since $A \cup B \in \mathcal{A}^*$ if $A, B \in \mathcal{A}^*$, (11.8) follows. This completes the proof.

For the remainder of this section we will always be dealing with a nonsingular walk. We already know via Theorem 10.1 that for any $\varphi \in \Phi$ the potential $A_b\varphi$ determines φ. We will now show that a stronger uniqueness principle is available.

THEOREM 11.2. *Let μ generate a nonsingular walk. Let $f \in \Phi$ have support on $B \in \mathcal{B}$, $|B| > 0$. If $k_b(B) \neq 0$ then the value of $A_b f$ a.e. on B determines $A_b f$ a.e. on $\mathfrak{G}$, and consequently f a.e.*

Proof. Suppose $E \subset B$, and $|E| = 0$. Then $\int_{\mathfrak{G}} P^n(x, E)\, dx = |E| = 0$, so $P^n(x, E) = 0$, a.e., and thus $P_x\{S_n \in E \text{ for some } n > 0\} = 0$ a.e. x.

To demonstrate the theorem we need to show that if $A_b f = 0$ a.e. on $B = \text{support}\,(f)$, then $f = 0$ a.e. Set $E = \{x : A_b f(x) \neq 0\}$. Then as $G_B f(x) \equiv 0$,

$$A_b f(x) = \int_E H_B(x, dz)\, A_b f(z) + J(f)\, \mathfrak{L}_B(x). \tag{11.10}$$

Since $H_B(x, E) = 0$ a.e. on B' we see that

$$A_b f(x) = J(f)\, \mathfrak{L}_B(x) \quad \text{a.e.}, \tag{11.11}$$

and thus $(\tilde{l}_B, A_b f) = J(f)(\tilde{l}_B, \mathfrak{L}_B) = 0$. Consequently

$$0 = J(f)\, k_b(B),$$

and thus $J(f) = 0$. It follows from (11.10) that $A_b f(x) = 0$ a.e., and thus by Theorem 10.1 $f = 0$ a.e.

Examples show that if $k_b(B) = 0$ then the above uniqueness principle fails. However, since we may always choose b so that $k_b(B) \neq 0$ we always have the following

COROLLARY 11.1. *Assume* $B \in \mathcal{B}$, $|B| > 0$. *If* φ_1 *and* φ_2 *have support on* B *and* $A\varphi_1 = A\varphi_2$ *a.e. on* B *and* $J(\varphi_1) = J(\varphi_2)$ *then* $A\varphi_1 = A\varphi_2$ *a.e. on* $\mathfrak{G}$ *and* $\varphi_1 = \varphi_2$ *a.e.*

We may now establish the analogue of equilibrium principle.

THEOREM 11.3. *Assume* μ *generates a nonsingular walk. Then* l_B *is the essentially unique element of* $\mathfrak{M}(B)$ *whose potential* Al_B *is essentially constant on* B. *The constant is the Robin's constant* $k(B)$ *of* B.

Proof. In view of Proposition 11.1, where $\mathfrak{M}(B)$ is defined, we need only establish uniqueness. Suppose then that $\varphi \in \mathfrak{M}(B)$ and $A\varphi = c$, a.e. on B. Then

$$c = (\tilde{l}_B, A\varphi) = (\tilde{A}\tilde{l}_B, \varphi) = k(B)$$

so $c = k(B)$. Since $J(\varphi) = J(l_B)$ the result now follows from Corollary 11.1.

We will next establish an analogue of the minimum principle.

PROPOSITION 11.2. *Let* μ *generate a nonsingular walk. Let* $f \in \Phi$ *have support on* B, $|B| > 0$, *and assume* $J(f) \geq 0$. *If* $A_b f \geq \alpha$ *a.e. on* B *then* $A_b f \geq \alpha$ *a.e. on* $\mathfrak{G}$.

Proof. Let $E = \{x : A_b f(x) < \alpha\}$. Since

$$A_b f = H_B A_b f + J(f) L_B(x)$$

and $H_B(x, E) = 0$ a.e. we see that

$$A_b f(x) \geq \alpha + J(f) L_B(x) \geq \alpha \quad \text{a.e. } x \in \mathfrak{G}.$$

Another basic principle is the principle of domination, the analogue of which is the following

THEOREM 11.4. *Let $f \in \Phi$ have support on B, and assume b such that $k_b(B) > 0$. Let $h \in \Phi^+$. If for some $\alpha \geqslant 0$,*

$$A_b f \geqslant A_b h + \alpha \quad \text{a.e. on } B, \tag{11.12}$$

then $J(f) \geqslant J(h)$ and

$$A_b f \geqslant A_b h + \alpha \quad \text{a.e. on } \mathfrak{G}. \tag{11.13}$$

Proof. To begin, observe that it follows easily from (11.12) that

$$k_b(B)J(f) = (\tilde{l}_B, A_b f) \geqslant (\tilde{l}_B, A_b h) + \alpha \geqslant k_b(B)J(h)$$

and thus $J(f) \geqslant J(h)$. Next observe that

$$A_b h = H_B A_b h + J(h)\,\mathfrak{L}_B - G_B h \leqslant H_B A_b h + J(h)\,\mathfrak{L}_B$$

while
$$A_b f = H_B A_b f + J(f)\,\mathfrak{L}_B.$$

Thus setting $E = \{x : A_b f(x) < A_b h(x) + \alpha\}$ we see that

$$A_b f - A_b h - \alpha \geqslant \int_E H_B(x, dz)\,[A_b f(z) - A_b h(z) - \alpha] \geqslant 0 \quad \text{a.e.}$$

since $H_B(x, E) = 0$ a.e. Thus (11.13) holds.

Remark. It is clear from the proof of Proposition 11.2 that if we know that $Af \geqslant \alpha$ everywhere on $B = \text{support } (f)$, then we may conclude that $Af \geqslant \alpha$ everywhere on $\mathfrak{G}$. Likewise, if in Theorem 11.4 we know that (11.12) holds everywhere on B we may conclude that (11.13) holds everywhere on $\mathfrak{G}$.

PROPOSITION 11.3. *Let μ generate a nonsingular walk. Assume $B \in \mathfrak{B}$, $|B| > 0$. Then $k(B)$ is the unique constant c such that for any $f \in \Phi$ having support on B such that $J(f) = 1$,*

$$\operatorname*{ess\,inf}_{x \in B} Af(x) \leqslant c \leqslant \operatorname*{ess\,sup}_{x \in B} Af(x). \tag{11.14}$$

Proof. If there is a c satisfying (11.14) then choosing $f = l_B$ shows that $c = k(B)$. Suppose $\operatorname{ess\,inf}_{x \in B} Af(x) > k(B)$. Since $J(f) = 1$ we then see that $\operatorname{ess\,inf}_{x \in B} A_b f(x) > k_b(B)$. Choosing b such that $k_b(B) > 0$ we see that there is a t, $0 < t < 1$ such that $tA_b f \geqslant k_b(B)$ a.e. on B. Hence $t(\tilde{l}_B, A_b f) \geqslant k_b(B)$. But $(\tilde{l}_B, A_b f) = (\tilde{A}_b \tilde{l}_B, f) = k_b(B)J(f)$ and thus $t \geqslant 1$, a contradiction. The other inequality is proved similarly.

Our final result in this section will be to establish an analogue of the Balyage principle. Another proof of this fact will be given as a corollary to Theorem 12.2.

THEOREM 11.5. *Assume $B \in \mathcal{B}$, $|B|>0$ and $k_b(B)>0$. Let $f \in \Phi^+$. Then there is an essentially unique $\varphi \in \Phi^+$ having support on B whose potential $A_b\varphi(x) = A_b f(x)$ a.e. on B. Moreover, $A_b\varphi \geqslant A_b f$ a.e. on $\mathfrak{G}$. The charge φ may be computed by*

$$\varphi = f + (P-I)\left\{\frac{(\tilde{\mathfrak{L}}_B, f)}{k_b(B)}\mathfrak{L}_B + G_B f\right\}$$

and the corresponding potential is

$$A_b\varphi = A_b f + \frac{(\mathfrak{L}_B, f)}{k_b(B)}\mathfrak{L}_B + G_B f. \tag{11.15}$$

Proof. Since we will give another proof in the next section we will only establish the result here for a type one walk although a similar argument would also work in the type two case. A simple computation shows that $\varphi \geqslant 0$ and has support on B. What needs to be proved is that the potential of φ is given by the right-hand side of (11.15). Set $c = (\tilde{\mathfrak{L}}_B, f)/k_b(B)$. Then $c\mathfrak{L}_B + G_B f \geqslant 0$ is a solution of $(P-I)h = \varphi - f$ and it follows from Theorem 10.1 and (10.3) that $h = A_b(\varphi - f) + \beta$ a.e. so

$$c\mathfrak{L}_B + G_B f = A_b(\varphi - f) + \beta \quad \text{a.e.}$$

Let $g \in \Phi^+$, $J(g) = 1$. Then

$$c[\mathfrak{L}_B - Ag] + G_B f - [A_b(\varphi - f) - J(\varphi - f)Ag] = [J(\varphi - f) - c]Ag + \beta, \quad \text{a.e.}$$

Since the left-hand side converges to

$$-ck(B) + (\tilde{\mathfrak{L}}_B, f) - bJ(\varphi - f)$$

as $x \to \infty$ it must be that $J(\varphi - f) = c$ and thus

$$\beta = -ck_b(B) + (\tilde{\mathfrak{L}}_B, f) = 0.$$

This completes the proof.

12. Approximations by potentials

Throughout this section we will assume that the random walk generated by μ is recurrent and *nonsingular*. Let $N = \{f \in C_c(\mathfrak{G}) : J(f) = 0\}$. Define the space $C_0(\mathfrak{G})$ as the collection of all continuous functions f on $\mathfrak{G}$ such that $\lim_x f(x) = 0$. For a type one random walk, $C_0(\mathfrak{G})$ is then just the usual space of continuous functions vanishing at ∞. For a type two walk $C_0(\mathfrak{G})$ is the closed subspace of the continuous functions on the *two* point compactification of $\mathfrak{G}$ which is the kernel of the linear functional $f \to f(+\infty) + f(-\infty)$. Let $\chi = C_0(\mathfrak{G}) \oplus \{a(-x)\}$ denote the direct sum space of the space $C_0(\mathfrak{G})$ with the one dimensional space of multiples of $a(-x)$, where $a(x)$ is the function defined in § 7.

Consider the Poisson equation $(P-I)f=\varphi$. It follows from Theorem 10.3 that if $\varphi \in C_c(\mathfrak{G})$ then the potential $A\varphi$ is the unique solution of this equation in the space χ. Moreover, if also $J(\varphi)=0$ then $A\varphi$ is the unique solution in $C_0(\mathfrak{G})$. The main result of this section is to show that χ is in fact the correct range space of potentials of functions in $C_c(\mathfrak{G})$.

THEOREM 12.1. *The set* $\{A\varphi : \varphi \in C_c(\mathfrak{G})\}$ *is dense in* χ. *The subset* $\{A\varphi : \varphi \in N\}$ *is dense in the closed subspace* $C_0(\mathfrak{G})$.

A continuous linear functional γ^* on $C_0(\mathfrak{G})$ can be identified with a bounded signed measure γ on $\mathfrak{G}$ in the type one case and with a pair (γ, α) in the type two case, where in the type two case,

$$(\gamma^*, f) = \int_{\mathfrak{G}} f(x)\, \gamma(dx) + \alpha f(+\infty). \tag{12.1}$$

Henceforth we will carry out the proof only for the type two case. The proof for the type one case can be carried out by following the same argument and just omitting all terms involving α. To proceed we will need the following

LEMMA 12.1. *If* $(\gamma^*, A\varphi)=0$ *for all* $\varphi \in N$ *then* $(\gamma^*, Ah)=0$ *for all* $h \in \Phi$ *such that* $J(h)=0$.

Proof. Let $\eta(dx)=dx+|\gamma U_2|(dx)$, where U_2 is the measure defined in § 7. Let $\varepsilon>0$ be given. By Lusin's theorem there is an $h' \in C_c(\mathfrak{G})$ such that $h(x)=h'(x)$ except on a set D, $\|h'\|_\infty \leqslant \|h\|_\infty$, and $\eta(D)<\varepsilon$. Observe that

$$|J(h')| \leqslant \int_D |h'(x)-h(x)|\, dx \leqslant K_1 |D| \leqslant K_1 \varepsilon, \tag{12.2}$$

where here and in the following, K_i will denote constants. Choose $g \in C_c^+(\mathfrak{G})$ such that $J(g)=1$ and set $f=h'-J(h')g$. Then $f \in N$ and for $x \notin D$,

$$|h(x)-f(x)| = |h(x)-h'(x)+J(h')g(x)| = |J(h')||g(x)| \leqslant K_2 \varepsilon. \tag{12.3}$$

Since γ^* also acts on Ah we may write

$$(\gamma^*, Ah) = (\gamma^*, A(h-f)) + (\gamma^*, Af) = (\gamma^*, A(h-f)) = \int_{\mathfrak{G}} A(h-f)(x)\, \gamma(dx) + \alpha A(h-f)(+\infty).$$

From results in § 7 we know that $\lim_{x\to\pm\infty} a(y-x)-a(-x) = \mp(\psi(y))/\sigma^2$, uniformly on compacts, and thus as $a(x)$ is continuous, we see that for any compact set E there is a constant K (dependent on E in general) such that $|a(y-x)-a(-x)| \leqslant K$, $x \in \mathfrak{G}$, $y \in E$. It follows from Urysohn's lemma that we may assume that the supports of the h' are all contained in some fixed compact set. Let E be the union of the supports of f, h and the h'. Then

$$|A(h-f)(+\infty)| = \left| \lim_{x\to+\infty} \int_{\mathfrak{G}} [a(y-x)-a(-x)][h(y)-f(y)]\,dy - \lim_{x\to+\infty} U_2(h-f)(x) \right|$$
$$= \left| \frac{1}{\sigma^2} \int_{\mathfrak{G}} \psi(y)[h(y)-f(y)] \right| dy \leqslant \frac{1}{\sigma^2}\int_D |\psi(y)|\,|h(y)-f(y)|\,dy + \frac{1}{\sigma^2}\int_{E-D} |\psi(y)|\,|h(y)-f(y)|\,dy.$$

In view of (12.2) and (12.3) we then see that

$$|A(h-f)(+\infty)| \leqslant K_3|D| + K_4\varepsilon \leqslant K_5\varepsilon. \tag{12.4}$$

In a similar way,

$$\left| \int_{\mathfrak{G}} \gamma(dx)\, A(h-f)(x) \right| \leqslant \left| \int_{\mathfrak{G}} \gamma(dx) \int_D [a(y-x)-a(-x)][h(y)-f(y)]\,dy \right.$$
$$\left. - \int_D \gamma U_2(dy)[h(y)-f(y)] \right| + \left| \int_{\mathfrak{G}} \gamma(dx) \int_{E-D} [a(y-x)-a(-x)][h(y)-f(y)]\,dy \right.$$
$$\left. - \int_{E-D} \gamma U_2(dy)[h(y)-f(y)] \right| \leqslant K_6\eta(D) + K_7\varepsilon \leqslant K_8\varepsilon. \tag{12.5}$$

The lemma now follows from (12.4) and (12.5).

We may now establish the theorem.

Proof of theorem. We will establish the second assertion of the theorem first. Suppose then that γ^* annihilates Af for all $f \in N$. Choose $g \in C_c^+(\mathfrak{G})$ such that $J(g)=1$. Then for any $\varphi \in \Phi$, $\varphi - J(\varphi)g$ is a null function and the lemma shows $(\gamma^*, A(\varphi - J(\varphi)g)) = 0$. Thus for any $\varphi \in \Phi$,

$$0 = (\gamma^*, A(\varphi - J(\varphi)g)) = (\gamma^*, H_B A\varphi - G_B\varphi + J(\varphi)(\mathfrak{L}_B - Ag)) \tag{12.6}$$
$$= (\gamma^*, H_B A\varphi) - (\gamma^*, G_B\varphi) + J(\varphi)(\gamma^*, \mathfrak{L}_B - Ag).$$

For any measurable function f and compact set B, let $I_B f(x) = 1_B(x) f(x)$, and set $P^B f(x) = 1_B(x) P H_B f(x)$. Then if f is bounded on B,

$$PH_B f = H_B f + (P^B - I_B) f$$

and by iteration
$$P^{n+1} H_B f = H_B f + \sum_{l=0}^{n} P^l (P^B - I_B) f,$$

and thus by Lemma 10.6
$$(\bar{l}_B, f) = H_B f - A(P^B - I_B) f. \tag{12.7}$$

Since $(P^B - I_B)f$ is a null function with support B, it follows from (12.7) with $f = A\varphi$ that

$$[\gamma(\mathfrak{G})+\alpha](\tilde{l}_B, A\varphi) = (\gamma^*, H_B A\varphi).$$

Thus we may write (12.6) as

$$0 = [\gamma(\mathfrak{G})+\alpha](\tilde{l}_B, A\varphi) - (\gamma^*, G_B\varphi) + J(\varphi)(\gamma^*, \mathfrak{L}_B - Ag). \tag{12.8}$$

Let B be compact and $|B| > 0$. Choose $\varphi \in \Phi$ so that φ has support on B and $J(\varphi) \neq 0$. Then from (12.8) we see that

$$0 = [\gamma(\mathfrak{G})+\alpha](\tilde{l}_B, A\varphi) + J(\varphi)(\gamma^*, \mathfrak{L}_B - Ag).$$

Since $(\tilde{l}_B, A\varphi) = (\tilde{A}\tilde{l}_B, \varphi) = k(B)J(\varphi) + (\tilde{\mathfrak{L}}_B, \varphi)$, we see that as φ has support on B,

$$[\gamma(\mathfrak{G})+\alpha]k(B) + (\gamma^*, \mathfrak{L}_B - Ag) = 0.$$

Thus for an arbitrary $\varphi \in \Phi$ equation (12.8) and the above shows that

$$[\gamma(\mathfrak{G})+\alpha](\tilde{\mathfrak{L}}_B, \varphi) = (\gamma^*, G_B\varphi) = \int_{\mathfrak{G}} G_B\varphi(x)\,\gamma(dx) + \alpha G_B\varphi(+\infty).$$

Now γG_B is a well-defined signed measure on compacts and

$$G_B\varphi(+\infty) = (\tilde{\mathfrak{L}}_B^+, \varphi).$$

Thus
$$([\gamma(\mathfrak{G})+\alpha]\tilde{\mathfrak{L}}_B - \alpha\tilde{\mathfrak{L}}_B^+, \varphi) = (\gamma G_B, \varphi).$$

Hence
$$\{[\gamma(\mathfrak{G})+\alpha]\tilde{\mathfrak{L}}_B(x) - \alpha\tilde{\mathfrak{L}}_B^+(x)\}\,dx = \gamma G_B(dx). \tag{12.9}$$

Theorem 10.2 and a simple duality argument shows that for $y \in \mathfrak{G} - B$, on the one hand,

$$\int \tilde{\mathfrak{L}}_B(x)\,Q_B(x, dy)\,dx = \tilde{\mathfrak{L}}_B(y)\,dy,$$

and
$$\int \tilde{\mathfrak{L}}_B^+(x)\,Q_B(x, dy)\,dx = \tilde{\mathfrak{L}}_B^+(y)\,dy.$$

On the other hand, on $\mathfrak{G} - B$,

$$\gamma G_B Q_B = \gamma G_B - \gamma.$$

Operating on both sides of (12.9) by Q_B then yields the relation

$$\{[\gamma(\mathfrak{G})+\alpha]\tilde{\mathfrak{L}}_B(x) - \alpha\tilde{\mathfrak{L}}_B^+(x)\}\,dx = \gamma G_B(dx) - \gamma(dx), \quad x \notin B.$$

Thus by (12.9) we see that γ must be the 0 measure on $\mathfrak{G} - B$. Owing to the arbitrariness of B, γ must be 0. Then for $\varphi \in N$

$$0=(\gamma^*, A\varphi)=\alpha A\varphi(-\infty)=-\frac{\alpha}{\sigma^2}\int_{\mathfrak{G}}\psi(y)\varphi(y)\,dy.$$

Taking φ such that $\int\psi(y)\varphi(y)\neq 0$ we see that $\alpha=0$, and thus $\gamma^*=0$. This establishes the second assertion of the theorem.

To establish the first assertion of the theorem we can proceed as follows. A continuous linear functional ψ^* on χ is a pair (γ^*, b), where γ^* is a continuous linear functional on $C_0(\mathfrak{G})$ and b is a real number, such that for $f=(f_0, s)\in\chi$, $f_0\in C_0(\mathfrak{G})$, $(\psi^*, f)=(\gamma^*, f_0)+sb$. Thus if ψ^* annihilates $\{A\varphi:\varphi\in C_c(\mathfrak{G})\}$, then

$$0=(\psi^*, A\varphi)=(\gamma^*, A\varphi-J(\varphi)a(-x))+J(\varphi)b.$$

In particular for $\varphi\in N$, $(\psi^*, A\varphi)=(\gamma^*, A\varphi)=0$. Hence by what has already been proved $\gamma^*=0$. Choosing φ such that $J(\varphi)\neq 0$ then shows $b=0$. Thus $\psi^*=0$. This completes the proof.

We will now examine the potential operator restricted to a relatively compact set. Theorem 12.2 given below will be extension to arbitrary $\mathfrak{G}$ of the fundamental fact that for the groups Z^d, A restricted to a finite set B is a bijection of R^B onto R^B, whenever $k(B)\neq 0$.

Let $B\in\mathcal{B}$, $|B|>0$ and let $\Phi(B)$ denote the collection of all bounded measurable functions on B with the essential sup as norm. (Two functions equal a.e. on B are identified.) Define the mapping $T_B:\Phi(B)\to\Phi(B)$ by $T_B\varphi(x)=A_b\varphi(x)$.

THEOREM 12.2. *Assume $k_b(B)\neq 0$. Then T_B is a topological isomorphism of $\Phi(B)$ onto $\Phi(B)$ having inverse K_B defined by*

$$K_B\varphi=(P^B-I_B)\varphi+\frac{(\tilde{l}_B, \varphi)}{k_b(B)}\,l_B. \tag{12.10}$$

Proof. The fact that T_B is a bounded linear map of $\Phi(B)$ into $\Phi(B)$ follows from the fact that

$$|T_B\varphi(x)|\leq\int_B|a(y-x)\varphi(y)|\,dy+|U_2\varphi(x)|+b|J(\varphi)|,$$

and the fact that $a(x)$ is continuous and U_2 a bounded measure on $\mathfrak{G}$. That K_B is a bounded linear operator is clear. Suppose that $T_B\varphi=0$. Then $J(\varphi)k_b(B)=0$, so $J(\varphi)=0$, and it follows from the uniqueness principle that $\varphi=0$ a.e. on $\mathfrak{G}$. Hence T_B is $1-1$. By (12.7) we see that for $x\in B$,

$$A_b(P^B-I_B)\varphi(x)=\varphi(x)-(\tilde{l}_B, \varphi).$$

Since $A_b l_B(x)=k_b(B)$ a.e. on B we see that for any $\varphi\in\Phi$, $T_BK_B\varphi=\varphi$. Setting $\varphi=T_B\psi$ for

$\psi \in \Phi(B)$ we see that $T_B K_B T_B \psi = T_B \psi$ so it must be that $K_B T_B \psi = \psi$. Thus K_B is a two-sided inverse. This completes the proof.

Remark. If $k_b(B) = 0$ then it is clear that T_B is not a bijection. It is neither 1–1 nor onto for l_B is annihilated by T_B and $\tilde{l}_B$ annihilates the closure of the range of T_B.

Using Theorem 12.2 we may easily give another proof of the Balyage principle. Indeed from the theorem we know that

$$\psi = K_B A_b f = (P^B - I_B)\, A_b f + \frac{(\tilde{l}_B, Af)}{k_b(B)}\, l_B$$

is the essentially unique element of Φ having support on B whose potential $A_b \psi = A_b f$ a.e. on B. What needs to be demonstrated is that $\psi \geq 0$ a.e. To see that this is true let $h \in \Phi^+$ be arbitrary and note (12.7) applied to the dual walk yields

$$(l_B, h)\, J(f) = (f, \tilde{H}_b h) - (f, \tilde{A}_b (\tilde{P}_B - I_B)\, h) \geq \int_B f(x)\, h(x) - ((P^B - I_B)\, A_b f, h).$$

Thus a.e. on B
$$(P^B - I_B)\, A_b f \geq f - J(f)\, l_B.$$

But then
$$K_B A_b f \geq f + \frac{(\tilde{\mathfrak{L}}, f)}{k_b(B)} \geq f \quad \text{a.e. on } B.$$

Since $f \geq 0$, we have the desired result. An easy computation then shows that $A_b \psi$ is given by the right-hand side of (11.15).

13. Time dependent behavior

Throughout this section we will assume that μ generates a recurrent random walk on $\mathfrak{G}$. Define the class $\mathcal{B}^*$ as $\mathcal{B}$ if the walk is nonsingular and as the subset $\{B \in \mathcal{B} : \text{int } B \neq \emptyset\}$ in general. Set

$$E_B(n) = \int_{\mathfrak{G}} P_x(V_B \leq n)\, dx.$$

We start our investigation with the following

THEOREM 13.1. *Assume $A, B \in \mathcal{B}^*$, $|A|, |B| > 0$. Then $E_B(n) \sim E_A(n)$, $n \to \infty$.*

Proof. It suffices to establish the result for $A \subset B$. Set $\varphi_A^n(x) = P_x(V_A > n)\, 1_{B-A}(x)$. Then

$$E_A(n) = E_B(n) - \int_{\mathfrak{G}} dx \int_B \sum_{j=1}^{n} {}_BP^j(x, dz)\, \varphi_A^{n-j}(z) = E_B(n) - \sum_{j=1}^{n} ({}_BP^j \varphi_A^{n-j}, 1_{\mathfrak{G}})$$

$$= E_B(n) - \sum_{j=1}^{n} (\varphi_A^{n-j}, {}_B\tilde{P}^j 1_{\mathfrak{G}}) = E_B(n) - \sum_{j=1}^{n} \int_{B-A} P_x(V_A > n-j)\, \tilde{P}_x(V_B \geqslant j)\, dx. \quad (13.1)$$

Set $a_n = \sup_{x \in B} P_x(V_A > n)$ and $b_n = \int_B \tilde{P}_x(V_B > n)\, dx$. Then

$$\sum_{j=1}^{n} \int_{B-A} dx\, P_x(V_A > n-j)\, \tilde{P}_x(V_B \geqslant j) \leqslant \sum_{j=0}^{n-1} a_{n-1-j} b_j.$$

Since $a_n \to 0$ and

$$E_B(n) = ({}_BP^n 1_B, 1_{\mathfrak{G}}) = (1_B, {}_B\tilde{P}^n 1_{\mathfrak{G}}) = \int_B \tilde{P}_x(V_B > n-1)\, dx = b_{n-1}, \quad (13.2)$$

a simple summability argument shows that

$$\lim_{n \to \infty} \frac{1}{E_B(n)} \sum_{j=1}^{n} \int_{B-A} P_x(V_A > n-j)\, \tilde{P}_x(V_B \geqslant j)\, dx = 0.$$

This establishes the result.

Using (13.2) and taking duals we obtain the following

COROLLARY 13.1. *Let* $A, B \in \mathcal{B}^*$, $|A|, |B| > 0$. *Then*

$$\lim_{n \to \infty} \frac{\sum_{j=0}^{n} \int_B P_x(V_B > j)\, dx}{\sum_{j=0}^{n} \int_A P_x(V_A > j)\, dx} = 1. \quad (13.3)$$

The quantity $E_B(n)$ has several different interpretations. Let $B_n^* = \{x : x \in S_j + B$ for some j, $1 \leqslant j \leqslant n\}$. Then $E|B_n^*| = E_B(n)$ so that $E_B(n)$ is the expect volume swept out by time n in translating the set B by the random walk. This interpretation was introduced by Spitzer [13]. A second interpretation of $E_B(n)$ was given by Port [9]. Distribute particles in $\mathfrak{G}$ according to a point process with rate dx, and allow each particle to move independently according to the random walk. Then $E_B(n)$ is the expected number of distinct particles to visit B by time n. It will be shown subsequently that $E_B(n)$ is connected in a natural way with the Robin's constant $k(B)$. For the moment however we return to the study of $P_x(V_B > n)$.

We will now show how the passage times to B are linked with the recurrence times to B.

THEOREM 13.2. *Let* $B \in \mathcal{A}^*$, $|B| > 0$, *and let* $A \in \mathcal{B}$ *be such that* $|\partial A| = 0$. *Then in any type one random walk*

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} \int_A P_x(V_B > j)\, dx}{\sum_{j=0}^{n} \int_B P_x(V_B > j)\, dx} = \int_A L_B(x)\, dx. \tag{13.4}$$

If the walk is also nonsingular we may improve this to

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} P_x(V_B > j)}{\sum_{j=0}^{n} \int_B P_x(V_B > j)\, dx} = L_B(x), \tag{13.5}$$

uniformly in x on compacts.

The proof of this fact is the same as that of Theorem 5.4 and 5.5 in [11] and will therefore be omitted.

Remark. If the stronger limits

$$\lim_{n\to\infty} \frac{P_x(V_B > n)}{\int_B P_x(V_B > n)\, dx}$$

and

$$\lim_{n\to\infty} \frac{\int_B P_x(V_B > n)\, dx}{\int_A P_x(V_A > n)\, dx}$$

exist, then the above theorems show that their values must be $L_B(x)$ and 1 respectively. For random walks on Z^1 or Z^2 the existence of such limits were demonstrated by Kesten and Spitzer [6]. In a far reaching extension of this result Ornstein [8] demonstrated the existence of such limits for all recurrent random walks on R^1 or R^2 when B and A are intervals. The existence of such limits for an arbitrary recurrent random walk on a locally compact group is one of the remaining major open problems of the theory.

If we have some regularity properties then we can assert that these stronger limits exist. Choose $F \in \mathcal{A}^*$, $|F| > 0$, and set $q_n = \int_F P_x(V_F > n)dx$.

PROPOSITION 13.1. *Suppose for some g in $\mathfrak{J}$,*

$$U^\lambda g(0) \sim (1-\lambda)^{-\alpha} H\left(\frac{1}{1-\lambda}\right), \quad \lambda \uparrow 1 \tag{13.6}$$

for some slowly varying function H and constant α, $0 \leqslant \alpha > 1$. Then

$$q_n \sim \frac{n^{-\alpha}}{H(n)\,\Gamma(1-\alpha)}, \quad n \to \infty, \tag{13.7}$$

and

$$\lim_{n\to\infty} \frac{P_x(V_B > n)}{q_n} = L_B(x). \tag{13.8}$$

Proof. If such a g exists, then choosing it to be the g in the definition of D^λ yields

$$\lim_{\lambda \uparrow 1} (1-\lambda)\, U^\lambda g(0) \sum_{n=0}^{\infty} P_x(V_B > n)\, \lambda^n = L_B(x)$$

uniformly in x on compacts. The monotonicity of the q_n and Karamata's theorem then shows that uniformly in x on compacts,

$$P_x(V_B > n) \sim \frac{L_B(x)}{\Gamma(1-\alpha)} \frac{n^{-\alpha}}{H(n)}.$$

For a type two walk we always have very strong results.

THEOREM 13.3. *Let $B \in \mathcal{B}^*$, $|B| > 0$. Then in any type two random walk*

$$\lim_{n\to\infty} P_x(V_B > n)\sqrt{n} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sigma L_B(x) \tag{13.9}$$

uniformly in x on compacts.

Proof. Since the random walk is of type two we may identify $\mathfrak{G}$ with either $R^1 \oplus H$ or $Z^1 \oplus H$. A character θ is then of the form $\theta_1 + \theta_2$ where $\theta_1 \in \text{dual}\,(R^1)$ and $\theta_2 \in \text{dual}\,(H)$. Choose g on $\mathfrak{G}$ such that $\hat{g}(\theta) = \hat{f}(\theta_1)\delta(\theta_2, 0)$ where f is a symmetric function on R^1 or Z^1, $J(f) = 1$, and $\hat{f}(\theta_1)$ has support on $[-\delta, \delta]$ where $\hat{\mu}'(\theta_1) \neq 0$, $\theta_1 \in [-\delta, \delta] - \{0\}$. Here μ' is the measure induced on R^1 or Z^1 by μ. Then

$$U^\lambda g(0) = (2\pi)^{-1} \int_{-\delta}^{\delta} \frac{\hat{f}(\theta_1)\, d\theta_1}{1 - \lambda \hat{\mu}^1(\theta_1)}$$

$$\sim 2(2\pi)^{-1}(1-\lambda)^{-1} \int_0^{\delta} \frac{\hat{f}(\theta_1)\, d\theta_1}{1 + \dfrac{\lambda}{1-\lambda}\dfrac{\sigma^2}{2}\theta_1^2} \sim (2\sigma^2)^{-\frac{1}{2}}(1-\lambda)^{-\frac{1}{2}}.$$

Thus $U^\lambda g(0)$ satisfies the requirement of Proposition 13.1 and applying that proposition we find that (13.9) holds.

The above result admits an immediate extension.

COROLLARY 13.2. *Let $N_n(B) = \sum_{k \leqslant n} 1_B(S_K)$, and let $B \in \mathcal{B}^*$. Then in any type two random walk, for any $k \geqslant 1$,*

$$P_x(N_n(B)=k) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sigma n^{-\frac{1}{2}} (\Pi_B)^k L_B(x).$$

Proof. Let
$$H_k^\lambda(x) = \sum_{n=0}^{\infty} \lambda^n P_x(N_n(B)=k).$$

Then $H_k^\lambda(x) = \Pi_B^\lambda H_{k-1}^\lambda(x)$, and an easy induction argument shows that

$$\lim_{\lambda \uparrow 1} (1-\lambda)\, G^\lambda g(0)\, H_k^\lambda(x) = (\Pi_B)^k L_B(x).$$

Thus
$$\sum_{n=0}^{\infty} P_x(N_n(B) \leqslant k)\, \lambda^n \sim \left[\sum_{j=0}^{k} (\Pi_B)^j\right] L_B(x)\, [G^\lambda g(0)\,(1-\lambda)]^{-1}$$

$$\sim \left[\sum_{j=0}^{k} (\Pi_B)^j\right] L_B(x)\, (2\,\sigma^2)^{\frac{1}{2}} (1-\lambda)^{-\frac{1}{2}}.$$

Monotonicity of $P_x(N_n(B) \leqslant k)$ in n and Karamata's theorem now yields

$$P_x(N_n(B) \leqslant k) \sim \left[\sum_{j=0}^{k} (\Pi_j)^j\right] L_B(x) \left[\frac{2}{\pi}\right]^{-\frac{1}{2}} \sigma n^{-\frac{1}{2}},$$

and the result follows.

Let
$$E_B(n; A) = \int P_x(V_B \leqslant n; S_{V_B} \in A)\, dx$$

then
$$E_B(n; A) = \sum_{j=1}^{n} \int_B \tilde{P}_y(V_B \geqslant j)\, dy.$$

It follows from Theorems 13.2 and 13.3 that the following holds

COROLLARY 13.3. *Let $A, B \in \mathcal{A}^*$, $|B| > 0$. Then*

$$E_B(n; A) \sim \left[\int_A \tilde{l}_B(y)\, dy\right] E_B(n), \quad n \to \infty.$$

We will now return to the study of $E_B(n)$ and show there is a very interesting connection between $E_B(n)$ and the Robin's constant $k(B)$.

THEOREM 13.4. *Let A and $B \in \mathcal{A}^*$, $|A|$, $|B| > 0$, and assume μ generates a type two random walk. Then*

$$\lim_{n \to \infty} [E_B(n) - E_A(n)] = [k(B) - k(A)]\, 2\,\sigma^2 \tag{13.10}$$

Proof. It suffices to establish the result for $A \subset B$. By (13.1)

$$E_B(n)-E_A(n)=\sum_{j=0}^{n-1}\int_{B-A}P_x(V_A>n-1-j)\,\tilde{P}_x(V_B>j)\,dx. \tag{13.11}$$

Theorem 13.3 and a simple Abelian argument shows that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\int_{B-A}P_x(V_A>n-1-j)\,\tilde{P}_x(V_B>j)\,dx=2\,\sigma^2\int_{B-A}L_A(x)\,\tilde{L}_B(x)\,dx,$$

uniformly on compacts. Hence

$$\lim_{n\to\infty}[E_B(n)-E_A(n)]=2\,\sigma^2\int_{B-A}L_A(x)\tilde{L}_B(x)\,dx.$$

It follows from Lemma 11.1 that the right-hand side is just $[k(B)-k(A)]2\sigma^2$. This completes the proof.

THEOREM 13.5. *In any nonsingular type one random walk for any two sets A, $B\in\mathcal{B}$ having nonzero measure,*

$$\lim_{n\to\infty}\frac{\sum_{m=1}^{n}[E_B(m)-E_A(m)]}{\sum_{m=0}^{n-1}\left(\sum_{j=0}^{m}q_j\,q_{m-j}\right)}=k(B)-k(A). \tag{13.12}$$

Moreover, if for some g, $U^\lambda g$ satisfies the condition in Proposition 13.1, then

$$\lim_{n\to\infty}\frac{E_B(n)-E_A(n)}{\sum_{j=0}^{n-1}q_j q_{n-1-j}}=k(B)-k(A), \tag{13.13}$$

and

$$\sum_{j=0}^{n-1}q_j q_{n-j}\sim\frac{n^{-2\alpha+1}}{H(n)^2\,\Gamma(2-2\,\alpha)}.$$

Proof. It suffices to consider the case $A\subset B$. Then from (13.11)

$$\sum_{m=1}^{n}[E_B(m)-E_A(m)]=\int_{B-A}dx\left[\sum_{m=0}^{n-1}\sum_{j=0}^{m}\tilde{P}_x(V_B>j)\,P_x(V_A>m-j)\right]. \tag{13.14}$$

By Corollary 13.1 and Theorem 13.2 we know that uniformly on B,

$$\sum_{j=0}^{m}\tilde{P}_x(V_B>j)\sim\tilde{L}_B(x)\sum_{j=0}^{m}q_j$$

$$\sum_{j=0}^{m}P_x(V_A>j)\sim L_A(x)\sum_{j=0}^{m}q_j.$$

An Abelian type argument (see Lemma 3.2 of [9] for details) then shows that the right-hand side of (13.14) is asymptotic to

$$\left[\int_{B-A} \tilde{L}_B(x)\, L_A(x)\, dx\right] \sum_{m=0}^{n} \sum_{j=0}^{m} q_{m-j}\, q_j$$

and (13.12) follows from this fact and the identification of the integral as $k(B)-k(A)$ given in Lemma 11.1. Finally if $U^\lambda g(0)$ satisfies the condition of Proposition 13.1, then by (13.8), (13.11) and a familiar Abelian theorem we see that

$$\sum_{j=0}^{n} \int_{B-A} \tilde{P}_x(V_B > j)\, P_x(V_A > n-j)\, dx \sim \left(\sum_{j=0}^{n} q_j\, q_{n-j}\right) [k(B)-k(A)]$$

$$\sim [k(B)-k(A)] \frac{n^{-2\alpha+1}}{H(n)^2} \frac{1}{\Gamma(2-2\alpha)}$$

and (13.13) follows.

Remark. The stronger form (13.13) was shown to be valid by Port [9] for every recurrent random walk on Z^1 or Z^2. A similar argument and the strong result of Ornstein on the behavior of $P_x(V_B > n)$ will show that (13.13) also holds for every recurrent random walk on R^1 or R^2 at least when A and B are intervals. The behavior of $E_B(n)$ is also of interest for transient random walks. Here the behavior is quite different and the analysis of $E_B(n)$ in that case can be found in our joint paper [12].

For a general singular walk we can only establish a weaker version of the above result.

PROPOSITION 13.2. *Let A, $B \in \mathcal{A}^*$, $|A|$, $|B| > 0$. Then*

$$\lim_{\lambda \uparrow 1} (1-\lambda)^2\, U^\lambda g(0)\, \tilde{U}^\lambda g(0) \sum_{n=0}^{\infty} \lambda^n [E_B(n) - E_A(n)] = k(B) - k(A). \tag{13.15}$$

Proof. It suffices to consider the case when $A \subset B$. From (13.11) we see that

$$\sum_{n=0}^{\infty} \lambda^n [E_B(n) - E_A(n)] = \int_{B-A} \tilde{Q}_B^\lambda(x)\, Q_A^\lambda(x)$$

and it follows that the limit on the left in (13.15) is $\int_{B-A} \tilde{L}_B(x) L_A(x)\, dx$. The desired result now follows from Lemma 11.1.

14. The compact case

In this final section we will consider a random walk on a compact group $\mathfrak{G}$ and discuss the analogues of the preceeding results for it. Of course, all the limit theorems are out (since there is no infinity) so the bulk of the difficult portions of the noncompact theory evaporate. Still, there are some of the potential theoretic results which are of interest but here too things are far simpler than before. We will not state any theorems as such in the compact case but merely sketch the pertinent facts.

The main difference in the compact case is that for any $f \in C(\mathfrak{G})$, $n^{-1}\sum_{j=0}^{n} P^j f \to J(f)$ uniformly on $\mathfrak{G}$. (This well-known fact is an easy consequence of the ratio limit theorem of Stone [18]). The first thing to establish is the existence of a potential operator. Since the group $\mathfrak{G}$ is discrete in this case it easily follows from the Fourier analysis that if $\mathfrak{J}$ is the class of all f such that $\hat{f}$ is compact, then for $g \in \mathfrak{J}$ and $\hat{g}(0) = \delta(x, 0)$

$$Af(x) = \lim_{\lambda \uparrow 1} A^{\lambda} f = \lim_{\lambda \uparrow 1} [G^{\lambda} g(0)\, J(f) - G^{\lambda} f] \tag{14.1}$$

exists and the convergence is uniform on $\mathfrak{G}$. In the nonsingular case it easily follows from the ratio theorem of Stone [18] and Doeblin's condition that the limit in [14.1] also exists for all bounded f when say $g = 1$. [In the nonsingular case we will take $g = 1$ in the definition of A. Note however that unless $S(\mu) - S(\mu)$ generates $\mathfrak{G}$ these Abel limits cannot be replaced with ordinary convergence.]

Using the identity

$$A^{\lambda} f - H_B A^{\lambda} f = -G_B^{\lambda} f + J(f)\, \mathcal{L}_B^{\lambda}(x) \tag{14.2}$$

and the fact that for B having nonempty interior, or in the nonsingular case, just positive measure

$$\lim_{\lambda \uparrow 1} \mathcal{L}_B^{\lambda}(x) = E_x T_B,$$

we find that for $f \in \mathfrak{J}$ (or Φ in the nonsingular case) that

$$Af - H_B Af = -G_B f + J(f)\, E_x T_B, \tag{14.3}$$

which is what the basic identity becomes in the present context.

For a singular walk on $\mathfrak{G}$ there is very little more that can be said so from now on we will consider a nonsingular walk.

Let $\tilde{l}_B(x)\,dx = \int_{\mathfrak{G}} H_B(y, dx)\,dy$. Then it follows from (14.3) and an easy duality argument that

$$\tilde{A}\tilde{l}_B(x) = \tilde{G}_B 1(x) - \int_{\mathfrak{G}} (E_x T_B)\, dx \quad \text{a.e.}$$

Setting $ET_B = \int_{\mathfrak{G}} (E_x T_B)\,dx$ we see from the above that

$$\tilde{A}\tilde{l}_B = E_x \tilde{T}_B - ET_B \quad \text{a.e.} \tag{14.4}$$

and, in particular, $\tilde{A}\tilde{l}_B(x) = -ET_B$ a.e. on B. Thus $-ET_B$ is the dual Robin's constant $k(B)$ of B. Since $E\tilde{T}_B = ET_B$ we see that $\tilde{k}(B) = k(B)$ and

$$Al_B(x) = E_x T_B - ET_B \quad \text{a.e.} \tag{14.5}$$

If we use the operator $A_b\varphi = A\varphi + bJ(\varphi)$, then l_B, $\tilde{l}_B$ are still the equilibrium charges, but now the equilibrium potential is

$$Al_B(x) = E_x T_B - (ET_B - b) \quad \text{a.e.}$$

so that the b-Robin's constant is $k_b(B) = b - ET_B$.

It easily follows from (14.2) that

$$(P-I)A_b\varphi = A_b(P-I)\varphi = \varphi - J(\varphi), \tag{14.6}$$

from which it can easily be deduced that if $A_b\varphi_1 = A_b\varphi_2$ a.e. and $J(\varphi_1) = J(\varphi_2)$, then $\varphi_1 = \varphi_2$ a.e. The analogue of the uniqueness, minimum, and domination principles then easily follow from this fact and (14.3) just as in § 11.

The mapping A_b of the bounded measurable functions on $\mathfrak{G}$ into themselves is a bijection for $b \neq 0$. Indeed by (14.6) given any φ the function $(P-I)\varphi + (J(\varphi))/b = \psi$ has potential $A\psi = \varphi$, so that for A_b, $b \neq 0$ the range of A_b is the entire space. This is a sharper version of Theorem 12.1. To see that it is 1–1 note that if $A_bf = A_bg$ a.e., then $bJ(f) = (1, A_bf) = (1, A_bg) = bJ(g)$, so $J(f) = J(g)$ and the conclusion follows from the uniqueness principle stated above. Both of these facts are clearly false if $b = 0$ because then $A1 = 0$ and 1 annihilates the range of A.

Let B be such that $|B| > 0$ and $k_b(B) \neq 0$. Then Theorem 12.2 is valid for functions in $\Phi(B)$. The proof is the same as in the noncompact case. Using the second proof of the Balyage principle following Theorem 12.2 we see that this principle is also valid in the compact case.

Finally, consider the Poisson equation

$$(P-I)f = \psi. \tag{14.7}$$

Then

$$P^{n+1}f = f + \sum_{j=0}^{n} P^j\psi$$

and as $1/n \sum_{j=0}^{n} P^j f \to J(f)$ it follows that $\lim_{\lambda \uparrow 1} G^\lambda \psi$ exists. But that is only possible if $J(\psi) = 0$. Thus in order that there exist a solution of (14.7) at all it is necessary that ψ be a null function. It follows at once from (14.6) that in that case $A\psi$ is a solution and that the only other solutions are of the form $A\psi + \beta$ for constants β. Since all our functions are bounded we see that this result is the same as in the case of a noncompact $\mathfrak{G}$.

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