# COHOMOLOGY OF OPERATOR ALGEBRAS

## I. *Type I von Neumann algebras*

BY

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### 1. Introduction

The cohomology theory of associative linear algebras over an arbitrary field was initiated and developed by Hochschild [5, 6, 7]. With $\mathfrak{A}$ an algebra and $\mathcal{M}$ a two-sided $\mathfrak{A}$-module, the linear space $C^n(\mathfrak{A}, \mathcal{M})$ of *n-cochains* consists of all $n$-linear mappings from $\mathfrak{A} \times \mathfrak{A} \times \ldots \times \mathfrak{A}$ into $\mathcal{M}$. The *coboundary operator* $\Delta$ maps $C^n(\mathfrak{A}, \mathcal{M})$ linearly into $C^{n+1}(\mathfrak{A}, \mathcal{M})$ for each $n = 0, 1, 2, \ldots$, and satisfies $\Delta^2 = 0$. With $Z^n(\mathfrak{A}, \mathcal{M})$ the null-space of $\Delta$ in $C^n(\mathfrak{A}, \mathcal{M})$, and $B^{n+1}(\mathfrak{A}, \mathcal{M})$ the range of $\Delta$ in $C^{n+1}(\mathfrak{A}, \mathcal{M})$, we have $B^n(\mathfrak{A}, \mathcal{M}) \subseteq Z^n(\mathfrak{A}, \mathcal{M})$ $(n = 1, 2, \ldots)$. The quotient space $Z^n(\mathfrak{A}, \mathcal{M})$ is called the *n-dimensional cohomology group* of $\mathfrak{A}$, with coefficients in $\mathcal{M}$, and is denoted by $H^n(\mathfrak{A}, \mathcal{M})$.

The present paper is concerned with cohomology groups of operator algebras. For such algebras there are several possible cohomology theories, closely analogous to the Hochschild theory in algebraic structure, but differing from one another in the nature and extent of the topological properties required of the module $\mathcal{M}$, the action of $\mathfrak{A}$ on $\mathcal{M}$, and the $n$-linear mappings which are admitted as $n$-cochains. The Hochschild theory itself is available but, with one important exception, the problem of computing the (purely algebraic) cohomology groups $H^n(\mathfrak{A}, \mathcal{M})$, with $\mathfrak{A}$ a $C^*$-algebra and $\mathcal{M}$ a two-sided $\mathfrak{A}$-module, seems intractable. The exceptional case, which has provided much of the motivation for the work in this paper, arises from the fact that a von Neumann algebra $\mathcal{R}$ has no outer derivations ([10, 13]; for a later proof see [9]). This result can easily be reformulated, in terms of the Hochschild cohomology theory, as the assertion that $H^1(\mathcal{R}, \mathcal{R}) = 0$. A number of other problems concerning derivations of operator algebras can be expressed in cohomological terms. The present paper treats the case in which the module $\mathcal{M}$ is the dual space of some Banach space, and the bilinear mappings $(A, m) \to Am$ and $(A, m) \to mA$ (from $\mathfrak{A} \times \mathcal{M}$ into

$\mathcal{M}$) which arise from the left and right action of $\mathfrak{A}$ on $\mathcal{M}$ are required to be norm continuous in both variables and weak * continuous in the second. Only bounded $n$-linear mappings are admitted, in this theory, as $n$-cochains. The cohomology groups which arise in this setting are denoted by $H_c^n(\mathfrak{A}, \mathcal{M})$, $n=1, 2, \ldots$ Preparatory results concerning the centre-adjustment of norm continuous cocycles are proved in § 3, and in Theorem 4.4 we show that $H_c^n(\mathcal{R}, \mathcal{R})=0$ $(n=1, 2, \ldots)$ when $\mathcal{R}$ is a type I von Neumann algebra. Further results concerning more general von Neumann algebras will be published subsequently.

We consider briefly another (*normal*) cohomology theory applicable to a $C^*$-algebra $\mathfrak{A}$ represented as operators acting on a Hilbert space $\mathcal{H}$. In this theory, in addition to the conditions outlined above, we require ultraweak continuity properties of both the action of $\mathfrak{A}$ on the module $\mathcal{M}$ and also the $n$-linear mappings which are admitted as $n$-cochains. The cohomology groups which arise in this context are denoted by $H_w^n(\mathfrak{A}, \mathcal{M})$, $n=1, 2, \ldots$; in Corollary 4.6 we show that $H_w^n(\mathcal{R}, \mathcal{R})=0$ when $\mathcal{R}$ is a type I factor.

Cohomology groups of commutative Banach algebras have previously been studied by Kamowitz [11]; more recently, Johnson [8] has considered the cohomology theory of general Banach algebras, and (by methods quite different from ours) has obtained a number of results including a proof of our Theorem 4.4.

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## 2. Terminology and notation

Throughout this paper, the term *algebra* is understood to refer to an associative linear algebra over the complex field, and vector spaces always have complex scalars. We recall that a *left module* for a unital algebra $\mathfrak{A}$ is a vector space $\mathcal{M}$ equipped with a bilinear mapping $(A, m) \to Am$: $\mathfrak{A} \times \mathcal{M} \to \mathcal{M}$ such that $Im=m$, $A_1(A_2m)=(A_1A_2)m$ whenever $A_1$, $A_2 \in \mathfrak{A}$ and $m \in \mathcal{M}$; the concept of *right module* is defined similarly. A *two-sided $\mathfrak{A}$-module* is a vector space $\mathcal{M}$ which is both a left $\mathfrak{A}$-module and a right $\mathfrak{A}$-module, the left and right actions of $\mathfrak{A}$ on $\mathcal{M}$ being related by the condition $(A_1m)A_2=A_1(mA_2)$.

By a *two-sided Banach module* for a Banach algebra $\mathfrak{A}$ we mean a Banach space $\mathcal{M}$ which is a two-sided $\mathfrak{A}$-module for which the bilinear mappings $(A, m)\to Am$, $(A, m)\to mA$ from $\mathfrak{A}\times\mathcal{M}$ into $\mathcal{M}$ are bounded. If, further, $\mathcal{M}$ is (isometrically isomorphic to) the dual space of a Banach space $\mathcal{M}_*$ and, for each $A$ in $\mathfrak{A}$, the mappings $m\to Am$ and $m\to mA$: $\mathcal{M}\to\mathcal{M}$ are weak * continuous, we refer to $\mathcal{M}$ as a two-sided *dual* $\mathfrak{A}$-module.

With $\mathfrak{A}$ a Banach algebra, $\mathcal{M}$ a two-sided Banach $\mathfrak{A}$-module and $n$ a positive integer, we denote by $C_c^n(\mathfrak{A}, \mathcal{M})$ the linear space of all bounded $n$-linear mappings from $\mathfrak{A}\times\mathfrak{A}\times \dots\times\mathfrak{A}$ into $\mathcal{M}$, and refer to these mappings as (continuous) *n-cochains*. The *coboundary operator* $\Delta$, from $C_c^n(\mathfrak{A}, \mathcal{M})$ into $C_c^{n+1}(\mathfrak{A}, \mathcal{M})$, is defined by

$$(\Delta\varrho)(A_0, \dots, A_n) = A_0\varrho(A_1, \dots, A_n) + \sum_{j=1}^{n}(-1)^j\varrho(A_0, \dots, A_{j-2}, A_{j-1}A_j, A_{j+1}, \dots, A_n) + (-1)^{n+1}\varrho(A_0, \dots, A_{n-1})A_n. \tag{1}$$

By convention, $C_c^0(\mathfrak{A}, \mathcal{M})$ is $\mathcal{M}$, and $\Delta$: $C_c^0(\mathfrak{A}, \mathcal{M})\rightarrow C_c^1(\mathfrak{A}, \mathcal{M})$ is defined by $(\Delta m)(A) = Am - mA$, for $A$ in $\mathfrak{A}$ and $m$ in $\mathcal{M}$. For $n=0, 1, 2, \dots$, the range of $\Delta$ in $C_c^{n+1}(\mathfrak{A}, \mathcal{M})$ is a linear space denoted by $B_c^{n+1}(\mathfrak{A}, \mathcal{M})$, the space of $(n+1)$-*coboundaries*; and the nullspace of $\Delta$ in $C_c^n(\mathfrak{A}, \mathcal{M})$ is denoted by $Z_c^n(\mathfrak{A}, \mathcal{M})$, the space of *n-cocycles*. It is only the continuity conditions which distinguish these concepts from the analogous ones introduced by Hochschild [5] in the purely algebraic context; and, just as in the algebraic case, it can be shown that $\Delta^2 = 0$. From this, it follows that $B_c^n(\mathfrak{A}, \mathcal{M})\subseteq Z_c^n(\mathfrak{A}, \mathcal{M})$ $(n=1, 2, \dots)$; the quotient space $Z_c^n(\mathfrak{A}, \mathcal{M})/B_c^n(\mathfrak{A}, \mathcal{M})$ is denoted by $H_c^n(\mathfrak{A}, \mathcal{M})$ and called the $n$-dimensional (continuous) *cohomology group* (of $\mathfrak{A}$, with coefficients in $\mathcal{M}$).

The simplest example of a two-sided Banach module for a Banach algebra $\mathfrak{A}$ is obtained by taking $\mathcal{M}=\mathfrak{A}$, with $Am$ and $mA$ interpreted as products in $\mathfrak{A}$ when $A, m\in\mathfrak{A}$. The coboundary of an element $B$ of $\mathfrak{A}(=C_c^0(\mathfrak{A}, \mathfrak{A}))$ is the inner derivation $A\rightarrow AB-BA$ of $\mathfrak{A}$, while $Z_c^1(\mathfrak{A}, \mathfrak{A})$ consists of all continuous derivations of $\mathfrak{A}$; thus $H_c^1(\mathfrak{A}, \mathfrak{A})=0$ if and only if $\mathfrak{A}$ has no continuous outer derivations.

If $\mathfrak{A}$ is a Banach algebra with centre $\mathcal{C}$, $\mathcal{M}$ is a two-sided Banach $\mathfrak{A}$-module and $n\geqslant 1$, we denote by $NC_c^n(\mathfrak{A}, \mathcal{M})$ the class of all cochains $\varrho$ in $C_c^n(\mathfrak{A}, \mathcal{M})$ which satisfy

$$\varrho(A_1, \dots, A_{j-1}, CA_j, A_{j+1}, \dots, A_n) = C\varrho(A_1, \dots, A_n) = \varrho(A_1, \dots, A_n)C$$

whenever $1\leqslant j\leqslant n$, $C\in\mathcal{C}$ and $A_1, \dots, A_n\in\mathfrak{A}$. By convention,

$$NC_c^0(\mathfrak{A}, \mathcal{M}) = \{m\in\mathcal{M}:\ Cm = mC \text{ for each } C \text{ in } \mathcal{C}\}.$$

With $NZ_c^n(\mathfrak{A}, \mathcal{M})$ defined to be $Z_c^n(\mathfrak{A}, \mathcal{M})\cap NC_c^n(\mathfrak{A}, \mathcal{M})$, it follows at once from the coboundary formula that $\Delta$ maps $NC_c^n(\mathfrak{A}, \mathcal{M})$ into $NZ_c^{n+1}(\mathfrak{A}, \mathcal{M})$ $(n=0, 1, 2, \dots)$. We prove in Corollary 3.5 that, if $\mathcal{M}$ is a two-sided dual module for a $C^*$-algebra $\mathfrak{A}$, then each $\varrho$ in $Z_c^n(\mathfrak{A}, \mathcal{M})$ is cohomologous to (that is, differs by a coboundary from) an element of $NZ_c^n(\mathfrak{A}, \mathcal{M})$. It turns out that, for $C^*$-algebras, elements of $NZ_c^n(\mathfrak{A}, \mathfrak{A})$ are in some respects more easily handled than general norm-continuous cocycles—a point that is illustrated by our proof, in section 4, that $H_c^n(\mathcal{R}, \mathcal{R})=0$ $(n=1, 2, \ldots)$ when $\mathcal{R}$ is a type I von Neumann algebra.

We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators acting on the Hilbert space $\mathcal{H}$. If $\mathfrak{A}$ is a $C^*$-algebra acting on $\mathcal{H}$ and $\mathcal{M}$ is a normed closed subspace of $B(\mathcal{H})$, then $\mathcal{M}$ is a two-sided Banach $\mathfrak{A}$-module (with $Am$ and $mA$ the usual operator products) provided $Am \in \mathcal{M}$ and $mA \in \mathcal{M}$ whenever $A \in \mathfrak{A}$ and $m \in \mathcal{M}$. The cases in which $\mathcal{M}=\mathfrak{A}$, $\mathcal{M}=\mathfrak{A}^-$ (the weak operator closure of $\mathfrak{A}$), $\mathcal{M}=\mathcal{B}(\mathcal{H})$, or $\mathcal{M}$ is an ideal in $\mathfrak{A}$, are of particular interest. If $\mathcal{M}$ is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$ such that $Am \in \mathcal{M}$, $mA \in \mathcal{M}$ whenever $A \in \mathfrak{A}$ and $m \in \mathcal{M}$, then $\mathcal{M}$ is a two-sided dual $\mathfrak{A}$-module: for $\mathcal{M}$ can be identified with the dual space of the Banach space $\mathcal{M}_*$ of all ultraweakly continuous linear functionals on $\mathcal{M}$ [3: p. 38, Théorème 1], with the weak * topology corresponding to the ultraweak topology, and the mappings $m \to Am$, $m \to mA$: $\mathcal{M} \to \mathcal{M}$ are ultraweakly continuous, for each $A$ in $\mathfrak{A}$.

With $\mathfrak{A}$ a $C^*$-algebra acting on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ a two-sided dual $\mathfrak{A}$-module, we describe $\mathcal{M}$ as a two-sided dual *normal* $\mathfrak{A}$-module if, for each $m$ in $\mathcal{M}$, the mappings $A \to Am$, $A \to mA$ are continuous from $\mathfrak{A}$ (with the ultraweak topology) into $\mathcal{M}$ (with the weak * topology). Since operator multiplication is separately continuous in the ultraweak topology, the examples described at the end of the preceding paragraph are of this type. Given such a module, and a positive integer $n$, we denote by $C_w^n(\mathfrak{A}, \mathcal{M})$ the linear space of all $n$-linear mappings from $\mathfrak{A} \times \mathfrak{A} \times \ldots \times \mathfrak{A}$ into $\mathcal{M}$ which are continuous in each variable (separately) from $\mathfrak{A}$ (with the ultraweak topology) into $\mathcal{M}$ (with the weak * topology). We refer to elements of $C_w^n(\mathfrak{A}, \mathcal{M})$ as *normal* $n$-cochains. Bearing in mind the ultraweak-weak * continuity of the mappings $A \to Am$, $A \to mA$, it is apparent that the coboundary operator $\Delta$ (again defined by (1)) carries $C_w^n(\mathfrak{A}, \mathcal{M})$ into $C_w^{n+1}(\mathfrak{A}, \mathcal{M})$ for $n=1, 2, \ldots$, and, as before, $\Delta^2=0$. The same is true when $n=0$, with the convention that $C_w^0(\mathfrak{A}, \mathcal{M})$ is $\mathcal{M}$ and $(\Delta m)(A)=Am-mA$, for $A$ in $\mathfrak{A}$ and $m$ in $\mathcal{M}$. With $Z_w^n(\mathfrak{A}, \mathcal{M})$ the nullspace of $\Delta$ in $C_w^n(\mathfrak{A}, \mathcal{M})$, and $B_w^{n+1}(\mathfrak{A}, \mathcal{M}) (\subseteq C_w^{n+1}(\mathfrak{A}, \mathcal{M}))$ the image under $\Delta$ of $C_w^n(\mathfrak{A}, \mathcal{M})$, we have $B_w^n(\mathfrak{A}, \mathcal{M}) \subseteq Z_w^n(\mathfrak{A}, \mathcal{M})$ $(n=1, 2, \ldots)$; the quotient space $H_w^n(\mathfrak{A}, \mathcal{M})=Z_w^n(\mathfrak{A}, \mathcal{M})/B_w^n(\mathfrak{A}, \mathcal{M})$ is the *n-dimensional normal cohomology group* (of $\mathfrak{A}$, with coefficients in $\mathcal{M}$).

We observe that $C_w^n(\mathfrak{A}, \mathcal{M}) \subseteq C_c^n(\mathfrak{A}, \mathcal{M})$. With $\varrho$ in $C_w^n(\mathfrak{A}, \mathcal{M})$ and $A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n$ in $\mathfrak{A}$, the mapping $A_j \to \varrho(A_1, \ldots, A_n)$ from $\mathfrak{A}$ into $\mathcal{M}$ is ultraweak-weak * continuous; so it carries the (ultraweakly bounded) unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$ onto a subset of $\mathcal{M}$ which is weak * bounded, hence norm bounded. Thus $\varrho$ is norm continuous in each of its variables (separately) and, from [4: p. 70, Exercise 4] (essentially the uniform boundedness principle), $\varrho$ is a bounded $n$-linear mapping; so $\varrho \in C_c^n(\mathfrak{A}, \mathcal{M})$.

## 3. Centre normalisation of cocycles

The main result of this section, Theorem 3.4, is concerned with norm continuous cocycles with coefficients in a dual module. The first two lemmas, although stated here in terms of norm continuous cocycles, are valid in a purely algebraic context.

LEMMA 3.1. *If $\mathfrak{A}$ is a Banach algebra with centre $\mathcal{C}$, $\mathcal{M}$ is a two-sided Banach $\mathfrak{A}$-module, $1 \leqslant k \leqslant n$ and $\varrho$ in $Z_c^n(\mathfrak{A}, \mathcal{M})$ vanishes whenever any of its first $k$ arguments lies in $\mathcal{C}$, then*

$$\varrho(A_1, ..., A_{j-1}, CA_j, A_{j+1}, ..., A_n) = C\varrho(A_1, ..., A_n)$$

*whenever $1 \leqslant j \leqslant k$, $C \in \mathcal{C}$ and $A_1, ..., A_n \in \mathfrak{A}$.*

*Proof.* Since

$$\begin{aligned}0 = (\Delta\varrho)(C, A_1, ..., A_n) &= C\varrho(A_1, ..., A_n) - \varrho(CA_1, A_2, ..., A_n) + \varrho(C, A_1A_2, A_3, ..., A_n)\\ &\quad - ... \pm \varrho(C, A_1, ..., A_{n-2}, A_{n-1}A_n) \mp \varrho(C, A_1, ..., A_{n-1})A_n\\ &= C\varrho(A_1, ..., A_n) - \varrho(CA_1, A_2, ..., A_n),\end{aligned}$$

we have $\varrho(CA_1, A_2, ..., A_n) = C\varrho(A_1, ..., A_n)$ whenever $C \in \mathcal{C}$ and $A_1, ..., A_n \in \mathfrak{A}$. In addition, if $1 < j \leqslant k$,

$$\begin{aligned}0 &= (\Delta\varrho)(A_1, ..., A_{j-1}, C, A_j, ..., A_n)\\ &= A_1\varrho(A_2, ..., A_{j-1}, C, A_j, ..., A_n) - \varrho(A_1A_2, A_3, ..., A_{j-1}, C, A_j, ..., A_n)\\ &\quad + ... \mp \varrho(A_1, ..., A_{j-2}, A_{j-1}C, A_j, ..., A_n) \pm \varrho(A_1, ..., A_{j-1}, CA_j, A_{j+1}, ..., A_n)\\ &\quad \pm ... + (-1)^{n+1}\varrho(A_1, ..., A_{j-1}, C, A_j, ..., A_{n-1})A_n\\ &= \pm[\varrho(A_1, ..., A_{j-2}, CA_{j-1}, A_j, ..., A_n) - \varrho(A_1, ..., A_{j-1}, CA_j, A_{j+1}, ..., A_n)].\end{aligned}$$

Thus

$$\begin{aligned}\varrho(A_1, ..., A_{j-1}, CA_j, A_{j+1}, ..., A_n) &= \varrho(A_1, ..., A_{j-2}, CA_{j-1}, A_j, ..., A_n)\\ &= ... = \varrho(CA_1, A_2, ..., A_n) = C\varrho(A_1, ..., A_n).\end{aligned}$$

LEMMA 3.2. *If $\mathfrak{A}$ is a Banach algebra with centre $\mathcal{C}$, $\mathcal{M}$ is a two-sided Banach $\mathfrak{A}$-module, $n \geqslant 1$ and $\varrho$ in $Z_c^n(\mathfrak{A}, \mathcal{M})$ vanishes whenever any of its arguments lies in $\mathcal{C}$, then $\varrho \in NZ_c^n(\mathfrak{A}, \mathcal{M})$.*

*Proof.* By Lemma 3.1,

$$\varrho(A_1, ..., A_{j-1}, CA_j, A_{j+1}, ..., A_n) = C\varrho(A_1, ..., A_n)$$

whenever $1 \leqslant j \leqslant n$, $C \in \mathcal{C}$ and $A_1, ..., A_n \in \mathfrak{A}$. Furthermore, the first $n$ terms in the expansion of $(\Delta\varrho)(A_1, ..., A_n, C)$ are zero, so

$$0=(\Delta\varrho)(A_1,\dots,A_n,C)=\varrho(A_1,\dots,A_{n-1},A_nC)-\varrho(A_1,\dots,A_n)C$$
$$=\varrho(A_1,\dots,A_{n-1},CA_n)-\varrho(A_1,\dots,A_n)C$$
$$=C\varrho(A_1,\dots,A_n)-\varrho(A_1,\dots,A_n)C.$$

Suppose that $\mathcal{V}$ is a discrete group and $l_\infty(\mathcal{V})$ is the Banach space of all bounded complex-valued functions on $\mathcal{V}$. We recall [1: p. 515] that $\mathcal{V}$ is said to be *amenable* if it has a two-sided *invariant mean*, that is, a linear functional $\mu$ on $l_\infty(\mathcal{V})$ such that

(i) $\mu(f)\geqslant 0$ if $f\in l_\infty(\mathcal{V})$ and $f(V)\geqslant 0$ for each $V$ in $\mathcal{V}$;

(ii) $\mu(f_W)=\mu({}_Wf)=\mu(f)$ whenever $W\in\mathcal{V}$ and $f\in l_\infty(\mathcal{V})$, where $f_W(V)=f(VW)$, ${}_Wf(V)=f(WV)$;

(iii) $\mu(f)=1$ if $f(V)=1$ for all $V$ in $\mathcal{V}$.

Conditions (i) and (iii) imply that $\mu$ is a bounded linear functional, with $\|\mu\|=1$.

For completeness, we give a proof of the following lemma, which is of familiar type.

LEMMA 3.3. *Suppose that $\mathcal{M}$ is a two-sided dual module for a unital Banach algebra $\mathfrak{A}$, $\mathcal{G}$ is the multiplicative group of all invertible elements of $\mathfrak{A}$, $\mathcal{V}$ is a subgroup of $\mathcal{G}$ with an invariant mean $\mu$, and $l_\infty(\mathcal{V},\mathcal{M})$ is the Banach space of all bounded $\mathcal{M}$-valued functions $\varphi$ on $\mathcal{V}$, with $\|\varphi\|=\sup\{\|\varphi(V)\|: V\in\mathcal{V}\}$. Then there is a norm-decreasing linear mapping $\bar\mu$ from $l_\infty(\mathcal{V},\mathcal{M})$ into $\mathcal{M}$ such that*

(i) *if $A, B\in\mathfrak{A}$, $\varphi\in l_\infty(\mathcal{V},\mathcal{M})$ and $\varphi_1(V)=A\varphi(V)B$ for all $V$ in $\mathcal{V}$, then $\bar\mu(\varphi_1)=A\bar\mu(\varphi)B$;*

(ii) *if $W\in\mathcal{V}$, $\varphi\in l_\infty(\mathcal{V},\mathcal{M})$ and $\varphi_W(V)=\varphi(VW)$ for all $V$ in $\mathcal{V}$, then $\bar\mu(\varphi_W)=\bar\mu(\varphi)$;*

(iii) *$\bar\mu(\varphi)=m$ if $\varphi(V)=m(\in\mathcal{M})$ for each $V$ in $\mathcal{V}$.*

*Proof.* We can identify $\mathcal{M}$ with the dual of a Banach space $\mathcal{M}_*$, denoting by $\langle m, m_*\rangle$ the canonical bilinear form on $\mathcal{M}\times\mathcal{M}_*$. For each $\varphi$ in $l_\infty(\mathcal{V},\mathcal{M})$ and $m_*$ in $\mathcal{M}_*$, the mapping $V\to\langle\varphi(V), m_*\rangle$ is in $l_\infty(\mathcal{V})$, with norm not exceeding $\|\varphi\|\,\|m_*\|$. Its mean, $f(\varphi, m_*)$, satisfies $|f(\varphi,m_*)|\leqslant\|\varphi\|\,\|m_*\|$; so, for each fixed $\varphi$ in $l_\infty(\mathcal{V},\mathcal{M})$, the mapping $\bar\mu(\varphi)$: $m_*\to f(\varphi,m_*)$ is a bounded linear functional on $\mathcal{M}_*$, with $\|\bar\mu(\varphi)\|\leqslant\|\varphi\|$. Thus $\bar\mu$ is a norm-decreasing mapping from $l_\infty(\mathcal{V},\mathcal{M})$ into $\mathcal{M}$, and is clearly linear. Since the mean of the constant mapping $V\to\langle m, m_*\rangle$ is $\langle m, m_*\rangle$, $\langle\bar\mu(\varphi), m_*\rangle=\langle m, m_*\rangle$ for each $m_*$ in $\mathcal{M}_*$ (and thus $\bar\mu(\varphi)=m$) when $\varphi(V)=m$ for every $V$ in $\mathcal{V}$. From invariance of $\mu$, the mean $\langle\bar\mu(\varphi_W), m_*\rangle$ of the mapping $V\to\langle\varphi_W(V), m_*\rangle=\langle\varphi(VW), m_*\rangle$ is $\langle\bar\mu(\varphi), m_*\rangle$; so $\bar\mu(\varphi_W)=\bar\mu(\varphi)$.

For $A$, $B$ in $\mathfrak{A}$ and $m_*$ in $\mathcal{M}_*$, the mappings $m\to Am$, $m\to mB$ from $\mathcal{M}$ into $\mathcal{M}$ are weak * continuous, so $m\to\langle AmB, m_*\rangle$ is a weak * continuous linear functional on $\mathcal{M}$. By Phillips' theorem [4: Theorem 9, p. 421] there is an element $n_*$ of $\mathcal{M}_*$ such that $\langle AmB, m_*\rangle=\langle m, n_*\rangle$ for every $m$ in $\mathcal{M}$. With $\varphi$ and $\varphi_1$ as in (i), $\langle\bar\mu(\varphi_1), m_*\rangle$ is the mean of the mapping

$$V \to \langle \varphi_1(V), m_* \rangle = \langle A\varphi(V)B, m_* \rangle = \langle \varphi(V), n_* \rangle;$$

so $\langle \bar{\mu}(\varphi_1), m_* \rangle = \langle \bar{\mu}(\varphi), n_* \rangle = \langle A\bar{\mu}(\varphi)B, m_* \rangle$, for each $m_*$ in $\mathfrak{M}_*$. Thus $\bar{\mu}(\varphi_1) = A\bar{\mu}(\varphi)B$.

THEOREM 3.4. *If $\mathfrak{A}$ is a unital C*-algebra with centre $\mathcal{C}$, $\mathfrak{M}$ is a two-sided dual $\mathfrak{A}$-module, $n \geq 1$ and $\varrho \in Z_c^n(\mathfrak{A}, \mathfrak{M})$, there is a $\xi$ in $C_c^{n-1}(\mathfrak{A}, \mathfrak{M})$ such that $(\varrho - \Delta\xi)(A_1, \ldots, A_n) = 0$ if some $A_j \in \mathcal{C}$.*

*Proof.* We shall prove by induction on $k$ that, for $k = 1, \ldots, n$, there is a $\xi_k$ in $C_c^{n-1}(\mathfrak{A}, \mathfrak{M})$ such that $(\varrho - \Delta\xi_k)(A_1, \ldots, A_n) = 0$ if any one of $A_1, \ldots, A_k$ lies in $\mathcal{C}$. The theorem then follows, with $\xi = \xi_n$.

With $\mathcal{V}$ the unitary group of $\mathcal{C}$, the linear span of $\mathcal{V}$ is $\mathcal{C}$; and, since $\mathcal{V}$ is abelian, it has a two-sided invariant mean $\mu$ [12; p. 79: see also 2; p. 406]. We introduce the norm-decreasing mapping $\bar{\mu}$, from $l_\infty(\mathcal{V}, \mathfrak{M})$ into $\mathfrak{M}$, as in Lemma 3.3, and refer to $\bar{\mu}(\varphi)$ as the mean of $\varphi(\in l_\infty(\mathcal{V}, \mathfrak{M}))$.

With $A_1, \ldots, A_{n-1}$ in $\mathfrak{A}$, the mapping $V \to V^*\varrho(V, A_1, \ldots, A_{n-1})$ from $\mathcal{V}$ into $\mathfrak{M}$ is an element of $l_\infty(\mathcal{V}, \mathfrak{M})$, with norm not exceeding $K\|\varrho\|\|A_1\|\ldots\|A_{n-1}\|$, where $K$ is the bound of the bilinear mapping $(A, m) \to Am$: $\mathfrak{A} \times \mathfrak{M} \to \mathfrak{M}$. Its mean $\xi_1(A_1, \ldots, A_{n-1})$ is in $\mathfrak{M}$, and

$$\|\xi_1(A_1, \ldots, A_{n-1})\| \leq K\|\varrho\|\|A_1\|\ldots\|A_{n-1}\|;$$

it is clear that $\xi_1$ is multilinear, so $\xi_1 \in C_c^{n-1}(\mathfrak{A}, \mathfrak{M})$. Furthermore, since

$$(\Delta\xi_1)(A_1, \ldots, A_n) = A_1\xi_1(A_2, \ldots, A_n) + \sum_{j=1}^{n-1} (-1)^j \xi_1(A_1, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_n)$$
$$+ (-1)^n \xi_1(A_1, \ldots, A_{n-1})A_n,$$

it follows from Lemma 3.3 (i) and the definition of $\xi_1$ that $(\Delta\xi_1)(A_1, \ldots, A_n)$ is the mean of the mapping

$$V \to A_1 V^*\varrho(V, A_2, \ldots, A_n) + \sum_{j=1}^{n-1} (-1)^j V^*\varrho(V, A_1, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_n)$$
$$+ (-1)^n V^*\varrho(V, A_1, \ldots, A_{n-1})A_n = A_1 V^*\varrho(V, A_2, \ldots, A_n)$$
$$- V^*(\Delta\varrho)(V, A_1, \ldots, A_n) + V^*[V\varrho(A_1, \ldots, A_n) - \varrho(VA_1, A_2, \ldots, A_n)]$$
$$= \varrho(A_1, \ldots, A_n) + A_1 V^*\varrho(V, A_2, \ldots, A_n) - V^*\varrho(VA_1, A_2, \ldots, A_n).$$

When $A_1 \in \mathcal{V}$, this mapping is

$$V \to \varrho(A_1, \ldots, A_n) + A_1 V^*\varrho(V, A_2, \ldots, A_n) - A_1(VA_1)^*\varrho(VA_1, A_2, \ldots, A_n).$$

By Lemma 3.3 (iii) and (ii) (with $W = A_1$), its mean $(\Delta\xi_1)(A_1, \ldots, A_n)$ is $\varrho(A_1, \ldots, A_n)$. Thus $(\varrho - \Delta\xi_1)(A_1, \ldots, A_n) = 0$ whenever $A_1 \in \mathcal{V}$; by linearity, the same is true for all $A_1$ in $\mathcal{C}$. This proves the existence of a suitable cochain $\xi_1$ in $C_c^{n-1}(\mathfrak{A}, \mathfrak{M})$.

To continue the inductive process, suppose that $1 \leqslant k < n$ and a suitable cochain $\xi_k$ has been constructed. With $\sigma$ the element $\varrho - \Delta\xi_k$ of $Z_c^n(\mathfrak{A}, \mathcal{M})$,

$$\sigma(A_1, \ldots, A_n) = 0 \text{ if any one of } A_1, \ldots, A_k \text{ lies in } \mathcal{C}. \tag{2}$$

By Lemma 3.1,

$$\sigma(A_1, \ldots, A_{j-1}, CA_j, A_{j+1}, \ldots, A_n) = C\sigma(A_1, \ldots, A_n) \quad \text{if } 1 \leqslant j \leqslant k \text{ and } C \in \mathcal{C}. \tag{3}$$

With $A_1, \ldots, A_{n-1}$ in $\mathfrak{A}$, the mapping

$$V \to V^*\sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1}) \tag{4}$$

from $\mathcal{V}$ into $\mathcal{M}$ is an element of $l_\infty(\mathcal{V}, \mathcal{M})$, with norm not exceeding $K\|\sigma\|\,\|A_1\|\ldots\|A_{n-1}\|$. Its mean $\eta(A_1, \ldots, A_{n-1})$ is in $\mathcal{M}$, and

$$\|\eta(A_1, \ldots, A_{n-1})\| \leqslant K\|\sigma\|\,\|A_1\|\ldots\|A_{n-1}\|.$$

It is clear that $\eta$ is multilinear, so $\eta \in C_c^{n-1}(\mathfrak{A}, \mathcal{M})$.

We assert that

$$\eta(A_1, \ldots, A_{n-1}) = 0 \text{ if any one of } A_1, \ldots, A_k \text{ lies in } \mathcal{C}, \tag{5}$$

and that

$$\eta(A_1, \ldots, A_{j-1}, CA_j, A_{j+1}, \ldots, A_{n-1}) = C\eta(A_1, \ldots, A_{n-1}) \quad \text{if } 1 \leqslant j \leqslant k \text{ and } C \in \mathcal{C}. \tag{6}$$

For this, note that $\eta(A_1, \ldots, A_{n-1})$ is the mean of the mapping (4), which by (2) is the zero mapping under the conditions specified in (5). Furthermore, if $1 \leqslant j \leqslant k$ and $C \in \mathcal{C}$, the left-hand side of (6) is the mean of the mapping

$$V \to V^*\sigma(A_1, \ldots, A_{j-1}, CA_j, A_{j+1}, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1});$$

which, by (3), is the mapping

$$V \to C\,V^*\sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1}).$$

This last has mean $C\eta(A_1, \ldots, A_{n-1})$ by Lemma 3.3 (i).

It results from (5) and (6) that $(\Delta\eta)(A_1, \ldots, A_n) = 0$ if $A_j \in \mathcal{C}$ for some $j$ such that $1 \leqslant j \leqslant k$; for all terms except the $j$th and $(j+1)$st in the expansion of $(\Delta\eta)(A_1, \ldots, A_n)$ are zero by (5), while (6) implies that the two remaining terms are equal to $(-1)^{j-1}T$, $(-1)^jT$, where $T = A_j\eta(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n)$. This, together with (2), shows that

$$(\sigma \pm \Delta\eta)(A_1, \ldots, A_n) = 0 \text{ if any one of } A_1, \ldots, A_k \text{ lies in } \mathcal{C}. \tag{7}$$

We prove next that

$$(\sigma - (-1)^k\Delta\eta)(A_1, \ldots, A_n) = 0 \text{ if } A_{k+1} \in \mathcal{C}. \tag{8}$$

By linearity, it suffices to consider the case in which $A_{k+1} \in \mathfrak{V}$. Now the first $k$ terms in the expansion of $(\Delta\eta)(A_1, \ldots, A_n)$ are zero by (5), since $A_{k+1} \in \mathfrak{C}$; so

$$(\Delta\eta)(A_1, \ldots, A_n) = \sum_{j=k}^{n-1} (-1)^j \eta(A_1, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_n) + (-1)^n \eta(A_1, \ldots, A_{n-1}) A_n.$$

By Lemma 3.3 (i) and the definition of $\eta$, $(\Delta\eta)(A_1, \ldots, A_n)$ is the mean of the mapping $\varphi$ from $\mathfrak{V}$ into $\mathcal{M}$ which is defined by

$$\begin{aligned}\varphi(V) = (-1)^k V^* \sigma(A_1, \ldots, A_{k-1}, A_k A_{k+1}, V, A_{k+2}, \ldots, A_n) \\ + \sum_{j=k+1}^{n-1} (-1)^j V^* \sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_n) \\ + (-1)^n V^* \sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1}) A_n. \end{aligned} \tag{9}$$

In the expansion of $(\Delta\sigma)(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_n)$, the first $k$ terms are zero, for all $V$ in $\mathfrak{V}$ $(\subseteq \mathfrak{C})$, by (2): so

$$\begin{aligned} 0 &= V^*(\Delta\sigma)(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_n) \\ &= (-1)^k V^* \sigma(A_1, \ldots, A_{k-1}, A_k V, A_{k+1}, \ldots, A_n) \\ &\quad + (-1)^{k+1} V^* \sigma(A_1, \ldots, A_k, V A_{k+1}, A_{k+2}, \ldots, A_n) \\ &\quad + \sum_{=k+1}^{n-1} (-1)^{j+1} V^* \sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_n) \\ &\quad + (-1)^{n+1} V^* \sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1}) A_n. \end{aligned}$$

This, together with (9), gives

$$\begin{aligned} \varphi(V) = (-1)^k V^* \sigma(A_1, \ldots, A_{k-1}, A_k A_{k+1}, V, A_{k+2}, \ldots, A_n) \\ + (-1)^k V^* \sigma(A_1, \ldots, A_{k-1}, A_k V, A_{k+1}, \ldots, A_n) \\ + (-1)^{k+1} V^* \sigma(A_1, \ldots, A_k, V A_{k+1}, A_{k+2}, \ldots, A_n). \end{aligned}$$

Since $A_{k+1}$, $V \in \mathfrak{V} (\subseteq \mathfrak{C})$, it follows from (3) that

$$\begin{aligned} \varphi(V) = (-1)^k \{A_{k+1} V^* \sigma(A_1, \ldots, A_k, V, A_{k+2}, \ldots, A_n) + \sigma(A_1, \ldots, A_n) \\ - A_{k+1}(V A_{k+1})^* \sigma(A_1, \ldots, A_k, V A_{k+1}, A_{k+2}, \ldots, A_n)\}. \end{aligned}$$

By Lemma 3.3 (iii) and (ii) (with $W = A_{k+1}$), the mean $(\Delta\eta)(A_1, \ldots, A_n)$ of $\varphi$ is $(-1)^k \sigma(A_1, \ldots, A_n)$; so (8) is proved.

With $\xi_{k+1}$ the element $\xi_k + (-1)^k \eta$ of $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$, $\varrho - \Delta\xi_{k+1} = \varrho - \Delta\xi_k - (-1)^k \Delta\eta = \sigma - (-1)^k \Delta\eta$; thus $(\varrho - \Delta\xi_{k+1})(A_1, \ldots, A_n) = 0$ if any one of $A_1, \ldots, A_{k+1}$ lies in $\mathfrak{C}$, by (7) and (8). This completes the inductive construction of $\xi_1, \ldots, \xi_n$, and so proves the theorem.

COROLLARY 3.5. *If $\mathfrak{A}$ is a unital $C^*$-algebra, $\mathcal{M}$ is a two-sided dual $\mathfrak{A}$-module and $n \geqslant 1$, then $Z_c^n(\mathfrak{A}, \mathcal{M}) = B_c^n(\mathfrak{A}, \mathcal{M}) + NZ_c^n(\mathfrak{A}, \mathcal{M})$.*

*Proof.* With $\varrho$ in $Z_c^n(\mathfrak{A}, \mathcal{M})$, let $\xi$ in $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ satisfy the conclusion of Theorem 3.4. By Lemma 3.2, $\varrho - \Delta\xi \in NZ_c^n(\mathfrak{A}, \mathcal{M})$, so

$$\varrho = \Delta\xi + (\varrho - \Delta\xi) \in B_c^n(\mathfrak{A}, \mathcal{M}) + NZ_c^n(\mathfrak{A}, \mathcal{M}).$$

This shows that $Z_c^n(\mathfrak{A}, \mathcal{M}) \subseteq B_c^n(\mathfrak{A}, \mathcal{M}) + NZ_c^n(\mathfrak{A}, \mathcal{M})$. The reverse inclusion is apparent.

## 4. Cohomology of type I von Neuman algebras

This section is devoted primarily to a proof that $H_c^n(\mathcal{R}, \mathcal{R}) = 0$ $(n = 1, 2, \ldots)$ when $\mathcal{R}$ is a type I von Neumann algebra. We employ an idea used by Hochschild in showing that the cohomology groups (with coefficients in any module $\mathcal{M}$) of a full matrix ring $M_q(\Gamma)$ are all zero. With $\{e_{jk}: j, k = 1, \ldots, q\}$ the usual matrix units in $M_q(\Gamma)$, and $\varrho$ in $Z^n(M_q(\Gamma), \mathcal{M})$, it can be shown that $\varrho = \Delta\xi$, where $\xi$ in $C^{n-1}(M_q(\Gamma), \mathcal{M})$ is defined by

$$\xi(a_1, \ldots, a_{n-1}) = \sum_{j=1}^{q} e_{j1}\, \varrho(e_{1j}, a_1, \ldots, a_{n-1}) \tag{10}$$

[5: see p. 61 for the case $n = 1$; the general case then follows from Theorem 3.1 and the preceding discussion]. A similar construction of $\xi$ is possible when $\mathcal{R}$ is a type I von Neumann algebra and $\varrho \in NZ_c^n(\mathcal{R}, \mathcal{R})$; but in this context, (10) is replaced by an infinite series, and suitable convergence has to be established.

LEMMA 4.1. *Suppose that $\mathcal{R}$ is a von Neumann algebra of type $I_m$ (where the cardinal $m$ need not be finite), $\mathcal{C}$ is the centre of $\mathcal{R}$, $\{E_{jk}: j, k \in J\}$ is a self-adjoint system of matrix units in $\mathcal{R}$ with each projection $E_{jj}$ abelian and $\sum_{j \in J} E_{jj} = I$, and $\varrho$ in $C_c^1(\mathcal{R}, \mathcal{R})$ satisfies $\varrho(CA) = C\varrho(A)$ for all $C$ in $\mathcal{C}$ and $A$ in $\mathcal{R}$. Then, for each $h$ in $J$ and $A$ in $\mathcal{R}$,*

$$\sum_{j \in J} E_{jh}\, \varrho(E_{hj} A) \tag{11}$$

*converges in the ultrastrong topology to an element $\sigma(A)$ of $\mathcal{R}$, and $\|\sigma(A)\| \leqslant \|\varrho\| \, \|A\|$.*

*Proof.* Since * isomorphisms between von Neumann algebras are ultrastrongly bicontinuous, it is sufficient to prove the result for some von Neumann algebra which is * isomorphic to $\mathcal{R}$; accordingly, we assume that $\mathcal{R}$ has an abelian commutant, whence $\mathcal{R}' = \mathcal{C}$. The abelian von Neumann algebra $E_{hh}\mathcal{R}E_{hh} (= \mathcal{C}E_{hh})$ has abelian commutant $\mathcal{R}'E_{hh} (= \mathcal{C}E_{hh})$ and is therefore maximal abelian as an algebra of operators acting on the range space of $E_{hh}$.

We shall show that, if $x$ is a unit vector and $A \in \mathcal{R}$, then

$$\sum_{j \in J} \| E_{jh} \varrho(E_{hj} A) x \|^2 \leqslant \| \varrho \|^2 \| A \|^2. \tag{12}$$

From this, it follows that the series $\sum_{j \in J} E_{jh} \varrho(E_{hj} A) x$, which consists of pairwise orthogonal vectors, converges strongly to a vector $\sigma(A)x$ such that $\|\sigma(A)x\| \leqslant \|\varrho\| \|A\|$. This in turn establishes the ultrastrong convergence of (11) to a bounded operator $\sigma(A)$, in $\mathcal{R}$, satisfying $\|\sigma(A)\| \leqslant \|\varrho\| \|A\|$.

In order to prove (12), let $T_j = \varrho(E_{hj} A)$ and $x_j = E_{hh} T_j x$. Note that, since $E_{jh}$ is a partial isometry,

$$\| E_{jh} \varrho(E_{hj} A) x \| = \| E_{jh} T_j x \| = \| E_{hh} T_j x \| = \| x_j \|. \tag{13}$$

With $F$ a finite subset of $J$, the positive normal functional $(\sum_{j \in F} \omega_{x_j}) \mid E_{hh} \mathcal{R} E_{hh}$ has the form $\omega_y \mid E_{hh} \mathcal{R} E_{hh}$ for some $y$ in the range of $E_{hh}$ [3: p. 223, Corollaire]. Since, for each $j$ in $F$,

$$\omega_{x_j} \mid E_{hh} \mathcal{R} E_{hh} \leqslant \omega_y \mid E_{hh} \mathcal{R} E_{hh},$$

it follows that $x_j = S_j y$ for some $S_j$ in the commutant $\mathcal{C}E_{hh}$ of $E_{hh} \mathcal{R} E_{hh}$; thus $x_j = C_j y$ for some $C_j$ in $\mathcal{C}$. With $K$ in $\mathcal{R}$ and $H$ in $\mathcal{C}$ defined by

$$K = \sum_{j \in F} C_j^* E_{hj}, \qquad H = \Big(\sum_{j \in F} C_j^* C_j\Big)^{\frac{1}{2}},$$

we have
$$KK^* = \sum_{j \in F} C_j^* E_{hj} E_{jh} C_j = H^2 E_{hh},$$

and so $K$ has polar decomposition $K = H E_{hh} V$ for some partial isometry $V$ in $\mathcal{R}$. Since $x_j = E_{hh} T_j x = C_j y$ for each $j$ in $F$,

$$\begin{aligned}
\sum_{j \in F} \| x_j \|^2 &= \sum_{j \in F} \langle E_{hh} T_j x, C_j y \rangle = \sum_{j \in F} \langle E_{hh} C_j^* \varrho(E_{hj} A)\, x, y \rangle = \langle E_{hh} \varrho\Big(\sum_{j \in F} C_j^* E_{hj} A\Big) x, y \rangle \\
&= \langle E_{hh} \varrho(KA)\, x, y \rangle = \langle E_{hh} \varrho(H E_{hh} V A)\, x, y \rangle = \langle E_{hh} \varrho(E_{hh} V A)\, x, Hy \rangle \\
&\leqslant \| \varrho \| \, \| A \| \, \| Hy \|,
\end{aligned}$$

since $\|x\| = 1$. Now

$$\| Hy \|^2 = \langle H^2 y, y \rangle = \sum_{j \in F} \langle C_j^* C_j y, y \rangle = \sum_{j \in F} \| C_j y \|^2 = \sum_{j \in F} \| x_j \|^2;$$

and this, with the preceding inequality, gives

$$\sum_{j \in F} \| x_j \|^2 \leqslant \| \varrho \|^2 \, \| A \|^2.$$

Since $F$ is an arbitrary finite subset of $J$,

$$\sum_{j\in J}\|x_j\|^2 \leqslant \|\varrho\|^2\,\|A\|^2$$

which, in view of (13), completes the proof of (12).

LEMMA 4.2. *Under the conditions of Lemma 4.1, $\sigma(A)=A\sigma(I)$ for each $A$ in $\mathcal{R}$.*

*Proof.* With $\mathcal{H}$ the Hilbert space on which $\mathcal{R}$ acts, and $\mathcal{H}_0$ the everywhere dense subspace defined by

$$\mathcal{H}_0=\{u\in\mathcal{H}:\ u=\textstyle\sum_{j\in F}E_{jj}u \text{ for some finite subset } F \text{ of } J\}, \tag{14}$$

it is sufficient to prove that

$$\langle\sigma(A)x,\,y\rangle=\langle A\sigma(I)x,\,y\rangle \tag{15}$$

whenever $x,\ y\in\mathcal{H}_0$ and $A\in\mathcal{R}$. Since $E_{hj}AE_{kh}\in E_{hh}\mathcal{R}E_{hh}=\mathcal{C}E_{hh}$, there exists $C_{jk}$ in $\mathcal{C}$ such that $E_{hj}AE_{kh}=C_{jk}E_{hh}$. Thus

$$E_{hj}A=\sum_{k\in J}E_{hj}AE_{kh}E_{hk}=\sum_{k\in J}C_{jk}E_{hk}, \tag{16}$$

$$AE_{kh}=\sum_{j\in J}E_{jh}E_{hj}AE_{kh}=\sum_{j\in J}E_{jh}C_{jk}, \tag{17}$$

with convergence in the strong * topology. Formal manipulation gives

$$\begin{aligned}\langle\sigma(A\rangle x,y\rangle&=\sum_{j\in J}\langle E_{jh}\varrho(E_{hj}A)x,y\rangle=\sum_{j\in J}\langle E_{jh}\varrho(\sum_{k\in J}C_{jk}E_{hk})x,y\rangle\\&=\sum_{j\in J}\sum_{k\in J}\langle E_{jh}\varrho(C_{jk}E_{hk})x,y\rangle=\sum_{j\in J}\sum_{k\in J}\langle E_{jh}C_{jk}\varrho(E_{hk})x,y\rangle \qquad (18)\\&=\sum_{k\in J}\sum_{j\in J}\langle E_{jh}C_{jk}\varrho(E_{hk})x,y\rangle=\sum_{k\in J}\langle AE_{kh}\varrho(E_{hk})x,y) \qquad (19)\\&=\langle A\sigma(I)x,y\rangle.\end{aligned}$$

This heuristic argument needs justification in two places; it is necessary to validate the equality

$$\langle E_{jh}\varrho(\sum_{k\in J}C_{jk}E_{hk})x,y\rangle=\sum_{k\in J}\langle E_{jh}\varrho(C_{jk}E_{hk})x,y\rangle \tag{20}$$

used at stage (18), and the change in order of summation at (19). Since $y\in\mathcal{H}_0$, there is a finite subset $F$ of $J$ such that $E_{jj}y=0$ for all $j$ in $J-F$; so all terms in which $j\in J-F$, in (19) and the preceding line, are zero. Thus $j$ can be restricted to the finite set $F$, and the change in order of summation is permissible. It remains to prove (20).

Let $\mathcal{F}$ denote the class of all finite subsets $F$ of $J$. From (16) and the continuity on bounded sets of the mapping $T\to TT^*$ in the strong * topology,

$$(\sum_{k\in J} C_{jk}E_{hk})(\sum_{k\in J} C_{jk}E_{hk})^* = \sum_{k\in J} C_{jk}C_{jk}^* E_{hh}. \tag{21}$$

Similarly, if $E\in\mathcal{F}$, then

$$(\sum_{k\in J-F} C_{jk}E_{hk})(\sum_{k\in J-F} C_{jk}E_{hk})^* = \sum_{k\in J-F} C_{jk}C_{jk}^* E_{hh}. \tag{22}$$

Since $E_{hh}$ has central carrier $I$, the mapping $C\to CE_{hh}$ is a *-isomorphism from $\mathcal{C}$ onto $\mathcal{C}E_{hh}$, and is therefore bicontinuous on bounded sets in the strong * topology. This, with (21), implies the strong * convergence of $\sum_{k\in J} C_{jk}C_{jk}^*$ to an element $C$ of $\mathcal{C}$.

With $\varepsilon$ a preassigned positive number, and $F$ in $\mathcal{F}$, let $P_F$ be the largest projection in $\mathcal{C}$ for which

$$(\sum_{k\in J-F} C_{jk}C_{jk}^*)P_F = (C-\sum_{k\in F} C_{jk}C_{jk}^*)P_F \geqslant \varepsilon^2 P_F \tag{23}$$

(so that $P_F$ is a spectral projection of $\sum_{k\in J-F} C_{jk}C_{jk}^*$). Note that, by maximality of $P_F$,

$$(\sum_{k\in J-F} C_{jk}C_{jk}^*)(I-P_F) \leqslant \varepsilon^2(I-P_F);$$

whence, by (22),

$$\begin{aligned}\|\sum_{k\in J-F} C_{jk}E_{hk}(I-P_F)\| &= \|(\sum_{k\in J-F} C_{jk}E_{hk})(\sum_{k\in J-F} C_{jk}E_{hk})^*(I-P_F)\|^{\frac12} \\ &= \|\sum_{k\in J-F} C_{jk}C_{jk}^*(I-P_F)E_{hh}\|^{\frac12} \leqslant \varepsilon. \end{aligned}\tag{24}$$

It is clear that $P_F$ decreases when $F$ increases, so $\lim_{F\in\mathcal{F}} P_F$ ($\mathcal{F}$ directed by $\subseteq$) exists in the strong topology, and is a projection $P$ in $\mathcal{C}$ such that $P\leqslant P_F$ for each $F$ in $\mathcal{F}$. From (23), multiplying by $P$,

$$(C-\sum_{k\in F} C_{jk}C_{jk}^*)P \geqslant \varepsilon^2 P$$

for each $F$ in $\mathcal{F}$; whence

$$0 = (C-\sum_{k\in J} C_{jk}C_{jk}^*)P \geqslant \varepsilon^2 P,$$

and so $\lim_{F\in\mathcal{F}} P_F = P = 0$.

By (24),

$$\begin{aligned}|\langle E_{jh}\varrho(\sum_{k\in J} C_{jk}E_{hk})x,y\rangle - \sum_{k\in F}\langle E_{jh}\varrho(C_{jk}E_{hk})x,y\rangle| &= |\langle E_{jh}\varrho(\sum_{k\in J-F} C_{jk}E_{hk})x,y\rangle| \\ \leqslant |\langle E_{jh}\varrho(\sum_{k\in J-F} C_{jk}E_{hk}(I-P_F))x,y\rangle| &+ |\langle E_{jh}\varrho(\sum_{k\in J-F} C_{jk}E_{hk}P_F)x,y\rangle|\end{aligned}$$

$$\leqslant \|\varrho\| \, \|\sum_{k\in J-F} C_{jk} E_{hk}(I-P_F)\| \, \|x\| \, \|y\| + |\langle E_{jh}\varrho(\sum_{k\in J-F} C_{jk}E_{hk})x, P_F y\rangle|$$

$$\leqslant \varepsilon\|\varrho\| \, \|x\| \, \|y\| + \|\varrho\| \, \|\sum_{k\in J-F} C_{jk}E_{hk}\| \, \|x\| \, \|P_F y\| \leqslant \|\varrho\| \, \|x\| \, [\varepsilon\|y\| + \|A\| \, \|P_F y\|]$$

for each $F$ in $\mathcal{J}$ (at the last step we have used the equality $\sum_{k\in J-F} C_{jk}E_{hk} = E_{hj}A\sum_{k\in J-F} E_{kk}$, which results from (16)). Since $\lim_{F\in\mathcal{J}} P_F = 0$, there is an $F_0$ in $\mathcal{J}$ such that $\|P_F y\| < \varepsilon$ whenever $F \supseteq F_0$; and

$$|\langle E_{jh}\varrho(\sum_{k\in J} C_{jk}E_{hk})x, y\rangle - \sum_{k\in F}\langle E_{jh}\varrho(C_{jh}E_{hk})x, y\rangle| \leqslant \varepsilon\|\varrho\| \, \|x\| \, [\|y\| + \|A\|]$$

whenever $F \supseteq F_0$. This completes the proof of (20).

LEMMA 4.3. *Suppose that $\mathcal{R}$ is a von Neumann algebra of type $I_m$ (where the cardinal $m$ need not be finite), $\mathcal{C}$ is the centre of $\mathcal{R}$, $\{E_{jk}: j, k\in J\}$ is a self-adjoint system of matrix units in $\mathcal{R}$ with each projection $E_{jj}$ abelian and $\sum_{j\in J} E_{jj} = I$, $n \geqslant 1$ and $\varrho$ in $C_c^n(\mathcal{R}, \mathcal{R})$ satisfies*

$$\varrho(CA_1, A_2, \ldots, A_n) = C\varrho(A_1, \ldots, A_n)$$

*whenever $C \in \mathcal{C}$ and $A_1, \ldots, A_n \in \mathcal{R}$. Then, for each $h$ in $J$ and $A_1, \ldots, A_n$ in $\mathcal{R}$,*

$$\sum_{j\in J} E_{jh}\,\varrho(E_{hj}A_1, A_2, \ldots, A_n) \tag{25}$$

*converges in the ultrastrong topology to an element $\sigma(A_1, \ldots, A_n)$ of $\mathcal{R}$. Moreover, $\sigma \in C_c^n(\mathcal{R}, \mathcal{R})$, $\|\sigma\| \leqslant \|\varrho\|$, and*

$$\sigma(A_1, \ldots, A_n) = A_1\sigma(I, A_2, \ldots, A_n). \tag{26}$$

*If the mapping $A_j \to \varrho(A_1, \ldots, A_n)$ is ultraweakly continuous, for some fixed $j$ and all $A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n$ in $\mathcal{R}$, then $A_j \to \sigma(A_1, \ldots, A_n)$ has the same continuity property.*

*Proof.* For each $A_2, \ldots, A_n$ in $\mathcal{R}$ the mapping $A \to \varrho(A, A_2, \ldots, A_n)$ is an element $\varrho_0$ of $C_c^1(\mathcal{R}, \mathcal{R})$, with $\|\varrho_0\| \leqslant \|\varrho\| \, \|A_2\| \ldots \|A_n\|$. By applying Lemmas 4.1 and 4.2 to $\varrho_0$, it follows that, for all $A_1, \ldots, A_n$ in $\mathcal{R}$, the series (25) converges ultrastrongly to an element $\sigma(A_1, \ldots, A_n)$ of $\mathcal{R}$ such that

$$\|\sigma(A_1, \ldots, A_n)\| \leqslant \|\varrho\| \, \|A_1\| \ldots \|A_n\|$$

and $\sigma(A_1, \ldots, A_n) = A_1\sigma(I, A_2, \ldots, A_n)$. The multilinearity of $\sigma$ is clear, so $\sigma \in C_c^n(\mathcal{R}, \mathcal{R})$ and $\|\sigma\| \leqslant \|\varrho\|$.

Suppose that $A_j \to \varrho(A_1, \ldots, A_n)$ is ultraweakly continuous, for some fixed $j$ and all $A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n$ in $\mathcal{R}$. With $x$ and $y$ in the everywhere dense subspace $\mathcal{H}_0$ introduced in (14) there is a finite set $F$ $(\subseteq J)$ such that $y = \sum_{j\in F} E_{jj}\, y$, and

$$\langle \sigma(A_1, \ldots, A_n)x, y\rangle = \sum_{j \in F} \langle E_{jh}\varrho(E_{hj}A_1, A_2, \ldots, A_n)x, y\rangle;$$

so $A_j \to \langle \sigma(A_1, ..., A_n)x, y\rangle$ is ultraweakly continuous. Since the ultraweakly continuous linear functionals on $\mathcal{R}$ form a norm closed subspace $\mathcal{R}_*$ of the Banach dual space of $\mathcal{R}$, and finite linear combinations of functionals $\omega_{xy}$: $T \to \langle Tx, y\rangle$, with $x$, $y$ in $\mathcal{H}_0$, form an everywhere dense subspace of $\mathcal{R}_*$ [3: p. 38, Théorème 1], it now follows that $A_j \to f(\sigma(A_1, ..., A_n))$ is ultraweakly continuous for each $f$ in $\mathcal{R}_*$; that is, $A_j \to \sigma(A_1, ..., A_n)$ is ultraweakly continuous.

THEOREM 4.4. *If $\mathcal{R}$ is a type I von Neumann algebra, then* $H_c^n(\mathcal{R}, \mathcal{R}) = 0$ $(n = 1, 2, ...)$.

*Proof.* By Corollary 3.5, each element of $Z_c^n(\mathcal{R}, \mathcal{R})$ is cohomologous to some $\varrho$ in $NZ_c^n(\mathcal{R}, \mathcal{R})$; and it suffices to prove that $\varrho \in B_c^n(\mathcal{R}, \mathcal{R})$.

In the first place we suppose that $\mathcal{R}$ is of type $\mathrm{I}_m$, for some cardinal $m$, and we select a self-adjoint system $\{E_{jk}: j, k \in J\}$ of matrix units in $\mathcal{R}$, in which each projection $E_{jj}$ is abelian and $\sum_{j \in J} E_{jj} = I$. With $\varrho$ in $NZ_c^n(\mathcal{R}, \mathcal{R})$ and $h$ in $J$, it follows from Lemma 4.3 that there is a $\sigma$ in $C_c^n(\mathcal{R}, \mathcal{R})$, with $\|\sigma\| \leq \|\varrho\|$, defined by

$$\sigma(A_1, \ldots, A_n) = \sum_{j \in J} E_{jh}\varrho(E_{hj}A_1, A_2, \ldots, A_n). \tag{27}$$

With $\xi$ in $C_c^{n-1}(\mathcal{R}, \mathcal{R})$ defined by

$$\xi(A_2, ..., A_n) = \sigma(I, A_2, ..., A_n), \tag{28}$$

$\|\xi\| \leq \|\sigma\| \leq \|\varrho\|$. Furthermore,

$$\begin{aligned}
(\Delta\xi)(A_1, \ldots, A_n) &= A_1\xi(A_2, \ldots, A_n) - \xi(A_1A_2, A_3 \ldots, A_n) + \xi(A_1, A_2A_3, A_4, \ldots, A_n) - \ldots \\
&\quad + (-1)^{n-1}\xi(A_1, \ldots, A_{n-2}, A_{n-1}A_n) + (-1)^n\xi(A_1, \ldots, A_{n-1})A_n \\
&= \sum_{j \in J}\{A_1E_{jh}\varrho(E_{hj}, A_2, \ldots, A_n) - E_{jh}\varrho(E_{hj}, A_1A_2, A_3, \ldots, A_n) \\
&\quad + E_{jh}\varrho(E_{hj}, A_1, A_2A_3, A_4, \ldots, A_n) - \ldots \\
&\quad + (-1)^{n-1}E_{jh}\varrho(E_{hj}, A_1, \ldots, A_{n-2}, A_{n-1}A_n) \\
&\quad + (-1)^nE_{jh}\varrho(E_{hj}, A_1, \ldots, A_{n-1})A_n\} \\
&= \sum_{j \in J}\{A_1E_{jh}\varrho(E_{hj}, A_2, \ldots, A_n) - E_{jh}(\Delta\varrho)(E_{hj}, A_1, \ldots, A_n) \\
&\quad + E_{jj}\varrho(A_1, \ldots, A_n) - E_{jh}\varrho(E_{hj}A_1, A_2, \ldots, A_n)\} \\
&= A_1\sigma(I, A_2, \ldots, A_n) + \varrho(A_1, \ldots, A_n) - \sigma(A_1, \ldots, A_n) = \varrho(A_1, \ldots, A_n),
\end{aligned}$$

by Lemma 4.3. We have proved that, if $\mathcal{R}$ is of type $\mathrm{I}_m$ and $\varrho \in NZ_c^n(\mathcal{R}, \mathcal{R})$, then there is a $\xi$ in $C_c^{n-1}(\mathcal{R}, \mathcal{R})$ satisfying $\|\xi\| \leqslant \|\varrho\|$, $\Delta\xi = \varrho$.

With $\mathcal{R}$ now a general type I von Neumann algebra, and $\varrho$ in $NZ_c^n(\mathcal{R}, \mathcal{R})$, let $\{Q_\alpha\}$ be a family of central projections in $\mathcal{R}$, with sum $I$, such that each $\mathcal{R}Q_\alpha$ is of type $\mathrm{I}_{m(\alpha)}$ for some cardinal $m(\alpha)$. With $A_1, \ldots, A_n$ in $\mathcal{R}$,

$$Q_\alpha \varrho(A_1, \ldots, A_n) = \varrho(A_1 Q_\alpha, \ldots, A_n Q_\alpha);$$

from this, it follows that the restriction $\varrho \,|\, \mathcal{R}Q_\alpha$ is an element $\varrho_\alpha$ of $NZ_c^n(\mathcal{R}Q_\alpha, \mathcal{R}Q_\alpha)$, with $\|\varrho_\alpha\| \leqslant \|\varrho\|$. By the result of the preceding paragraph, there is a $\xi_\alpha$ in $C_c^{n-1}(\mathcal{R}Q_\alpha, \mathcal{R}Q_\alpha)$ such that $\|\xi_\alpha\| \leqslant \|\varrho_\alpha\| (\leqslant \|\varrho\|)$ and $\Delta\xi_\alpha = \varrho_\alpha$. With $A_1, \ldots, A_{n-1}$ in $\mathcal{R}$, define

$$\xi(A_1, \ldots, A_{n-1}) = \sum_\alpha \xi_\alpha(A_1 Q_\alpha, \ldots, A_{n-1} Q_\alpha);$$

the series converges in strong operator topology, and $\|\xi(A_1, \ldots, A_{n-1})\| \leqslant \|\varrho\| \, \|A_1\| \ldots \|A_{n-1}\|$, since $\|\xi_\alpha(A_1 Q_\alpha, \ldots, A_{n-1} Q_\alpha)\| \leqslant \|\xi_\alpha\| \, \|A_1\| \ldots \|A_{n-1}\| \leqslant \|\varrho\| \, \|A_1\| \ldots \|A_{n-1}\|$. It is clear that $\xi$ is multilinear, so $\xi \in C_c^{n-1}(\mathcal{R}, \mathcal{R})$. For each $\alpha$,

$$Q_\alpha \, \xi(A_1, \ldots, A_{n-1}) = \xi_\alpha(A_1 Q_\alpha, \ldots, A_{n-1} Q_\alpha);$$

from this, and the coboundary formula, it follows easily that

$$\begin{aligned} Q_\alpha(\Delta\xi)(A_1, \ldots, A_n) = (\Delta\xi_\alpha)(A_1 Q_\alpha, \ldots, A_n Q_\alpha) = \varrho_\alpha(A_1 Q_\alpha, \ldots, A_n Q_\alpha) &= \varrho(A_1 Q_\alpha, \ldots, A_n Q_\alpha) \\ &= Q_\alpha \, \varrho(A_1, \ldots, A_n), \end{aligned}$$

for all $A_1, \ldots, A_n$ in $\mathcal{R}$. Summation with respect to $\alpha$ gives $(\Delta\xi)(A_1, \ldots, A_n) = \varrho(A_1, \ldots, A_n)$, so $\varrho = \Delta\xi \in B_c^n(\mathcal{R}, \mathcal{R})$.

THEOREM 4.5. *If $\mathcal{R}$ is a type I von Neumann algebra, then $NZ_c^n(\mathcal{R}, \mathcal{R}) \subseteq B_w^n(\mathcal{R}, \mathcal{R})$ $(n = 1, 2, \ldots)$.*

*Proof.* If $\varrho \in NZ_w^n(\mathcal{R}, \mathcal{R})$, then $\varrho \in NZ_c^n(\mathcal{R}, \mathcal{R})$, so the arguments used in proving Theorem 4.4 remain valid. We indicate the minor modifications needed to prove the present theorem.

If $\mathcal{R}$ is of type $\mathrm{I}_m$ for some cardinal $m$, $\varrho = \Delta\xi$ where $\xi$ in $C_c^{n-1}(\mathcal{R}, \mathcal{R})$ is defined by (28) and (27). Since $\varrho$ is ultraweakly continuous in each of its arguments (separately), it follows from Lemma 4.3 that the same is true of $\sigma$ and hence, also, of $\xi$. Thus $\xi \in C_w^{n-1}(\mathcal{R}, \mathcal{R})$ and $\varrho = \Delta\xi \in B_w^n(\mathcal{R}, \mathcal{R})$.

With $\mathcal{R}$ a general type I von Neumann algebra, we can select the central projections

$Q_\alpha$ and construct $\varrho_\alpha$, $\xi_\alpha$, $\xi$, just as in the proof of Theorem 4.4. Since $\varrho_\alpha = \varrho \mid \mathcal{R}Q_\alpha$, $\varrho_\alpha$ is ultraweakly continuous in each of its arguments (separately); and it follows from the preceding paragraph that $\xi_\alpha$ has the same continuity property. With $\mathcal{H}$ the Hilbert space on which $\mathcal{R}$ acts, let $\mathcal{H}_0$ be the everywhere dense subspace of vectors $x$ such that $Q_\alpha x = 0$ for all but a finite set of $\alpha$'s. When $x,\ y \in \mathcal{H}_0$

$$\langle \xi(A_1, \dots, A_{n-1})\,x, y\rangle = \sum_\alpha \langle \xi(A_1, \dots, A_{n-1})\,Q_\alpha x, y\rangle = \sum_\alpha \langle \xi_\alpha(A_1 Q_\alpha, \dots, A_{n-1} Q_\alpha)\,x, y\rangle,$$

and the summation can be restricted to the finite set $\{\alpha\colon Q_\alpha x \neq 0\}$. For $j = 1, \dots, n-1$, the mappings

$$A_j \to A_j Q_\alpha \to \xi_\alpha(A_1 Q_\alpha, \dots, A_{n-1} Q_\alpha)$$

are ultraweakly continuous; so the linear functional $A_j \to \langle \xi(A_1, \dots, A_{n-1})\,x,\ y\rangle$ is ultraweakly continuous on $\mathcal{R}$, whenever $x,\ y \in \mathcal{H}_0$. By the argument used at the end of the proof of Lemma 4.3, $A_j \to \xi(A_1, \dots, A_{n-1})$ is ultraweakly continuous; so $\xi \in C_w^{n-1}(\mathcal{R}, \mathcal{R})$ and $\varrho = \Delta\xi \in B_w^n(\mathcal{R}, \mathcal{R})$.

COROLLARY 4.6. *If $\mathcal{R}$ is a type I factor, $H_w^n(\mathcal{R}, \mathcal{R}) = 0$ $(n = 1, 2, \dots)$.*

*Proof.* When $\mathcal{R}$ is a factor, $Z_w^n(\mathcal{R}, \mathcal{R}) = NZ_w^n(\mathcal{R}, \mathcal{R})$, so the result follows from Theorem 4.5.

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