ON THE LIMITING DISTRIBUTION OF ADDITIVE ARITHMETIC FUNCTIONS

BY

P. D. T. A. ELLIOTT[1]

University of Colorado, Boulder, Colorado, USA

To Professor P.Erdös on his sixtieth birthday

1

Let $f(n)$ be a real-valued additive arithmetic function. Let $\alpha(x)$ and $\beta(x)$ be real-valued functions which are defined for all real numbers $x \geq 1$, and in such a way that $\beta(x) > 0$. For each real number z let $N(x, z)$ denote the number of integers n not exceeding x, for which the inequality

$$f(n) - \alpha(x) \leqslant z\beta(x)$$

is satisfied. Define the frequencies

$$\nu_x(n;\, f(n) - \alpha(x) \leqslant z\beta(x)) = x^{-1}N(x, z), \quad (x \geqslant 1).$$

In this paper we shall make certain restrictions upon the rate of growth of the renormalising functions $\alpha(x)$ and $\beta(x)$, and then give necessary and sufficient conditions in order that the above frequencies should converge weakly.

For simplicity of exposition only, we shall assume that the function $f(n)$ is strongly additive. In other words, for each prime p and positive integer m the relation $f(p^m) = f(p)$ is satisfied. *No other assumptions will be made concerning the function* $f(n)$. It also proves to be advantageous to consider frequencies which are defined in terms of a continuous parameter x.

In order to present our main result it is convenient to define independent random variables X_p, one for each prime p, by

$$X_p = \begin{cases} f(p) & \text{with probability } \dfrac{1}{p} \\ 0 & \text{with probability } 1 - \dfrac{1}{p}. \end{cases}$$

[1] Partially supported by N.S.F. Grant NSF-GP-33026 ×.

THEOREM. *Let $\beta(x)$ satisfy the conditions $\beta(x)\to\infty$, and*

$$\sup_{x^{\frac{1}{2}}\leqslant y\leqslant x}|\beta(x)-\beta(y)|=o(\beta(x)),\quad (x\to\infty).$$

PROPOSITION A. *There exists a real-valued function $\alpha(x)$, with the property that*

$$\sup_{x^{\frac{1}{2}}\leqslant y\leqslant x}|\alpha(x)-\alpha(y)|=o(\beta(x)),\quad (x\to\infty),$$

and such that the frequencies

$$\nu_x(n;\, f(n)-\alpha(x)\leqslant z\beta(x)),\quad (x\geqslant 2),$$

converge weakly as $x\to\infty$.

PROPOSITION B. *There exists a function $\alpha(x)$ so that the distributions*

$$P(\sum_{p\leqslant x}X_p-\alpha(x)\leqslant z\beta(x)),\quad (x\geqslant 2),$$

converge weakly as $x\to\infty$. Moreover, for each pair of positive real numbers ε and u the condition

$$\sum_{\substack{x^{\varepsilon}<p\leqslant x\\ |f(p)|>u\beta(x)}}\frac{1}{p}\to 0,\quad (x\to\infty)$$

is satisfied.

If, in addition, the function $\beta(x)$ is continuous for $x\geqslant 2$, then each of these two propositions is also equivalent to the following proposition C.

PROPOSITION C. *Set $\sigma=1+(\log x)^{-1}$. Then there exists a function $\alpha(x)$ so that for each pair of real numbers t and τ the limit*

$$w(t)=\lim_{x\to\infty}\left(\sum_p\frac{\log p}{p^{\sigma+i\tau(\sigma-1)}}\left\{\exp\left(\frac{itf(p)}{\beta(x)}\right)-1\right\}-it\frac{\alpha(x)}{\beta(x)}\right)$$

exists, and is independent of τ. The function $w(t)$ is continuous at $t=0$. Moreover, the limiting relation

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_p\frac{\log p}{p^{\sigma+i\tau(\sigma-1)}}\exp\left(\frac{itf(p)}{\beta(x)}\right)=\frac{1}{1+i\tau}$$

is satisfied.

The functions $\alpha(x)$ which can occur in these propositions are determined uniquely up to the addition of a function of the form $c\beta(x)+o(\beta(x))$, c a constant. In particular, if A (and so B) is satisfied we may choose the same function $\alpha(x)$ in propositions A and B, and the limit laws will coincide. If, moreover, $\beta(x)$ is continuous then we may choose the same function $\alpha(x)$ in all three of the propositions. The limit law will then have a characteristic function of the form $\exp(w(t))$.

Remarks. The addition to $\alpha(x)$ of a function which is of the form $c\beta(x)+o(\beta(x))$ merely convolutes the limit law with an improper law.

If we assume that $\beta(x)$ is a measurable function of x, then the second of the two hypotheses which we make upon its rate of growth is equivalent to the assertion that for each positive real number y, $\beta(x^y)\sim\beta(x)$, as $x\to\infty$. One can view $\beta(x)$ as a slowly oscillating function of $\log x$. For a study of the pertinent properties of measurable slowly oscillating functions we refer to the paper of van Aardenne-Ehrenfest, de Bruijn and Korevaar [1].

Although we give a detailed proof of the theorem for strongly additive functions $f(n)$ it is possible to prove that the theorem is valid for an additive function f if and only if it is valid for the strongly additive function whose value coincides with the value of f on the prime numbers. The limit laws will then also coincide.

The theorem exhibits a connection between the theory of those Dirichlet series which possess Euler products, and the limiting behaviour of sums of independent random variables.

In particular, the present result includes the well-known theorem of Kubilius ([5] Chapter 4, Theorem 4.1, p. 58), concerning the limiting behaviour of additive functions of class H.

We conclude this introduction with an historical example. Let $f(n)=\omega(n)$, the function which counts the number of distinct prime divisors of the integer n. Then Erdös and Kac [3] proved that as $x\to\infty$

$$\nu_x(n;\omega(n)-\log\log x\leqslant z\sqrt{\log\log x})\to G(z),$$

where $G(z)$ denotes the normal distribution, and which is defined by

$$G(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}w^2}dw.$$

It is easy to check that the choices $\alpha(x)=\log\log x$, $\beta(x)=(\log\log x)^{1/2}$ fall within the scope of the above theorem. By combining the equivalence of propositions A and B together with a well-known criterion of Gnedenko and Kolmogorov ([4] Chapter 5, Theorem 3, p. 130) we can assert the

COROLLARY. *In order that for a strongly additive function $f(n)$ the frequencies*

$$\nu_x(n;f(n)-\log\log x\leqslant z\sqrt{\log\log x}),\quad x\geqslant 1),$$

should converge to the normal law, it is both necessary and sufficient that for each positive number ε the limiting relations

$$\sum_{p \leqslant x, |f(p)| > \varepsilon\sqrt{\log\log x}} \frac{1}{p} \to 0,$$

$$\frac{1}{\log\log x} \sum_{p \leqslant x, |f(p)| \leqslant \varepsilon\sqrt{\log\log x}} \frac{f^2(p)}{p} \to 1, \quad \text{(both as } x \to \infty)$$

be satisfied.

Notation. It is convenient to retain and extend the notation which was introduced above. For any property ... we define the frequencies

$$\nu_x(n; \ldots) = x^{-1} \sum_{n \leqslant x}' 1.$$

Here ′ denotes that the summation is confined to those positive integers n for which the property ... is valid.

We shall use $c_1, c_2, \ldots$ to denote positive constants. These will be absolute unless otherwise stated.

2. Proof of the theorem

We shall give an essentially cyclic proof of the theorem. In order to do this it will be convenient to introduce a modified form of proposition C, namely:

PROPOSITION C_0. *Set* $\sigma_0 = 1 + (\log x)^{-1}$, *and* $s_0 = \sigma_0 + i\tau(\sigma_0 - 1)$. *Then there are functions* $\alpha(x)$, $x \geqslant 2$, *and* $\mu(t)$, *such that as* $x \to \infty$:

$$\exp\left(\sum_p \frac{1}{p^{s_0}} \left\{\exp\left(\frac{itf(p)}{\beta(x)}\right) - 1\right\} - \frac{it\alpha(x)}{\beta(x)}\right) = \mu(t) + o(1).$$

The function $\mu(t)$ *is independent of* τ, *and continuous at the point* $t = 0$. *Moreover, there is an interval* $|t| \leqslant t_0$ *about the origin* $t = 0$ *in which the limiting relation*

$$\lim_{x \to \infty} \frac{1}{\log x} \sum \frac{\log p}{p^{s_0}} \exp\left(\frac{itf(p)}{\beta(x)}\right) = \frac{1}{1 + i\tau}$$

is valid for every real number τ.

We shall prove the theorem by establishing the sequence of propositions $A \to C_0 \to B \to A$ and $C \leftrightarrow C_0$.

3. Proof that A implies C_0

In this section we consider Dirichlet series whose coefficients depend upon a real parameter x. We show that information concerning the behaviour of these coefficients under a transformation $x \to x^y$, where y is a positive real number, leads to information concerning a certain limiting behaviour of the Dirichlet series which they define, and conversely.

Let $s = \sigma + i\tau$ denote a complex variable. We shall set $\sigma_0 = 1 + (\log x)^{-1}$, as in Proposition C_0.

For each real number t define

$$g(n) = \exp\,(it\beta(x)^{-1}f(n))$$

so that $g(n)$ is a multiplicative function of n, which satisfies $|g(n)| \leqslant 1$ for $n = 1, 2, \ldots$ We define the associated Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} g(n)\, n^{-s}.$$

This series is absolutely convergent in the half-plane $\sigma > 1$.

Let y be a real number. We shall consider the contour integral

$$J(x, y) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{x^{sy}}{s}\, G(s)\, ds.$$

We begin by examining the behaviour of $J(x, y)$ as $x \to \infty$, for a fixed value of y. We shall need

LEMMA 3.1. *Let the frequencies*

$$\nu_x(n;\, f(n) - \alpha(x) \leqslant z\beta(x)), \quad (x \geqslant 2),$$

converge weakly to a distribution $F(z)$. Let $\varphi(t)$ denote the characteristic function of $F(z)$. We assert that as $x \to \infty$ we have

$$x^{-y} J(x, y) \exp\left(-it\frac{\alpha(x)}{\beta(x)}\right) \to \begin{cases} \varphi(t) & \textit{if } y > 0 \\ 0 & \textit{if } y < 0. \end{cases}$$

Proof. Since $G(s)$ converges absolutely for $\sigma > 1$, we may apply a standard theorem of Perron (see for example Titchmarsh [6] Chapter IX, p. 300), and deduce that if x^y is not an integer, then

$$J(x, y) = \sum_{n \leqslant x^y} g(n).$$

If $y < 0$ then $J(x, y) = 0$, and the second of the two assertions contained in Lemma 3.1 is immediate. Suppose, therefore, that $y > 0$. In this case, the same theorem of Perron assures that we can omit the condition that x^y be non-integral provided that we add to the sum over the $g(n)$ a term which is here absolutely bounded. Consider the expression

$$K = K(x, y) = \exp\left(-\frac{it\alpha(x)}{\beta(x)}\right) x^{-y} \sum_{n \leqslant x^y} g(n).$$

We shall estimate this function $K(x, y)$ by deforming it into an expression to which we can apply the hypothesis of Lemma 3.1. Let us first replace the function $\alpha(x)$ by $\alpha(x^y)$. Since, by the hypothesis of proposition A, $\alpha(x)-\alpha(x^y)=o(\beta(x))$ as $x\to\infty$, this changes the value of the sum K by at most

$$\left|\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)-\exp\left(-\frac{it\alpha(x^y)}{\beta(x)}\right)\right| x^{-y}\sum_{n\leqslant x^y}|g(n)|.$$

This sum in turn does not exceed

$$\left|\exp\left(-it\left\{\frac{\alpha(x)-\alpha(x^y)}{\beta(x)}\right\}\right)-1\right|\leqslant|t|\left|\frac{\alpha(x)-\alpha(x^y)}{\beta(x)}\right|=o(1),\quad (x\to\infty).$$

Hence
$$K(x,y)=x^{-y}\sum_{p\leqslant x^y}\exp\left(it\left\{\frac{f(n)-\alpha(x^y)}{\beta(x)}\right\}\right)+o(1)$$

as $x\to\infty$. We next replace $\beta(x)$ by $\beta(x^y)$. This is a little more complicated. Let ε be a positive real number. Choose a real number u, so large that for all sufficiently large values of x the inequality

$$\nu_{x^y}(n;|f(n)-\alpha(x^y)|\leqslant u\beta(x^y))>1-\varepsilon$$

is satisfied. That this can be done is assured by the second of the two assertions which occur in proposition A of the theorem. For a particular value of x, let the integers n which are counted in this last frequency be denoted by n_j, $(j=1, \ldots, r)$. We write

$$K(x,y)=x^{-y}\sum_{j=1}^{r}\exp\left(-it\left\{\frac{f(n_j)-\alpha(x^y)}{\beta(x)}\right\}\right)+x^{-y}\sum_{\substack{n\leqslant x^y\\ n\neq n_j}}(\ldots)=\Sigma_1+\Sigma_2,$$

say. We can obtain an upper bound for the second of these two sums at once by $|\Sigma_2|\leqslant\varepsilon$. In each of the terms in Σ_1 we replace $\beta(x)$ by $\beta(x^y)$. This will then change the value of Σ_1 by not more than

$$\begin{aligned}
&x^{-y}\sum_{j=1}^{r}\left|\exp\left(-it\left\{\frac{f(n_j)-\alpha(x^y)}{\beta(x)}\right\}\right)-\exp\left(-it\left\{\frac{f(n_j)-\alpha(x^y)}{\beta(x^y)}\right\}\right)\right|\\
&=x^{-y}\sum_{j=1}^{r}\left|\exp\left(-it\{f(n_j)-\alpha(x^y)\}\left\{\frac{1}{\beta(x)}-\frac{1}{\beta(x^y)}\right\}\right)-1\right|\\
&\leqslant x^{-y}\sum_{j=1}^{r}|t|\,|f(n_j)-\alpha(x^y)|\left|\frac{1}{\beta(x)}-\frac{1}{\beta(x^y)}\right|\leqslant u|t|\frac{|\beta(x^y)-\beta(x)|}{\beta(x)}=o(1),\quad (x\to\infty).
\end{aligned}$$

We have now proved that as $x\to\infty$

$$K(x, y)=x^{-y} \sum_{j=1}^{r} \exp \left(-it\left\{\frac{f(n_j)-\alpha(x^y)}{\beta(x^y)}\right\}\right)+o(1)+\theta \varepsilon, \quad (|\theta| \leqslant 1).$$

By adding to the sum which occurs in this equation the appropriate terms, we see that

$$\limsup_{x \rightarrow \infty}\left|K(x, y)-x^{-y} \sum_{n \leqslant x} \exp \left(-it\left\{\frac{f(n)-\alpha(x^y)}{\beta(x^y)}\right\}\right)\right| \leqslant 2 \varepsilon .$$

Since ε can be chosen arbitrarily small:

$$K(x, y)=\int_{-\infty}^{\infty} e^{itz} d\nu_{x^y}(n ; f(n)-\alpha(x^y) \leqslant z \beta(x^y))+o(1),$$

as $x \rightarrow \infty$. However, the integral which appears on the right hand side of this estimate is the characteristic function of a frequency which converges weakly to $F(z)$ as $x \rightarrow \infty$. By a standard theorem from the theory of probability we deduce that as $x \rightarrow \infty$

$$x^{-y} J(x, y) \exp \left(-\frac{it\alpha(x)}{\beta(x)}\right)=K(x, y) \rightarrow \varphi(t).$$

This completes the proof of Lemma 3.1.

To continue with our proof that proposition A implies C_0 it is convenient to transform the integral $J(x, y)$ by the substitution $\tau \rightarrow(\sigma_0-1) \tau$. We then obtain the representation

$$J(x, y)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{G(s)}{s} x^{sy}(\sigma_0-1)\, d \tau$$

where it is now to be understood that $s=s_0=\sigma_0+i \tau(\sigma_0-1)$.

Define

$$h(\tau)=\frac{G(s_0)}{2 \pi s_0}(\sigma_0-1).$$

Then since $x^{s_0 y}=x^y \exp (y\{s_0-1\} \log x)=x^y \exp (y\{1+i \tau\}(\sigma_0-1) \log x)=x^y \exp ((1+i \tau) y)$ we can write

$$(e x)^{-y} J(x, y)=\int_{-\infty}^{\infty} e^{i \tau y} h(\tau)\, d \tau .$$

It is pertinent at this point to note that

$$|h(\tau)|=\frac{|G(s_0)|(\sigma_0-1)}{2 \pi|\sigma_0+i \tau(\sigma_0-1)|} \leqslant \frac{\lambda(x)}{\sqrt{1+\tau^2}}$$

where the 'constant' $\lambda(x)$ depends upon x. For each fixed value of x the function $h(\tau)$ belongs to the class $L^2(-\infty, \infty)$.

It will be convenient in what follows to denote the fourier transform of a function h, by $h^{\wedge}$. In fact the fourier integral involving $h(\tau)$ can be proved to exist as an improper Riemann integral.

We have so far proved that

$$\left[h(\tau)\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)\right]^{\wedge}=(ex)^{-y}J(x,y)\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)=\varrho(x,y),$$

say. Since for positive values of $x\geqslant 1$ and y the inequality $|J(x,y)|\leqslant c_1x^y$ holds uniformly, the function which occurs on the right hand side of this equation belongs both to the class $L(-\infty,\infty)$ and $L^2(-\infty,\infty)$ with respect to the variable y. We can thus apply a fourier inversion to obtain the relation

$$h(\tau)\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\tau y}\varrho(x,y)\,dy.$$

For each fixed value of $y\neq 0$, we have proved in Lemma 3.1 that

$$\varrho(x,y)\to\begin{cases}e^{-y}\varphi(t) & \text{if } y>0\\ 0 & \text{if } y<0.\end{cases}$$

Moreover, $|\varrho(x,y)|\leqslant c_1e^{-y}$ holds uniformly for all values of $x\geqslant 1$. We may therefore apply Lebesgue's theorem on dominated convergence, to deduce that

$$\lim_{x\to\infty}h(\tau)\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\tau y}e^{-y}\varphi(t)\,dy=\frac{\varphi(t)}{2\pi(1+i\tau)}. \tag{3.2}$$

Let us examine this expression involving $h(\tau)$. Let $\zeta(s)$ denote the Riemann zeta-function, which is defined for $\sigma>1$ by

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}.$$

This function is well-known to be everywhere analytic except for a simple pole with residue 1 at the point $s=1$. We shall only need its properties in the neighbourhood of $s=1$. We write $h(\tau)$ in the form

$$h(\tau)=G(s_0)\,\zeta(s_0)^{-1}\left\{\frac{\sigma_0-1}{2\pi s_0}\,\zeta(s_0)\right\}.$$

As $x\to\infty$ the expression inside the curly brackets has the estimate

$$\frac{\sigma_0-1}{2\pi s_0}(1+o(1))\frac{1}{s_0-1}\sim\frac{\sigma_0-1}{2\pi s_0(\sigma_0-1+i\tau(\sigma_0-1))}\sim\frac{1}{2\pi(1+i\tau)}.$$

From our limiting relation (3.2) we deduce that

$$G(s_0)\,\zeta(s_0)^{-1}\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)\to\varphi(t),\quad (x\to\infty). \tag{3.3}$$

Both $G(s)$ and $\zeta(s)$ possess Euler products in the half-plane $\sigma>1$. In terms of these we we can write

$$G(s_0)\,\zeta(s_0)^{-1}=\prod_p\left(1+\sum_{m=1}^{\infty}g(p)\,p^{-s_0m}\right)(1-p^{-s_0}).$$

For each prime p define

$$\theta_p(x)=\log\,(1+g(p)\,p^{-s_0}(1-p^{-s_0})\{1-p^{-s_0}\})-\{g(p)-1\}\,p^{-s_0}.$$

Define also
$$\Theta(x)=\sum_p|\theta_p(x)|.$$

We assert that if the principal value of the logarithms are taken, then as $x\to\infty$, $\Theta(x)\to 0$. In fact if p is large, and we apply Taylor's theorem, then on the one hand

$$\theta_p(x)=g(p)\left\{p^{-2s_0}(1-p^{-s_0})^{-1}+\sum_{m=2}^{\infty}(-g(p)\,p^{-s_0}(1-p^{-s_0})^{-1})^m\frac{1}{m}-\sum_{m=2}^{\infty}p^{-ms_0}\frac{1}{m}\right\}=O(p^{-2}).$$

Hence for all absolutely large numbers P, uniformly in $x\geqslant 2$:

$$\sum_{p>P}|\theta_p(x)|=O(\sum_{p>P}p^{-2})=O(P^{-1}).$$

On the other hand, for each fixed prime p it is easy to see that $\theta_p(x)\to 0$ as $x\to\infty$. We deduce that

$$\limsup_{x\to\infty}\Theta(x)=O(P^{-1}).$$

But P can be chosen arbitrarily large, so that $\Theta(x)\to 0$ as $x\to\infty$, as was asserted.

Applying this result, and making use of (3.3) we see that

$$\exp\left(\sum_p(g(p)-1)\,p^{-s_0}-\frac{it\alpha(x)}{\beta(x)}\right)=\varphi(t)+o(1),\quad \text{as } x\to\infty.$$

Here the function $\varphi(t)$ is a characteristic function, and so is continuous for all values of t, and in particular at the point $t=0$.

This proves the validity of the first assertion made in proposition C_0, with $\mu(t)=\varphi(t)$.

To obtain the second part of proposition C_0 we carry out the same series of operations, but using $G'(s)=dG(s)/ds$ in place of $G(s)$. The integral corresponding to $J(x,y)$ then has an approximate representation

$$-\sum_{n\le x^y} g(n)\log n.$$

This introduces an extra factor of $y \log x$ into the calculations, but no further complications occur. Proceeding along the above lines we arrive at the asymptotic relation

$$\frac{G'(s_0)}{2\pi s_0}(\sigma_0-1)^2\exp\left(-\frac{it\alpha(x)}{\beta(x)}\right)\to\frac{1}{2\pi}\int_0^\infty e^{-i\tau y}ye^{-y}\varphi(t)\,dy=\frac{-\varphi(t)}{2\pi(1+i\tau)^2}.$$

Since $\varphi(t)$ is a characteristic function there is a proper interval about the origin, $|t|\le t_0$ say, in which $\varphi(t)$ does not vanish. For values of t in the interval $|t|\le t_0$ we apply the above asymptotic relation together with the (genuine) asymptotic relation (3.3), to deduce that

$$(\sigma_0-1)\frac{G'(s_0)}{G(s_0)}\to\frac{-1}{1+i\tau},\quad (x\to\infty).$$

By logarithmic differentiation of $G(s)$:

$$\frac{G'(s_0)}{G(s_0)}=-\sum_p g(p)p^{-s_0}\log p+O(1),$$

as $x\to\infty$, so that

$$-(\sigma_0-1)\frac{G'(s_0)}{G(s_0)}=\frac{1}{\log x}\sum_p\frac{\log p}{p^{s_0}}\exp\left(\frac{itf(p)}{\beta(x)}\right)+O\left(\frac{1}{\log x}\right).$$

This leads at once to the validity of the second assertion contained in proposition C_0, and we have completed a proof that $A\to C_0$.

4. Proof that C_0 implies B

It is convenient to begin by proving the second of the two assertions which we made in proposition B.

We consider the second of the two limiting relations of C_0, namely that if $|t|\le t_0$ then

$$\frac{1}{\log x}\sum_p p^{-s_0}\log p\exp\left(\frac{itf(p)}{\beta(x)}\right)\to\frac{1}{1+i\tau},\quad (x\to\infty).$$

From the theory of the Riemann zeta function, as $x\to\infty$ we have

$$\frac{1}{\log x}\sum_p p^{-s_0}\log p\sim-(\sigma_0-1)\frac{\zeta'(s_0)}{\zeta(s_0)}\sim\frac{1}{1+i\tau}$$

so that

$$\frac{1}{\log x}\sum_p p^{-s_0}(1-g(p))\log p\to 0,\quad (x\to\infty).$$

We set $\tau=0$, and take real parts. In this way we deduce that for $|t|\leqslant t_0$,

$$S(x)=\frac{1}{\log x}\sum_p p^{-\sigma_0}(1-\operatorname{Re} g(p))\log p\to 0,\quad (x\to\infty).$$

Here $1-\operatorname{Re} g(p)=2(\sin tf(p)/2\beta(x))^2$. By means of the inequality $|\sin mu|\leqslant m|\sin u|$ which is certainly valid for every positive integer m, and real number u, we can extend the validity of this last limiting relation to hold for each real number t. It is convenient to note at this point that since $|g(p)|=1$

$$S(x)\leqslant\frac{2}{\log x}\sum_p\frac{\log p}{p^{\sigma_0}}\ll 1$$

uniformly for all $x\geqslant 2$.

Let ε and u be positive real numbers. Set $T=2/u$. Then we easily obtain the chain of inequalities

$$\frac{1}{\log x}\sum_{\substack{x^\varepsilon<p\leqslant x^{1/\varepsilon}\\ |f(p)|>u\beta(x)}}\frac{\log p}{p^{\sigma_0}}\leqslant\frac{1}{\log x}\sum_{\substack{x^\varepsilon<p\leqslant x^{1/\varepsilon}\\ |f(p)|>u\beta(x)}}\frac{2\log p}{p^{\sigma_0}}\left(1-\frac{\sin\dfrac{Tf(p)}{\beta(x)}}{\dfrac{Tf(p)}{\beta(x)}}\right)$$

$$=\frac{2}{\log x}\sum_{x^\varepsilon<p\leq x^{1/\varepsilon}}\frac{\log p}{p^{\sigma_0}}\frac{1}{T}\int_0^T\sin^2\frac{tf(p)}{2\beta(x)}\,dt$$

$$\leqslant\frac{1}{T}\int_0^T\frac{1}{\log x}\sum_{p\leqslant x^{1/\varepsilon}}p^{-\sigma_0}(1-\operatorname{Re} g(p))\log p\,dt\leqslant\frac{1}{T}\int_0^T S(x)\,dt.$$

By applying Lebesgue's theorem on dominated convergence we see that the integral which occurs on the extreme right hand end of this chain of inequalities is $o(1)$ as $x\to\infty$. In the range $x^\varepsilon<p\leqslant x^{1/\varepsilon}$ we have $\log p>\varepsilon\log x$, and $p^{-1}<p^{-\sigma_0}\exp(1/\varepsilon)$. In particular, therefore, we have proved that

$$\lim_{x\to\infty}\sum_{\substack{x^\varepsilon<p\leqslant x\\ |f(p)|>u\beta(x)}}\frac{1}{p}=0.$$

This is the second of the two limiting relations which are asserted to be valid in proposition B.

We shall now apply this last relation to simplify the result that

$$\exp\left(\sum_p p^{-s_0}(g(p)-1)-\frac{it\alpha(x)}{\beta(x)}\right)\to\mu(t),\quad (x\to\infty).$$

Let ε and u be positive real numbers. We shall ultimately allow them decrease to zero. Then from the above results we can assert that as $x\to\infty$

$$\sum_{\substack{x^\varepsilon < p \leqslant x^{1/\varepsilon} \\ |f(p)| > u\beta(x)}} p^{-\sigma_0} |g(p)-1| \leqslant 2\exp(1/\varepsilon) \sum_{\substack{x^\varepsilon < p \leqslant x^{1/\varepsilon} \\ |f(p)| > u\beta(x)}} p^{-1} = o(1).$$

On the other hand, whenever $|f(p)| \leqslant u\beta(x)$ is satisfied we can assert that

$$|g(p)-1| \leqslant |f(p)\,\beta(x)^{-1}| \leqslant u.$$

Thus

$$\sum_{\substack{x^\varepsilon < p \leqslant x^{1/\varepsilon} \\ |f(p)| \leqslant u\beta(x)}} p^{-\sigma_0} |g(p)-1| \leqslant u \sum_{x^\varepsilon < p \leqslant x^{1/\varepsilon}} p^{-1} = 2u(-\log\varepsilon + o(1)),$$

as $x \to \infty$. From these last two estimates we deduce that

$$\limsup_{x\to\infty} \sum_{x^\varepsilon < p \leqslant x^{1/\varepsilon}} p^{-\sigma_0} |g(p)-1| \leqslant -2u\log\varepsilon. \tag{4.1}$$

Determine the unique integer k so that $2^k < x^{1/\varepsilon} \leqslant 2^{k+1}$. Consider the right hand side of the following inequality:

$$\sum_{p > x^{1/\varepsilon}} p^{-\sigma_0} |g(p)-1| \leqslant 2 \sum_{m=k}^{\infty} \sum_{2^m < p \leqslant 2^{m+1}} p^{-\sigma_0}.$$

For each integer m the innermost sum has the value

$$\sum_{2^m < p \leqslant 2^{m+1}} p^{-1} \exp(-(\sigma_0 - 1)\log p) \leqslant \sum_{2^m < p \leqslant 2^{m+1}} p^{-1} \exp\left(-\frac{m\log 2}{\log x}\right)$$

$$\leqslant m^{-1} c_2 \exp\left(-\frac{m\log 2}{\log x}\right) < c_3 k^{-1} \int_m^{m+1} \exp\left(-\frac{y\log 2}{\log x}\right) dy.$$

Here we have made use of the elementary estimate, which is uniform in all positive integers m:

$$\sum_{2^m < p \leqslant 2^{m+1}} \frac{1}{p} = \log\log 2^{m+1} - \log\log 2^m + O\left(\frac{1}{\log 2^m}\right) = O\left(\frac{1}{m}\right).$$

The constants c_2 and c_3 are absolute. Summing over $m = k, k+1, \ldots$ we obtain the upper bound

$$2c_3 k^{-1} \int_k^\infty \exp\left(-\frac{y\log 2}{\log x}\right) dy \leqslant \exp\left(-\frac{k\log 2}{\log x}\right) \frac{2c_3 \log x}{k\log 2}$$

From the definition of k it follows that $k+1 \geqslant \log x/\varepsilon \log 2$, so that if x is sufficiently large (in terms of ε) the right hand side of this inequality will not be more than $4c_3\varepsilon$. Putting this inequality together with that of (4.1) we see that

$$\limsup_{x\to\infty} \sum_{p > x^\varepsilon} p^{-\sigma_0} |g(p)-1| \leqslant -2u\log\varepsilon + 4c_3\varepsilon. \tag{4.2}$$

In particular we deduce that

$$\sum_{x^{\varepsilon}<p\leqslant x} p^{-1}|g(p)-1| \leqslant e \sum_{x^{\varepsilon}<p\leqslant x} p^{-\sigma_0}|g(p)-1| \leqslant -2eu\log\varepsilon + 4ec_3 + o(1), \quad (x\to\infty).$$

We shall need this result later.

Let us now examine the sum

$$\sum_{p\leqslant x^{\varepsilon}} p^{-s_0}(g(p)-1).$$

If we replace s_0 by 1 we change the value of this sum by at most

$$\sum_{p\leqslant x^{\varepsilon}} |(g(p)-1)\{p^{-s_0}-p^{-1}\}| \leqslant 2\sum_{p\leqslant x^{\varepsilon}} p^{-1}|\exp((s_0-1)\log p)-1|.$$

We note that since each prime p does not exceed x^{ε}, when x is large enough $|s_0-1|\log p \leqslant (1+\tau^2)^{\frac{1}{2}}\varepsilon \leqslant 1/2$, provided only that ε is sufficiently small in terms of τ. In these circumstances

$$|\exp((s_0-1)\log p)-1| \leqslant \sum_{m=1}^{\infty}(|s_0-1|\log p)^m\frac{1}{m!} \leqslant 2|s_0-1|\log p.$$

Hence the error term which we have presently introduced is not more than

$$2|s_0-1|\sum_{p\leqslant x^{\varepsilon}} p^{-1}\log p = 2|s_0-1|(\log x^{\varepsilon}+O(1)) < 3(1+\tau^2)^{\frac{1}{2}}\varepsilon.$$

Here we have made use of another elementary estimate from the theory of numbers, namely

$$\sum_{p\leqslant y}\frac{\log p}{p} = \log y + O(1),$$

which is valid for all real numbers $y \geqslant 2$.

Putting all of these inequalities together (with $\tau=0$) leads to the following inequality

$$\limsup_{x\to\infty}\left|\exp\left(\sum_{p\leqslant x} p^{-1}(g(p)-1)-\frac{it\alpha(x)}{\beta(x)}\right)-\mu(t)\right| < c_4(-u\log\varepsilon+\varepsilon).$$

valid for all sufficiently small but positive values of u and ε. Letting $u\to 0+$ and then $\varepsilon\to 0+$ we arrive at the limiting relation

$$\exp\left(\sum_{p\leqslant x} p^{-1}(g(p)-1)-\frac{it\alpha(x)}{\beta(x)}\right)\to\mu(t), \quad (x\to\infty). \tag{4.3}$$

Consider now the distributions

$$P(\sum_{p\leqslant x} X_p - \alpha(x) \leqslant z\beta(x)), \quad (x\geqslant 1).$$

Their associated characteristic functions $\varphi(x\ t)$ have the form

$$\varphi(x,t)=\exp\left(-it\frac{\alpha(x)}{\beta(x)}\right)\prod_{p\leqslant x}\left(1+\frac{1}{p}(g(p)-1)\right).$$

In a calculation very similar to that made in 3.3 concerning the function $h(\tau)$ one can prove that

$$\varphi(x,t)=\exp\left(\sum_{p\leqslant x}p^{-1}(g(p)-1)-\frac{it\alpha(x)}{\beta(x)}+o(1)\right).$$

From the limiting relation (4.3) we see at once that

$$\varphi(x,t)\to\mu(t),\quad (x\to\infty).$$

Since $\mu(t)$ is continuous at the point $t=0$, it must be a characteristic function, and the random variable

$$\beta(x)^{-1}\{\sum_{p\leqslant x}X_p-\alpha(x)\}$$

converges to its corresponding distribution.

This completes the first assertion of proposition B of the theorem, and also the proof that $C_0\to B$.

5. Proof that B implies A

In this section we shall make use of a representation theorem of Kubilius.

LEMMA 5.1. *Let x be a real number, $x\geqslant 2$. Let r be a further real number in the range $2\leqslant r\leqslant x$. Let $j(n)$ be a strongly additive function. Define independent random variables ξ_p, one for each rational prime p, by*

$$\xi_p=\begin{cases} j(p) & \text{with probability } \dfrac{1}{p}\\[2mm] 0 & \text{with probability } 1-\dfrac{1}{p}.\end{cases}$$

Then there is a positive absolute constant so that the inequality

$$\nu_x(n;\sum_{p\mid n,\,p\leqslant r}j(p)\leqslant z)=P(\sum_{p\leqslant r}\xi_r\leqslant z)+O\left(\exp\left(-\frac{c\log x}{\log r}\right)\right)$$

holds uniformly for all real numbers z, r $(2\leqslant r\leqslant x)$, and functions $j(n)$.

Proof. Kubilius proves this lemma in his monograph [5], Chapter 2, pp. 25–27. Our use of the real variable x where he has an integer n is not of great significance.

It is convenient to define distribution functions

$$G(x,z)=\nu_x(n;f(n)-\alpha(x)\leqslant z\beta(x)),$$

and

$$H(x, z) = P(\sum_{p \leqslant q} X_p - \alpha(x) \leqslant z\beta(x)),$$

for $x \geqslant 1$, where the random variables X_p are those which are introduced in the formulation of the main theorem.

Let ε be a positive real number, and let $\alpha(x)$ be a function so that the two assertions of proposition B are valid. Then we obtain at once the inequality

$$G(x, z) \leqslant \nu_x(n; f(n) - \alpha(x) \leqslant z\beta(x), \bar{\exists}\, p \mid n,\ x^\varepsilon < p \leqslant x,\ |f(p)| > \varepsilon^2\beta(x))$$
$$+ \nu_x(n; \exists\, p \mid n,\ x^\varepsilon < p \leqslant x,\ |f(p)| > \varepsilon^2\beta(x)).$$

The second of the two frequencies which occur on the right hand side of this inequality does not exceed

$$\sum_{\substack{x^\varepsilon < p \leqslant x \\ |f(p)| > \varepsilon^2\beta(x)}} \frac{1}{p}$$

and by the second part of proposition B this sum is $0(1)$ as $x \to \infty$. As for the first frequency on this same side, we note that if n is an integer which is counted in it, then

$$\sum_{p \mid n,\, p \leqslant x^\varepsilon} f(p) - \alpha(x) \leqslant f(n) - \alpha(x) + \sum_{x^\varepsilon < p \leqslant x,\, p \mid n} |f(p)| \leqslant (z + \varepsilon)\beta(x),$$

since n can have at most ε^{-1} distinct prime divisors p which lie in the interval $x^\varepsilon < p \leqslant x$. Hence we have proved that

$$G(x, z) \leqslant \nu_x(n; f(n) - \alpha(x) \leqslant (z + \varepsilon)\beta(x)) + o(1), \quad (x \to \infty). \tag{4.2}$$

We now apply Lemma 4.1 (Kubilius' representation theorem) with $r = x^\varepsilon$, and replace the expression on the right hand side of the inequality by

$$P(\sum_{p \leqslant x^\varepsilon} X_p - \alpha(x) \leqslant (z + \varepsilon)\beta(x)) + O(\exp(-c\varepsilon^{-1})) + o(1).$$

In turn, the probability which appears in this expression certainly does not exceed

$$P(\sum_{p \leqslant x} X_p - \alpha(x) \leqslant (z + 2\varepsilon)\beta(x)) + P(|\sum_{x^\varepsilon < p \leqslant x} X_p| > \varepsilon\beta(x)) = H(x, z + 2\varepsilon) + P_1,$$

say. We can majorise the probability P_1 by choosing a positive real number η, and introducing new independent variables Y_p, defined by

$$Y_p = \begin{cases} X_p & \text{if } |X_p| \leqslant \eta\beta(x) \\ 0 & \text{if } |X_p| > \eta\beta(x). \end{cases}$$

Then $$P_1 \leqslant P(\exists\, X_p \neq Y_p, x^\varepsilon < p \leqslant x) + P(|\sum_{x^\varepsilon < p \leqslant x} Y_p| > \varepsilon\beta(x)) = P_2 + P_3,$$

say. We can estimate P_2 at once by applying the second part of hypothesis B:

$$P_2 \leqslant \sum_{\substack{x^\varepsilon < p \leqslant x \\ |f(p)| > \eta\beta(x)}} \frac{1}{p} = o(1), \quad (x \to \infty).$$

To estimate P_3 we apply a standard argument of Tchebycheff:

$$\begin{aligned} P_3 &\leqslant (\varepsilon\beta(x))^{-2} \,\text{Expect}\,(\sum_{x^\varepsilon < p \leqslant x} Y_p)^2 = (\varepsilon\beta(x))^{-2}\{\text{Var}\,(\sum_{x^\varepsilon < p \leqslant x} Y_p) + (\text{Expect} \sum_{x^\varepsilon < p \leqslant x} Y_p)^2\} \\ &= (\varepsilon\beta(x))^{-2}\{\sum_{x^\varepsilon < p \leqslant x} \text{Var}\, Y_p + (\sum_{x^\varepsilon < p \leqslant x} \text{Expect}\, Y_p)^2\} \\ &\leqslant (\varepsilon\beta(x))^{-2}\left\{\sum_{\substack{x^\varepsilon < p \leqslant x \\ |f(p)| \leqslant \eta\beta(x)}} \frac{f^2(p)}{p} + \left(\sum_{\substack{x^\varepsilon < p \leqslant x \\ |f(p)| \leqslant \eta\beta(x)}} \frac{|f(p)|}{p}\right)^2\right\}, \\ &< \varepsilon^{-2}\eta^2\left(\sum_{x^\varepsilon < p \leqslant x} \frac{1}{p} + 1\right)^2 \\ &= \varepsilon^{-2}\eta^2(1 - \log\varepsilon + o(1))^2, \quad (x \to \infty). \end{aligned}$$

Altogether this proves that if η and ε are positive real numbers $0 < \varepsilon < 1$, then as $x \to \infty$

$$P_1 \leqslant \varepsilon^{-2}\eta^2(1 - \log\varepsilon)^2 + o(1).$$

We have therefore proved that the inequality

$$G(x, z) \leqslant H(x, z + 2\varepsilon) + \varepsilon^{-2}\eta^2(1 - \log\varepsilon)^2 + O(\exp(-c\varepsilon^{-1})) + o(1)$$

holds as $x \to \infty$, for any fixed pair of positive real numbers ε and u, $0 < \varepsilon < 1$, and uniformly for all real numbers z.

In a precisely similar way we can obtain the inequality

$$G(x, z) \geqslant H(x, z - 2\varepsilon) - \varepsilon^{-2}\eta^2(1 - \log\varepsilon)^2 + O(\exp(-c\varepsilon^{-1})) + o(1), \quad (x \to \infty).$$

We can express these two inequalities in a somewhat different manner.

We recall that if F and G are two distribution functions then their *Lévy-distance* $\varrho(F, G)$ is defined to be the infinum of those real numbers h for which the inequalities

$$F(z - h) - h \leqslant G(z) \leqslant F(z + h) + h$$

hold uniformly for all real numbers z. This defines a metric on the space of distribution functions; and a sequence of distribution functions F_n, $(n = 1, 2, \ldots)$ will converge weakly to a distribution function F if and only if $\varrho(F_n, F)\to 0$ as $n\to\infty$. In these terms our last two inequalities can be expressed thus:

$$\limsup_{x\to\infty} \varrho(G(x,z), H(x,z)) \leqslant 2\varepsilon + \varepsilon^{-2}\eta^2(1-\log\varepsilon)^2 + O(\exp(-c\varepsilon^{-1})).$$

We let $\eta\to 0+$ and then $\varepsilon\to 0+$ to deduce that

$$\varrho(G(x,z), H(x,z))\to 0, \quad (x\to\infty).$$

It is now clear that if the distribution functions $H(x, z)$ converge weakly as $x\to\infty$, then so will the distribution functions $G(x, z)$, and to the same limit law.

This establishes the first part of proposition A, and it remains to verify that

$$\sup_{x^{\frac{1}{2}}\leqslant y\leqslant x} |\alpha(x)-\alpha(y)| = o(\beta(x)), \quad (x\to\infty).$$

In order to do this we make use of the fact that $\alpha(x)$ occurs as a renormalising function which is restricted by the fact that the distributions $H(x, z)$ converge. We shall need a part of the following result of Gnedenko and Kolmogorov ([4], § 25 Theorem 1, pp. 116–121.).

LEMMA 4.3. *In order that for some suitably chosen constants A_n the distributions of the sums*

$$\xi_{n1} + \ldots + \xi_{nk_n} - A_n$$

of independent infinitesimal random variables converge to a limit, it is necessary and sufficient that there exist non-decreasing functions

$$M(u),\ (M(-\infty)=0),\ N(u),\ (N(+\infty)=0),$$

defined in the intervals $(-\infty, 0)$ and $(0,\infty)$ respectively, and a constant $\sigma\geqslant 0$, such that

(1) *At every continuity point of $M(u)$ and $N(u)$*

$$\lim_{x\to\infty} \sum_{k=1}^{k_n} P(\xi_{nk}<u) = M(u), \quad (u<0)$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \{P(\xi_{nk}<u)-1\} = N(u). \quad (u>0).$$

$$\text{(2)} \qquad \lim_{\varepsilon\to 0}\liminf_{n\to\infty} \sum_{k=1}^{k_n} \left\{\int_{|z|<\varepsilon} z^2 dP(\xi_{nz}<z) - \left(\int_{|z|<\varepsilon} z dP(\xi_{nk}<z)\right)^2\right\} = \sigma^2,$$

together with a similar relation obtained by replacing 'lim inf' by 'lim sup'.

The constants A_n may be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \int_{|z|<\tau} z dP(\xi_{nk} < z)$$

where $\pm\tau$ are continuity points of $M(u)$ and $N(u)$.

Remark. In their theorem, Gnedenko and Kolmogorov also determine the form of the limit law in terms of the representation theorem for infinitely divisible distributions of Lévy and Khinchine. We do not state this part of their result since we shall have no need of it.

In our present circumstances we shall be interested in the variables

$$\{\xi_{nk}, 1 \leqslant k \leqslant k_n\} = \{\beta(n)^{-1} X_p, p \leqslant n\}, \quad (n = 1, 2, \ldots).$$

Thus we shall be considering cumulative sums of independent random variables, so that any possible limit law will belong to the class L of Khinchine (see for example Gnedenko and Kolmogorov [4], Chapter 6, §§ 29, 30). Each of the functions $M(u)$ and $N(u)$ which occur in the formulation of Lemma 4.3 are then actually continuous, so that in the final assertion of that lemma any (fixed) positive value of τ may be taken.

Since the distributions $H(x, z)$, $(x \geqslant 2)$ and therefore $H(n, z)$, $(n = 1, 2, \ldots)$ are assumed to converge weakly, there exists a continuous function $k(u)$,defined for real numbers $u > 0$, so that

$$\sum_{\substack{p \leqslant n \\ |f(p)| > u\beta(n)}} \frac{1}{p} \to k(u), \quad (n \to \infty). \tag{4.4}$$

Assume now that the distributions $H(x, z)$ converge to a proper limit law. That is to say, a law whose characteristic function is not of the form exp (ict), c a constant. Consider the sequence of distributions $H(x, z)$, $n = 1, 2, \ldots$. Then, by Lemma 4.3 we may choose

$$A(n) = \sum_{\substack{p \leqslant n \\ |f(p)| < \tau\beta(n)}} \frac{f(p)}{p}$$

provided that τ is a fixed positive real number. Since the variables

$$\beta(n)^{-1} \{ \sum_{p \leqslant n} X_p - A(n) \}$$

$$\beta(n)^{-1} \{ \sum_{p \leqslant n} X_p - \alpha(n) \}$$

converge to the same proper law, an elementary result from the theory of probability (see for example Gnedenko and Kolmogorov [4] Chapter 2, Theorem 2, pp. 42–44) implies that as $n \to \infty$

$$\frac{A(n)-\alpha(n)}{\beta(n)} \to 0.$$

We next prove that as $n \to \infty$

$$\sup_{|y-n|\leqslant 1} |\alpha(y)-\alpha(n)| = o(\beta(n)) \tag{4.5}$$

To do this it suffices to note that if y lies in the interval $n-1<y<n+1$ then $G(n, z)$ and the frequency

$$\nu_n(m;\, f(m)-\alpha(y) \leqslant z\beta(y)), \quad (=n^{-1}yG(y,z))$$

both converge to a certain proper limit law. Hence, we deduce that both

$$\beta(y)/\beta(n) \to 1, \; |\alpha(y)-\alpha(n)|/\beta(n) \to 0,$$

as $n\to\infty$. The first of these limiting relations is of no present value to us, but the second is the relation which we wished to establish.

Let x and y be real numbers which satisfy $2 \leqslant x^{1/2} \leqslant y \leqslant x$. Define integers m and n so that $m \leqslant y < m+1$, and $n-1<x\leqslant n$. Thus the inequalities $2 \leqslant m < n \leqslant x+1$ are satisfied. As $x\to\infty$ we see from property (4.5) that

$$\begin{aligned}\alpha(x)-\alpha(y) &= \alpha(n)-\alpha(m)+o(\beta(x))\\ &= A(m)-A(n)+o(\beta(x))\end{aligned}$$

It is then convenient to write

$$A(m)-A(n) = \sum_{\substack{m<p\leqslant n\\ |f(p)|\leqslant \tau\beta(n)}} \frac{f(p)}{p} + \sum_{\substack{p\leqslant m,\, |f(p)|\leqslant \tau\beta(n)\\ |f(p)|>\tau\beta(m)}} \frac{f(p)}{p} - \sum_{\substack{p\leqslant m,\, |f(p)|\leqslant \tau\beta(m)\\ |f(p)|>\tau\beta(n)}} \frac{f(p)}{p} = \textstyle\sum_1 + \sum_2 + \sum_3,$$

say. Let ε be a positive real number. Then if x is large enough $\beta(n) \leqslant (1+\varepsilon)\,\beta(m)$, so that by condition (4.4) of the present section

$$\begin{aligned}\left|\textstyle\sum_2\right| &\leqslant \tau\beta(n)\left\{\sum_{p\leqslant m,\, |f(p)|\leqslant \tau(1+\varepsilon)\beta(m)} \frac{1}{p} - \sum_{p\leqslant m,\, |f(p)|\leqslant \tau\beta(m)} \frac{1}{p}\right\}\\ &\leqslant \tau\beta(n)\{-k(\tau+\varepsilon\tau)+k(\tau)+o(1)\}, \quad (x\to\infty).\end{aligned}$$

Hence (since $\beta(n) \sim \beta(x)$ as $x\to\infty$),

$$\limsup_{x\to\infty} \beta(x)^{-1} \left|\textstyle\sum_2\right| \leqslant -\tau\{k(\tau+\varepsilon\tau)-k(\tau)\}.$$

We let $\varepsilon \to 0+$, and recall that τ is a point of continuity of $k(u)$. In this way we can prove that $\Sigma_2 = o(\beta(x))$ as $x\to\infty$.

In exactly the same way we prove that $\Sigma_3 = o(\beta(x))$.

To consider Σ_1 we split the sum into two parts. Into the first part we put those terms involving primes p for which $|f(p)| > \varepsilon\beta(n)$. The contribution to Σ_1 which such terms make is at most

$$\tau\beta(n) \sum_{\substack{n^{\frac{1}{2}} < p \leqslant n \\ |f(p)| > \varepsilon\beta(n)}} \frac{1}{p}$$

and by the second condition of proposition B this expression is $o(\beta(x))$ as $x \to \infty$. On the other hand, those terms which remain in Σ_1 contribute not more than

$$\varepsilon\beta(n) \sum_{n^{\frac{1}{2}} < p \leqslant n} \frac{1}{p} = \varepsilon\beta(n)\{\log 2 + o(1)\}, \quad (n \to \infty).$$

Hence
$$\limsup_{x \to \infty} \beta(x)^{-1} \left|\sum\nolimits_1\right| \leqslant \varepsilon \log 2,$$

holds for every positive value of ε. Letting $\varepsilon \to 0+$ we see that $\Sigma_1 = o(\beta(x))$, and that

$$\alpha(x) - \alpha(y) = o(\beta(x)), \quad (x \to \infty)$$

holds uniformly for all values of y in the interval $x^{1/2} \leqslant y \leqslant x$.

This is the second assertion of proposition A, and so we have completed a proof that $B \to A$, in every case except that of when the distributions $H(x, z)$, and so $G(x, z)$, converge to an improper law. In this last case we are concerned with a form of the weak law of large numbers. This has been considered by the author on another occasion [2], and the arguments and results given there guarantee the validity of the inference B *true* $\to A$ *true* in this special case under hypotheses on $\alpha(x)$ and $\beta(x)$ which are considerably weaker than those which are assumed in the present theorem.

6. Proof that C_0 implies C, and conversely

It is immediate that $C \to C_0$. Assume, therefore, that proposition C_0 is valid. Then by the proofs of §§ 3–5 so are propositions A and B, with in fact the same $\alpha(x)$ as a possibility. Moreover, $\mu(t)$ is the characteristic function of a limit law for the sums

$$\beta(x)^{-1}\Big(\sum_{p \leqslant x} X_p - \alpha(x)\Big).$$

It is easy to see that the variables $\beta(x)^{-1}X_p$, $(2 \leqslant p \leqslant x)$, are infinitesimal, and since they are independent such a law must be infinitely divisible. In particular $\mu(t)$ will be non-zero for all real values of t (Gnedenko and Kolmogorov [4] Theorem 2 of § 24, Chapter 4, and

Theorem 1 of § 17, Chapter 3). Thus none of our earlier arguments need to be restricted to any particular interval of t-values, and the second assertion of proposition C_0 (and therefore of C) holds for each real value of t.

We now show how to obtain the first of the two assertions in proposition C. More exactly, let $\log \mu(t)$ be defined continuously from the principal value taken at $t=0$. (Since $\mu(t)$ is a characteristic function it will be a continuous function of t). Then we shall prove that as $x\to\infty$

$$\sum_p p^{-s_0}\left\{\exp\left(\frac{itf(p)}{\beta(x)}\right)-1\right\}\frac{it\alpha(x)}{\beta(x)}\to\log\mu(t).$$

For ease of presentation let us denote the expression which occurs here on the left hand side of the arrow by $w(x, s_0, t)$. Then the proof that $A\to C_0$ will also yield that $\exp w(x, s_0, t)\to \mu(t)$ uniformly over any (fixed) bounded sets of real numbers $|\tau|\leqslant\tau_0$, $|t|\leqslant t_0$. The only adjustment needed is that one obtains a form of Lemma 3.1 in which the convergence is uniform over any fixed interval of y-values $0<c_6<y\leqslant c_7$, and any bounded interval of t-values. This is easily obtained since uniformly for such an interval of y-values $\{\alpha(x^y)-\alpha(x)\}\beta(x)^{-1}\to 0$, as $x\to\infty$. We can therefore assert that for suitably chosen integers $n(x,\tau,t)$ we have

$$w(x, s_0, t)=\log\mu(t)+2\pi\, in\,(x,\tau,t)+o(1)$$

as $x\to\infty$, uniformly for $|\tau|\leqslant\tau_0$, $|t|\leqslant t_0$. Thus we can find a real number x_0 so that if $x>x_0$, then with these same uniformities

$$|w(x, s_0, t)-\log\mu(t)-2\pi\, in\,(x,\tau,t)|<\tfrac{1}{4}.$$

The function $w(x, s_0, t)$ is a continuous function of x. (Here $\beta(x)$ is now assumed to be a continuous function of x. The number s_0 also depends continuously upon x). This can be readily proved as follows. Let P be a positive real number. Let x_1 be a real number ($x_1\geqslant 2$), $s_1=1+(\log x_1)^{-1}+i\tau(\log x_1)^{-1}$. Then

$$|w(x, s_0, t)-w(x_1, s_1, t)|\leqslant 4\sum_{p>P}p^{-\sigma_0}+\left|\sum_{p\leqslant P}(p^{-s_0}-p^{-s_1})\left\{\exp\left(\frac{itf(p)}{\beta(x)}\right)-1\right\}\right|$$
$$+\left|\sum_{p\leqslant P}p^{-s_1}\left\{\exp\left(\frac{itf(p)}{\beta(x)}\right)-\exp\left(\frac{itf(p)}{\beta(x_1)}\right)\right\}\right|,$$

As $x_1\to x$ we have $s_1\to s_0$, $\beta(x_1)\to\beta(x)$, and deduce that

$$\limsup_{x_1\to x}|w(x, s_0, t)-w(x_1, s_1, t)|\leqslant 4\sum_{p>P}p^{-\sigma_0}.$$

Letting $P\to\infty$ we see that uniformly for $|\tau|\leqslant\tau_0$, $|t|\leqslant t_0$, we have $|n(x,\tau,t)-n(x_1,\tau,t)|<1$

provided only that x_1 is sufficiently near to x, and $x > x_0$. We can therefore assert that the integers $n(x, \tau, t)$ in fact do not depend upon the value of x. Let us therefore write $n(\tau, t)$ in place of $n(x, \tau, t)$. Then as $x \to \infty$

$$w(x, s_0, t) \to \log \mu(t) + 2\pi i n(\tau, t)$$

uniformly for $|\tau| \leq \tau_0$, $|t| \leq t_0$. Since $w(x, s_0, t)$ is continuous in τ and t and the convergence is suitably uniform the function $\lim w(x, s_0, t)$ $(x \to \infty)$ is also continuous in τ and t. In view of the continuous definition of $\log \mu(t)$ the integer $n(\tau, t)$ must be continuous, and so a constant for $|\tau| \leq \tau_0$, $|t| \leq t_0$. Therefore over this rectangle of values of τ and t we have $n(\tau, t) = n(0, 0) = 0$. This proves that as $x \to \infty$

$$w(x, s_0, t) \to \log \mu(t)$$

uniformly for any rectangle of (τ, t)-values.

We have now established the first assertion in proposition C, and so completed the proof of the theorem.

7. Concluding remarks

It is clear from the arguments of § 3 that the properties of $\alpha(x)$ and $\beta(x)$ which se assume are of a simple nature with respect to their behaviour under the transformations $x \to x^y$ $(y > 0)$. In fact, for our purposes these functions are essentially asymptotically invariant under such transformations. It is quite possible to consider other renormalising functions $\alpha(x)$ and $\beta(x)$ whose behaviour under these transformations is entirely different. The nature of the resulting conditions which are necessary in order that the frequencies $G(x, z)$ should converge weakly are then quite different. In particular, the function $f(n)$ need no longer behave like a sum of independent random variables.

In another direction, we can view the transformations $x \to x^y$ as forming a group Γ (with composition as a group law) which is isomorphic to the multiplicative group of positive real numbers. Our use of fourier analysis with respect to the variable y can thus be viewed as fourier analysis upon the group Γ. Accordingly, we can ask whether in certain circumstances one might not profitably use groups of transformations other than Γ with which to operate.

We intend to return to various such questions at a future date.

Note: Since this paper was accepted for publication it has come to the notice of the author that a form of necessary and sufficient condition in order that proposition A be valid has been established, inter alia, in a paper of Levin and Timofeev (B. V. Levin and N. M. Timofeev: An analytical method in probabilistic number theory. Transactions of the Vladimir State Pedagogical Institute of the Ministry of Culture, RSFSR. pp. 56–150, see pp. 113–117). The method that these authors use differs considerably from that of the present paper. In particular the above method is one which applies quite naturally in more general circumstances, as is indicated in this section.

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Received March 20, 1973