ZEROS OF THE DERIVATIVES OF THE RIEMANN ZETA-FUNCTION

BY

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1. Introduction and some results

Several diverse theorems concerning the zeros of $\zeta^{(k)}(s)$, the kth derivative, of the Riemann zeta function will be presented. Relationships with existing results, [1], [5–9], will be discussed.

THEOREM 1. *Let $N^-(T)$ be the number of zeros of $\zeta(s)$ in R: $0<t<T$, $0<\sigma<\frac{1}{2}$ where $s=\sigma+it$. Let $N_1^-(T)$ be the number of zeros of $\zeta'(s)$ in R. Then*

$$N_1^-(T)=N^-(T)+O(\log T). \tag{1.1}$$

Unless $N^-(T)>T/2$ for all large T there exists a sequence $\{T_j\}$, $T_j\to\infty$ as $j\to\infty$ such that

$$N_1^-(T_j)=N^-(T_j). \tag{1.2}$$

Theorem 1 can be regarded as stating that $\zeta(s)$ and $\zeta'(s)$ have the same number of zeros in $0<\sigma<\frac{1}{2}$. The following is essentially due to Speiser [5].

COROLLARY TO THEOREM 1. *The Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0<\sigma<\frac{1}{2}$.*

One half of the above, namely RH$=>\zeta'(s)$ is zero-free in $0<\sigma<\frac{1}{2}$ was rediscovered by Spira [9].

Let $N_k(T)$ be the number of non-real zeros of $\zeta^{(k)}(s)$ for $0<t<T$. Then it was shown by Berndt [1], and will also be a by-product of the proof of Theorem 2, that for $k\geqslant 1$

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$$N_k(T) = \frac{T}{2\pi}\left(\log\frac{T}{4\pi} - 1\right) + O(\log T). \tag{1.3}$$

THEOREM 2. *Denote the number of non-real zeros of* $\zeta^{(k)}(s)$ *in* $0<t<T$, $\sigma \leqslant c$ *by* $N_k^-(c, T)$ *and the number for* $\sigma \geqslant c$ *by* $N_k^+(c, T)$. *Then, for given* k, *uniformly for* $\delta > 0$

$$N_k^+(\tfrac{1}{2}+\delta, T) + N_k^-(\tfrac{1}{2}-\delta, T) \ll \delta^{-1} T \log\log T.$$

In view of (1.3)

$$N_k^+(\tfrac{1}{2}+\delta, T) + N_k^-(\tfrac{1}{2}-\delta, T) \ll \frac{N_k(T)\log\log T}{\delta \log T}.$$

Hence most of the zeros of $\zeta^{(k)}(s)$ are clustered around $\sigma = \frac{1}{2}$. It was proved by Spira [8] that most of the zeros of $\zeta^{(k)}(s)$ lie in $0 \leqslant \sigma \leqslant \frac{1}{2}+\delta$ for $\delta > 0$.

In proving Theorem 2 it will also be seen that the corresponding result is valid in $T<t<T+U$ where $U > T^{1/2}$. A consequence of this is that if $w(t) \to \infty$ as $t \to \infty$, then most of the zeros of $\zeta^{(k)}(s)$ lie in

$$|\sigma - \tfrac{1}{2}| \leqslant w(t) \log\log t / \log t.$$

Let $\varrho = \beta + i\gamma$ denote the non-real zeros of $\zeta(s)$ as usual. Let $\varrho' = \beta' + i\gamma'$ denote those of $\zeta'(s)$. Let $\varrho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ denote the non-real zeros of $\zeta^{(k)}(s)$, $k \geqslant 1$ (so that ϱ' and $\varrho^{(1)}$ are equivalent).

THEOREM 3. *For* $0 < U < T$

$$2\pi \sum_{T \leqslant \gamma^{(k)} \leqslant T+U} (\beta^{(k)} - \tfrac{1}{2}) = kU \log\log\frac{T}{2\pi} + U(\tfrac{1}{2}\log 2 - k\log\log 2)$$
$$+ O(U^2/(T\log T)) + O(\log T). \tag{1.4}$$

THEOREM 4. *Let* $U > \log T$. *Then*

$$\sum_{\substack{T<\gamma'<T+U \\ \beta<1/2}} (\tfrac{1}{2} - \beta') \leqslant \sum_{\substack{T<\gamma<T+U \\ \beta<1/2}} (\tfrac{1}{2} - \beta) + O(U).$$

COROLLARY. *By Selberg* [3], *if* $U \geqslant T^a$, $a > \frac{1}{2}$, *then*

$$\sum_{\substack{T<\gamma<T+U \\ \beta<1/2}} (\tfrac{1}{2} - \beta) = O(U),$$

and so it follows that

$$\sum_{\substack{T<\gamma'<T+U \\ \beta'<1/2}} (\tfrac{1}{2} - \beta') = O(U).$$

THEOREM 5. *For* $U \geqslant T^a$, $a > \frac{1}{2}$

$$\sum_{\substack{T\leqslant\gamma'\leqslant T+U \\ \beta'>1/2}} (\beta' - \tfrac{1}{2}) = \frac{U}{2\pi}\log\log\frac{T}{2\pi} + O(U). \tag{1.5}$$

THEOREM 6. *Let* $\frac{1}{2} < a \leqslant 1$. *Let* $\delta > C/\log T$ *where* C *is large* (*but independent of* T *and* a). *Let* $U = T^a$. *Then*

$$\sum_{\substack{T<\gamma'<T+U \\ \beta'<1/2-\delta}} (\tfrac{1}{2} - \delta - \beta') \ll (1 + \delta \log U)^2 U^{1-\delta(2-1/a)/4}. \tag{1.6}$$

Also there exists U_j $j = 1, 2$ *such that* $U/4 \leqslant U_j \leqslant U/2$

$$\sum_{\substack{T-U_1<\gamma'<T+U_2 \\ \beta'<1/2-2\delta}} (\tfrac{1}{2} - \beta') \ll \log \frac{1}{\delta} \sum_{\substack{T-U_1\leqslant\gamma\leqslant T+U_2 \\ \beta\geqslant 1/2+\delta}} (\beta - \tfrac{1}{2}).$$

COROLLARY. *If* $\delta = w(T)/\log T$ *where* $w(T) \to \infty$ as $T \to \infty$ *then*

$$N_1^-(\tfrac{1}{2} - \delta, T + U) - N_1^-(\tfrac{1}{2} - \delta, T) \ll w^2(T) \exp\{-(2a-1)\, w(t)/4\}\, U \log T. \tag{1.7}$$

Thus most of the complex zeros of $\zeta'(s)$ lie to the right of $\sigma = \frac{1}{2} - w(t)/\log t$ if $w(t) \to \infty$.

THEOREM 7. *Let* $m \geqslant 0$. *If* $\zeta^{(m)}(s)$ *has only a finite number of non-real zeros in* $\sigma < \frac{1}{2}$, *then* $\zeta^{(m+j)}(s)$ *has the same property for* $j \geqslant 1$.

COROLLARY. *The R.H. implies that* $\zeta^{(k)}(s)$ *has at most a finite number of non-real zeros in* $\sigma < \frac{1}{2}$ *for* $k \geqslant 1$.

THEOREM 8. *The R.H. implies that*

$$2\pi \sum_{\substack{0<\gamma_k\leqslant T \\ \beta^{(k)}>1/2}} (\beta^{(k)} - 1/2) = kT \log\log \frac{T}{2\pi} - 2\pi k \operatorname{Li}\left(\frac{T}{2\pi}\right) + T(\tfrac{1}{2}\log 2 - k \log\log 2) + O(\log T).$$

Here $\operatorname{Li}(x)$ is $\int_2^x dv/\log v$.

2. Proof of Theorem 1

With $\{\varrho\}$ the zeros of ζ in the critical strip

$$\operatorname{Re}\frac{\zeta'}{\zeta}(s) = -\operatorname{Re}\frac{1}{s-1} + \tfrac{1}{2}\log\pi - \tfrac{1}{2}\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \operatorname{Re}\sum\frac{1}{s-\varrho}. \tag{2.1}$$

From the functional equation if $\varrho = \beta + i\gamma$, $\beta < \frac{1}{2}$, then $1 - \bar{\varrho} = 1 - \beta + i\gamma$ is also a zero. With $\beta < \frac{1}{2}$

$$\operatorname{Re}\left(\frac{1}{s-\varrho} + \frac{1}{s-1+\bar{\varrho}}\right) = -2(\tfrac{1}{2} - \sigma)\frac{(t-\gamma)^2 + (\sigma-\frac{1}{2})^2 - (\frac{1}{2}-\beta)^2}{|s-\varrho|^2\,|s-1+\bar{\varrho}|^2}.$$

Let

$$I_1 = 2\sum_{\beta<1/2}\frac{(t-\gamma)^2 + (\sigma-\frac{1}{2})^2 - (\frac{1}{2}-\beta)^2}{|s-\varrho|^2\,|s-1+\bar{\varrho}|^2} + \sum_{\beta=1/2}\frac{1}{|s-\varrho|^2}. \tag{2.2}$$

Then
$$I = \operatorname{Re} \sum_{\varrho} \frac{1}{s-\varrho} = -(\tfrac{1}{2}-\sigma) I_1. \tag{2.3}$$

The Euler-Maclaurin sum formula for Γ'/Γ easily leads to

$$\frac{\Gamma'}{\Gamma}(w) = \log w - \frac{1}{2w} + R, \quad |R| \leqslant \frac{1}{10|w|^2}, \quad |w| \geqslant 2, u \geqslant 0,$$

where $w = u + iv$. Hence for $|s| \geqslant 3, \sigma \geqslant 0$,

$$\operatorname{Re} \frac{\Gamma}{\Gamma'}\left(\frac{s}{2}+1\right) = \log\left|1+\frac{s}{2}\right| - \frac{\sigma+2}{|s+2|^2} + R_1, \quad |R_1| \leqslant \frac{2}{5|s+2|^2}. \tag{2.4}$$

Using standard explicit estimates on $N(T)$, the number of zeros of $\zeta(s)$ in $0<\sigma<1$, $0<t<T$, and the fact that $\beta=\frac{1}{2}$ for $|\gamma|<1\,000$ it is easy to verify from (2.1), (2.2), (2.3) and (2.4) that $\operatorname{Re}\zeta'/\zeta<0$ for $t=10$, $0\leqslant\sigma\leqslant 1$.

For $\sigma=0$, it is obvious from (2.2) since $0<\beta\leqslant\frac{1}{2}$, that all terms in I_1 are positive for $\sigma=0$. Hence $I<0$ on $\sigma=0$. From (2.4) and (2.1) it then follows easily that $\operatorname{Re}\zeta'/\zeta<0$ on $\sigma=0$ for $t\geqslant 10$. On $\sigma=\frac{1}{2}$, except at zeros of $\zeta(\frac{1}{2}+it)$, it is evident that $I=0$. Let $\varrho_0=\beta_0+i\gamma_0$ be a zero with $\beta_0=\frac{1}{2}$. Then the single term $|s-\varrho_0|^{-2}$ can be made arbitrarily large for $|s-\varrho_0|$ small. Hence on a small semi-circle with center at ϱ_0 and $\sigma<\frac{1}{2}$, $I_1>0$ and so $I<0$. Thus on such a semi-circle $\operatorname{Re}\zeta'/\zeta<0$. Hence on an appropriately indented contour on $\sigma=\frac{1}{2}$, $\operatorname{Re}\zeta'/\zeta<0$ for $t\geqslant 10$. Suppose next that there is a sequence $\{T_j\}$, $T_j\to\infty$ as $j\to\infty$, such that $\operatorname{Re}\zeta'/\zeta<0$ on $t=T_j$ for $0<\sigma<\frac{1}{2}$. Then on the closed indented contour with vertices at $10i$, $\frac{1}{2}+10i$, $\frac{1}{2}+T_j i$, $T_j i$, $\operatorname{Re}\zeta'/\zeta<0$ and so the change in arg ζ'/ζ is 0 on the contour. Thus the number of zeros of ζ' and ζ are the same inside the contour proving (1.2).

Next suppose no such sequence $\{T_j\}$ exists. Then for sufficiently large t, $\operatorname{Re}\zeta'/\zeta$ is non-negative for some σ, $0<\sigma<\frac{1}{2}$. This can happen only where $I_1<0$. But $I_1<0$ only if at least one term in I_1 is negative. Hence for some $\beta<1/2$

$$(\tfrac{1}{2}-\beta)^2 > (t-\gamma)^2 + (\sigma-\tfrac{1}{2})^2,$$

which implies $|t-\gamma|<\frac{1}{2}$. In particular if t is taken as an integer n, then there is at least one zero ϱ with $\beta<\frac{1}{2}$ and $|\gamma-n|<1/2$. Thus in this case $N^-(T)\geqslant T+O(1)$.

Finally, to prove (1.1), by a standard use of Jensen's theorem it can be shown that the change in arg $\zeta(\sigma+it)$ and arg $\zeta'(\sigma+it)$ from $\sigma=1$ to $\sigma=0$ for large t is $O(\log t)$. This with the previous fact that $\operatorname{Re}\zeta'/\zeta(s)<0$ on $\sigma=0$, $t\geqslant 10$, $0\leqslant\sigma\leqslant 1$, $t=10$, and on the indented line $\sigma=\frac{1}{2}$, $t\geqslant 10$ proves (1.1) and completes the proof of the theorem.

It was proved by Spira [8] that for $|s|>165$ and $\sigma\leqslant 0$, $\zeta'(s)$ has only real zeros and exactly one in $(-1-2n, 1-2n)$. The following is an easy consequence of (2.1).

THEOREM 9. *For $n \geqslant 2$ there is a unique solution of $\zeta'(s)=0$ in the interval $(-2n, -2n+2)$ and there are no other zeros of $\zeta'(s)$ in $\sigma \leqslant 0$.*

Proof of Theorem 9. By direct consideration of $\zeta'(it)$ and $\zeta(it)$, or what is equivalent by the functional equation, of $\zeta'(1+it)$ and $\zeta(1+it)$ it follows that $\arg \zeta'/\zeta(it)$ changes by approximately -2π from $-6.25i$ to $6.25i$. On the remainder of the boundary of the rectangle with vertices at $-2N-1 \pm iN$, $\pm iN$ it follows from (2.1) that $\operatorname{Re} \zeta'/\zeta(s)<0$. Since $\zeta(s)$ has N real zeros in the rectangle, $\zeta'(s)$ must have at least $N-1$ real zeros by Rolle's theorem and by the change in argument of -2π it does indeed have exactly $N-1$ and so all of these are real.

A consequence of Theorems 1 and 9 is that RH$\Leftrightarrow\zeta'(s)$ has no non-real zeros for $\sigma<1/2$ [5].

Remark on the numerical location of zeros of $\zeta(s)$ off of $\sigma=\frac{1}{2}$: From the functional equation it is easy to show, as will be seen in § 4, that $\zeta'(s)$ and $J(1-s)$ have the same zeros for $0<\sigma<1$ where from (4.1).

$$J(s)=\zeta(s)+\zeta'(s)\left[\frac{h'(s)}{h(s)}+\frac{h'(1-s)}{h(1-s)}\right]^{-1}.$$

Here $h(s)=\pi^{-s/2}\Gamma(s/2)$. In view of Theorem 1 the number of zeros of $J(s)$ in $1>\sigma>\frac{1}{2}$ is equal to that of $\zeta(s)$. It follows easily from the fact that $\operatorname{Re} \zeta'/\zeta<0$ on $\sigma=\frac{1}{2}$, except at zeros of $\zeta(s)$, that $\zeta'(\frac{1}{2}+it)$ can be zero only where $\zeta(\frac{1}{2}+it)$ is zero. Hence except at multiple zeros of $\zeta(s)$, $\zeta'(s)$ and so $J(s)$ does not vanish on $\sigma=\frac{1}{2}$. Thus because $J(s)$ might be expected to vanish seldom if at all on $\sigma=\frac{1}{2}$, the determination of the number of zeros of $\zeta(s)$ in $\sigma>\frac{1}{2}$ can be conveniently ascertained from the variation of $\arg J(\frac{1}{2}+it)$.

The calculation of $J(\frac{1}{2}+it)$ and hence $\arg J(\frac{1}{2}+it)$ can be based on the asymptotic Riemann-Siegel formula for $\zeta(s)$. Indeed since $\zeta'(s)$ can be expressed in terms of $\zeta(s)$ by the Cauchy integral formula, differentiation of the asymptotic series is justified and represents $\zeta'(s)$ asymptotically.

For h'/h the standard Stirling formulas are available.

3. Proofs of Theorems 2 and 3

Here Littlewood's lemma is used in a familiar way [10, Chap. 9]. For $\sigma>1$

$$\zeta^{(k)}(s)=(-1)^k \sum (\log n)^k/n^s.$$

Let

$$Z_k(s)=(-1)^k 2^s (\log 2)^{-k} \zeta^{(k)}(s),$$

so that $Z_k(s) \to 1$ as $\sigma \to \infty$. Z_k is real on $t=0$ and $(s-1)^{k+1} Z_k(s)$ is entire.

It was shown by Spira [7] that the non-real zeros of $\zeta^{(k)}(s)$ lie in a vertical strip

$-b_k < \sigma < a_k$. This will also be evident below. Littlewood's lemma will be applied on the rectangle with vertices at $a+i$, $a+iT$, $-b+iT$, $-b+i$ where $a=a_k$ and $b=b_k$. It gives

$$\int_1^T \log|Z_k(-b+it)|\,dt - \int_1^T \log|Z_k(a+it)|\,dt - \int_{-b}^{a} \arg Z_k(\sigma+i)\,d\sigma + \int_{-b}^{a} \arg Z_k(\sigma+iT)\,d\sigma = 2\pi\sum(b+\beta^{(k)}) \quad (3.1)$$

where the zeros of $Z_k(s)$ in the rectangle are designated by $\varrho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$. As will be seen it is an easy consequence of the functional equation and Stirling's formula for $\log\Gamma(s)$ that as t increases the zeros of $Z_k(s)$ lie in $\sigma > -\delta$ for $\delta > 0$.

The $\arg Z_k(s)$ in (3.1) is obtained by continuation of $\log Z_k(s)$ leftward from the value 0 at $\sigma=\infty$. (If $Z_k(s)$ has a zero on $t=1$ the lower vertices of the rectangle should be moved a little.) The third integral in (3.1) is independent of T and so is $O(1)$. The fourth integral is handled in a familiar way by getting a bound on the number of zeros of $\operatorname{Re} Z_k(\sigma+iT)$ by use of Jensen's theorem. Since $\zeta^{(k)}(s)$ can be represented in terms of $\zeta(s)$ be Cauchy's integral formula the standard bounds on $\zeta(s)$ give $t^{2-\sigma}$ as a bound on $Z_k(s)$ for use here and leads to $O(\log T)$ as a bound on the fourth integral.

The second integral in (3.1) is also easy to deal with. Indeed if $a=a_k$ is chosen so that

$$\sum_3^\infty \left(\frac{\log n}{\log 2}\right)^k \left(\frac{2}{n}\right)^{a/2} < \tfrac{1}{2},$$

then for $\sigma \geqslant a$

$$|Z_k(s)-1| \leqslant \tfrac{1}{2}(\tfrac{2}{3})^{\sigma/2}. \quad (3.2)$$

Hence $\log Z_k(s)$ is analytic for $\sigma \geqslant a$. By Cauchy's theorem

$$\int_{a+i}^{a+Ti} \log Z_k(s)\,ds = \int_a^\infty \log Z_k(\sigma+i)\,d\sigma - \int_a^\infty \log Z_k(\sigma+iT)\,d\sigma.$$

By (3.2) the two integrals on the right are bounded independent of T. Thus (3.1) becomes

$$2\pi\sum(b+\beta^{(k)}) = I + O(\log T), \quad (3.3)$$

where

$$I = \int_1^T \log|Z_k(-b+it)|\,dt. \quad (3.4)$$

On the line $\sigma=-b$ use is made of the functional equation

$$\zeta(s) = F(s)\,\zeta(1-s);\ F(s) = 2^s\pi^{-1+s}\sin\frac{\pi s}{2}\,\Gamma(1-s).$$

Using Stirling's formula for $\Gamma(s)$, we find

$$F(s) = \exp\left(\frac{\pi i}{4} - 1 + f(s)\right)$$

where the analytic function

$$f(s) = (\tfrac{1}{2} - s)\log\frac{(1-s)\,i}{2\pi} + s + O\left(\frac{1}{s}\right), \tag{3.5}$$

in the sector $|\arg s - \pi/2| \leqslant \pi/4$ and

$$\begin{aligned} f^{(1)}(s) &= -\log\frac{(1-s)\,i}{2\pi} + O\left(\frac{1}{s}\right), \\ f^{(j)}(s) &= O\left(\frac{1}{s^{j-1}}\right) \quad j \geqslant 2. \end{aligned} \tag{3.6}$$

Hence $$F^{(j)}(s) = F(s)\,(f^{(1)}(s))^j \left\{1 + O\left(\frac{1}{t\log^2 t}\right)\right\}.$$

From the functional equation

$$\zeta^{(k)}(s) = F^{(k)}(t)\zeta(1-s) - \binom{k}{1}F^{(k-1)}(s)\zeta^{(1)}(1-s) + \binom{k}{2}F^{(k-2)}(s)\zeta^{(2)}(1-s) - \dots.$$

Hence for $\sigma < -\delta$, $\delta > 0$,

$$\begin{aligned} \zeta^{(k)}(s) &= F(s)\,(f^{(1)}(s))^k \zeta(1-s)\left\{1 + O\left(\frac{1}{t\log^2 t}\right)\right\} \\ &\quad \times \left[1 - \binom{k}{1}(f^{(1)}(s))^{-1}\frac{\zeta^{(1)}(1-s)}{\zeta(1-s)}\left(1 + O\left(\frac{1}{t\log^2 t}\right)\right)\right. \\ &\quad \left. + \binom{k}{2}(f^{(1)}(s))^{-2}\frac{\zeta^{(k)}(1-s)}{\zeta(1-s)}\left(1 + O\left(\frac{1}{t\log^2 t}\right)\right) + \dots\right] \\ &= F(s)\,(f^{(1)}(s))^k \zeta(1-s)\,F_k(s)\left(1 + O\left(\frac{1}{t\log^2 t}\right)\right), \end{aligned}$$

where $$F_k(s) = \sum(-1)^j\binom{k}{j}(f^{(1)}(s))^{-j}\frac{\zeta^{(j)}(1-s)}{\zeta(1-s)}, \tag{3.7}$$

and $F_k(s) = 1 + O(1/\log T)$ for $\sigma < -\delta$ and s in the sector.

(*Remark.* A result valid for $|\arg s - \pi| \leqslant \pi/2$ follows if $\sin \pi s/2$ is kept as a separate factor on the right of $F(s)$ in the above analysis and leads easily to the existence of b_k.)

Hence

$$Z_k(s) = (-1)^k 2^s (\log 2)^{-k} \exp\left(\frac{\pi i}{4} - 1 + f(s)\right)(f^{(1)}(s))^k \zeta(1-s)\,F_k(s)\left(1 + O\left(\frac{1}{t\log^2 t}\right)\right). \tag{3.8}$$

From the asymptotic behavior of f, $f^{(1)}$ and of F_k as $t \to \infty$ it is clear that the zeros of $\zeta^{(k)}(s)$ must lie to the right of $\sigma = -\delta$ for $\delta > 0$. From (3.8)

$$\log|Z_k(-b+it)| = -b\log 2 - k\log\log 2 - 1 + \operatorname{Re} f(-b+it) + k\log|f^{(1)}(-b+it)| + \log|\zeta(1+b-it)| + \log|F_k(-b+it)| + O\left(\frac{1}{t\log^2 t}\right). \tag{3.9}$$

Let

$$\operatorname{Li}(t) = \int_2^t dv/\log v.$$

Then using (3.5) and (3.6) I in (3.4) can be computed from (3.9) to give

$$I = (\tfrac{1}{2}+b)\,T\log\frac{T}{2\pi} + kT\log\log\frac{T}{2\pi} - T(\tfrac{1}{2}+b+b\log 2+k\log\log 2) - 2\pi k\operatorname{Li}\left(\frac{T}{2\pi}\right) + O(\log T) + I_1 + I_2, \tag{3.10}$$

where

$$I_1 = \int_1^T \log|\zeta(1+b-it)|\,dt,\quad I_2 = \int_1^T \log|F_k(-b+it)|\,dt.$$

Proceeding much as below (3.2), but more simply, $I_1 = O(1)$.

To treat I_2 use is made of (3.7) to get

$$F_k(\sigma+it) = 1 + O(2^\sigma), \tag{3.11}$$

for $-\sigma$ large and $3\pi/4 \geq \arg s \geq \pi/2$. Using Cauchy's theorem on $\log F_k(s)$ on the triangle with vertices at $-b+ib$, $-T+iT$, $-b+iT$, it follows from (3.11) that $I_2 = O(1)$. Hence from (3.3) and (3.10) now follows

LEMMA 3.1.

$$2\pi \sum_{1<\gamma_k<T} (b+\beta^{(k)}) = (\tfrac{1}{2}+b)\,T\log\frac{T}{2\pi} + kT\log\log\frac{T}{2\pi} - T(\tfrac{1}{2}+b+b\log 2+k\log\log 2) - 2\pi k\operatorname{Li}\left(\frac{T}{2\pi}\right) + O(\log T). \tag{3.12}$$

If $N_k(T)$ is the number of non-real of $\zeta^{(k)}(s)$ with $0<t<T$ then increasing b to $b+1$ in (3.12) and subtracting the case b from $b+1$ gives [1]

$$N_k(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1 - \log 2\right) + O(\log T). \tag{3.13}$$

A familiar approximate formula for $\zeta(s)$, [10, 4.11], using Cauchy's integral formula for $\zeta^{(k)}$ in terms of ζ gives

$$\zeta^{(k)}(s) = \sum(-\log n)^k n^{-s} + O\{(\log t)^k t^{-\sigma}\},$$

where Σ is for $n \leq t$. In a standard way this leads to

$$\int_1^T |\zeta^{(k)}(\tfrac{1}{2}+it)|^2 dt = O(T\log^{2k+1} T),$$

which in turn yields

$$\int_1^T \log|\zeta^{(k)}(\tfrac{1}{2}+it)|\,dt = O(T\log\log T).$$

By Littlewood's lemma this in turn yields

$$\sum_{\substack{\beta^{(k)}\geqslant 1/2\\ 1<\gamma^{(k)}<T}} (\beta^{(k)}-\tfrac{1}{2}) = O(T\log\log T). \tag{3.14}$$

Subtracting (3.14) from (3.12) gives

$$\sum_{\substack{\beta^{(k)}<1/2\\ 1<\gamma^{(k)}<T}} (b+\beta^{(k)}) + \sum_{\substack{\beta^{(k)}\geqslant 1/2\\ 1<\gamma^{(k)}<T}} (b+\tfrac{1}{2}) = (\tfrac{1}{2}+b)\frac{T}{2\pi}\log\frac{T}{2\pi} + O(T\log\log T).$$

Denote the number of zeros of $\zeta^{(k)}(s)$ in $0<t<T$ and $\sigma\leqslant c$ by $N_k^-(c, T)$ and the number of zeros in $0<t<T$ and $\sigma\geqslant c$ by $N^+(c, T)$. The above yields for any $\delta>0$

$$(b+\tfrac{1}{2}-\delta)\,N_k^-(\tfrac{1}{2}-\delta, T) + (b+\tfrac{1}{2})\,(N_k(T)-N_k^-(\tfrac{1}{2}-\delta, T))$$
$$\geqslant (\tfrac{1}{2}+b)\frac{T}{2\pi}\log\frac{T}{2\pi} + O(T\log\log T).$$

Using (3.13) with the above yields

$$\delta N_k^-(\tfrac{1}{2}-\delta, T) = O(T\log\log T).$$

From (3.14) follows, for $\delta>0$,

$$\delta N_k^+(\tfrac{1}{2}+\delta, T) = O(T\log\log T),$$

and these two results prove Theorem 2. A more refined result than the above can be obtained which justifies the statement below Theorem 2 concerning $T<t<T+U$. Using the approximate functional equation for $\zeta^{(k)}(s)$ which, by Cauchy's integral formula for $\zeta^{(k)}$ in terms of ζ, follows from that for $\zeta(s)$ gives in crude form

$$|\zeta^{(k)}(\tfrac{1}{2}+it)| \leqslant \left|\sum{}' \frac{\log^k n}{n^{1/2+it}}\right| + \log^k t \sum_{j\leqslant k}\left|\sum{}' \frac{\log^j n}{n^{1/2-it}}\right| + O(t^{-\frac{1}{4}}\log^k t),$$

where Σ' is for $n\leqslant(t/2\pi)^{1/2}$. For $U\geqslant T^{1/2}$ this leads to

$$\int_T^{T+U} |\zeta^{(k)}(\tfrac{1}{2}+it)|^2 dt = O(U\log^{4k+1} T),$$

which then yields results in $(T, T+U)$.

If $2\pi(b+\frac{1}{2})\,N_k(T)$, given in (3.13), is subtracted from (3.12) then we obtain.

THEOREM 10.

$$2\pi \sum_{0<\gamma_k\leq T} (\beta^{(k)}-\tfrac{1}{2}) = kT \log\log\frac{T}{2\pi} - 2\pi k \ \mathrm{Li}\left(\frac{T}{2\pi}\right)$$
$$+ T(\tfrac{1}{2}\log 2 - k \log\log 2) + O(\log T), \tag{3.15}$$

and this yields Theorem 3 *because*

$$\log\log\frac{T+U}{2\pi} - \log\log\frac{T}{2\pi} = \log\left(1+\frac{\log(1+U/T)}{\log T/2\pi}\right) = \frac{U}{T\log T/2\pi} + O\left(\frac{U^2}{T^2\log T}\right),$$

and

$$\mathrm{Li}\left(\frac{T+U}{2\pi}\right) - \mathrm{Li}\left(\frac{T}{2\pi}\right) = \frac{U}{2\pi\log T/2\pi} - \int_{T/2\pi}^{(T+U)/2\pi}\left(\frac{1}{\log T/2\pi} - \frac{1}{\log x}\right)dx$$
$$= \frac{U}{2\pi\log T/2\pi} + O\left(\frac{U^2}{T\log^2 T}\right).$$

4. Proofs of Theorems 4 and 5

By the functional equation

$$\zeta(s) = \frac{h(1-s)}{h(s)}\,\zeta(1-s),$$

where $h(s)$ is defined near the end of § 2. Hence

$$\zeta'(s) = \frac{h(1-s)}{h(s)}\left\{\left(\frac{h'(s)}{h(s)} + \frac{h'(1-s)}{h(1-s)}\right)\zeta(1-s) + \zeta'(1-s)\right\}.$$

By Stirling's formula

$$\frac{h'(s)}{h(s)} + \frac{h'(1-s)}{h(1-s)} = \log\frac{t}{2\pi} + O\left(\frac{1}{t}\right),$$

in $|\sigma|<2$ and so has no zeros in the strip for large $|t|$. Thus if

$$J(s) = \zeta(s) + \left[\frac{h'(s)}{h(s)} + \frac{h'(1-s)}{h(1-s)}\right]^{-1}\zeta'(s), \tag{4.1}$$

then the complex zeros of $\zeta'(s)$ and $J(1-s)$ coincide at least for large $|t|$. Hence using Littlewood's lemma to the right of $\sigma=\frac{1}{2}$ gives

$$I = \frac{1}{2\pi}\int_T^{T+U}\log\left|\frac{J(1/2+it)}{\zeta(1/2+it)}\right|dt$$
$$= \sum_{\substack{T<\gamma'<T+U\\ \beta'<\frac{1}{2}}}(1/2-\beta') - \sum_{\substack{T<\gamma<T+U\\ \beta>\frac{1}{2}}}(\beta-1/2) + O\left(\frac{U}{\log T}\right) + O(\log T). \tag{4.2}$$

Since
$$|1+z| \leqslant 1+|z| \leqslant \exp(|z|^{1/2})$$
$$I \leqslant \frac{1}{2\pi}\int_T^{T+U}\left|\frac{J}{\zeta}(\tfrac{1}{2}+it)-1\right|^{1/2}dt.$$

By (4.1)
$$\left|\frac{J}{\zeta}-1\right| \leqslant \frac{2}{\log t/2\pi}\left|\frac{\zeta'}{\zeta}\right|,$$

and so
$$I \leqslant \frac{1}{\pi}\frac{2}{(\log T)^{1/2}}\int_T^{T+U}\left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it)\right|^{1/2}dt. \tag{4.3}$$

As is well known [10, 9.6] for $|t-n| \leqslant 1$ and $0<\sigma<1$
$$\frac{\zeta'}{\zeta}(s)+\sum_{|\gamma-n|<2}\frac{1}{s-\varrho}+O(\log t).$$

If now Σ is for $|\gamma-n|<2$ then
$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it)\right|^{1/2}dt \leqslant \int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\left|\sum\frac{1}{1/2+it-\varrho}\right|^{1/2}dt+O((\log n)^{1/2})$$
$$\leqslant P_n+2\,Q_n+O((\log n)^{1/2}), \tag{4.4}$$

where
$$P_n=\int_{n-2}^{n+2}\left|\sum_{\beta=\frac{1}{2}}\frac{1}{t-\gamma}\right|^{1/2}dt,\ Q_n=\int_{n-2}^{n+2}\left|\sum_{\beta<\frac{1}{2}}\frac{1}{t-\gamma+i(\frac{1}{2}-\beta)}\right|^{1/2}dt.$$

Now the following lemma is required [2, Chap. 4].

LEMMA 4.1. *Let* $-2 \leqslant a_j \leqslant 2, b_j \geqslant 0, c_j>0$ *and let*
$$f(x)=\sum\frac{c_j}{x-a_j+ib_j},$$

where Σ *is a finite sum. Suppose* $0<p<1$. *Then*
$$\int_{-2}^{2}|f(x)|^p dx \leqslant \frac{8}{1-p}|\sum c_j|^p.$$

The proof is given below.

If Σ is now again for $|\gamma-n|<2$, then using the lemma above,
$$P_n \leqslant \frac{8}{1-\frac{1}{2}}\left(\sum_{\beta=\frac{1}{2}}1\right)^{\frac{1}{2}}=O((\log n)^{\frac{1}{2}}),$$

since the number of poles of $\zeta'/\zeta(s)$ in $|\gamma-n|<2$ is $O(\log n)$. A similar result holds for Q_n. Hence by (4.4)
$$\int_T^{T+U}\left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it)\right|^{\frac{1}{2}}dt=U\,O((\log T)^{1/2}). \tag{4.5}$$

Therefore $I=O(U)$ and by (4.2) Theorem 4 is proved.

Proof of Lemma 4.1. Let

$$H(z)=i\sum\frac{c_j}{z-a_j+ib_j}.$$

Then
$$\operatorname{Re} H(z)=\sum\frac{c_j(y+b_j)}{|z-a_j+ib_j|^2}>0,\quad y>0.$$

So $|\arg H(z)|<\pi/2$ for $y>0$ and so

$$|H(z)|^p\leqslant\frac{\operatorname{Re}(H(z))^p}{\cos\pi p/2}\leqslant\frac{\operatorname{Re}(H(z))^p}{1-p}. \tag{4.6}$$

Let $\varepsilon>0$ be small. Integrating $H^p(z)$ around the rectangle with vertices $-2+i\varepsilon$, $3+i\varepsilon$, $3+i$, $-3+i$, shows

$$\left|\int_{-3}^{3}(H(x+i\varepsilon))^p\,dx\right|\leqslant 8(\sum c_j)^p.$$

Using (4.6), this gives

$$\int_{-2}^{2}|H(x+i\varepsilon)|^p\,dx\leqslant\frac{8}{1-p}(\sum c_j)^p.$$

Letting $\varepsilon\to 0$ now yields the result.

Proof of Theorem 5. From Theorem 3 with $k=1$

we get
$$2\pi\sum_{\substack{T\leqslant\gamma'\leqslant T+U\\ \beta'>1/2}}(\beta'-\tfrac{1}{2})=U\log\log\frac{T}{2\pi}+S_1+O(U),$$

where
$$S_1=2\pi\sum_{\substack{T\leqslant\gamma'\leqslant T+U\\ \beta'<1/2}}(\tfrac{1}{2}-\beta').$$

By the corollary to Theorem 4, $S_1=O(U)$ and so Theorem 5 is proved.

5. Proof of Theorem 6

By the symmetry of the roots of $\zeta(s)$, (2.2) and (2.3) can be written as

$$I=\operatorname{Re}\sum\frac{1}{s-\varrho}=(\sigma-\tfrac{1}{2})\,I_1, \tag{5.1}$$

where
$$I_1=2\sum_{\beta>1/2}\frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\beta-\frac{1}{2})^2}{|s-\varrho|^2\,|s-1+\bar{\varrho}|^2}+\sum_{\beta=1/2}\frac{1}{|s-\varrho|^2}, \tag{5.2}$$

and so from (2.1) and (2.4)

$$\operatorname{Re}\frac{\zeta'}{\zeta}(s)=(\sigma-\tfrac{1}{2})\,I_1-\tfrac{1}{2}\log\left|\frac{s}{2\pi}\right|+O\left(\frac{1}{s}\right). \tag{5.3}$$

By (4.1) and the formula above it, for t positive

$$\frac{J}{\zeta}(s)=1+\frac{1}{\log t/2\pi}\frac{\zeta'}{\zeta}(s)\left(1+O\left(\frac{1}{t\log t}\right)\right),$$

where it will be recalled that $\zeta'(s)$ and $J(1-s)$ have their complex zeros for large $|t|$ in common. For $-1<\sigma<2$, [10, 9.6]

$$\frac{\zeta'}{\zeta}(s)=O(\log t)+\sum_{|t-\gamma|<1}\frac{1}{s-\varrho},$$

and so if $\min|s-\varrho|\geq 1/(10t)$, since the number of zeros in $|t-\gamma|<1$ is $O(\log t)$,

$$\left|\frac{\zeta'}{\zeta}(s)\right|\ll t\log t.$$

Therefore
$$\frac{J}{\zeta}(s)=1+\frac{1}{\log t/2\pi}\frac{\zeta'}{\zeta}(s)+O\left(\frac{1}{\log t}\right).$$

Thus by (5.3), for $|s-\varrho|\geq 1/(10t)$

$$\operatorname{Re}\frac{J}{\zeta}(s)=\tfrac{1}{2}+\frac{\sigma-1/2}{\log t/2\pi}I_1+O\left(\frac{1}{\log t}\right). \tag{5.4}$$

Fix T and let $T/2\leq t\leq 3T/2$. By Selberg [4, Lemma 8] if $H=T^a/10$, $\frac{1}{2}<a\leq 1$, $\delta>0$

$$\sum_{\substack{t-H\leq\gamma\leq t+H\\ \beta>1/2+\delta/2}}\left(\beta-\tfrac{1}{2}-\frac{\delta}{2}\right)\ll H^{1-(2-1/a)\delta/8}.$$

Since $\beta-\frac{1}{2}\leq 3(\beta-\frac{1}{2}-\frac{1}{2}\delta)$ for $\beta\geq\frac{1}{2}+\delta-1/T$ and $\delta>1/\log T$

$$\sum_{\substack{t-H\leq\gamma\leq t+H\\ \beta\geq 1/2+\delta-1/T}}(\beta-\tfrac{1}{2})\ll H^{1-(2-1/a)\delta/8},$$

and so if $\delta>C\log T$, C sufficiently large

$$\sum_{\substack{t-H\leq\gamma\leq t+H\\ \beta\geq 1/2+\delta-1/T}}(\beta-\tfrac{1}{2})<\frac{H}{20}. \tag{5.5}$$

For each $\beta>\frac{1}{2}$ let B_ϱ be the open box $\frac{1}{2}<\sigma<\beta+1/T$, $|t-\gamma|<\beta-\frac{1}{2}$. Note that by (5.2) $I_1\geq 0$ if s is not inside of any box and $\sigma\geq\frac{1}{2}$. Let s be inside of no box and let $\sigma-\frac{1}{2}\geq\delta>1/\log T$. Then $|s-\varrho|\geq 1/T$ and so by (5.4)

$$\operatorname{Re}\frac{J}{\zeta}(s)\geq\tfrac{1}{3};\ s\notin B_\varrho,\ \sigma\geq\tfrac{1}{2}+\delta. \tag{5.6}$$

Consider next only those boxes which protrude to the right of $\delta+\frac{1}{2}$. A *chain* consists of a sequence of protruding boxes each of which has points in common with such a box above it except for the last which is separated from the next protruding box above it. Moreover there is a lowest box in a chain which is separated from the next protruding box below it. The sum of the heights of the boxes in a chain is at most $2\Sigma(\beta-\frac{1}{2})$ for $\beta-\frac{1}{2}>\delta-1/T$ and so by (5.5) with $t=T+3U/8$, where $U=T^a$, a chain must terminate in the interval $(T+U/4,\ T+U/2)$ say at $T+U_2$ where $U/4\leqslant U_2\leqslant U/2$ (unless that interval has no protruding boxes). Similarly a chain must commence at $T-U_1$ where $U/4\leqslant U_1\leqslant U/2$.

Next consider a chain, if there is one, in $(T-U_1,\ T+U_2)$ consisting of the boxes $B_{\varrho_1}, B_{\varrho_2}, \ldots, B_{\varrho_k}$ where $\gamma_1\leqslant\gamma_2\leqslant\ldots\leqslant\gamma_k$. For $1\leqslant j\leqslant k$ let

$$\delta_m=\max(\beta_j-\tfrac{1}{2})+1/T,$$

and let
$$t_1=\min\gamma_j-(\beta_j+\tfrac{1}{2});\quad t_2=\max(\gamma_j+\beta_j-\tfrac{1}{2}).$$

Apply Littlewood's lemma [10, Chap. 9] to J/ζ in the rectangle with vertices at $\delta+it_1$, δ_m+it_1, $\delta+it_2$, δ_m+it_2. By (5.6), $|\arg J/\zeta|\leqslant\pi/2$ on the upper and lower sides of the rectangle. On the right side, by (5.6),

$$-\log|J/\zeta|\leqslant-\log|\operatorname{Re} J/\zeta|\leqslant\log 3.$$

Moreover since $\delta>1/\log T$, $\delta_m-\delta\leqslant\max(\beta_j-\frac{1}{2})$. Hence the contribution of these three sides of the rectangle to the integrals in Littlewood's lemma is at most

$$2\max(\beta_j-\tfrac{1}{2})\,\pi/2+(t_2-t_1)\log 3<10\sum_{1\leqslant j\leqslant k}(\beta_j-\tfrac{1}{2})$$

because $t_2-t_1\leqslant 2\Sigma(\beta_j-\frac{1}{2})$. Summing over the three sides of the rectangles associated with the several chains, the total is dominated by

$$10\sum_{\substack{T-U_1<\gamma<T+U_2\\ \beta>1/2+\delta-1/T}}(\beta-\tfrac{1}{2}). \tag{5.7}$$

For the left side of the rectangle the contribution is, for integer M,

$$\int_{t_1}^{t_2}\log\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right|dt\leqslant 2M\int_{t_1}^{t_2}\left|\frac{J}{\zeta}-1\right|^{1/(2M)}dt, \tag{5.8}$$

because $|1+z|\leqslant 1+|z|\leqslant\exp(2M|z|^{1/(2M)})$, since $(2M)^{2M}>(2M)!$. Denoting the sum of the left side of (5.8) over the left sides of the rectangles for the chains in $(T-U_1,\ T+U_2)$ by Φ and denoting the left sides themselves by L.S.

$$\Phi=\sum\int_{\text{L.S.}}\log\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right|dt\leqslant 2M\sum\int_{\text{L.S.}}\left|\frac{J}{\zeta}-1\right|^{1/(2M)}dt$$
$$\leqslant 2M\left(\int_{T-U_1}^{T+U_2}\left|\frac{J}{\zeta}-1\right|^{1/2}dt\right)^{1/M}(\sum\text{length of L.S.})^{1-1/M}.$$

Since $\left|\frac{J}{\zeta}-1\right| \leqslant 2\left|\frac{\zeta'}{\zeta}\right|/\log t$ the procedure below (4.3) which yields (4.5) now gives, since $U_1+U_2 \leqslant U$,

$$\Phi \ll MU^{1/M}\Big(2 \sum_{\substack{T-U_1<\gamma<T+U_2 \\ \beta>1/2+\delta-1/T}} (\beta-\tfrac{1}{2})\Big)^{1-1/M}. \tag{5.9}$$

By [4, Lemma 8], if $\varepsilon>0, H=T^a/2$, it follows easily that

$$\sum_{\substack{T-H<\gamma<T+H \\ \beta>1/2+\delta-\varepsilon-1/T}} (\beta-\tfrac{1}{2}-\delta+\varepsilon) \ll H^{1-(2-1/a)(\delta-\varepsilon)/4},$$

because $\log H/T<1$. For $\beta \geqslant \frac{1}{2}+\delta-1/T$, $\varepsilon=1/\log H$, $\beta-\frac{1}{2} \leqslant (1+\delta/\varepsilon)(\beta-\frac{1}{2}-\delta+\varepsilon)$, and so

$$\sum_{\substack{T-H<\gamma<T+H \\ \beta>1/2+\delta-1/T}} (\beta-\tfrac{1}{2}) \ll (1+\delta/\varepsilon)\, H^{1-(2-1/a)(\delta-\varepsilon)/4}. \tag{5.10}$$

Note $H=U/2$. With $\varepsilon=1/\log H$, the above in (5.9) gives

$$\begin{aligned}\Phi &\ll M(1+\delta\log U)\, U^{1/M}\big(U^{1-(2-1/a)\delta/4}\big)^{1-1/M} \\ &\ll M(1+\delta\log U)\, U^{1-(2-1/a)\delta/4} \quad U^{(2-1/a)\delta/(4M)}.\end{aligned}$$

Let $M=[\delta\log U]$, where $[x]$ represents the integer part of x, to get

$$\Phi \ll (1+\delta\log U)^2\, U^{1-(2-1/a)\delta/4}. \tag{5.11}$$

By (5.6) there are no zeros of $J(s)$ in $T-U_1<t<T+U_2$, $\sigma>1/2+\delta$, except in the several rectangles. Hence applying Littlewood's lemma, recalling that the zeros of $J(1-s)$ and $\zeta'(s)$ coincide, and using (5.7) on the three sides of the rectangles and (5.11) on the left side

$$\begin{aligned}&\sum_{\substack{T-U_1<\gamma'<T+U_2 \\ \beta'<1/2-\delta}} (\tfrac{1}{2}-\delta-\beta') - \sum_{\substack{T-U_1<\gamma<T+U_2 \\ \beta>1/2+\delta}} (\beta-\tfrac{1}{2}-\delta) \\ &\qquad \ll \sum_{\substack{T-U_1<\gamma<T+U_2 \\ \beta>1/2+\delta-1/T}} (\beta-\tfrac{1}{2}) + (1+\delta\log U)^2\, U^{1-(2-1/a)\delta/4},\end{aligned} \tag{5.12}$$

and by (5.10), with $\varepsilon=1/\log H$,

$$\sum_{\substack{T-U_1<\gamma'<T+U_2 \\ \beta'<1/2-\delta}} (\tfrac{1}{2}-\delta-\beta') \ll (1-\delta\log U)^2\, U^{1-(2-1/a)\delta/4}.$$

Several applications of the above yields the first result of Theorem 6.

To get the second result in Theorem 6 the procedure in (5.8) is changed. Now use is made of

$$\int_{t_1}^{t_2} \log\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right| dt \leqslant 2(t_2-t_1)\log\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right|^{1/2} dt\right).$$

Since by (4.1) and the formula above it, for large t,

$$\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right|^{1/2} \leqslant 1+\left(\frac{2}{\log t/2\pi}\right)^{1/2}\left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+\delta+it)\right|^{1/2},$$

the procedure in (4.4) that led to (4.5) here gives

$$\int_{t_0}^{t_2}\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right|^{1/2} dt \ll (t_2-t_1)+1,$$

and so, since $t_2-t_1 \geqslant 2(\beta_j-1/2)>\delta$ for a protruding box,

$$\int_{t_1}^{t_2}\log\left|\frac{J}{\zeta}(\tfrac{1}{2}+\delta+it)\right| \ll (t_2-t_1)\log 1/\delta$$
$$\ll \sum_{1\leqslant j\leqslant k}(\beta_j-1/2)\log 1/\delta.$$

Adding this for the left sides of the several rectangles and using it instead of Φ on the right side of (5.12) leads to

$$\sum_{\substack{T-U_1<\gamma'<T+U_2\\ \beta'<1/2-\delta}}(\tfrac{1}{2}-\delta-\zeta') \ll \sum_{\substack{T-U_1<\gamma<T+U_2\\ \beta>1/2+\delta-1/T}}(\beta-\tfrac{1}{2})\log 1/\delta,$$

from which the second result of Theorem 6 follows by first replacing δ by $\delta+1/T$ and then using

$$\left(\tfrac{1}{2}-\delta-\frac{1}{T}-\beta'\right)>\tfrac{1}{3}(\tfrac{1}{2}-\beta') \text{ for } \tfrac{1}{2}-\beta'\geqslant 2\delta.$$

Proof of the Corollary to Theorem 6.

Replace δ in (1.6) by $\delta-1/\log U$ to get

$$N_1^-(\tfrac{1}{2}-\delta, T+U)-N_1^-(\tfrac{1}{2}-\delta, T) \ll (1+\delta\log U)^2 (U\log U)\, U^{-(2-1/a)\delta/4}.$$

Now let $\delta=w(T)/\log T$. Then since $\log U=a\log T$ (1.7) is proved. Because $N_1(T+U)-N_1(T)\sim 2\pi U\log T$ the statement below (1.7) follows.

6. Proofs of Theorems 7 and 8

For fixed m denote the real zeros of $\zeta^{(m)}(s)$ by $-a_j$. Spira [7] showed that $a_j=2j+O(1)$. It was also shown [7] that there exists an A_k such that $\zeta^{(k)}(s)$ has no non-real zeros for $|\sigma|\geqslant A_k$. Denote the non-real zeros of $\zeta^{(m)}(s)$ by $p_j\pm iq_j$, $q_j>0$. Then

$$\frac{\zeta^{(m+1)}}{\zeta^{(m)}}(s)=c+\sum\left(\frac{i}{s-p_j-iq_j}+\frac{i}{s-p_j+iq_j}\right)+O\left(\frac{1}{|s-1|}\right)+\sum\left(\frac{1}{s+a_j}-\frac{1}{a_j}\right)\cdot \qquad (6.1)$$

where c is a constant (and the second sum is modified if an a_j is zero). Hence

$$\operatorname{Re}\frac{\zeta^{(m+1)}}{\zeta^{(m)}}(s)=c+2\sum\nolimits_1\frac{\sigma-p_j}{|s-p_j-iq_j|^2}+O\left(\frac{1}{|s-1|}\right)$$
$$+2\sum\nolimits_2\frac{\sigma-p_j}{|s-p_j-iq_j|^2}+\sum\left(\frac{\sigma+a_j}{|s+a_j|^2}-\frac{1}{a_j}\right),$$

where Σ_1 is for $p_j\geqslant\frac{1}{2}$ and Σ_2 is for $p_j<\frac{1}{2}$. The hypothesis is that Σ_2 is a finite sum. For $-A_m<\sigma<\frac{1}{2}$ it follows that Σ_1 is negative. If furthermore t is large, Σ_2 is bounded. Therefore

$$\operatorname{Re}\frac{\zeta^{(m+1)}}{\zeta^{(m)}}(s)\leqslant O(1)+J_1,$$

where J_1, the last sum in (6.1), is given by

$$J_1=-|s|^2\sum\frac{1}{a_j|s+a_j|^2}-\sigma\sum\frac{1}{|s+a_j|^2}.$$

Since $-A_m<\sigma<1/2$ the last sum above is $O(1)$ for large t. For $|a_j|<|s|/2$, $|s+a_j|\leqslant 3|s|/2$, and so

$$J_1\leqslant-\tfrac{4}{9}\sum_{|a_j|\leqslant|s|/2}\frac{1}{a_j}+O(1).$$

Since $a_j=2j+O(1)$, $J_1\leqslant-2(\log|s|)/9+O(1)$.

Thus
$$\operatorname{Re}\frac{\zeta^{(m+1)}}{\zeta^{(m)}}(s)\leqslant-\tfrac{2}{9}\log|s|+O(1),$$

which means $\zeta^{(m+1)}(s)\neq 0$ for t large and $\sigma<\frac{1}{2}$. This proves the theorem for $j=1$ and the rest follows by induction.

Proof of Theorem 8. Theorem 8 follows from (3.15) and the corollary to Theorem 7 which shows that the number of $\beta^{(k)}<\frac{1}{2}$ is finite.

References

[1]. Berndt, B. C., The number of zeros for $\zeta^{(k)}(s)$. *J. London Math. Soc.*, 2 (1970), 577–580.
[2]. Levinson, N., *Gap and Density Theorems.* Am. Math. Soc. Colloq. Publ. Vol. 26, 1940.
[3]. Selberg, A., The zeros of Riemann's zeta-function. *Skr. Norske Vid.-Akad. Oslo*, (1942) no. 10.
[4]. —— Contributions to the theory of the Riemann zeta-function. *Arch. Math. og Naturvid. Oslo*, 48 (1946) no. 5.
[5]. Speiser, A., Geometrisches zur Riemannschen Zetafunktion. *Math. Ann.*, 110 (1934), 514–521.
[6]. Spira, R., Zero-free regions of $\zeta^{(k)}(s)$. *J. London Math. Soc.*, 40 (1965), 677–682.
[7]. —— Another zero-free region for $\zeta^{(k)}(s)$. *Proc. Amer. Math. Soc.*, 26 (1970), 246–247.
[8]. —— Zeros of $\zeta'(s)$ in the critical strip. *Proc. Amer. Math. Soc.*, 35 (1972), 59–60.
[9]. —— Zeros of $\zeta'(s)$ and the Riemann hypothesis. *Illinois J. Math.*, 17 (1973), 147–152.
[10]. Titchmarsh, E. C., *The theory of the Riemann zeta-function.* Oxford 1951.

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