A CLASS OF IDEMPOTENT MEASURES ON COMPACT NILMANIFOLDS

BY

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Introduction

If G is a locally compact topological group and D a cocompact discrete subgroup, it would be interesting to be able to classify the bounded Borel measures on the compact homogeneous space $D\backslash G$ in terms of the representation theory of G and the structure of D. In the case of abelian groups, this is accomplished by means of the Fourier-Stieltjes transform. In Theorem 1.1 of this paper, we take G to be unimodular, and we show that the continuous projections in $L^2(D\backslash G)$ which commute with all right translations and map continuous functions into continuous functions correspond one-to-one with those two-sided D-invariant Borel measures on $D\backslash G$ which are idempotent. Although idempotence is normally defined only for measures on spaces having a well-defined multiplication of points (such as groups and semi-groups), the concept can be readily extended to two-sided D-invariant measures on homogeneous spaces $D\backslash G$.

After section 1, we restrict our attention to finite dimensional, real, connected, simply connected nilpotent Lie groups N with cocompact discrete subgroups D. Corollary (3.5) presents our basic tool for the study of two-sided D-invariant Borel measures on $D\backslash N$. We map the measure v on $D\backslash N$ into a measure v_F on a torus of the same dimension, and we show that the Fourier-Stieltjes transform $\hat{v}_F$ can be evaluated by finding the value at a

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special point of $T_v\varphi$, where φ is a certain function on $D\backslash N$ and T_v is the transformation in $L^2(D\backslash N)$ corresponding to v. In order to be able to use the structure theory of $L^2(D\backslash N)$ to evaluate $T_v\varphi$ at a specific point, we show in Theorem (3.6) that $T_v\varphi$ enjoys a weakened form of continuity at that point. The main results of section 3 are contained in Theorems (3.9)–(3.12), classifying all the idempotent measures corresponding to irreducible representations induced from characters of normal subgroups. In Theorem (4.1) we show that if the projection T_v corresponding to the idempotent measure v projects $L^2(D\backslash N)$ orthogonally onto an N-invariant subspace H, and if $V: H\to H^1$ is a unitary equivalence, then V induces a transformation of measures which carries v into a measure which projects onto H^1. It is interesting that mutually orthogonal projections can be thus interrelated.

It is important to note that nilpotence is used only in (3.1)–(3.3) to obtain suitable global coordinates on N, and in (3.6), where the polynomial multiplication which is characteristic of nilpotent Lie groups is used to prove the "semi-continuity" of $T_v\varphi$ at one point. In section 5, we present four special hypotheses subject to which our theorems hold on compact solvmanifolds. We call such special solvmanifolds type F, and, to illustrate the fact that compact nilmanifolds are not the only type F solvmanifolds, we show that many three dimensional compact solvmanifolds are type F. Our theorems (3.9)–(3.12) then classify all those idempotent measures on three dimensional compact solvmanifolds which correspond to projections onto irreducible translation-invariant subspaces.

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We lean heavily on the multiplicity theory and L^2-structure theory for compact nilmanifolds. The first results on multiplicities were obtained by C. C. Moore in [8], and exact multiplicity formulas were discovered independently by R. Howe in [5] and also in [11]. The structure theory in [11] has been extended to compact solvmanifolds by L. Auslander and J. Brezin in [1], while L. Corwin and F. Greenleaf have obtained new results concerning multiplicities in [3], as have C. C. Moore and J. Wolf in [9].

§1. Idempotent measures on compact homogeneous spaces

Suppose that G is an arbitrary locally compact unimodular topological group and D a closed unimodular cocompact subgroup. Then the compact homogeneous space $D\backslash G$ will possess a right G-invariant measure [10]. Suppose that T is an arbitrary continuous projection onto a right-translation invariant subspace of $L^2(D\backslash G)$. Thus $T(g\cdot f)=g\cdot(Tf)$, where $(g\cdot f)(Dg_0)=f(Dg_0g)$. Suppose also that $T: C(D\backslash G)\to C(D\backslash G)$, the space of continuous functions on $D\backslash G$. The mapping T, being a continuous projection, is continuous in the L^2-norm. Now suppose $f_n\to f$ and $Tf_n\to g$ in the sup-norm, where f_n, f, g_n and g are all continuous. Then, since sup-norm convergence implies L^2-norm convergence, and since T is continuous in the L^2-norm, $Tf=g$. But then T is also continuous in the sup-norm, by the closed graph theorem.

Define $E: C(D\backslash G)\to \mathbf{C}$, the complex numbers, by $Ef=f(De)$, where e is the identity of G. Then $E\circ T$ is a continuous linear functional on $C(D\backslash G)$ in the sup-norm. By the Riesz-Markov-Kakutani Theorem, there is a bounded measure v on $D\backslash G$ such that

$$(E\circ T)f=(Tf)(De)=\int_{D\backslash G} f(Dg)\,dv(Dg).$$

But, since T commutes with right translations, we have

$$(TF)(Dg_0)=(g_0\cdot(Tf))(De)=(T(g_0\cdot f))(De)=\int_{D\backslash G}(g_0\cdot f)(Dg)\,dv(Dg)=\int_{D\backslash G} f(Dgg_0)\,dv(Dg).$$

(The above argument is similar to the proof of Wendel's theorem for locally compact abelian groups in [13].)

Now we will make use of the fact that f and Tf are both well-defined on $D\backslash G$. Thus, if $d\in D$, $(Tf)(Ddg_0)=(Tf)(Dg_0)$, so that

$$\int_{D\backslash G} f(Dgg_0)\,dv(Dg)=\int_{D\backslash G} f(Dgdg_0)\,dv(Dg)=\int_{D\backslash G} f(Dgg_0)\,dv(Dgd^{-1}),$$

for all d in D and for all $f\in C(D\backslash G)$. Thus $v(E)=v(Ed)$, for all $d\in D$ and for all Borel sets $E\subset D\backslash G$.

The fact that v must be two-sided D-invariant enables us to define a natural *convolution* of v with any one-sided D-invariant measure w on $D\backslash G$. Namely, $(v*w)(E)=\int_{D\backslash G} v(Eg^{-1})\,dw(Dg)$, which is well-defined since v is right D-invariant. We can now verify that if v corresponds to the projection T, then v must be idempotent. Recall that $T^2=T$, and let ψ_E denote the characteristic function of the Borel set $E\subset D\backslash G$. Observe that

$$(T\psi_E)(Dg_0)=\int_{D\backslash G}\psi_E(Dgg_0)\,dv(Dg)=v(Eg_0^{-1}),$$

and that

$$(T^2\psi_E)(Dg_0)=\int_{D\backslash G}(T\psi_E)(Dgg_0)\,dv(Dg)=\int_{D\backslash G} v(Eg_0^{-1}g^{-1})\,dv(Dg)=(T\psi_E)(Dg_0)=v(Eg_0^{-1}).$$

Thus v is idempotent: i.e., $v*v=v$.

(1.1) THEOREM. *Let G be a locally compact unimodular topological group and D a cocompact discrete subgroup. Then $T: L^2(D\backslash G) \to L^2(D\backslash G)$ is a continuity-preserving continuous projection which commutes with all right translations by elements of G if and only if $(Tf)(Dg_0) = \int_{D\backslash G} f(Dgg_0)\,dv(Dg)$ for some two-sided D-invariant idempotent Borel measure v on $D\backslash G$. In this case we write $T = T_v$.*

Proof. We have already proved that if T is as stated, then $T = T_v$. Conversely, suppose v is idempotent and

$$(T_v f)(Dg_0) = \int_{D\backslash G} f(Dgg_0)\,dv(Dg),$$

and suppose $f \in C(D\backslash G)$. Since $D\backslash G$ is compact, if $g_n \to g_0$ in $D\backslash G$, then $f(Dgg_n) \to f(Dgg_0)$ uniformly. Since v is bounded, $(Tf)(Dg_n) \to (Tf)(Dg_0)$; thus $Tf \in C(D\backslash G)$. Also, T commutes with right G-translations, since left and right translations commute, and $T^2 = T$, since v is idempotent. Finally, to show that T_v is L^2-continuous, observe first that $D\backslash G$ has a precompact Borel section F contained in G. Then, since D is discrete, $FF = \{gg_0 \mid (g, g_0) \in F \times F\}$ is contained in the union of a finite subcollection of the set $\{dF \mid d \in D\}$. Let p denote the number of sets of the form dF needed to cover FF. Then it is easy to show that $\|T_v\| \leqslant \sqrt{p}\|v\|$. Hence T_v is L^2-continuous, and the proof of the theorem is complete.

If D is discrete, let us denote by $(D\backslash G)^\wedge$ the set of all those equivalence classes of irreducible unitary representations of G which occur in the decomposition of $L^2(D\backslash G)$ into a direct sum of mutually orthogonal irreducible translation invariant subspaces. Then, for each $\pi \in (D\backslash G)^\wedge$, the multiplicity with which π occurs in $L^2(D\backslash G)$ is finite [4]. We will call any irreducible translation invariant subspace corresponding to $\pi \in (D\backslash G)^\wedge$ an *irreducible π-space*. We will call the closed linear span of all irreducible π-spaces the *π-primary summand.*

(1.2) COROLLARY. *If T_v is as in Theorem* (1.1), *and $T_v(L^2(D\backslash G))$ is an irreducible π-space H for some $\pi \in (D\backslash G)^\wedge$, then $T_v h = 0$ for all $h \in L^2(D\backslash G)$ such that h is orthogonal to the π-primary summand $\mathcal{H}_\pi$.*

Proof. We can decompose $\mathcal{H}_\pi^\perp = \oplus_{n\in\mathbf{Z}} H_n$ into an orthogonal direct sum of irreducible subspaces. Then $h = \sum_{n\in\mathbf{Z}} h_n$, $h_i \in H_i$ for each i. But $T_v(g \cdot h) = g \cdot (T_v h)$, which implies that $T_v(H_i) = 0$, since H is *not* unitarily equivalent to H_i. Thus $T_v h = 0$.

We remark that Corollary (1.2) is trivial for abelian groups, since multiplicities never exceed one in such cases. Thus, in the case in which G is abelian, the projection T_v is zero throughout the orthogonal complement of the irreducible π-space $T_v(L^2(D\backslash G))$.

§2. Structure of L^2 of a compact nilmanifold

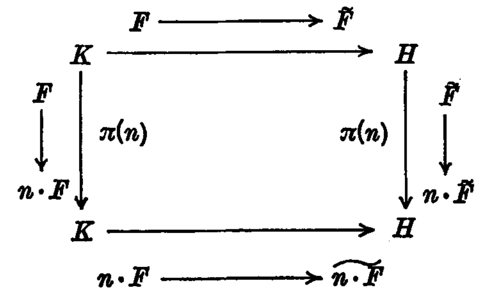
Let N be any real finite dimensional connected simply connected nilpotent Lie group and D a cocompact discrete subgroup. We will describe in greater detail the decomposition of $L^2(D\backslash N)$ alluded to in section one for general compact homogeneous spaces. All the results in this section, except for Lemma (2.1), are contained in [11].

A character χ of a subgroup M of N is given by $\chi(m) = \exp 2\pi i\lambda (\log m)$, where λ is a linear functional on the Lie algebra $\mathfrak{n}$ of N, and log is the inverse of the exponential map, which is one-to-one and onto. The condition that χ is a character is equivalent to the condition that λ: $[\mathfrak{m}, \mathfrak{m}]\to 0$. M is called maximal (relative to χ) if and only if $\mathfrak{m}$ is of maximal dimension so that λ: $[\mathfrak{m}, \mathfrak{m}]\to 0$. An *integral maximal character* is a pair (χ, M), where χ is a character of M, χ: $D_M \to 1$, where $D_M = D \cap M$ is *cocompact* in M, and M is maximal. (M is thus a rational subgroup of N.) It is known that $\pi \in (D\backslash N)^\wedge$ if and only if π is induced, in the sense of Mackey, by an integral maximal character.

We define an action of the group N on (χ, M) by $(\chi, M)\cdot n = (\chi^n, {}^{n^{-1}}M)$, $n\in N$, where $\chi^n(p) = \chi(npn^{-1})$ and ${}^{n^{-1}}M = n^{-1}Mn$. If (χ, M) and $(\chi, M)\cdot n$ are both integral maximal characters, then we call n an integral point of N. M acts trivially on (χ, M), and D maps integral maximal characters into integral maximal characters. If we denote the set of integral points of N by $(M\backslash N)_D$, then the number of distinct orbits of D in $(M\backslash N)_D$ is known to be the multiplicity with which π occurs in the decomposition of $L^2(D\backslash N)$ into a direct sum of irreducible translation invariant subspaces. We define $\underline{\text{Int}\,(\chi, M)} = (M\backslash N)_D/D$, the set of distinct D-orbits in $(M\backslash N)_D$. Int (χ, M) is always a finite set, since multiplicities are finite.

If $\pi \in (D\backslash N)^\wedge$, we can construct a full complement of irreducible subspaces of the π-primary summand as follows. Let (χ, M) be an integral maximal character inducing π, and let K be the set of all functions F: $N\to \mathbf{C}$, the complex numbers, such that $F(mn) = \chi(m)F(n)$ for all $m\in M$, and such that $|F| \in L^2(M\backslash N)$ and $|F|$ has compact support in $M\backslash N$. Let H be the linear span of function of the form $\tilde{F}(Dn) = \sum_{d\in D_M\backslash D}(F\cdot d)(n)$, where $F\in K$ and thus $\tilde{F}$ is well-defined on $D\backslash N$. Then Fig. (2.1) is a commutative diagram, and the unitary map $F\to \tilde{F}$ can be completed, making H an irreducible N-invariant subspace of $L^2(D\backslash N)$.

Now, let Int $(\chi, M) = \{x_0, x_1, ..., x_{n-1}\}$, $x_0 = e$, the identity of N. Apply the above map $F\to \tilde{F}$, called the *lift map* to each $(\chi, M)\cdot x_i$, to obtain a *lift space* H_i. It is known that H_i is independent of the choice of integral maximal character in $(\chi, M)\cdot x_i D$. Then $\{H_0, H_1, ..., H_{n-1}\}$ is a set of mutually orthogonal irreducible π-spaces which span the entire π-primary summand. $\{H_0, H_1, ..., H_{n-1}\}$ is called a *constructible basis* for the π-primary summand.



(Fig. 2.1).

Let us note that there exist at most countably many constructible bases for the π-primary summand since there exist at most countably many rational subgroups M of N. Yet there exists a whole continuum of irreducible π-spaces whenever the multiplicity of π exceeds one. The following lemma will be useful for dealing with this difficulty.

(2.1) LEMMA. *Suppose the orthogonal irreducible π-spaces $H_0, ..., H_{n-1}$ generate the π-primary summand of $\pi \in (D\backslash N)^{\wedge}$. Suppose V_k: $H_0 \to H_k$ is a unitary equivalence, $k=0, ..., n-1$. Suppose T is the orthogonal projection of $L^2(D\backslash N)$ onto an irreducible π-space H, and T_k: $L^2(D\backslash N) \to H_k$ is an orthogonal projection, $k=0, 1, ..., n-1$. Then there exists a complex vector $c=(c_0, ..., c_{n-1}) \in \mathbf{C}^n$, $|c|=1$, such that*

$$T\varphi = \sum_{k,l=0}^{n-1} c_l \bar{c}_k V_l V_k^{-1} T_k \varphi, \quad \text{for all} \quad \varphi \in L^2(D\backslash N).$$

Furthermore, the family of all such subspaces H can be identified with the points of the complex projective space CP^{n-1}.

Proof. Since H must have a non-trivial projection onto at least one of the irreducible subspaces $H_0, H_1, ..., H_{n-1}$, we can assume without loss of generality that $T_0(H) \neq 0$, and, since T_0 commutes with right-translations, T_0 must be onto. It follows from Schur's lemma that $T_i f = \lambda_i V_i T_0 f$, for all $f \in H$, for some $\lambda_i \in \mathbf{C}$. Thus

$$H = \{T_0 f + \lambda_1 V_1 T_0 f + ... + \lambda_{n-1} V_{n-1} T_0 f \mid f \in H\},$$

or

$$H = \{f + \lambda_1 V_1 f + ... + \lambda_{n-1} V_{n-1} f \mid f \in H_0\}.$$

Recalling our initial hypothesis, the set of all such subspaces $H \subset H_0 \oplus H_1 \oplus ... \oplus H_{n-1}$ can be paired with the set of straight lines through the origin in $\mathbf{C}^n$. We can then choose a vector c of length one in the direction of the line through the origin of $\mathbf{C}^n$ to designate the subspace H_c. ($-c$ would do just as well.)

To evaluate $T\varphi$, for each $\varphi \in L^2(D\backslash N)$, we note that $T\varphi = T(T_0\varphi + T_1\varphi + \dots + T_{n-1}\varphi)$ and we evaluate $TT_i\varphi$ by finding an $h \in H$, $\|h\| = 1$, such that $\langle T_i\varphi, h\rangle$ is maximized, so that $TT_i\varphi = \langle T_i\varphi, h\rangle h$. Thus

$$h = \frac{1}{\|T_i\varphi\|}(c_0 V_i^{-1}T_i\varphi + \dots + c_i T_i\varphi + \dots + c_{n-1}V_{n-1}V_i^{-1}T_i\varphi),$$

since $\|T_i h\|$ must be $|c_i|$, if $h \in H$ and $|c| = 1$.

Hence

$$TT_i\varphi = \bar{c}_i \|T_i\varphi\| h = \bar{c}_i(c_0 V_i^{-1}T_i\varphi + \dots + c_i T_i\varphi + \dots + c_{n-1}V_{n-1}V_i^{-1}T_i\varphi).$$

Therefore,

$$T\varphi = \sum_{i=0}^{n-1} TT_i\varphi = c_0\sum_{i=0}^{n-1}\bar{c}_i V_i^{-1}T_i\varphi + \dots + c_{n-1}\sum_{i=0}^{n-1}\bar{c}_i V_{n-1}V_i^{-1}T_i\varphi,$$

or

$$T\varphi = \sum_{l,i=0}^{n-1} c_l\bar{c}_i V_l V_i^{-1}T_i\varphi.$$

This proves the lemma.

§ 3. Irreducible idempotent measures

An idempotent measure v on $D\backslash N$ will be called *irreducible* if and only if the corresponding projection T_v maps $L^2(D\backslash N)$ onto an irreducible, N-invariant subspace H of $L^2(D\backslash N)$. (This definition of irreducibility is not related to the concept of the same name in abelian harmonic analysis.)

Now we will outline a technique whereby any Borel measure v on an l-dimensional compact nilmanifold can be identified with a Borel measure v_F on an l-dimensional torus T^l, and any Borel measurable function φ on $D\backslash N$ can be identified with a Borel measurable function φ_F on T^l.

In [7], Malcev proved that if M is any rational, *normal* Lie subgroup of N, then N has a system of one-parameter-subgroups $d_1(t), \dots, d_l(t)$; $t \in \mathbf{R}$, the real numbers, which can be described as follows.

(3.1) *Malcev coordinates* corresponding to a rational normal Lie subgroup M of N:

$$\{d_l(t_l)d_{l-1}(t_{l-1})\cdot \dots \cdot d_{j+1}(t_{j+1}) \mid (t_l, \dots, t_{j+1}) \in \mathbf{R}^{l-j}\} = M,$$

$$\{d_l(n_l)d_{l-1}(n_{l-1})\cdot \dots \cdot d_{j+1}(n_{j+1}) \mid (n_l, \dots, n_{j+1}) \in \mathbf{Z}^{l-j}\} = D_M = D \cap M,$$

where $\mathbf{Z}$ denotes the integers

$$\{d_l(t_l)d_{l-1}(t_{l-1})\cdot\ldots\cdot d_1(t_1)\,|\,(t_l,\ldots,t_1)\in\mathbf{R}^l\}=N,$$

and

$$\{d_l(n_l)d_{l-1}(n_{l-1})\cdot\ldots\cdot d_1(n_1)\,|\,(n_l,\ldots,n_1)\in\mathbf{Z}^l\}=D.$$

Also, if

$$N_i=\{d_l(t_l)d_{l-1}(t_{l-1})\cdot\ldots\cdot d_i(t_i)\,|\,(t_l,\ldots,t_i)\in\mathbf{R}^{l-i}\},$$

then N_i is normal in N for each $i=l,\ldots,2$.

(3.2) LEMMA. *Let* $F=d_l[0,1)\cdot d_{l-1}[0,1)\cdot\ldots\cdot d_1[0,1)$, *where* $d_1,\ldots,d_l$ *are as in* (3.1). *Then F is a fundamental domain for $D\backslash N$. Furthermore,* $F_M=d_l[0,1)\cdot\ldots\cdot d_{j+1}[0,1)$ *is a fundamental domain for* $D_M\backslash M$.

Proof. Clearly, $d_l[0,1)$ is a fundamental domain for $(D\cap N_\lambda)\backslash N_\lambda$. Suppose inductively that $d_l[0,1)\cdot\ldots\cdot d_i[0,1)$ is a fundamental domain for $(D\cap N_i)\backslash N_i$. We need only show that $d_l[0,1)\cdot\ldots\cdot d_i[0,1)d_{i-1}[0,1)$ is a fundamental domain for $(D\cap N_{i-1})\backslash N_{i-1}$. Let $n=n_id_{i-1}(t_{i-1})\in N_{i-1}$, where $n_i\in N_i$. Then $d_{i-1}(t_{i-1})=d_{i-1}(p_{i-1})d_{i-1}(s_{i-1})$ for some $p_{i-1}\in\mathbf{Z}$ and $s_{i-1}\in[0,1)$. Thus

$$n=d_{i-1}(p_{i-1})d_{i-1}^{-1}(p_{i-1})n_id_{i-1}(p_{i-1})d_{i-1}(s_{i-1}),$$

and $d_{i-1}^{-1}(p_{i-1})n_id_{i-1}(p_{i-1})=d_id_l(s_l)\cdot\ldots\cdot d_i(s_i)$ for some $d_i\in D\cap N_i$ and $(s_l,\ldots,s_i)\in[0,1)^{l-i}$, since N_i is normal in N_{i-1}. This completes the proof.

We will often identify the cube F in $\mathbf{R}^l$ with a torus T^l, since F is clearly a fundamental domain for T^l as well as for $D\backslash N$. It is proved in [7] that $(t_l,\ldots,t_1)\to d_l(t_l)\cdot\ldots\cdot d_1(t_1)$ is a diffeomorphism of $\mathbf{R}^l$ onto N. It follows that the one-to-one pointwise correspondence between $D\backslash N$ and T^l determined by the fundamental domain F carries Borel sets to Borel sets and enables us to identify any Borel measure v on $D\backslash N$ with a Borel measure v_F on T^l and any Borel measurable function φ on $D\backslash N$ with a Borel measurable function φ_F on T^l.

(3.3) LEMMA. *If m denotes the normalized right N-invariant measure which $D\backslash N$ inherits from Haar measure on N, then m_F is Lebesgue measure on the torus T^l.*

Proof. Writing $N=\{d_l(t_l)\cdot\ldots\cdot d_1(t_1)\}$, we need only show that $dt_l dt_{l-1}\ldots dt_1$ is right N-invariant on N. If $l=1$, this is trivial. Suppose inductively that the result is true when the dimension of N is less than l. Hence it is true for $N_2=\{d_l(t_l)\cdot\ldots\cdot d_2(t_2)\}$, which is normal in N. Let E be any Borel set in M and let φ_E be the characteristic function of E. Then, if $n_2\in N_2$, $n_1\in d_1(\mathbf{R})$,

$$\int_N \psi_E(d_l(t_l)\cdot\ldots\cdot d_2(t_2)\,d_1(t_1)\,n_2 n_1)\,dt_l\ldots dt_2\,dt_1$$
$$= \int_N \psi_E(d_l(t_l)\cdot\ldots\cdot d_2(t_2)[d_1(t_1)\,n_2\,d_1^{-1}(t_1)]\,d_1(t_1)\,n_1)\,dt_l\ldots dt_2\,dt_1$$
$$= \int_N \psi_E(d_l(t_l)\cdot\ldots\cdot d_2(t_2)\cdot d_1(t_1))\,dt_l\ldots dt_2\,dt_1,$$

since N_2 is normal in N and $d_1(t_1)$ and n_1 both lie in $d_1(\mathbf{R})$. This proves the lemma.

We note that the coordinates of (3.1)–(3.3) are a special case of those constructed for non-normal M in [3] and [11].

Next, we will develop the fundamental connection between v_F and T_v, where v is any two sided D-invariant Borel measure on $D\backslash N$. Namely, we will relate the Fourier-Stieltjes transform $\hat{v}_F$ to the action of T_v in $L^2(D\backslash N)$.

(3.4) *Definition.* Let n denote any vector $(n_1, \ldots, n_l)$ where $n_i \in \mathbf{Z}$ for each $i=1, 2, \ldots, l$. Let φ_n denote that unique function defined at each point of $D\backslash N$ such that $(\varphi_n)_F(t_l, \ldots, t_1) = e(n_l t_l + \ldots + n_1 t_1)$, where $e(a) = \exp(2\pi i a)$. (We are using the coordinates of (3.1)–(3.3).)

(3.5) COROLLARY. *If v is any two sided D-invariant Borel measure on $D\backslash N$, then* $\hat{v}_F(n_l, \ldots, n_1) = (T_v \varphi_n)(Dd_l(0)\ldots d_1(0))$.

Proof. Note that $T_v\varphi_n$ is defined for each point of $D\backslash N$. Recall that

$$(T_v\varphi_n)(Dd_l(t_l')\cdot\ldots\cdot d_1(t_1'))$$
$$= \int_{D\backslash N} \varphi_n(Dd_l(t_l)\cdot\ldots\cdot d_1(t_1)\cdot d_l(t_l')\cdot\ldots\cdot d_1(t_1'))\,dv(Dd_l(t_l)\cdot\ldots\cdot d_1(t_1))$$
$$= \int_{T^l} e(n_l t_l'' + \ldots + n_1 t_1'')\,dv_F(t_l, \ldots, t_1),$$

where t_i'' is some polynomial function of $t_l, \ldots, t_1$ and $t_l', \ldots, t_1'$. Substituting $t_l' = \ldots = t_1' = 0$ makes $t_i'' = t_i$, $i = 1, \ldots, l$. This completes the proof.

It is extremely important to note that although $(\varphi_n)_F$ is continuous on the torus T^l, φ_n is *not* continuous on $D\backslash N$. Thus $T_v\varphi_n$ need *not* be continuous on $D\backslash N$. However, $T_v\varphi_n$ does have a property at $Dd_l(0)\cdot\ldots\cdot d_1(0)$ which is a form of semi-continuity, and we will use this property heavily in this paper.

Define an $a\varepsilon$-slab in $d_l[0, 1)\cdot\ldots\cdot d_1[0, 1)$ to be a set $d_l(\varepsilon/2, \varepsilon)\cdot d_{l-1}(a_{l-1}(\varepsilon/2), a_{l-1}\varepsilon)\cdot\ldots\cdot d_1(a_1(\varepsilon/2), a_1\varepsilon)$, where $a = (1, a_{l-1}, \ldots, a_1)$.

(3.6) THEOREM. *There exists a vector a such that*

$$\lim_{\varepsilon\to 0} (T_v\varphi_n)(Dd_l(t_l')\cdot\ldots\cdot d_1(t_1')) = (T_v\varphi_n)(Dd_l(0)\cdot\ldots\cdot d_1(0)),$$

where t' is restricted to the $a\varepsilon$-slab.

Proof. Denote $f(t') = (T_v\varphi_n)(Dd_l(t_l')\cdot\ldots\cdot d_1(t_1'))$, where $t' = (t_l', \ldots, t_1')$.

Using the 1-parameter coordinates of (3.1)–(3.3), we write $d_l(t_l)\cdot\ldots\cdot d_1(t_1)\cdot d_l(t_l')\cdot\ldots\cdot d_1(t_1') = d_l(s_l)\cdot\ldots\cdot d_1(s_1)$, where $s_i = t_i + t_i' + P_i(t_{i-1}', \ldots, t_1; t_{i-1}', \ldots, t_1')$ and P_i is a polynomial having only terms with mixed products of t and t' coordinates [7]. By making $1 \gg a_{l-1} \gg \ldots \gg a_1 \gg 0$ we can guarantee that, if t' lies in an $a\varepsilon$-slab and if t has all its coordinates between 0 and 1, then s has all its coordinates non-negative.

Now we will prove that, as $\varepsilon\to 0$, $f(t')\to f(0)$, if t' lies in the $a\varepsilon$-slabs, where a is restricted as above. We must show that

$$\int_F \varphi_n(d_l(s_l')\ldots d_1(s_1'))\,dv_F(d_l(t_l)\ldots d_1(t_1)) \to \int_F \varphi_n(d_l(t_l)\ldots d_1(t_1))\,dv_F(d_l(t_l)\ldots d_1(t_1)),$$

as $\varepsilon\to 0$, t' in the $a\varepsilon$-slabs, where $d_l(s_l')\ldots d_1(s_1')$ is the unique representative in F of $d_l(s_l)\ldots d_1(s_1)$.

Now, if the t_i's are all in $[0, 1)$ and bounded away from 1, then our choice of a guarantees that, for small ε, $s' = s \in F$, and $\varphi_n(s')$ is uniformly close to $\varphi_n(t)$. On the other hand,

$$|v_F|\,(d_l[0,1)\cdot\ldots\cdot d_{i+1}[0,1)\,d_i(1-\delta, 1)\,d_{i-1}[0,1)\ldots d_1[0,1))\to 0,$$

as $\delta\to 0$, since these sets form a descending chain of Borel sets with empty intersection. Hence $f(t')\to f(0)$ as $\varepsilon\to 0$, t' restricted to the $a\varepsilon$-slabs.

This completes the proof of the theorem.

Next we turn our attention to the problem of classifying all the irreducible idempotent measures on a compact nilmanifold $D\backslash N$. The following theorem suggests that this problem is equivalent to the problem of classifying *all* idempotent measures on $D_1\backslash N_1$, where $N_1 \subset N$ is a rational subgroup of codimension one and $D_1 = D \cap N_1$.

(3.7) THEOREM. *Suppose the orthogonal projection onto the lift space H corresponding to (χ, M), which induces $\pi \in (D\backslash N)^\wedge$, is given by T_v, where v is some irreducible idempotent measure on $D\backslash N$ as in* (1.1). *Then there exists a rational subgroup $N_1 \subset N$ with codimension one and a one-parameter subgroup $d_1(\mathbf{R})$ such that $N = N_1\cdot d_1(\mathbf{R})$ (semi-direct product) and there exists an idempotent measure v_1 on $D_1\backslash N_1$, $D_1 = D\cap N_1$, such that v is the Cartesian product measure $v_1 \times \delta_0$, where δ_0 is the unit mass at the identity of $d_1(\mathbf{R})$.*

Proof. M must be contained in a rational (normal) subgroup N_1 of codimension one in N [8, 11]. We can apply (3.1)–(3.3) using N_1 in place of M to decompose $N = N_1 \cdot d_1(\mathbf{R})$ where $N_1 = d_l(\mathbf{R}) \cdot \ldots \cdot d_2(\mathbf{R})$. Since v_F is a Borel measure on the torus T^l, we can show that v_F is a Cartesian product of a measure on T^{l-1} with the unit mass at zero in $[0, 1)$ by showing that $\hat{v}_F(n_l, \ldots, n_2, n_1)$ is independent of n_1, where $\hat{v}_F$ is defined on $\mathbf{Z}^l = (T^l)^\wedge$. Recalling the description of the lift map from section 2, and writing $n = n_1 x$, where $n_1 \in N_1$ and $x \in d_1(\mathbf{R})$, a preimage under the lift map for a typical generating element of H is $F(n) = F(n_1 x)$ where $F(n_1 x) = f_1(n_1) f(x)$, $f \in L^2(\mathbf{R})$ has compact support, $f_1(mn_1) = \chi(m) f_1(n_1)$ for each $m \in M$, and $|f_1| \in L^2(M \backslash N_1)$ has compact support in $M \backslash N_1$. Then the typical generating element $\tilde{F}$ of H is given by

$$\tilde{F}(Dn) = \tilde{F}(Dn_1 x) = \sum_{\substack{d_1 d \in D_M \backslash D \\ d_1 \in D_1,\, d \in d(\mathbf{R}) \cap D}} (F \cdot d_1 d)(n_1 x) = \sum_{d_1 d} f_1(d_1(dn_1 d^{-1}))\, f(dx).$$

Since $f_1(d_1(dn_1 d^{-1}))$ is independent of x, and since $f \in L^2(\mathbf{R})$ is arbitrary, it follows that H is the closed linear span of $H_1 \times L^2[0, 1)$, where H_1 is some subspace of $L^2(D_1 \backslash N_1)$.

Define φ_n as in (3.4) and invoke (3.5). Now T_v is the orthogonal projection of $L^2(D \backslash N)$ onto H = closed linear span of $H_1 \times L^2[0, 1)$, and, by Lemma (3.3), inner products in $L^2(D \backslash N)$ are carried into inner products in $L^2(T^l)$ by the pointwise map $D \backslash N \to T^l$. Thus

$$e(n_1 t_1')\, T_v\, \varphi_{(n_l, \ldots, n_2, 0)}(t_l', \ldots, t_1') = f(t'),$$

and $T_v \varphi_n(t_l', \ldots, t_1') = g(t')$ are square integrable functions which are equal almost everywhere. We can conclude that $\hat{v}_F(n)$ is independent of n_1 by proving that $f(0) = g(0)$. This follows however, from (3.6), since if $f(0) \neq g(0)$ f and g would be unequal on a set of positive measure. Hence $\hat{v}_F$ is independent of n_1 and v and v_F can be decomposed into a Cartesian product measure as required: $v = v_1 \times \delta_0$. It is necessary only to prove that v_1 is idempotent on $D_1 \backslash N_1$.

Let $E = E_1 \times E_x$, $E_1 \subset D_1 \backslash N_1$, $E_x = d[0, 1)$, be a product of Borel sets, and let ψ_E be the characteristic function of E. Then

$$\int_{D \backslash N} \int_{D \backslash N} \psi_E(Drs)\, dv(Dr)\, dv(Ds)$$

$$= \int_{DN_1 \backslash N} \int_{D_1 \backslash N_1} \int_{DN_1 \backslash N} \int_{D_1 \backslash N_1} \psi_E(Dr_1 x_r s_1 x_s)\, dv_1(D_1 r_1)\, d\delta_0(x_r)\, dv_1(D_1 s_1)\, d\delta_0(x_s)$$

$$= \int_{D_1 \backslash N_1} \int_{D_1 \backslash N_1} \psi_{E_1}(D_1 r_1 s_1)\, dv_1(Dr_1)\, dv_1(Ds_1)$$

$$= \int_{DN_1 \backslash N} \int_{D_1 \backslash M} \psi_E(Dr_1 x_r)\, dv_1(D_1 r_1)\, d\delta_0(x_r)$$

$$= \int_{D_1 \backslash N_1} \psi_{E_1}(D_1 r_1)\, dv_1(D_1 r_1).$$

Thus v_1 is idempotent on $D_1 \backslash N_1$ since v is idempotent on $D \backslash N$.

This completes the proof of the theorem.

(3.8) THEOREM. *Let v be any irreducible idempotent measure on $D \backslash N$ such that T_v projects $L^2(D \backslash N)$ orthogonally onto an irreducible π-space, where $\pi \in (D \backslash N)^\wedge$ is induced by (χ, M) and M is normal in N. Then v_F is a finite linear combination of idempotent measures on T^l.*

Proof. It is sufficient to show that $\hat{v}_F$ is only finitely many valued on $\mathbf{Z}^l = (T^l)^\wedge$. Adopt the coordinates of (3.1)–(3.3). Let Int $(\chi, M) = \{x_0, x_1, \ldots, x_{n-1}\}$, where $x_0 = e$, as in section 2.

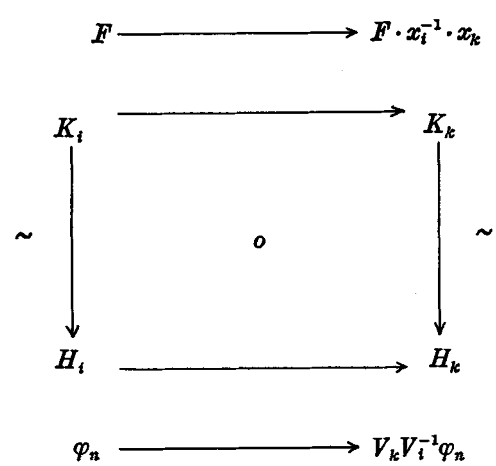
Denote $T_v = T_c = \sum_{k,i=0}^{n-1} c_k \bar{c}_i V_k V_i^{-1} T_i$, where T_i is the orthogonal projection onto H_i, the lift space corresponding to $(\chi, M) \cdot x_i$, as in Lemma (2.1). Recall that H_i is generated by functions

$$\varphi_n(d_l(t_l) \cdot \ldots \cdot d_1(t_1)) = e(n_l t_l + \ldots + n_1 t_1),$$

such that

$$e(n_l t_l + \ldots + n_{j+1} t_{j+1}) \in \{\chi^{x_i d} \mid d = d_j(p_j) \cdot \ldots \cdot d_1(p_1) \text{ for } (p_j, \ldots, p_1) \in \mathbf{Z}^j\}.$$

Also, as in (3.5) $\hat{v}_F(n) = (T_v \varphi_n)(Dd_l(0) \cdot \ldots \cdot d_1(0))$. The problem is to calculate $V_k V_i^{-1} \varphi_n$ for φ_n a generator of H_i, by tracing φ_n around the following diagram of the lift map in fig. 3.1.



(Fig. 3.1).

We will construct a preimage under the lift map for φ_n. Let the fundamental domain of (3.2) for $D\backslash N$ be denoted by E, and let $E' = d_j[0,1)\cdot\ldots\cdot d_1[0,1)$. Let $F \in K_i$ be such that $F(n)=0$ if $n \notin Md_0E'$ and $F(md_0d_j(t_j)\cdot\ldots\cdot d_1(t_1)) = \chi^{x_i}(m)e(n_jt_j+\ldots+n_1t_1)$ where $d_l(t_l)\cdot\ldots\cdot d_1(t_1)\in E$. Then $\tilde{F}=\varphi_n$, where φ_n is a typical generator of H_i.

It suffices to show that $(F\cdot x_i^{-1}\cdot x_k)^{\sim}(Dd_l(0)\cdot\ldots\cdot d_1(0))$ achieves only finitely many distinct values as n varies in $\mathbf{Z}^l$ such that φ_n agrees with $\chi^{x_id_0}$ on $D\backslash DM$ and as d_0 varies in $D_M\backslash D$.

We begin by showing that if $x=d_j(r_j)\cdot\ldots\cdot d_1(r_1)\in \operatorname{Int}(\chi, M)$ then r_i is rational for all $i=j,\ldots,1$. In fact, since (χ, M) is an integral maximal character, it is shown in [11] that there corresponds to $d_1(1)$ a rational point y_1 in M such that N_2 centralizes y_1 but the commutant of $d_1(1)$ and y_1 is not in the kernel of χ. It follows that r_1 is rational. We can procede similarly for $r_2,\ldots,r_j$.

Now, $(F\cdot x_i^{-1}x_k)^{\sim}(De)=\sum_{d\in D_M\backslash D}F(x_i^{-1}x_kd)$, where this is actually a finite sum over all d such that $x_i^{-1}x_kd\in Md_0E'$. Of course, the finite set of d's involved in such a sum will vary with d_0. Observe that $x_i^{-1}x_kd=d_0d_l(s_l)\cdot\ldots\cdot d_1(s_1)$ such that $s_j=S_j(x_k;x_i;d;d_0)$, a polynomial with rational coefficients in the coordinates of x_k, x_i, d, and d_0, for each $j=1,\ldots,l$. Even as d and d_0 vary in D, S_j can achieve only finitely many distinct values modulo one. Therefore, as n and d_0 vary, $\sum_d F(x_i^{-1}x_kd)$ achieves only finitely many distinct values. This completes the proof of Theorem (3.8).

In Theorem (3.9) we will utilize a certain Boolean ring of subsets of the character group of a torus. In particular, the *coset ring* in any discrete group is the smallest family of subsets of that group which contains all cosets of all subgroups and which is closed under finitely many applications of the operations of taking unions, intersections, and complements.

(3.9) THEOREM. *Suppose $\pi\in(D\backslash N)^{\wedge}$ is induced by a maximal integral character (χ, M), where M is normal in N. Let T be the orthogonal projection of $L^2(D\backslash N)$ onto H, where H is the lift space corresponding to (χ, M) as in section 2. Then T preserves the continuity of functions if and only if $\{\chi^d \mid d\in D_M\backslash D\}$ lies in the coset ring of the character group $(D_MM_1\backslash M)^{\wedge}$, where M_1 is the commutator subgroup of M and $D_M=D\cap M$.*

Proof. Following (3.1)–(3.3) coordinatize N with one-parameter subgroups $d_i(t)$, $i=1,\ldots,l$, in such a way that $\{d_l(t_l)d_{l-1}(t_{l-1})\ldots d_{k+1}(t_{k+1}) \mid (t_l,\ldots,t_{k+1})\in\mathbf{R}^{l-k}\}=M_1$, $\{d_l(t_l)d_{l-1}(t_{l-1})\ldots d_{j+1}(t_{j+1}) \mid (t_l,\ldots,t_{j+1})\in\mathbf{R}^{l-j}\}=M$, and $\{d_l(t_l)d_{l-1}(t_{l-1})\ldots d_1(t_1) \mid (t_l,\ldots,t_1)\in\mathbf{R}^{l}\}=N$. We may also assume that

$$D=\{d_l(n_l)d_{l-1}(n_{l-1})\ldots d_1(n_1) \mid (n_l,\ldots,n_1)\in\mathbf{Z}^l\}.$$

Then the set $F=d_l[0,1)\cdot d_{l-1}[0,1)\cdot\ldots\cdot d_1[0,1)$ is a fundamental domain for $D\backslash N$. Also, $F_1=d_l[0,1)\cdot d_{l-1}[0,1)\ldots d_{j+1}[0,1)$ is a fundamental domain for $D_M\backslash M$.

Next, recall that H, as described in section 2, is the closed linear span of the set

$$\{\chi^d(m)e(n_jt_j+\ldots+n_1t_1)\,|\,(n_j,\ldots,n_1)\in\mathbf{Z}^j,\ d\in D_M\backslash D\},$$

so that every function in H is constant on M_1-cosets, since this is the case for χ^d. Working on T^l, we have

$$f_F(t_l,\ldots,t_1)=\sum_{(n_l,\ldots,n_1)\in\mathbf{Z}^l}\hat{f}_F(n_l,\ldots,n_1)\,e(n_lt_l+\ldots+n_1t_1).$$

Now, $(Tf)_F$ corresponds to the subseries of the series for f_F with terms of the form $[e(n_1t_1+\ldots+n_jt_j)]\chi^d(m)$, $d\in D$, where we note that $\chi^d(m)$ can be identified with a trigonometric monomial on F_1, since χ^d is constant on M_1-cosets and $D_MM_1\backslash M$ is a $(j-k)$-dimensional (abelian) torus. Thus, H can be regarded as a direct sum of irreducible subspaces of an abelian torus; a convenient phenomenon which we will exploit.

Suppose T does preserve the continuity of functions on $D\backslash N$, so that $T=T_v$ for some idempotent measure v on $D\backslash N$, in the sense of section 1. We will show that v can be written as a Cartesian product measure $v(t_l,\ldots,t_1)=v_1(t_l,\ldots,t_{j+1})\times\delta_0(t_j,\ldots,t_1)$, where δ_0 is the unit mass at the identity, and v_1 is a bounded Borel measure on $D_M\backslash M$. To do this, it suffices to show that, if we view v_F on T^l via the natural $1-1$ pointwise map between F and $D\backslash N$, then $\hat{v}_F(n_l,\ldots,n_1)$ is independent of $n_j,\ldots,n_1$. Defining φ_n as in (3.4), we have $\hat{v}_F(n_l,\ldots,n_1)=(T_v\varphi_n)(d_l(0)\cdot\ldots\cdot d_1(0))$. But $T_v\varphi_n=\varphi_n$ if $e(n_lt_l+\ldots+n_{j+1}t_{j+1})=\chi^d$ for some $d\in D_M\backslash D$, and $T_v\varphi_n=0$ otherwise. Thus, using (3.6) as we did in (3.7), $\hat{v}_F(n_l,\ldots,n_1)=(T_v\varphi_n)(Dd_l(0)\cdot\ldots\cdot d_1(0))=1$ if $e(n_lt_l+\ldots+n_1t_1)\in\{\chi^d\,|\,d\in D_M\backslash D\}$ and $\hat{v}_F(n_l,\ldots,n_1)=0$ otherwise, independent of $n_j,\ldots,n_1$.

Thus $v=v_1\times\delta_0$, where v_1 is a bounded measure on $D_M\backslash M$, and $\hat{v}_{1_F}$ is also zero unless $n_{k+1}=\ldots=n_l=0$, since χ^d is trivial on M_1 for all $d\in D$. Thus $v=m\times w\times\delta_0$, where m is the translation-invariant measure on $D_{M_1}\backslash M_1$ derived from Haar measure on M_1, and w is a bounded Borel measure on $D_MM_1\backslash M$, determined by the Fourier-Stieltjes transform of v_{1_F}. We will prove that w is idempotent on the torus $D_MM_1\backslash M$, which will prove that the support of $\hat{w}$, namely $\{\chi^d\,|\,d\in D_M\backslash D\}$, lies in the coset-ring of $(D_MM_1\backslash M)^\wedge$, by the Helson–Rudin–Cohen idempotent measure theorem [12].

Recall that $m\times w\times\delta_0$ is idempotent on $D\backslash N$, and that this measure is 2-sided D-invariant. Pick an arbitrary Borel set $H\subset D_MM_1\backslash M$, and let $E=(D_{M_1}\backslash M_1)\times H$. Let ψ_E denote the characteristic function of E, and note that $m(D_{M_1}\backslash M_1)=1$. Also, if $x\in D_M\backslash M$, write $x=Dx_1x_m$, where $x_1\in M$, and x_m has the form $d_k(t_k)\cdot\ldots\cdot d_{j+1}(t_{j+1})$. Then, we show that w is idempotent on $D_MM_1\backslash M$ by using Fubini's Theorem several times as follows:

$$\int_{D_M M_1\backslash M}\psi_H(D_M M_1 x_m)\,dw(x_m)=\int_{D_M\backslash M}\psi_E(D_M x_1 x_m)\,dm(x_1)\,dw(x_m)$$
$$=\int_{D_M\backslash M}\psi_E(D_M x)\,d(m\times w)(x)=\int_{(D_M\backslash M)^2}\psi_E(D_M xy)\,d(m\times w)(x)\,d(m\times w)(y)$$
$$=\int_{(D_M\backslash M)^2}\psi_E(D_M x_1 x_m y_1 y_m)\,dm(x_1)\,dw(x_m)\,dm(y_1)\,dw(y_m)$$
$$=\int_{(D_M\backslash M)^2}\psi_E(D_M x_1(x_m y_1 x_m^{-1})\,x_m y_m)\,dm(x_1)\,dm(y_1)\,dw(x_m)\,dw(y_m)$$
$$=\int_{(D_M M_1\backslash M)^2}\psi_H(D_M M_1 x_m y_m)\,dw(x_m)\,dw(y_m),$$

since $y_1\to x_m y_1 x_m^{-1}$ is an automorphism of M_1 and leaves m invariant because it has Jacobian 1, as can be easily computed relative to a Jordan-Holder basis.

Conversely, suppose $\{\chi^d \mid d\in D_M\backslash D\}$ lies in the coset-ring of $(D_M M_1\backslash M)^\wedge$. Define the idempotent measure w on $D_M M_1\backslash M$ by requiring that $\hat{w}$ be the characteristic function of this set. Then, we must show that $m\times w\times\delta_0$ is an idempotent measure on $D\backslash N$ yielding T, where m and δ_0 are as before, and $m\times w\times\delta_0$ on F yields a measure of the same name on $D\backslash N$. Define φ_n as in (3.4). We can check both the right D-invariance of $m\times w\times\delta_0$, and the fact that $T_{m\times w\times\delta_0}=T$, by examining

$$\int_{D\backslash N}\varphi_n(Dd_l(t_l)\dots d_1(t_1)\,d_l(t_l')\dots d_1(t_1')\,d(m\times w\times\delta_0)(t),$$

where $t=(t_l, \dots, t_{k+1}; t_k, \dots, t_{j+1}; t_j, \dots, t_1)$. We integrate first with respect to δ_0 to reduce to an integral over D_M/M, with $t_j=\dots=t_1=0$, we note that M_1 is normal in M and that the integral with respect to m is zero unless $n_l=\dots=n_{k+1}=0$, and we are left with either zero, or, if $n_l=\dots=n_{k+1}=0$, we get

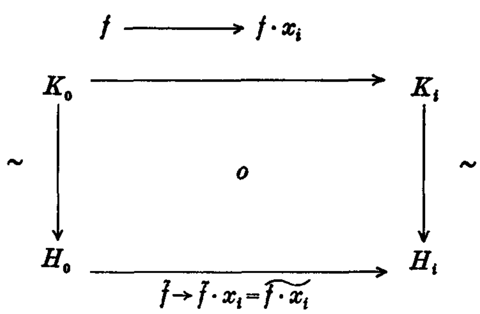
$$\int_{D_M M_1\backslash M}e(n_k t_k+\dots+n_{j+1}t_{j+1})\,dw(t_k, \dots, t_{j+1}).$$

If $t_l', \dots, t_1'$ are integers, we see that $m\times w\times\delta_0$ is right D-invariant. And we see that $T_{m\times w\times\delta_0}=T$.

This completes the proof. Examples appear in (3.15a–c).

(3.10) THEOREM. *If $\pi\in(D\backslash N)^\wedge$ is induced by (χ, M), where M is normal and $N=M\cdot X$ (semi-direct product) with X an abelian Lie subgroup of N, then the following two statements are equivalent:*

(i) *$\{\chi^d \mid d\in D_M\backslash D\}$ is in the coset-ring of $(D_M M_1\backslash M)^\wedge$.*
(ii) *Every projection orthogonally onto any irreducible π-space preserves continuity.*



(Fig. 3.2).

Proof. (ii) $\Rightarrow$ (i) by Theorem (3.9). We need only prove that (i) $\Rightarrow$ (ii).

Let $H_0, \ldots, H_{n-1}$ be a constructed basis for the π-primary summand corresponding to Int $(\chi, M) = \{x_0, \ldots, x_{n-1}\}$ with $x_0 = e$. Let $T_i: L^2(D\backslash N) \to H_i$ be an orthogonal projection, $i = 0, 1, \ldots, n-1$. To show that each T_i preserves continuity, it is necessary and sufficient to prove that $\{\chi^{x_i d} \mid d \in D_M\backslash D\} = \{\chi^{d x_i} \mid d \in D_M\backslash D\}$ lies in the coset-ring of $(D_M M_1\backslash M)^\wedge$. However, the mapping $\wedge \to \wedge^{x_i}$ in $(D_M M_1\backslash M)^\wedge$ carries cosets of subgroups onto cosets of subgroups: For example, $\wedge + S \to \wedge^{x_i} + S^{x_i}$, where S^{x_i} is a subgroup. The same applies to finite unions, intersections, and complementations. Hence T_i preserves continuity, for each $i = 0, 1, \ldots, n-1$.

Next, suppose $T: L^2(D\backslash N) \to H_c$, where H_c is an arbitrary irreducible π-space and T is an orthogonal projection. Then, by Lemma (2.1), $T\varphi = \sum_{l,i=0}^{n-1} c_l \bar{c}_i V_l V_i^{-1} T_i \varphi$, for each $\varphi \in L^2(D\backslash N)$. We need the following lemma.

(3.11) Lemma. *If K_0 and K_i are the pre-images of H_0 and H_i, respectively, under the lift map, and if $N = M \cdot X$ (semidirect), M normal, X abelian, then the diagram in fig.* 3.2 *is commutative.*

Proof. For each $d \in D$, we write $d = d_M d_1$ where $d_M \in D_M$ and $d_1 \in D \cap X$. Recalling that X is abelian, we have

$$(F\cdot x_i)^\sim(Dn) = \sum_{D_M\backslash D} (F\cdot x_i)(dn) = \sum_{D_M\backslash D} F(x_i dn) = \sum_{D_M\backslash D} F(x_i d_M x_i^{-1} x_i d_1 n)$$

$$= \sum_{D_M\backslash D} F(x_i d_1 n) = \sum_{D_M\backslash D} F(d_1 x_i n) = \tilde{F}(Dx_i n) = (\tilde{F}\cdot x_i)(Dn),$$

which is thus well-defined.

Thus $V_i f = f\cdot x_i$ for each $f \in H_0$. To complete the proof of (3.10), it is sufficient to show that each V_i and V_j^{-1} preserves continuity. But this follows from the fact that V_i and V_j^{-1} are essentially left-translations, by (3.11).

To be precise, any continuous function on $D\backslash N$ can be regarded as a left D-invariant continuous function on N. So viewed, V_i and V_j^{-1} act as left translations having the special property of leaving functions in either H_0 or H_j left D-invariant, and thus still continuous when viewed on $D\backslash N$.

This completes the proof of Theorem (3.10).

Next, we consider the delicate question of when the existance of an arbitrary idempotent measure v corresponding to $\pi \in (D\backslash N)^\wedge$ implies that every projection onto an irreducible subspace corresponding to π is given by some idempotent measure. Very few irreducible subspaces are constructible, in the sense of section 2. It is only for these that theorems (3.9) and (3.10) answer this question in the affirmative.

We suppose again that M is normal and $N = M \cdot X$ (s.d.) with X abelian, and (χ, M) induces $\pi \in (D\backslash N)^\wedge$. We will adopt the coordinatizing Malcev subgroups of (3.1). Let $\operatorname{Int}(\chi, M) = \{x_0, ..., x_{n-1}\}$, with $x_0 = e$, and write $x_i = d_j(x_j^{(i)}) \cdot ... \cdot d_1(x_1^{(i)})$, where $0 \leqslant x_j^{(i)}, ..., x_1^{(i)} < 1$. Denote the lift space corresponding to x_i by H_i. We can designate any irreducible π-space $H \subset H_0 \oplus ... \oplus H_{n-1}$ as H_c, where $c = (c_0, ..., c_{n-1})$ is a unit vector in $\mathbf{C}^n$, by Lemma (2.1). We will call H_c *singular relative to* $\{H_0, ..., H_{n-1}\} \Leftrightarrow$ the following condition holds:

If we let $A(n)$ be the complex conjugate of $\sum_{l=0}^{n-1} c_l e(n_j x_j^{(l)} + \ldots + n_1 x_1^{(l)})$, then for each $k = 0, ..., n-1$, there exists an $l \neq k$ such that

$$A(n)\, c_l e(n_j x_j^{(l)} + ... + n_1 x_1^{(l)}) = A(n)\, c_k e(n_j' x_j^{(k)} + ... + n_1' x_1^{(k)})$$

for some $(n_j, ..., n_1)$ and $(n_j', ..., n_1') \in \mathbf{Z}^j$.

We will call an irreducible π-space *singular* $\Leftrightarrow$ it is singular relative to every constructible basis for the π-primary summand. Using the ordinary (hemispherical) measure on CP^{n-1}, we see that the set of singular subspaces of the π-primary summand has measure zero.

(3.12) THEOREM. *Suppose (χ, M) induces $\pi \in (D\backslash N)^\wedge$, M is normal and $N = M \cdot X$ (s.d.) with X abelian. Suppose H is any non-singular irreducible π-space and v is an idempotent measure such that T_v: $L^2(D\backslash N) \to H$ is an orthogonal projection. Then every orthogonal projection onto any irreducible π-space is given by an idempotent measure*

Proof. Pick a constructible basis $\{H_0, ..., H_{n-1}\}$ relative to which H is not singular, where H_i is the lift space corresponding to the integral point $x_i \in \operatorname{Int}(\chi, M)$. Denote $H_c = H$, where $c = (c_0, ..., c_{n-1})$. We will use the coördinates of (3.1)–(3.3) and define φ_n as in (3.4). Then, as in Theorem (3.5),

$$\hat{v}_F(n_l, ..., n_1) = (T_v \varphi_n)(d_l(0) \cdot ... \cdot d_1(0)).$$

Recall from Lemma (2.1) that

$$T_v \varphi_n = \sum_{l,i=0}^{n-1} c_l \bar{c}_i V_l V_i^{-1} T_i \varphi_n.$$

Furthermore,

$$T_i \varphi_n = \varphi_n \quad \text{iff} \quad e(n_l t_l + \dots + n_{j+1} t_{j+1}) \in \{\chi^{x_i d} \mid d \in D_M \backslash D\},$$

and $T_i \varphi_n = 0$ otherwise. Recall also the description of V_i in Lemma (3.11). Then, denoting $x_i = (x_j^{(i)}, \dots, x_1^{(i)}) = d_j(x_j^{(i)}) \cdot \dots \cdot d_1(x_1^{(i)})$, where all $x_k^{(i)}$ are necessarily rational, and recalling that X is abelian, we have

$$\hat{v}_F(n_l, \dots, n_1) = \sum_{l=0}^{n-1} c_l \bar{c}_i \varphi_n(d_l(0) \cdot \dots \cdot d_{j+1}(0)\, d_j(x_j^{(l)} - x_j^{(i)}) \cdot \dots \cdot d_1(x_1^{(l)} - x_1^{(i)}) \quad \text{iff}$$

$$e(n_l t_l + \dots + n_{j+1} t_{j+1}) \in \{\chi^{x_i d} \mid d \in D_M \backslash D\}, \quad \text{or } 0, \text{ otherwise.}$$

Now, $\hat{v}_F$ has only finitely many distinct values. In particular, if $n \in \mathbf{Z}^j$ and if $T_i \varphi_n \neq 0$, then

$$\hat{v}_F(n_l, \dots, n_1) = V_i^n = \bar{c}_i \exp\left[-2\pi i (n_j x_j^{(i)} + \dots + n_1 x_1^{(i)})\right] \sum_{l=0}^{n-1} c_l \exp 2\pi i (n_j x_j^{(l)} + \dots + n_1 x_1^{(l)})$$

which, for each $i = 0, \dots, n-1$, runs through only finitely many distinct values for $n \in \mathbf{Z}^j$ since $x_j^{(k)}$ is rational for each j, k. Thus v_F is a finite linear combination of measures which are idempotent on the torus T^l.

By hypothesis, H_c is not singular, so we can pick an i such that $V_i^n \neq V_j^{n'}$ for all n, n' and $j \neq i$. Then the subset of the support of $\hat{v}_F$ on which $V_i^n = \hat{v}_F(n)$ must lie in the coset-ring of $\mathbf{Z}^l$. Hence $\{\chi^{x_i d} \mid d \in D_M \backslash D\}$ lies in the coset-ring of $(D_M M_1 \backslash M)^\wedge$. Now we apply Theorems (3.9) and (3.10) and the proof is complete.

(3.13) THEOREM. *Suppose (χ, M) induces $\pi \in (D \backslash N)^\wedge$, where M has codimension one in N. Then the following four statements are equivalent.*

(1) *$\{\chi^d \mid d \in D_M \backslash D\}$ lies in the coset ring of $(D_M M_1 \backslash M)^\wedge$.*

(2) *Orthogonal projections onto all irreducible π-spaces preserve continuity.*

(3) *The orthogonal projection onto the π-primary summand preserves continuity.*

(4) *The orthogonal projection onto at least one irreducible π-space preserves continuity.*

Note: We will make specific use of the polynomial multiplication in N, so that this theorem is not listed in section 5 as being extendable to suitable solvmanifolds.

Proof. (1)$\Leftrightarrow$(2), (1)$\Rightarrow$(3), and (1)$\Rightarrow$(4) by Theorems (3.9) and (3.10). We need the following lemma.

(3.14) LEMMA. *Let $p_1, ..., p_l$ be polynomials and let $x_0 = 0, x_1, ..., x_n$ be rational numbers. If $\bigcup_{i=0}^{n} \{(p_1(n+x_i), ..., p_l(n+x_i)) \mid n \in \mathbf{Z}\}$ lies in the coset-ring of $\mathbf{Z}^l$ then $\{(p_1(n), ..., p_l(n)) \mid n \in \mathbf{Z}\}$ must also lie in the coset-ring of $\mathbf{Z}^l$.*

Proof of lemma. If each polynomial p_i is linear then we are done. Suppose some p_j has degree greater than one. Since projections onto the coordinate axes map the coset-ring of $\mathbf{Z}^l$ onto the coset-ring of $\mathbf{Z}$, we have $\bigcup_{i=0}^{n} \{p_j(n+x_i) \mid n \in \mathbf{Z}\}$ in the coset-ring. Note that the gaps between successive elements of this set approaches infinity as $n \to \infty$. This is a contradiction, since subsets of $\mathbf{Z}$ in the coset ring are essentially equal except at finitely many points to periodic sequences. This proves the lemma. (Unfortunately, if the variable n has a multidimensional lattice for its domain, then the condition on the degree of p_j is false.)

(3) ⇒ (1). Since the orthogonal projection onto the π-primary summand of $L^2(D \backslash N)$ preserves continuity, we can use the same argument as in the proof of Theorem (3.9) to conclude that $\bigcup_{x_i \in \mathrm{Int}(\chi, M)} \{\chi^{dx_i} \mid d \in D_M \backslash D\}$ lies in the coset-ring of $(D_M M_1 \backslash M)^\wedge$. But $D_M \backslash D \cong \mathbf{Z}$, so we can apply Lemma (3.14) to conclude that $\{\chi^d \mid d \in D_M \backslash D\}$ is also in the coset-ring, the polynomials coming from the Campbell-Hausdorff formula [6].

(4) ⇒ (1). Suppose v is some irreducible idempotent measure corresponding to $\pi \in (D \backslash N)^\wedge$. It is shown in Theorem (3.8) that v_F is a finite linear combination of idempotent measures on a torus, so that $\hat{v}_F$ has its support essentially of the form of a union of sets $\{\chi^{dx_i} \mid d \in D_M \backslash D\}$, this union lying in the coset-ring. It follows from the lemma that $\{\chi^d \mid d \in D_M \backslash D\}$ also must lie in the coset ring.

This completes the proof.

(3.15) *Examples.* (a) Let N_3 be $\mathbf{R}^3$ equipped with the multiplication $(x, y, z)(x', y', z') = (x+x', y+y', z+z'+xy')$. Let D be the integral lattice points $\mathbf{Z}^3$ in N_3. Then any infinite dimensional $\pi \in (D \backslash N_3)^\wedge$ is induced by a character χ_λ of $M = \{(0, y, z)\}$, where $\chi_\lambda(0, y, z) = e(\lambda z)$, for some $\lambda \in \mathbf{Z}$. Then the set $\{\chi_\lambda^d \mid d \in D_M \backslash D\}$ lies in the coset-ring of $\mathbf{Z}^2$, so that every irreducible π-space is the image of an orthogonal projection given by an idempotent measure.

(b) Let N_4 be $\mathbf{R}^4$ equipped with the multiplication $(w, x, y, z)(w', x', y', z') = (w+w', x+x', y+y'+2wx', z+z'+2wy'+2w^2x')$. Let $D = \mathbf{Z}^4$ and $M = \{(0, x, y, z)\}$. Then every infinite dimensional $\pi \in (D \backslash N_4)^\wedge$ is induced by a character $\chi_{(\alpha, \beta, \gamma)}$ on M, where

$$\chi_{(\alpha, \beta, \gamma)}(0, x, y, z) = e(\alpha x + \beta y + \gamma z), \quad (\alpha, \beta, \gamma) \in \mathbf{Z}^3.$$

Furthermore, if π is non-trivial on the center Z, then $\gamma \neq 0$. (If $\pi | Z = I$, then we can factor Z out and the situation is reduced to example (a).) Then $\chi_{(\alpha, \beta, \gamma)}^{(n, 0, 0, 0)} = \chi_{(\alpha + 2n\beta + 2n^2\gamma, \beta + 2n\gamma, \gamma)}$. Hence $\{\chi_{(\alpha, \beta, \gamma)}^d \mid d \in D_M \backslash D\}$ is not in the coset-ring, so that there does not exist any irreducible idempotent measure corresponding to π, by Theorem (3.13).

(c) Now we give an example of a non-Heisenberg group N, $\pi \in (D\backslash N)^\wedge$, $\pi|Z \neq I$, and such that every orthogonal projection onto any irreducible π-space preserves continuity. Let N be $\mathbf{R}^5$ equipped with the multiplication

$$(x_1, x_2, y_1, y_2, z)(x_1', x_2', y_1'. y_2', z')$$
$$= (x_1 + x_1', x_2 + x_2', y_1 + y_1', y_2 + y_2' + 2x_2 y_1', z + z' + 2x_2 y_2' + 2x_2^2 y_1' + 2x_1 y_1').$$

Let $D = \mathbf{Z}^5$ and $(\chi, M)\uparrow\pi \in (D\backslash N)^\wedge$, where $\chi|Z \neq I$, $M = \{(0, 0, y_1, y_2, z)\}$. Then $\{\chi^d | d \in D_M\backslash D\}$ lies in the coset-ring of $\mathbf{Z}^3$.

§ 4. Transformations of measures on $D\backslash N$

Throughout this section we will make the hypothesis that M is normal in N and that $N = M \cdot X$ (semi-direct product) where X is an abelian Lie subgroup of N.

Suppose $\pi \in (D\backslash N)^\wedge$ is induced by (χ, M) and $\operatorname{Int}(\chi, M) = \{x_0, x_1, ..., x_{n-1}\}$ where $x_0 = e$. Let $H_0, ..., H_{n-1}$ be a constructed basis for the π-primary summand corresponding to $x_0, ..., x_{n-1}$, and let T_i: $L^2(D\backslash N) \to H_i$ be an orthogonal projection, $i = 0, ..., n-1$. Then, if $f \in L^2(D\backslash N)$, $T_0 f$ and $T_i f$ are *not* related to each other, since H_0 and H_i are orthogonal subspaces of $L^2(D\backslash N)$. However, we will see that, if T_0 preserves continuity, then the unitary equivalence given in (3.11) between H_0 and H_i induces a transformation of measures which carries the idempotent measure v_0 which corresponds to T_0 into the idempotent measure v_i which corresponds to T_i.

First, we let F denote the fundamental domain of (3.1). If E is any Borel set in $D\backslash N$, we define $xE = \{Dxn | Dn \in E \text{ and } n \in F\}$. Then xE is also a Borel set, and if v is any Borel measure on $D\backslash N$, we define $v^x(E) = v(x(Ex^{-1}))$. This seemingly artificial transformation yields canonical, well-defined measures, under the hypotheses of the next theorem.

(4.1) THEOREM. *Suppose $\pi \in (D\backslash N)^\wedge$ is induced by (χ, M), and $N = M \cdot X$ (s.d.), X abelian and M normal. Denote* $\operatorname{Int}(\chi, M) = \{x_0, ..., x_{n-1}\}$, *where $x_0 = e$. Let H_i be the lift space corresponding to $(\chi, M) \cdot x_i$, as in section 2, and suppose T_v: $L^2(D\backslash N) \to H_0$ is an orthogonal projection, where v is an idempotent measure. Then v^{x_i} is that unique idempotent measure such that $T_{v^{x_i}}$: $L^2(D\backslash N) \to H_i$, an orthogonal projection.*

In order to prove the above theorem we first need the following lemma.

(4.2) LEMMA. *If $g = d_l(g_l) \cdot ... \cdot d_1(g_1) \in N$, using the coordinates of* (3.1), *and if φ_g: $D\backslash N \to D\backslash N$ by $\varphi_g(Dn) = Dgn'$, where $n' \in F \cap Dn$, then φ_g is an automorphism of the Borel structure of $D\backslash N$.*

Proof of lemma. It is necessary only to show that φ_g is one-to-one and onto. Thus, given $a \in N$, we must show there exist unique $d \in D$ and $f \in F$ such that $dgf = a$. Denote $d = d_l(n_l) \dots d_1(n_1)$, where $(n_l, \dots, n_1) \in \mathbf{Z}^l$. Then we must show there is a unique solution to the equation

$$d_l(n_l) \dots d_1(n_1) d_l(g_l) \dots d_1(g_1) d_l(f_l) \dots d_1(f_1) = d_l(a_l) \dots d_1(a_1),$$

having $0 \leqslant f_l, \dots, f_1 < 1$. But it is proved in [7] that there exist polynomials $P_l, \dots, P_1$ such that

$$\begin{aligned} & d_l(n_l) \dots d_1(n_1) d_l(g_l) \dots d_1(g_1) d_l(f_l) \dots d_1(f_1) \\ & \quad = d_l(n_l + g_l + f_l + P_l(n_{l-1}, \dots, n_1; g_{l-1}, \dots, g_1; f_{l-1}, \dots, f_1)) \cdot \dots \cdot d_1(n_1 + x_1 + f_1). \end{aligned}$$

Clearly, n_1 and f_1 are uniquely determined. But then n_2 and f_2 are uniquely determined. We procede until the lemma is proved.

It is a simple consequence of the above lemma that

$$\int_{D \setminus N} (f \circ \varphi_g)(Dn)\, d(v \circ \varphi_g(Dn) = \int_{D \setminus N} f(Dn)\, dv(Dn)$$

for any Borel measure v and for any function $f \in L^2(D \setminus N)$.

Proof of Theorem (4.1). To complete the proof it will suffice to show that

$$(v_F^{x_i})^\wedge(n) = \begin{cases} 1 \text{ if } e(n_l t_l + \dots + n_{j+1} t_{j+1}) \in \{\chi^{dx_i} \mid d \in D_M \setminus D\}. \\ 0 \text{ otherwise,} \end{cases}$$

since this characteristic function has already been identified in the proof of Theorem (3.10) as the transform of the measure corresponding to $T_i: L^2(D \setminus N) \to H_i$. Note that if $(\varphi_n)_F = e(n_l t_l + \dots + n_1 t_1)$ such that $e(n_l t_l + \dots + n_{j+1} t_{j+1}) \in \{\chi^d \mid d \in D_M \setminus D\}$ then $(\varphi_n^{x_i})_F = (x_i^{-1} \cdot (\varphi_n \circ \varphi_{x_i}))_F = e(n_l' t_l + \dots + n_1' t_1)$ such that

$$e(n_l' t_l + \dots + n_{j+1}' t_{j+1}) \in \{\chi^{dx_i} \mid d \in D_M \setminus D\},$$

and every element of the latter set arises in this manner. Hence

$$\hat{v}_F(n) = \int \varphi_n(Dn)\, dv(Dn) = \int \varphi_n^{x_i}(Dn)\, dv^{x_i}(Dn) = (v_F^{x_i})^\wedge(n') = 1,$$

if

$$e(n_l' t_l + \dots + n_{j+1}' t_{j+1}) \in \{\chi^{dx_i} \mid d \in D_M \setminus D\},$$

where we have applied lemma (4.2). Similarly, $(v_F^{x_i})^\wedge(n') = 0$ otherwise. This completes the proof of the theorem.

(4.3) COROLLARY. *If v is the measure in Theorem* (4.1), *then v is both left and right ${}^{x_i}D$-invariant.*

Proof. As a result of Theorem (4.1), v^{x_i} is right D-invariant. But, v^{x_i} is right D-invariant if and only if, for each $d \in D$, $v^{x_i}(Ed) = v^{x_i}(E)$, for each Borel set $E \subset D\backslash N$. However, $v^{x_i}(Ed) = v\{Dx_i d'ndx_i^{-1} \mid Dn \in E, d'ndx_i^{-1} \in F\} = v\{D^{x_i}d'({}^{x_i}n)^{x_i}d \mid Dn \in E, \; d'ndx_i^{-1} \in F\} = v^{x_i}(E)$ if and only if v is both left and right ${}^{x_i}D$-invariant. This proves the corollary.

Note that the right ${}^{x_i}D$-invariance of v is also an easy consequence of the fact that T_v projects $L^2(D\backslash N)$ onto a space of left ${}^{x_i}D$-invariant functions. However, the left ${}^{x_i}D$-invariance of v is not so easy, and the above proof uses the strength of the coset-ring theorem.

(4.4) Corollary. *Under the hypotheses of Theorem* (4.1), *the measure* v_c *corresponding to* T_c: $L^2(D\backslash N) \to H_c$, *as described in* (2.1) *is given by*

$$v_c = \sum_{l,i=0}^{n-1} c_l \bar{c}_i (x_l^{-1} x_i) \cdot v_i,$$

where $(x \cdot v_i)(E) = v_i(Ex)$, *and* T_{v_i} *projects* $L^2(D\backslash N)$ *onto* H_i.

Proof. We need only note that

$$(V_l V_i^{-1} T_i \varphi)(Dn) = V_l V_i^{-1} \int_{D\backslash N} \varphi(Dn_1 n)\, dv_i(Dn_1)$$
$$= \int_{D\backslash N} \varphi(Dn_1 x_l x_i^{-1} n)\, dv_i(Dn_1) = \int_{D\backslash N} \varphi(Dn_1 n)\, dv_i(Dn_1 x_l^{-1} x_i).$$

This completes the proof.

(4.5) Corollary. *Under the hypotheses of Theorem* (4.1), *if* T_0: $L^2(D\backslash N) \to H_0$ *and* T_i: $L^2(D\backslash N) \to H_i$ *are orthogonal projections, we have* $(T_i f) = (T_0(f \cdot x_i^{-1})) \cdot x_i$, *for all* $f \in L^2(D\backslash N)$.

Proof. $(T_i f)(Dn) = (T_v{}^{x_i} f)(Dn)$, where $T_v = T_0$. But

$$(T_0(f \cdot x_i^{-1}) \cdot x_i)(Dn) = (T_v(f \cdot x_i^{-1}))(Dx_i n) = \int_{D\backslash N} f(Dx_i^{-1} n_1 x_i n)\, dv(Dn_1)$$
$$= \int_{D\backslash N} f(Dn_1 n)\, dv^{x_i}(Dn_1) = ((T_{v^{x_i}}) f)(Dn).$$

This completes the proof.

Now we will generalize Corollary (4.5) by eliminating the hypothesis that T_0 is given by a measure.

(4.6) Theorem. *Suppose* (χ, M) *induces* $\pi \in (D\backslash N)^\wedge$, *where* M *is normal and* $N = M \cdot X$ *(semi-direct), with* X *abelian. Let* $\{H_0, \ldots, H_{n-1}\}$ *be a constructed basis for the* π*-primary* *summand corresponding to* Int $(\chi, M)=\{e=x_0, x_1, \ldots, x_{n-1}\}$, *and let* $T_i: L^2(D\backslash N)\to H_i$ *be an orthogonal projection*, $i=0, \ldots, n-1$. *Then, letting* $f\cdot x_i$ *denote* $f\circ\varphi_{x_i}$ *(as in Lemma* (4.2)), *we have* $(T_i f)=(T_0(f\cdot x_i^{-1}))\cdot x_i$.

Proof. Denote $S(f)=(T_0(f\cdot x_i^{-1}))\cdot x_i$. Clearly S acts as the identity on H_i, so we need prove only that S: $H_i^{\perp}\to 0$. It suffices to show that $f\in H_i^{\perp}$ implies $f\cdot x_i^{-1}\in H_0^{\perp}$ or that

$$\int_{D\backslash N}(f\cdot x_i^{-1})(Dn)\,h_0(Dn)\,d\mu(Dn)=\int_{D\backslash N}f(Dn)(h_0\cdot x_i)(Dn)\,d\mu(Dn)=0$$

for each $f\in H_i^{\perp}$ and $h_0\in H_0$, where μ is the translation-invariant measure on $D\backslash N$. But this follows from Lemma (4.2). Specifically, we need only note that μ is invariant under φ_{x_i} since N is unimodular. This completes the proof.

§5. Applications to compact solvmanifolds

We will show in this section that the methods and results developed in section 3 for compact nilmanifolds are also true on suitable compact solvmanifolds. The author is indebted to J. Brezin for pointing out the generalizations in this section.

The compact solvmanifolds which we are able to treat must possess global coordinates similar to those in (3.1)–(3.3), so that (3.4) and (3.5) will remain true. Also, the fundamental domain F for the compact solvmanifold $D\backslash S$ will have to have a rather delicate relationship to the multiplication in S, enabling us to prove (3.6). To be specific, suppose S is a connected, simply connected, solvable Lie group and D a cocompact discrete subgroup. Suppose $\pi\in(D\backslash S)^\wedge$ is induced by a character χ of a normal subgroup M such that $(M\cap D)\backslash M$ is compact. Then the π-primary summand of $L^2(D\backslash S)$ is constructed by means of lift maps exactly like those of section 2 [1].

(5.1) *Definition.* We will call $D\backslash S$ a *type F* solvmanifold relative to M provided that $D\backslash S$ has the following four properties.

(1) There exist one-parameter subgroups $d_l(t), \ldots, d_1(t)$ of S such that $S=d_l(\mathbf{R})\cdot\ldots\cdot d_1(\mathbf{R})$ and $S_i=d_l(\mathbf{R})\cdot\ldots\cdot d_i(\mathbf{R})$ is normal in S_{i-1} for each $i=l, \ldots, 2$.

(2) $D=d_l(\mathbf{Z})\cdot\ldots\cdot d_1(\mathbf{Z})$, so that $d_l[0,1)\cdot\ldots\cdot d_1[0,1)$ is a fundamental domain for $D\backslash S$.

(3) There exist integers j and k such that $[M,M]=S_j$ and $M=S_k$, $0\leqslant k<j\leqslant l$.

(4) If $0<\delta_1<1$ and $0<\delta_2<1$, then there exists a set $S_{\delta_1,\delta_2}\subset[0,\delta_2)^l$, having positive measure in the invariant measure of $D\backslash S$, such that, if $t\in[0,1-\delta_1)^l$ and $t'\in S_{\delta_1,\delta_2}$, then

$$d_l(t_l)\cdot\ldots\cdot d_1(t_1)\,d_l(t_l')\cdot\ldots\cdot d_1(t_1')=d_l(t_l'')\cdot\ldots\cdot d_1(t_1''),$$

where $t_i''\geqslant 0$ for each $i=l, \ldots, 1$.

Properties (1)–(3) enable us to use (3.1)–(3.5), exactly as before. However, in order to use (3.5) effectively, it will be necessary to have a result very similar to Theorem (3.6). That is, the formula $\hat{v}_F(n)=(T_v\varphi_n)(Dd_l(0)\cdot\ldots\cdot d_1(0))$ is not computationally useful by itself since our knowledge of the structure of $L^2(D\backslash S)$ can determine $T_v\varphi_n$ only almost everywhere—not at the specific point $Dd_l(0)\cdot\ldots\cdot d_1(0)$. But (3.6) provides a "semi-continuity" property at this point which enables us to determine $\hat{v}_F(n)$ from the structure of $L^2(D\backslash S)$. The purpose of property (4) is to enable us to prove a theorem very similar to (3.6), except that S_{δ_1,δ_2} replaces the $a\varepsilon$-slab which worked when the multiplication was given by polynomials. Since S_{δ_1,δ_2} has positive measure, regardless of how small we choose δ_1 and δ_2, our new version of Theorem (3.6) is just as useful as the old version, and the proof requires no further changes and need not be duplicated here. It is then elementary to check that Theorems (3.9)–(3.12) apply just as well to type F solvmanifolds as to compact nilmanifolds.

We will show that many three dimensional compact solvmanifolds are type F, and we will use Theorems (3.9)–(3.12) to classify all the irreducible idempotent measures on these manifolds. It is proved in [2] that there are only two types of three dimensional compact solvmanifolds which are not nilmanifolds. Every such solvmanifold comes from a solvable Lie group which can be identified with a semi-direct product $\mathbf{R}^2\cdot\mathbf{R}$, where $\mathbf{R}^2$ is normal in $\mathbf{R}^2\cdot\mathbf{R}$ under an action of $\mathbf{R}$ on $\mathbf{R}^2$ given by a one-parameter subgroup A^t of $SL(2,\mathbf{R})$, and where D can be identified with the integral lattice points of $\mathbf{R}^3$. If A is a matrix in $SL(2,\mathbf{Z})$ having positive unequal eigenvalues p and p^{-1} then we will call the group S_1. If

$$A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

then we will call the group S_2. In either case, the group multiplication is given by $(v;t)\cdot(u;s)=(v+A^tu;\,t+s)$. In either case, all the infinite dimensional $\pi\in(D\backslash S)^\wedge$ are induced by integral characters of $M=\{(v;0)\}$. [1].

(5.2) THEOREM. *$D\backslash S_1$ and $D\backslash S_2$ are both type F solvmanifolds, relative to the normal subgroup $M=\{(v;0)\}$.*

Proof. Properties (1)–(3) are trivial, so we will concentrate on property (4).

Let us consider $D\backslash S_1$ first. The matrices A^t have eigenspaces $\mathbf{R}w_1$ and $\mathbf{R}w_2$ corresponding to the eigenvalues p^t and p^{-t} respectively. By choosing the one-parameter coordinate subgroups $d_3(t)$ and $d_2(t)$ in $\mathbf{R}^2$ sensibly, we can insure that one eigenspace extends into the interior of the first quadrant $d_3[0,\infty)\cdot d_2[0,\infty)$ or else both lie along the axes bordering the first quadrant. In either case, it is clear that, for any first quadrant vector u, lying between two suitable first quadrant rays A^tu will again lie in the first quadrant. Thus (4) is clearly satisfied, provided only that we make a sensible choice of $d_3(t)$ and $d_2(t)$.

The compact solvmanifold $D\backslash S_2$ is also type F, but the verification of property (4) is more delicate. In this case, $(v_1, v_2; t)(u_1, u_2; s) = (v_1 + u_1 \cos \frac{1}{2}\pi t + u_2 \sin \frac{1}{2}\pi t,\ v_2 + u_2 \cos \frac{1}{2}\pi t - u_1 \sin \frac{1}{2}\pi t; t+s)$. Given $0<\delta_1<1$ and $0<\delta_2<1$, let us restrict t to $[0, 1-\delta_1)$ and see whether there are acceptable conditions on (u_1, u_2) which will guarantee that

$$u_1 \cos \frac{\pi}{2} t + u_2 \sin \frac{\pi}{2} t \geqslant 0, \tag{a}$$

and

$$u_2 \cos \frac{\pi}{2} t - u_1 \sin \frac{\pi}{2} t \geqslant 0. \tag{b}$$

Note that $\sin \frac{1}{2}\pi t$ is bounded away from 1 and $\cos \frac{1}{2}\pi t$ is bounded away from 0 for $0 \leqslant t < 1-\delta_1$. Thus, any first quadrant vector u will satisfy (a) and, if $u_1 \ll u_2$, (b) is also satisfied. Hence the existence of S_{δ_1, δ_2} is assured, and (4) is satisfied. This completes the proof.

(5.3) *Example.* Suppose (χ, M) induces $\pi \in (D\backslash S_1)^\wedge$, an infinite dimensional irreducible representation. We will show that $\{\chi^d \mid d \in D_M\backslash D\}$ does not lie in the coset ring of $\mathbf{Z}^2 = (D_M\backslash M)^\wedge$, so that, by Theorem 3.12, the orthogonal projections onto all non-singular irreducible π-spaces fail to preserve continuity and hence cannot be given by idempotent measures. In particular, let us write $\chi = n = (n_1, n_2)$, where $\chi(u; 0) = e(n_1 u_1 + n_2 u_2)$. Then $\chi^d = ('A)^d n$, where $'A$ denotes the transpose of A, as may be easily calculated. But there is a non-singular linear transformation W such that

$$('A^d)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = W^{-1}\begin{pmatrix} p^d & 0 \\ 0 & p^{-d} \end{pmatrix} W \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

which does not have bounded gap in each coordinate. Hence $\{\chi^d \mid d \in \mathbf{Z} = D_M\backslash D\}$ does not lie in the coset-ring of $\mathbf{Z}^2$.

(5.4) *Example.* Let (χ, M) induce $\pi \in (D\backslash S_2)^\wedge$, an infinite dimensional irreducible representation. We will show that $\{\chi^d \mid d \in D_M\backslash D\}$ is in the coset-ring of $\mathbf{Z}^2$, so that every orthogonal projection onto any irreducible π-space is given by an idempotent measure. Let us denote χ again by $n = (n_1, n_2)$. Then the set $\{\chi^d \mid d \in D_M\backslash D\}$ is finite, and hence in the coset-ring, since

$$('A)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{because } A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus we have shown that on compact solvmanifolds $D\backslash S_1$, there are essentially no irreducible idempotent measures, whereas on compact solvmanifolds $D\backslash S_2$, there are as many irreducible idempotent measures as one could desire.

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