A BEURLING-TYPE THEOREM

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§ 1. Introduction and statement of the main result

In this paper we shall be concerned primarily with the linear topological space $A^{-\infty}$ whose elements are holomorphic functions

$$f(z) = \sum_0^\infty a_\nu z^\nu$$

in the unit disk $U = \{z: |z| < 1\}$ satisfying

$$|f(z)| \leqslant C_f(1-|z|)^{-n_f} \quad (z \in U) \tag{1.1}$$

or equivalently,

$$\log^+|a_\nu| = O(\log \nu) \quad (\nu \to \infty).$$

$A^{-\infty}$ can be thought of as the union of Banach spaces A^{-n} $(n > 0)$, the norm in each A^{-n} being defined as follows:

$$\|f\|_{-n} = \sup_{z \in U}\{|f(z)|(1-|z|)^n\} < \infty. \tag{1.2}$$

The topology in $A^{-\infty}$ is introduced in a standard way [6]. Clearly, $A^{-\infty}$ is a topological algebra under pointwise multiplication. It is the smallest algebra containing the disk algebra A (1) and closed under differentiation.

The dual of $A^{-\infty}$ is the topological algebra A^∞ whose elements are functions $F(z)$ holomorphic in U and infinitely differentiable in $\bar{U}$:

$$F(z) = \sum_0^\infty b_\nu z^\nu \quad (b = O(\nu^{-k}) \quad \forall k > 0). \tag{1.3}$$

The linear functionals in $A^{-\infty}$ are given by the formula

$$F(f) = \frac{1}{2\pi i} \lim_{r \to 1-} \int_{\partial U} \bar{F}(\zeta) f(r\zeta) \frac{d\zeta}{\zeta} = \sum_0^\infty \bar{b}_\nu a_\nu. \tag{1.4}$$

(1) A is the algebra of all functions continuous in $\bar{U}$ and analytic in U with sup-norm and pointwise multiplication.

Let T denote the (continuous) linear operator in $A^{-\infty}$ of multiplication by the argument:

$$(Tf)(z) = zf(z) \quad (f \in A^{-\infty}). \tag{1.5}$$

Since the set of all polynomials is dense in $A^{-\infty}$ it is readily seen that every invariant subspace for the operator T is a closed ideal in the algebra $A^{-\infty}$, and *vice versa.*

For every element $0 \neq f \in A^{-\infty}$ let Z_f denote the sequence $\{\alpha_\nu\}$ of its zeros, $0 \leqslant |\alpha_1| \leqslant |\alpha_2| \leqslant ... < 1$, each zero repeated according to its multiplicity. Z_f will be called the *zero set* of f. For every closed ideal $0 \neq I \subset A^{-\infty}$ the zero set Z_I is defined as the sequence of common zeros for *all* elements $f \in I$, each zero repeated according to its minimal multiplicity. For a complete description of $A^{-\infty}$-zero sets see [6] where a certain condition (T) was established and proved to be necessary and sufficient for a sequence $\alpha = \{\alpha_\nu\} \subset U$ to be an $A^{-\infty}$-zero set. This condition (T) implies in particular that ever subset of an $A^{-\infty}$-zero set is an $A^{-\infty}$-zero set itself. Therefore sets Z_I are in fact not different from Z_f: for every ideal $0 \neq I \subset A^{-\infty}$ there is an element $f \in A^{-\infty}$such that $Z_f = Z_I$.

Our main result (Theorem 1.1) concerns the description of closed ideals $0 \neq I \subset A^{-\infty}$. It states that every such ideal is uniquely determined by its zero set Z_I and by its so-called *$\varkappa$-singular measure* σ_I. Now, the notion of a $\varkappa$-singular measure σ_f associated with functions $f(z)$ of the class $A^{-\infty}$ (and, for that matter, with those of the larger class $\mathcal{N} = A^{-\infty}/A^{-\infty}$) was introduced in [6], but the definition adopted there depended heavily on a number of other concepts, in particular on that of a *premeasure of bounded $\varkappa$-variation.* There is, however, an alternative definition for σ_f $(f \in A^{-\infty})$ which is (at least formally) quite independent of the theory expounded in [6]. We shall use that definition to state our main result (Theorem 1.1) but we do not see how to prove it without making extensive use of the results from [6].

In this section we confine ourselves only to those preliminary notions and porpositions which are indispensible for introducing the concept of a $\varkappa$-singular measure and for formulating Theorem 1.1.

Definition 1.1. A subset F of the circumference ∂U is called a *Beurling–Carleson* (B.–C.) *set* if

(i) F is closed;

(ii) F is of Lebesgue measure zero, $|F| = 0$;

(iii) $\sum_\nu |I_\nu| \log \dfrac{1}{|I_\nu|} < \infty$, where $\{I_\nu\}$ are the complementary arcs of F (i.e. the components of $\partial U \setminus F$) and $|I_\nu|$ is the length of I_ν.

It is well known [2; 3] that B.–C. sets coincide with null sets for the classes $A^n = \{f: f^{(n)} \in A\}$ $(n=1, 2, \ldots)$. Moreover [7; 8], if F is a B.–C. set, then an outer function $\Phi(z) \in A^\infty$ exists such that $F = \{\zeta \in \partial U: \Phi^{(n)}(\zeta) = 0 \, \forall n \geqslant 0\}$.

The set of all B.–C. sets will be denoted $\mathfrak{J}$, and the set of all Borel sets B such that $\bar{B} \in \mathfrak{J}$ will be denoted $\mathcal{B}$.

Definition 1.2. A function σ: $\mathcal{B} \to \mathbf{R}$ is called a *$\varkappa$-singular measure* ($\varkappa$-s.m.) if

(i) σ is a finite Borel measure on every B.–C. set $F \subset \partial U$;

(ii) there is a constant $C > 0$ such that

$$|\sigma(F)| \leqslant C \sum_\nu |I_\nu| \left(\log \frac{2\pi}{|I_\nu|} + 1\right) \quad (\forall F \in \mathfrak{J}), \tag{1.6}$$

where $\{I_\nu\}$ are the conplementary arcs of F.

It is clear that a $\varkappa$-singular measure σ is completely determined by the values $\sigma(F)\,(F \in \mathfrak{J})$; in other words, a function σ: $\mathfrak{J} \to \mathbf{R}$ possesses (if at all) only one extension to a $\varkappa$-s.m.

The total variation $|\sigma|$ of a $\varkappa$-s.m. σ satisfying (1.6) is a non-negative $\varkappa$-s.m. with the constant not exceeding 2C.

Notations like $\max\{\sigma_1, \sigma_2\}$, $\min\{\sigma_1, \sigma_2\}$, l.u.b. $\{\sigma_\nu\}$, $\sigma_1 \geqslant \sigma_2$ have their familiar meaning accepted in the measure theory.

PROPOSITION 1.1. *Let $0 \neq f \in A^{-\infty}$, $F \in \mathfrak{J}$. Let further $\Phi \in A^\infty$ be an outer function such that $F = \{\zeta \in \partial U: \Phi^{(n)}(\zeta) = 0 \;\forall n \geqslant 0\}$ and μ be a non-negative Borel measure on F. Consider*

$$f_{F,\mu}(z) = f(z)\,\Phi(z) \exp\left\{\int_F \frac{\zeta + z}{\zeta - z} \mu(|d\zeta|)\right\} \quad (z \in U) \tag{1.7}$$

and define

$$\mathfrak{M}_{F,f} = \{\mu: f_{F,\mu} \in A^{-\infty}\}. \tag{1.8}$$

Then

(i) *$\mathfrak{M}_{F,f}$ does not depend on Φ, i.e. for any given F, μ all functions* (1.7) *(with different Φ) either belong to $A^{-\infty}$ or are outside $A^{-\infty}$;*

(ii) *$\mathfrak{M}_{F,f}$ has a maximal element μ_0, so that*

$$\mathfrak{M}_{F,f} = \{\mu: 0 \leqslant \mu \leqslant \mu_0\};$$

(iii) *there is a constant C such that*

$$\mu_0(F) \leqslant C \sum_\nu |I_\nu| \left(\log \frac{2\pi}{|I_\nu|} + 1\right) \quad (\forall F \in \mathfrak{J}). \tag{1.9}$$

This proposition will be proved in section 4 in the course of proving Theorem 1.1.

Definition 1.3. With every element $0 \neq f \in A^{-\infty}$ a non-positive $\varkappa$-singular measure σ_f will be associated defined as follows:

$$\sigma_f(F) = -\mu_0(F) = -\max_{\mu \in \mathfrak{M}_{F,f}} \mu(F) \quad (\forall F \in \mathfrak{F}). \tag{1.10}$$

For $f=0$ we set formally $\sigma_0(F) = -\infty (\forall F \in \mathfrak{F})$.

For every closed ideal $0 \neq I \in A^{-\infty}$ we define

$$\sigma_I = \underset{f \in I}{\text{l.u.b.}}\, \sigma_f. \tag{1.11}$$

In section 4 when proving Theorem 1.1 it will be shown that Definition 1.3 is equivalent to another definition of σ_f as the $\varkappa$-singular part of a premeasure [6].

We are now in a position to formulate our main result.

THEOREM 1.1. *Let $I \neq \{0\}$ be a closed ideal in $A^{-\infty}$; let Z_I and σ_I be respectively its zero set and its $\varkappa$-singular measure. Then*

$$I = \{f \in A^{-\infty}: Z_f \supseteq Z_1, \sigma_f \leqslant \sigma_I\}. \tag{1.12}$$

Conversely, let $\alpha = \{\alpha_\nu\}$ be an arbitrary $A^{-\infty}$-zero set and σ_0 be an arbitrary non-positive $\varkappa$-singular measure. Then

$$I(\alpha; \sigma_0) = \{f \in A^{-\infty}: Z_f \supseteq \alpha, \sigma_f \leqslant \sigma_0\} \tag{1.13}$$

is a non-trivial closed ideal in $A^{-\infty}$.

COROLLARY 1.1.1. *The necassary and sufficient condition for an element $f \in A^{-\infty}$ to be cyclic*(1) *is $Z_f = \emptyset$, $\sigma_f = 0$.*

CORALLARY 1.1.2. *Every closed ideal in $A^{-\infty}$ is principal, i.e. generated by a single element.*

CORALLARY 1.1.3. *The only "maximal" ideals in $A^{-\infty}$ are those of the form $I_{z_0} = \{f \in A^{-\infty}: f(z_0) = 0\}$ $(z_0 \in U)$. A closed ideal I such that $Z_I = \emptyset$, $\sigma_I \neq 0$ is not contained in any maximal ideal.*

In the succeeding pages we shall first (in section 2) carry out a more thorough study of $\varkappa$-singular measures and their relationship to premeasures of bounded $\varkappa$-variation [6]. In particular, we shall establish the following facts:

(a) Every $\varkappa$-singular measure is concentrated on a $\varkappa F_\sigma$-set, i.e. on the union of a countable set of B.–C. sets.

(b) For every non-positive $\varkappa$-s.m.σ there is an element $f \in A^{-\infty}$ such that $\sigma = \sigma_f$.

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(1) i.e. for $fA^{-\infty}$ to be dense in $A^{-\infty}$.

In section 3 we shall prove, using purely real-variable argument, a crucial approximation theorem for premeasures of bounded $\varkappa$-variation. Essentially, this theorem shows that in regard to some general measure-theoretical properties premeasure with a vanishing $\varkappa$-singular part comport themselves in some ways like absolutely continuous measures in the classical theory.

Finally, in section 4 we shall prove Theorem 1.1 using the above-mentioned approximation theorem, some standard functional-analytic argument involving the dual space A^∞ and the notion of annihilator, and results from [6] concerning holomorphic and meromorphic functions of the class $\mathcal{N}=A^{-\infty}/A^{-\infty}$ and their generalized Nevanlinna factorization. Incidentally we shall prove Proposition 1.1 and equivalence of the two definitions for σ_f.

We shall adhere throughout to the following notation: The letter C will be used to denote various positive constants which may differ from one formula to the next. The complement of a set $S\subseteq\partial U$ will be denoted $S^c=\partial U\setminus S$. $|S|$ is always used to designate Lebesgue measure of a set $S\subseteq\partial U$.

The author wishes to thank Lennart Carleson for valuable discussions.

§ 2. Classes of harmonic functions and premeasures.

For the reader's convenience we shall give here (in a slightly modified form) some definitions and results from [6] which will be used later. These results center round the representation of harmonic functions by means of generalized Poisson integrals involving so-called premeasures. Once such a representation is established, the problem arises to describe the class of harmonic functions under consideration in terms of premeasures. For the class $\mathcal{H}$ (see below) of harmonic functions a downright isomorphism exists between $\mathcal{H}$ and the corresponding space of premeasures. It is clear that such a close relationship should make it possible to treat many problems concerning harmonic (and analytic) functions by purely real-variable means.

Note that, in the light of some recent results of W. K. Hayman and the present author [5], it is highly probable that a similar relationship exists for much wider classes of harmonic functions than $\mathcal{H}$.

Definition 2.1. A real-valued harmonic function $u(z)$ $(z\in U)$, $u(0)=0$, is called *$\varkappa$-bounded above* (or just *$\varkappa$-bounded*) if

$$-\infty < u(z) \leqslant C \log\frac{1}{1-|z|} \quad (z\in U). \tag{2.1}$$

The least constant C in (2.1) will be called the *upper $\varkappa$-bound* of u and will be denoted $\|u\|^*$.

Clearly $\|u\|^* \geqslant 0$, and $\|u\|^*=0$ implies $u(z)\equiv 0$. The class of all $\varkappa$-bounded harmonic functions will be denoted $\mathcal{H}^+$.

Definition 2.2. $\mathcal{H}=\mathcal{H}^+-\mathcal{H}^+$, i.e. every $u(z)\in\mathcal{H}$ possesses a representation

$$u(z) = u_1(z) - u_2(z) \quad (u_1,\ u_2 \in \mathcal{H}^+). \tag{2.2}$$

PROPOSITION 2.1. *$\mathcal{H}$ becomes a Banach space if it is provided with the norm*

$$\|u\| = \min\,(\|u_1\|^* + \|u_2\|^*), \tag{2.3}$$

where minimum is taken over all the representations (2.2).

The proof is immediate, by the use of simple compactness theorems for harmonic functions. There is at least one minimal representation such that $\|u\| = \|u_1\|^* + \|u_2\|^*$.

Next we turn to the notion of a premeasure.

Definition 2.3. Let $\mathcal{K}$ be the set of all open, closed and halfclosed arcs of the circumference ∂U, including all one-point sets, ∂U and $\emptyset$. A function u: $\mathcal{K}\to\mathbf{R}$ is called a *premeasure* if

(i) $\mu(I_1\cup I_2)=\mu(I_1)+\mu(I_2)$ for all $I_1,\ I_2\in\mathcal{K}$ such that

$$I_1\cup I_2\in\mathcal{K},\ I_1\cap I_2=\emptyset;$$

(ii) $\mu(\partial U)=0$;

(iii) $\lim_{\nu\to\infty}\mu(I_\nu)=0$ whenever $I_\nu\in\mathcal{K}$, $I_1\supset I_2\supset\ldots,\bigcap_\nu I_\nu=\emptyset$.

Clearly, every premeasure is immediately extended by finite additivity to the class of sets

$$S=\bigcup_{\nu=1}^{n} I_\nu \quad (I_\nu\in\mathcal{K}).$$

With every premeasure μ a function $\hat\mu(\theta)=\mu(I_\theta)$ $(0<\theta\leqslant 2\pi)$ will be associated, where $I_\theta=\{e^{it}\colon 0\leqslant t<\theta$. Thus a $1-1$ correspondence is established between the set of premeasures and the set of all real functions $\hat\mu(\theta)$ $(0<\theta\leqslant 2\pi)$ satisfying the following conditions:

(a) $\hat\mu(\theta-)$ $(0<\theta\leqslant 2\pi)$ and $\hat\mu(\theta+)$ $(0\leqslant\theta<2\pi)$ exist;

(b) $\hat\mu(\theta-)=\hat\mu(\theta)$ $(0<\theta\leqslant 2\pi)$;

(c) $\hat\mu(2\pi)=0$.

Clearly, $\hat\mu(\theta)$ has at most a countable set of points of discontinuity, all of them jumps.

In what follows we shall adhere to the following notation: the *distance* between two points $\zeta_1, \zeta_2 \in \partial U$ is

$$d(\zeta_1, \zeta_2) = \frac{1}{\pi} \min\left\{\arg\frac{\zeta_2}{\zeta_1}, \arg\frac{\zeta_1}{\zeta_2}\right\} \quad (0 \leqslant \arg\zeta < 2\pi \quad \forall \zeta \in \partial U),$$

so that $0 \leqslant d(\zeta_1, \zeta_2) \leqslant 1$ $(\forall \zeta_1, \zeta_2 \in \partial U)$; the distance of a point $\zeta \in \partial U$ from a set $F \subset \partial U$ is

$$d(\zeta, F) = \inf_{\zeta' \in F} d(\zeta, \zeta').$$

The *δ-neighbourhood* of a set $S \subset \partial U$ is $S^\delta = \{\zeta \in \partial U : d(\zeta, S) < \delta\}$.

Definition 2.4. We shall assign to every open set $G \subset \partial U$ the quantity (which may be $+\infty$)

$$\varkappa(G) = \frac{1}{2\pi} \int_G |\log d(\zeta, G^c)| \cdot |d\zeta|; \tag{2.4}$$

Further, we define $\varkappa(\emptyset) = \varkappa(\partial U) = 0$.

A straightforward computation hows that

$$\varkappa(G) = \sum_\nu \frac{|I_\nu|}{2\pi}\left(\log\frac{2\pi}{|I_\nu|} + 1\right), \tag{2.5}$$

$\{I_\nu\}$ being the set of components of G.(1)

Definition 2.5. The *entropy* $\hat{\varkappa}_G(F)$ *of a closed set* F *with respect to an open set* $G \supset F$ is defined as

$$\hat{\varkappa}_G(F) = \frac{1}{2\pi}\int_G |\log d(\zeta, F \cup G^c)| \cdot |d\zeta| - \varkappa(G) = \frac{1}{2\pi}\int_G \left|\log\frac{d(\zeta, F \cup G^c)}{d(\zeta, G^c)}\right| \cdot |d\zeta|. \tag{2.6}$$

The entropy with respect to ∂U will be called simply *entropy* and will be denoted $\hat{\varkappa}(F)$:

$$\hat{\varkappa}(F) = \frac{1}{2\pi}\int_{\partial U} |\log d(\zeta, F)| \cdot |d\zeta|.$$

We have $\hat{\varkappa}_G(\emptyset) = 0$. If $|F| = 0$, then $\hat{\varkappa}_G(F) = \varkappa(G \setminus F)$. According to Definition 1.1 the B.–C. sets are exactly those sets F with $\hat{\varkappa}(F) < \infty$. From (2.6) follows easily that if $F_1, F_2 \subset G$, then

$$\hat{\varkappa}_G(F_1 \cup F_2) \leqslant \hat{\varkappa}_G(F_1) + \hat{\varkappa}_G(F_2). \tag{2.7}$$

(1) Sometimes we shall use notation (2.5) also for sets G, not necessarily open, consisting of a finite number of components $I_\nu \in \mathcal{K}$; we set

$$\frac{|I|}{2\pi}\left(\log\frac{2\pi}{|I|} + 1\right) = 0 \text{ if } |I| = 0.$$

Definition 2.6. A premeasure μ (and the associated function $\hat{\mu}(\theta)$) is said to be *ϰ-boundeed above* (or simply ϰ-bounded) if for all open arcs $I \subset \partial U$

$$\mu(I) \leqslant C\varkappa(I) = \frac{C|I|}{2\pi}\left(\log\frac{2\pi}{|I|}+1\right). \tag{2.8}$$

The least constant C in (2.8) will be called the *upper ϰ-bound* of the premeasure μ and will be denoted $\|\mu\|^+$. The set of all ϰ-bounded premeasures will be denoted $\varkappa B^+$.

Clearly $\|\mu\|^+ \geqslant 0$, and $\|\mu\|^+ = 0$ if and only if $\mu = 0$.

Definition 2.7. A premeasure μ (and the associated function $\hat{\mu}(\theta)$) is said to be of *bounded ϰ-variation* if for every finite set $\{I_\nu\}$ of non-overlapping open arcs such that $\cup_\nu I_\nu = \partial U$

$$\sum_\nu |\mu(I_\nu)| \leqslant C \sum_\nu \varkappa(I_\nu) = C \sum_\nu \frac{|I_\nu|}{2\pi}\left(\log\frac{2\pi}{|I_\nu|}+1\right). \tag{2.9}$$

The minimal constant C in (2.9) will be called the *ϰ-variation* of μ and will be denoted $\varkappa \operatorname{Var} \mu$. The set of all premeasures of bounded ϰ-variation will be denoted ϰV.

PROPOSITION 2.2 [6]. *Every ϰ-bounded premeasure is of bounded ϰ-variation and*

$$\varkappa \operatorname{Var} \mu \leqslant 2\|\mu\|^+. \tag{2.10}$$

PROPOSITION 2.3. [6]. *Every $\mu \in \varkappa V$ is the difference of two ϰ-bounded premeasures $\mu = \mu_1 - \mu_2$ with*

$$\|\mu_j\|^+ \leqslant a \cdot \varkappa \operatorname{Var} \mu \quad (j = 1, 2), \tag{2.11}$$

where a is an absolute constant.

PROPOSITION 2.4. *ϰV becomes a Banach space if provided with the norm*

$$\|\mu\| = \varkappa \operatorname{Var} \mu. \tag{2.12}$$

The proof is immediate.

Next comes a theorem which, though not stated explicitely in [6], follows directly from the results of that paper.

THEOREM 2.1. *There exists a linear operator $u = \mathcal{P}\mu$ (the generalized Poisson operator) which maps ϰV onto $\mathcal{H}$:*

$$u(z) = (\mathcal{P}\mu)(z) = \int_{\partial U} P(\zeta, z)\,\mu(|d\zeta|) \quad (z \in U), \tag{2.13}$$

where $P(\zeta, z) = \operatorname{Re}(\zeta + z)/(\zeta - z)$ $(\zeta \in \partial U,\ z \in U)$ is the Poisson kernel and the integral is understood either as a Riemann-Stieltjes integral

$$u(z) = \int_0^{2\pi} P(e^{i\theta}, z)\, d\hat{\mu}(\theta) \qquad (2.13')$$

or as a Riemann integral

$$u(z) = -\int_0^{2\pi} \hat{\mu}(\theta) \left[\frac{d}{d\theta} P(e^{i\theta}, z)\right] d\theta. \qquad (2.13'')$$

The inverse operator $\mu = \mathcal{D}^{-1} u$ *is given by*

$$\tfrac{1}{2}[\mu(\bar{I}) + \mu(I)] = \lim_{r \to 1-} \frac{1}{2\pi} \int_I u(r\zeta)|d\zeta|, \qquad (2.14)$$

where $I \subset \partial U$ *is an arbitrary open arc.*

COROLLARY 2.1.1. *There are two positive constants* λ_1 *and* λ_2 *such that*

$$\lambda_1 \cdot \varkappa \operatorname{Var} \mu \leqslant \|\mathcal{D}\mu\|_\varkappa \leqslant \lambda_2 \cdot \varkappa \operatorname{Var} \mu \quad (\forall \mu \in \varkappa V),$$

$$\lambda_1 \cdot \|\mu\|^+ \leqslant \|\mathcal{D}\mu\|^* \leqslant \lambda_2 \|\mu\|^+ \quad (\forall \mu \in kB^+).$$

Remark. The existence of the limit in (2.14) for $u \in \mathcal{H}^+$ is the crucial point in the proof of Theorem 2.1. Recently W. K. Hayman and the present author proved [5] that the limit in (2.14) exists for every harmonic function $u(z)$ $(z \in U)$, $u(0) = 0$, such that

$$\int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr < \infty,$$

where

$$k(r) = \max_{|z|=r} u(z).$$

PROPOSITION 2.5 [6]. *Let* μ *be a premeasure of bounded* $\varkappa$*-variation. Define for every B.-C. set* F

$$\sigma(F) = -\sum_\nu \mu(I_\nu), \qquad (2.15)$$

I_ν *being the complementary arcs of* F.[1] *Then* σ *possesses a unique extension to a* $\varkappa$*-singular measure. Moreover,*

$$|\sigma(F)| \leqslant \varkappa \operatorname{Var} \mu \cdot \hat{\varkappa}(F) \quad (\forall F \in \mathfrak{J}). \qquad (2.16)$$

If $\mu \in \varkappa B^+$, *then* $\sigma \leqslant 0$.

Definition 2.8. σ will be called the *$\varkappa$-singular part* of the premeasure μ.

We prove now a somewhat different form of (2.15) which will be needed later.

PROPOSITION 2.6. *Let* μ *be a premeasure of bounded* $\varkappa$*-variation and let* σ *be its* $\varkappa$*-singular part. Then for every* $F \in \mathfrak{J}$

$$\sigma(F) = \lim_{\delta \to 0} \mu(F^\delta) = \lim_{\delta \to 0} \mu(\bar{F}^\delta). \qquad (2.17)$$

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[1] The series (2.15) is absolutely convergent (cf. [6]).

Proof. It is enough to prove the former equality. For every complementary interval I_ν of F set

$$I_{\nu\delta} = \{\zeta \in \partial U:\ d(\zeta, I_\nu^c) \geqslant \delta\} = ((I_\nu^c)^\delta)^c.$$

Clearly,

$$\mu(F^\delta) = -\sum_{|I_\nu|\geqslant 2\delta} \mu(I_{\nu\delta}).$$

This together with (2.15) yields

$$\mu(F^\delta) - \sigma(F) = \sum_{|I_\nu|<2\delta} \mu(I_\nu) + \sum_{|I_\nu|\geqslant 2\delta} \mu(I_\nu \setminus I_{\nu\delta}).$$

The first sum tends to zero as $\delta \to 0$; what remains to be proved is that

$$\lim_{\delta\to 0} \sum_{|I_\nu|\geqslant 2\delta} \mu(I_\nu \setminus I_{\nu\delta}) = 0.$$

If we assume the contrary, then there is sequence $\delta_n \downarrow 0$ such that

(i) $|\mu(G_n)| \geqslant \varepsilon > 0 \quad (n = 1, 2, \ldots)$,

where

$$G_n = \bigcup_{2\delta_n > |I_\nu| \geqslant 2\delta_{n+1}} (I_\nu \setminus I_{\nu\delta_{n+1}});$$

(ii) $\varkappa(G_n) \leqslant 2^{-n} \quad (n = 1, 2, \ldots)$.

Consider now

$$G^{(N)} = \bigcup_{n=1}^{N} G_n.$$

$G^{(N)}$ is composed of a finite number of disjoint open arcs, say, $A_\nu^{(N)}$, and it is easily seen that for the complementary arcs $B_\nu^{(N)}$

$$\sum_\nu \varkappa(B_\nu^{(N)}) \leqslant \hat{\varkappa}(F) < \infty.$$

Therefore

$$\sum_\nu \varkappa(A_\nu^{(N)}) + \sum_\nu \varkappa(B_\nu^{(N)}) \leqslant \hat{\varkappa}(F) + 1.$$

On the other hand

$$\sum_\nu |\mu(A_\nu^{(N)})| + \sum_\nu |\mu(B_\nu^{(N)})| \geqslant \sum_{n=1}^{N} |\mu(G_n)| \geqslant N\varepsilon \to \infty \quad (N \to \infty).$$

This clearly contradicts our assumption that $\varkappa \operatorname{Var} \mu < \infty$. Thus Proposition 2.6 is proved.

Our next task is to prove that every $\varkappa$-s.m. (cf. Definition 1.2) is concentrated on a $\varkappa F_\sigma$-set, i.e. on union of a countable set of B.–C. sets.

THEOREM 2.2. *Let σ be a $\varkappa$-s.m. Then there is a sequence $\{F_\nu\}_1^\infty$ of B.–C. sets, $F_1 \subseteq F_2 \subseteq \ldots$, such that for every $F \in \mathfrak{F}$*

$$\sigma(F) = \lim_{\nu\to\infty} \sigma(F \cap F_\nu), \quad |\sigma|(F) = \lim_{\nu\to\infty} |\sigma|(F \cap F_\nu). \tag{2.18}$$

Proof. It is enough to prove the latter equality. Since $|\sigma|$ is a $\varkappa$-s.m., it satisfies (1.6) or, equivalently,

$$|\sigma|(F) \leqslant C\hat{\varkappa}(F) \quad (\forall F \in \mathfrak{F}). \tag{2.19}$$

We shall prove the theorem by organizing a transfinite "process of exhaustion" and by showing that this process stops after a countable number of steps. With this goal in view we shall introduce certain parameters associated with a $\varkappa$-s.m.

Let $G \subset \partial U$ be an open set such that $\bar{G} \setminus G \in \mathfrak{F}$, or equivalently $|\bar{G} \setminus G| = 0$, $\varkappa(G) < \infty$, $\varkappa(\partial U \setminus \bar{G}) < \infty$. Define

$$m(\sigma;\, G) = \sup_{G \supset F \in \mathfrak{F}} \{|\sigma|(F) - C\hat{\varkappa}_G(F)\}, \tag{2.20}$$

where C is the constant in (2.19). In view of (2.19) we have

$$|\sigma|(F) + |\sigma|(\partial G) \leqslant C\hat{\varkappa}(F \cup \partial G) = C[\hat{\varkappa}_G(F) + \varkappa(G) + \varkappa(\partial U \setminus \bar{G})]$$

for $F \subset G$, $F \in \mathfrak{F}$, where $\partial G = \bar{G} \setminus G$. Therefore

$$m(\sigma;\, G) \leqslant C[\varkappa(G) + \varkappa(\partial U \setminus \bar{G})] - |\sigma|(\partial G).$$

On the other hand, putting in (2.20) $F = \varnothing$ we get $m(\sigma;\, G) \geqslant 0$. Thus

$$0 \leqslant m(\sigma;\, G) \leqslant C\hat{\varkappa}(\partial G) - |\sigma|(\partial G). \tag{2.21}$$

To proceed further with the proof we need three simple lemmas. But first introduce the following

Definition 2.9. An open set $G \subset \partial U$ will be called *regular* if $\hat{\varkappa}(\partial G) < \infty$. The set of all regular sets G will be denoted $\mathcal{G}$.

LEMMA 2.2.1. *Let $\{G_\nu\}_1^\infty$ be a sequence of regular sets. Let the following hypotheses hold:*

(i) $\bar{G}_\nu \subset G_{\nu+1} \quad (\nu = 1, 2, \ldots)$;

(ii) $G = \bigcup_{\nu=1}^{\infty} G_\nu \in \mathcal{G}$;

(iii) $\hat{\varkappa}_G(\partial G_\nu) \to 0 \quad (\nu \to \infty)$.

Then

$$m(\sigma;\, G_\nu) \to m(\sigma;\, G) \quad (\nu \to \infty). \tag{2.22}$$

Proof. Let $F_\nu = \partial G_\nu = \bar{G}_\nu \setminus G_\nu$. We have

$$\begin{aligned} m(\sigma;\, G_\nu) &= \sup_{G_\nu \supset F \in \mathfrak{F}} \{|\sigma|(F) - C\hat{\varkappa}_{G_\nu}(F)\} \\ &= \sup_{G_\nu \supset F \in \mathfrak{F}} \{|\sigma|(F \cup F_\nu) - C\hat{\varkappa}_G(F \cup F_\nu)\} - |\sigma|(F_\nu) + C[\hat{\varkappa}_G(F \cup F_\nu) - \hat{\varkappa}_{G_\nu}(F)] \\ &\leqslant m(\sigma;\, G) - |\sigma|(F_\nu) + C[\hat{\varkappa}_G(F \cup F_\nu) - \hat{\varkappa}_{G_\nu}(F)] \quad (\forall G_\nu \supset F \in \mathfrak{F}), \end{aligned}$$

because the latter expression in brackets does not depend on F:

$$\hat{\varkappa}_G(F \cup F_\nu) - \hat{\varkappa}_{G_\nu}(F) = \varkappa(G \setminus (F \cup F_\nu)) - \varkappa(G) - \varkappa(G_\nu \setminus F) + \varkappa(G_\nu) = \hat{\varkappa}_G(F_\nu), \quad (2.23)$$

and therefore

$$\overline{\lim_{\nu\to\infty}}\, m(\sigma; G_\nu) \leqslant m(\sigma; G). \quad (2.24)$$

On the other hand, in view of (2.20) there is for every $\varepsilon > 0$ an $F \in \mathfrak{J}$, $F \subset G$, such that

$$|\sigma|(F) \geqslant m(\sigma; G) + C\hat{\varkappa}_G(F) - \varepsilon.$$

If ν is large enough, then $F \subset G_\nu$ and

$$m(\sigma; G_\nu) \geqslant |\sigma|(F) - C\hat{\varkappa}_{G_\nu}(F) \geqslant m(\sigma; G) - \varepsilon + C[\hat{\varkappa}_G(F) - \hat{\varkappa}_{G_\nu}(F)]. \quad (2.25)$$

Using (2.23) we find

$$\begin{aligned}\hat{\varkappa}_G(F) - \hat{\varkappa}_{G_\nu}(F) &= [\hat{\varkappa}_G(F \cup F_\nu) - \hat{\varkappa}_{G_\nu}(F)] - [\hat{\varkappa}_G(F \cup F_\nu) - \hat{\varkappa}_G(F)] \\ &= \hat{\varkappa}_G(F_\nu) + \hat{\varkappa}_G(F) - \hat{\varkappa}_G(F \cup F_\nu) \geqslant 0;\end{aligned}$$

therefore from (2.25) follows

$$\underline{\lim_{\nu\to\infty}}\, m(\sigma; G_\nu) \geqslant m(\sigma; G),$$

which together with (2.24) yields (2.22).

Lemma 2.2.2

$$\lim_{\delta\to 0} m(\sigma, F^\delta) = |\sigma|(F) \quad (\forall F \in \mathfrak{J}). \quad (2.26)$$

Proof. Let $\{I_\nu\}$, $|I_1| \geqslant |I_2| \geqslant \ldots$, be the complementary arcs of F. We have

$$\varkappa(F^\delta \setminus F) = \frac{\delta}{\pi}\left(\log\frac{2\pi}{\delta} + 1\right) \sum_{|I_\nu| \geqslant 2\delta} 1 + \sum_{|I_\nu| < 2\delta} \frac{|I_\nu|}{2\pi}\left(\log\frac{2\pi}{|I_\nu|} + 1\right).$$

Hence

$$\varkappa(F \setminus_\delta F) \to 0,\ \hat{\varkappa}_{F_\delta}(F) = \varkappa(F_\delta \setminus F) - \varkappa(F_\delta) \to 0 \quad (\delta \to 0),$$

and

$$\underline{\lim_{\delta\to 0}}\, m(\sigma; F_\delta) \geqslant |\sigma|(F). \quad (2.27)$$

Now we have to show that

$$\overline{\lim_{\delta\to 0}}\, m(\sigma; F_\delta) \leqslant |\sigma|(F). \quad (2.28)$$

Assuming the contrary, a sequence $\delta_1 > \delta_2 > \ldots$ and the corresponding sequence $\{F_\nu\}$ of B.-C. sets, $F_\nu \subset F^{\delta_\nu}$, could be chosen so that

$$|\sigma|(F_\nu) \geqslant C\hat{\varkappa}_{F^{\delta_\nu}}(F_\nu) + |\sigma|(F) + a \quad (a > 0;\ \nu = 1, 2, \ldots),$$

C being the constant in (2.19). This implies

$$|\sigma|(F_\nu \setminus F) \geq C\hat{\varkappa}_{F^{\delta_\nu}}(F_\nu) + a$$

and consequently

$$|\sigma|(F_\nu \setminus F^{\delta_{\nu'}}) \geq C\hat{\varkappa}_{F^{\delta_\nu}}(F_\nu) + \frac{a}{2} \geq C\hat{\varkappa}_{F^{\delta_\nu}}(F_\nu \setminus F^{\delta_{\nu'}}) + \frac{a}{2}$$

for sufficiently large $\nu' > \nu$. Taking $S_\delta = F_{\nu_k} \setminus F^{\delta_{\nu_{k+1}}}$ with a sufficiently sparse subsequence $\{\nu_k\}$ and a suitable $\delta_0 > 0$ we can therefore construct a sequence $\{S_k\}$ of disjoint B.–C. sets, all contained in some F^{δ_0}, such that

$$|\sigma|(S_k) \geq C\hat{\varkappa}_{F^{\delta_0}}(S_k) + \frac{a}{4}.$$

Hence,

$$|\sigma|\left(\bigcup_{k=1}^{n} S_k\right) \geq C \sum_{k=1}^{n} \hat{\varkappa}_{F^{\delta_0}}(S_k) + \frac{na}{4} \geq C\hat{\varkappa}_{F^{\delta_0}}\left(\bigcup_{k=1}^{n} S_k\right) + \frac{na}{4}$$

and therefore

$$m(\sigma;\ F^{\delta_0}) = \infty,$$

which contradicts (2.21). Thus our lemma is proved.

LEMMA 2.2.3. *Let $\mathcal{G}_r$ denote the set of all open sets $G \subset \partial U$ composed of a finite number of open arcs with rational end points: $G = \bigcup_{\nu=1}^{n} I_\nu$, $I_\nu = \{\zeta \in \partial U\colon \alpha_\nu < \arg \zeta < \beta_\nu\}$, α_ν and β_ν rational. Then for every pair σ_1, σ_2 of $\varkappa$-singular measures such that $|\sigma_1| > |\sigma_2|$ there is at least one $G \in \mathcal{G}_r$ such that*

$$m(\sigma_1;\ G) > m(\sigma_2;\ G). \tag{2.29}$$

Proof. There is a $F \in \mathfrak{F}$ such that $|\sigma_1|(F) > |\sigma_2|(F)$. Lemma 2.2.2 implies that there is a $\delta > 0$ such that $m(\sigma_1;\ F^\delta) > m(\sigma_2;\ F^\delta)$. Moving slightly the end points of the components of F^δ we can, in view of Lemma 2.2.1, replace F^δ by a $G \in \mathcal{G}_r$ so that (2.29) should hold.

We are now in a position to complete the proof of Theorem 2.2. Let $\sigma = \sigma_1 \neq 0$ be a $\varkappa$-s.m. Take a $F_1 \in \mathfrak{F}$ such that $|\sigma|(F_1) > 0$ and define $\sigma_2(F) = \sigma_1(F) - \sigma_1(F \cap F_1)$ ($\forall F \in \mathfrak{F}$). Clearly, $|\sigma_1| > |\sigma_2|$.

We define now σ_α, F_α by induction for all countable transfinite numbers α. Assume the σ_β and F_β have already been defined for all $\beta < \alpha$. If $|\sigma|(F) = \sup_{\beta<\alpha} |\sigma|(F_\beta \cap F)$ ($\forall F \in \mathfrak{F}$) set $\sigma_\alpha = 0$ and $\sigma_\gamma = 0$ for all $\gamma > \alpha$; if otherwise, take any $S \in \mathfrak{F}$ such that $|\sigma|(S) - \sup_{\beta<\alpha} |\sigma|(F_\beta \cap S) > 0$ and set $F_\alpha = (\bigcup_{\beta<\alpha} F_\beta) \cup S$, $\sigma_\alpha(F) = \sigma(F) - \sigma(F \cap F_\alpha)$. We have thus constructed a decreasing transfinite system of $\varkappa$.-s. measures $\{|\sigma_\alpha|\}$. Since $m(\sigma_\alpha;\ G) \leq m(\sigma_\beta;\ G)$ for $\alpha > \beta$, $G \in \mathcal{G}_r$ and since $\mathcal{G}_r$ is countable, there must be a countable transfinite γ such that $m(\sigma_\gamma;\ G) = 0$ ($\forall G \in \mathcal{G}_r$). Lemma 2.2.3 yields that $\sigma_\gamma = 0$; therefore

$$|\sigma|(F)=\sup_{\beta<\gamma}|\sigma|(F_\beta\cap F)\quad (\forall F\in\mathfrak{J}),$$

which is equivalent to the assertion of Theorem 2.2 because the set

$$F_\sigma=\bigcup_{\beta<\gamma}F_\beta$$

is union of a countable set of B.–C. sets.

THEOREM 2.3. *Let σ be a non-positive $\varkappa$-singular measure and let*

$$0\geqslant\sigma(F)\geqslant -C\hat{\varkappa}(F)\quad (\forall F\in\mathfrak{J}). \tag{2.30}$$

Then there is a premeasure μ such that

(i) *μ is $\varkappa$-bounded above and*

$$\|\mu\|\leqslant aC, \tag{2.31}$$

a being an absolute constant;

(ii) *the $\varkappa$-singular part of μ coincides with σ.*

Proof. The proof is broken into a number of steps. First we consider the simplest case when σ is concentrated on a finite set of points.

LEMMA 2.3.1. *Let $F_0=\{\zeta_\nu\}_1^n\subset\partial U$, $\sigma(\{\zeta_\nu\})=-\sigma_\nu\leqslant 0$ $(\nu=1,2,\dots,n)$,*

$$\sum_{\zeta_\nu\in F}\sigma_\nu\leqslant C\hat{\varkappa}(F)\quad (\forall F\subseteq F_0). \tag{2.32}$$

Then a non-negative piecewise constant function $p(\zeta)$ exists defined and continuous on $G=\partial U\setminus F_0$ and such that

$$\text{(i)}\quad \int_{\partial U}p(\zeta)|d\zeta|-\sum_1^n\sigma_\nu=0; \tag{2.33}$$

$$\text{(ii)}\quad \mu(I)=\int_I p(\zeta)|d\zeta|-\sum_{\zeta_\nu\in I}\sigma_\nu\leqslant aC\varkappa(I) \tag{2.34}$$

for all open arcs $I\subset\theta U$.

Proof. If $e^{i\theta_1}$ and $e^{i\theta_2}$ are the end points of I then $\mu(I)$ is linear in θ_i $(i=1,2)$ on every complementary arc of F_0. On the other hand, $\varkappa(I)$ being concave in θ_i $(i=1,2)$, the inequality (2.34) has to be ensured only for those I's with the end points in F_0, because it will then hold for all other I's automatically.

Assume that the points ζ_ν are arranged on ∂U counterclockwise and let I_{kl} $(1\leqslant k< l\leqslant n+1)$ be the open arc between ζ_k and ζ_l $(\zeta_{n+1}=\zeta_1)$. Write those inequalities (2.34) which correspond to the arcs I_{kl} in the form

$$\sum_{\nu=k}^{l} p_\nu |I_{\nu,\nu+1}| \leqslant \sum_{\nu=k+1}^{l} \sigma_\nu + aC\varkappa(I_{k,l+1}) \quad (1 \leqslant k \leqslant l \leqslant n); \tag{2.35}$$

(2.33) will then assume the form

$$\sum_{\nu=1}^{n} p_\nu |I_{\nu,\nu+1}| = \sum_{\nu=1}^{n} \sigma_\nu. \tag{2.36}$$

Moreover,

$$p_\nu \geqslant 0 \quad (\nu = 1, 2, \ldots, n). \tag{2.37}$$

We shall show that the system composed of (2.35), (2.36) and (2.37) is consistent provided (2.32) holds and $a=1$. By a well-known compatibility criterion for inequalities we have to verify that for every finite system of arcs $\{I_j\}$, $I_j = I_{k_j l_j}$, and for corresponding positive numbers $\{\lambda_j\}$ such that

$$\sum_j \lambda_j \chi_{I_j}(\zeta) \geqslant 1 \quad (\zeta \in \partial U \setminus F_0) \tag{2.38}$$

the following inequality holds:

$$\sum_{\nu=1}^{n} \sigma_\nu \leqslant \sum_j \lambda_j \sum_{\zeta_\nu \in I_j} \sigma_\nu + C \sum_j \lambda_j \varkappa(I_j), \tag{2.39}$$

$\chi_I(\zeta)$ being the characteristic function of an arc I. Clearly, we can confine ourselves to the case when all the λ_j are rational. Moreover, replacing some of the arcs I_j by shorter ones or discarding them altogether we can reduce (2.38) to an equality. Multiplying then (2.38) and (2.39) by the common denominator of the λ_j and replacing $\{I_j\}$ by another system of arcs (with some arcs repeated several times, if necessary), we shall give the required result the following form: for any system of open arcs $\{I_j\}$ which have their end points in F_0, do not contain ζ_1 and cover F_0^c exactly n times,

$$\sum_j \chi_{I_j}(\zeta) = n \quad (\zeta \in F_0^c),$$

the inequality holds

$$n \sum_{\nu=1}^{n} \sigma_\nu \leqslant \sum_j \sum_{\zeta_\nu \in I_j} \sigma_\nu + C \sum_j \varkappa(I_j) \tag{2.40}$$

or, equivalently,

$$\sum_{\nu=1}^{n} \Big(n - \sum_{\zeta_\nu \in I_j} 1\Big) \sigma_\nu \leqslant C \sum_j \varkappa(I_j). \tag{2.41}$$

For $n=1$ this is certainly true because (2.41) is then equivalent to (2.32), F being the set of end points of all the arcs I_j. The general case is proved by induction which is made possible by the fact that every n-covering $\{I_j\}$ of $\partial U \setminus F_0$ (with ζ_1 not covered at all) can be split up into n simple coverings.

Thus we have proved the existence of a function $p(\zeta)$ which satisfies (2.33) and (2.34) for any I not containing $\zeta_1 (a=1)$. If $\zeta_1 \in I$, write (2.34) for the two components of $I \setminus \{\zeta_l\}$:

$$\mu(I_1) \leqslant C\varkappa(I_1), \quad \mu(I_2) \leqslant C\varkappa(I_2).$$

Then

$$\mu(I) = \mu(I_1) + \mu(I_2) - \sigma_1 \leqslant C[\varkappa(I_1) + \varkappa(I_2)] \leqslant C\left[\varkappa(I) + \frac{|I|\log 2}{2\pi}\right] \leqslant aC\varkappa(I)$$

with $s = 1 + \log 2$. Thus (2.34) has been proved with $a = 1 + \log 2$.

Next in the proof of Theorem 2.3 comes the case when σ is supported by a B.–C. set F_0. Let I_ν $(\nu = 1, 2, \dots)$ be the components of F_0^c and let (2.30) hold for any closed $F \in F_0$. Consider the closed set

$$S_n = \partial U \setminus \bigcup_{\nu=1}^{n} I_\nu = \bigcup_{\nu=1}^{n} J_\nu^{(n)} \quad (n \geqslant 1)$$

which has exactly n components $J_\nu^{(n)}$ $(\nu = 1, 2, \dots, n)$ that are either points or closed arcs. Choose in each $J_\nu^{(n)}$ one point $\zeta_\nu^{(n)} \in F_0$ and let $F^{(n)} = \{\zeta_\nu^{(n)}\}_{\nu=1}^{n}$. Place at each $\zeta_\nu^{(n)}$ the mass $-\sigma_\nu^{(n)} = \sigma(F_0 \cap J_\nu^{(n)})$ and apply Lemma 2.3.1. First check condition (2.32). For any subset. $F \subseteq F^{(n)}$ let M_F denote the union of all those $J_\nu^{(n)}$ that have non-void intersection with F Then

$$\sum_{\zeta_\nu^{(n)} \in F} \sigma_\nu^{(n)} = -\sigma(F_0 \cap M_F) \leqslant C\hat{\varkappa}(F_0 \cap M_F). \tag{2.42}$$

On the other hand,

$$\hat{\varkappa}(F_0 \cap M_F) \leqslant \hat{\varkappa}(F) + \sum_{\nu=n+1}^{\infty} \varkappa(I_\nu),$$

and since

$$\lim_{n\to\infty} \sum_{\nu=n+1}^{\infty} \varkappa(I_\nu) = 0$$

we obtain

$$\sum_{\zeta_\nu^{(n)} \in F} \sigma_\nu^{(n)} \leqslant (C + \varepsilon)\,\hat{\varkappa}(F) \quad (\forall F \subseteq F^{(n)}),$$

$\varepsilon > 0$ being arbitrarily small if n is large enough. Now we can apply Lemma 2.3.1. and find a premeasure (in fact, a measure) $\mu^{(n)}$ with contant non-negative densities between the points $\zeta^{(n)}$ such that

$$\mu^{(n)}(\{\zeta_\nu^{(n)}\}) = -\sigma_\nu^{(n)} = \sigma(F_0 \cap J_\nu^{(n)}), \; \|\mu^{(n)}\|^+ \leqslant (C + \varepsilon)(1 + \log 2).$$

Using a Helly-type selection theorem [6] (or just a self-evident diagonal process) we can find a weakly convergent subsequence $\{\mu^{(n_s)}\}$ such that for every arc $I \subset \partial U$ whose end points are outside F_0

$$\lim_{s\to\infty} \mu^{(n_s)}(I) = \mu(I),$$

where μ is a measure with constant non-negative density on each I_ν. Clearly, $\|\mu\|^+ \leqslant C(1+\log 2)$. It remains to prove that $\mu_\sigma = \sigma$.

Let $F \subset F_0$ be a closed set. Using Proposition 2.6 we find that

$$\mu_\sigma(F) = \lim_{\delta\to 0} \mu(F^\delta). \tag{2.43}$$

On the other hand,

$$\mu(F^\delta) = \lim_{s\to\infty} \mu^{(n_s)}(F^\delta)$$

if δ is such that the end points of the components of F^δ are outside F_0. We can now estimate $\mu^{(n)}(F^\delta)$ as follows:

$$-\sum_{\zeta_\nu^{(n)}\in F^\delta} \sigma_\nu^{(n)} \leqslant \mu^{(n)}(F^\delta) \leqslant -\sum_{\zeta_\nu^{(n)}\in F^\delta} \sigma_\nu^{(n)} + (C+\varepsilon)(1+\log 2)[\hat{\varkappa}_{F_0^\delta}(F_0) + \varkappa(F_0^\delta)].$$

Letting $n\to\infty$ we get

$$\sigma(F_0 \cap F^\delta) \leqslant \mu(F^\delta) \leqslant \sigma(F_0 \cap F^\delta) + C(1+\log 2)[\hat{\varkappa}_{F_0^\delta}(F_0) + \varkappa(F_0^\delta)].$$

Since $F_0 \in \mathfrak{F}$ we obtain

$$\lim_{\delta\to 0} \hat{\varkappa}_{F_0^\delta}(F_0) = 0, \quad \lim_{\delta\to 0} \varkappa(F_0^\delta) = 0$$

and hence, bearing in mind (2.43),

$$\lim_{\delta\to 0} \mu(F^\delta) = \mu_\sigma(F) = \sigma(F) \quad (\forall F \in \mathfrak{F}). \tag{2.44}$$

To complete the proof of Theorem 2.3 we have to consider the general case when, according to Theorem 2.2, there is a sequence $F_1 \subseteq F_2 \subseteq \dots$ of B.–C. sets such that $\sigma(F) = \lim_{n\to\infty} \sigma(F \cap F_n)$ $(\forall F \in \mathfrak{F})$. For every F_n there is a (pre)measure $\mu^{(n)}$, $\|\mu^{(n)}\|^+ \leqslant C(1+\log 2)$, which has non-negative piecewise constant density on F_n^c and whose singular part is

$$\mu_\sigma^{(n)}(F) = \sigma_n(F) = \sigma(F \cap F_n) \quad (\forall F \in \mathfrak{F}).$$

Using again the Helly-type selection theorem [6] we can extract a subsequence $\{\mu^{(n_s)}\}$ which converges weakly to a premeasure μ. Repeating the same argument we used in proving (2.44) we shall arrive at the following conclusion:

$$\mu_\sigma(F) = \lim_{n\to\infty} \sigma_n(F) = \sigma(F) \quad (\forall F \in \mathfrak{F}).$$

Thus Theorem 2.3 has been proved.

We shall later need the following result which can be proved using the same technique:

COROLLARY 2.3.1. *Let $\sigma_1 \leqslant \sigma_2 \leqslant \ldots \leqslant 0$ be a sequence of $\varkappa$-singular measures and let σ_1 (and consequently all the σ_ν) satisfy condition of the type* (2.30). *If in addition*

$$\lim_{\nu\to\infty} \sigma_\nu(F) = 0 \quad (\forall F \in \mathfrak{F}), \tag{2.45}$$

then there is a sequence $\{\mu_\nu\}_1^\infty$ of premeasures such that

(i) $\|\mu_\nu\|^+ \leqslant aC$;

(ii) *the $\varkappa$-singular part of μ_ν is equal to σ_ν;*

(iii) $\sup_{I\in\mathfrak{X}} |\mu_\nu(I)| \to 0 \quad (\nu \to \infty)$. (2.46)

§ 3. An approximation theorem for premeasures

Definition 3.1. A premeasure μ of bounded $\varkappa$-variation is said to be *$\varkappa$-absolutely continuous below* if there is a sequence $\{\mu_\nu\}_1^\infty$ of premeasures, $\mu_\nu \in \varkappa B^+$, such that

(i) $\mu + \mu_\nu \in \varkappa B^+$, $\|\mu + \mu_\nu\|^+ \leqslant C \quad (\forall \nu)$; (3.1)

(ii) $\sup_{I\in\mathfrak{X}} |(\mu + \mu_\nu)(I)| \to 0 \quad (\nu \to \infty)$. (3.2)

THEOREM 3.1. *A premeasure $\mu \in \varkappa V$ is $\varkappa$-absolutely continuous below if and only if its $\varkappa$-singular part is non-negative:*

$$\mu_\sigma \geqslant 0. \tag{3.3}$$

Proof.

A. *Necessity.* Let μ be $\varkappa$-absolutely continuous below, i.e. let there be a sequence $\{\mu_\nu\}$ satisfying (3.1) and (3.2). Take an arbitrary set $F \in \mathfrak{F}$ and let $\{I_n\}$ be its complementary arcs. We have

$$-(\mu + \mu_\nu)_\sigma(F) = \sum_n (\mu + \mu_\nu)(I_n) = \sum_{n\leqslant N} (\mu + \mu_\nu)(I_n) + \sum_{n>N} (\mu + \mu_\nu)(I_n)$$
$$\leqslant \sum_{n\leqslant N} (\mu + \mu_\nu)(I_n) + C \sum_{n>N} \varkappa(I_n).$$

Using (3.2) we get

$$-\varliminf_{\nu\to\infty} (\mu + \mu_\nu)_\sigma(F) \leqslant C \sum_{n>N} \varkappa(I_n) \to 0 \quad (N \to \infty),$$

because $\hat{\varkappa}(F) < \infty$. Thus

$$\varliminf_{\nu\to\infty} (\mu + \mu_\nu)_\sigma(F) \geqslant 0. \tag{3.4}$$

Since $\mu_\nu \in \varkappa B^+$ its $\varkappa$-singular part is non-positive; therefore

$$(\mu + \mu_\nu)_\sigma(F) \leqslant \mu(F) \quad (\forall F \in \mathfrak{F}). \tag{3.5}$$

From (3.4) and (3.5) follows

$$\mu_\sigma(F) \geqslant 0 \quad (\forall F \in \mathfrak{F})$$

which proves (3.3)

B. *Sufficiency.* Let $N \geqslant 1$ be entire. Consider the set $\mathfrak{L}_N$ of half-open arc $I_{kl} = \{e^{i\theta}: (2\pi k)/N \leqslant \theta < (2\pi l)/N\}$ $(0 \leqslant k < l \leqslant N)$; let $\mu(I_{kl}) = \mu_{kl}$. If μ is $\varkappa$-absolutely continuous below then (3.1) and (3.2) imply that the following system of inequalities and equations is consistent:

$$\left.\begin{aligned} & x_{kl} \leqslant M\varkappa(I_{kl}), \\ & \mu_{kl} + x_{kl} \leqslant \min\{C\varkappa(I_{kl}), \varepsilon\}, \\ & x_{kl} = \sum_{s=k}^{l-1} x_{s,\,s+1},\ x_{0N} = 0 \quad (0 \leqslant k < l < N) \end{aligned}\right\} \tag{3.6}$$

for any $\varepsilon > 0$ and some $M = M_\varepsilon$. In fact, setting $x_{kl} = \mu_\nu(I_{kl})$ and writing out all the requirements of Definition 3.1 regarding the intervals $I \in \mathfrak{L}_N$ as well as all the additivity conditions and $\mu_\nu(\partial U) = 0$, we obtain (3.6). Conversely, if for any $\varepsilon > 0$ and for some $M = M_\varepsilon$ (3.6) has solutions for $N = 1, 2, \ldots$, then μ is $\varkappa$-absolutely continuous below. To prove this we have to form for every solution $\{x_{kl}\}$ of (3.6) a measure x having constant density $x_{s,\,s+1}/|I_{s,\,s+1}|$ over every $I_{s,\,s+1}$. Using then the Helly-type selection theorem for premeasures [6] and effecting transition to the limit with $N \to \infty$ we shall obtain a premeasure x which meets the following conditions:

$$x(I) \leqslant M\varkappa(I); \quad \mu(I) + x(I) \leqslant \min\{C\varkappa(I), \varepsilon\}$$

for all open arcs $I \subset \partial U$ which do not contain the point $\zeta = 1$, the last restriction being easily removed if $(1 + \log 2)\,C$, 2ε is substituted for C, ε respectively (cf. the proof of Lemma 2.3.1). Consequently, if μ is not $\varkappa$-absolutely continuous below then for every $C > 0$ there is an $\varepsilon > 0$ such that, however large M, (3.5) has no solutions for some N. Repeating the argument used in the proof of Lemma 2.3.1 we shall arrive at the conclusion that for such combination of C, ε, M there is a covering of ∂U by a finite system of disjoint half-closed arcs $\{I_\nu\}$ such that

$$\textstyle\sum_\nu \min\{\mu(I_\nu) + M\varkappa(I_\nu),\ C\varkappa(I_\nu),\ \varepsilon\} < 0.$$

Let $\{I'_\nu\}$ be those arcs among $\{I_\nu\}$ for which

$$\min\{\mu(I_\nu) + M\varkappa(I_\nu),\ C\varkappa(I_\nu),\ \varepsilon\} = \mu(I_\nu) + M\varkappa(I_\nu),$$

and let $\{I''_\nu\} = \{I_\nu\} \setminus \{I'_\nu\}$. Clearly, $\mu(I'_\nu) < 0$. Setting $F_M = \bigcup_\nu I'_\nu$ we find

$$\mu(F_M) < -(M - C)\,\varkappa(F_M) - C\varkappa(F_M) - C \sum_{|I''_\nu| < \delta} \varkappa(I''_\nu) - \varepsilon \sum_{|I''_\nu| \geqslant \delta} 1, \tag{3.7}$$

where δ is defined by the equation.

$$\frac{C\delta}{2\pi}\left(\log\frac{2\pi}{\delta} + 1\right) = \varepsilon.$$

Put now $C=2\varkappa \operatorname{Var}\mu$ and let $M\to\infty$. Bearing in mind the definition of $\varkappa$-variation we easily arrive at the following conclusion:

(a) $\{I''_\nu : |I''_\nu| \geqslant \delta\} \neq \emptyset \quad \text{for } M>2C$;

(b) $\sum \varkappa(I''_\nu)=O(1) \quad (M\to\infty)$;

(c) $\varkappa(F_M)\to 0 \quad (M\to\infty)$;

(d) $$\mu(F_M)\leqslant -2\varkappa \operatorname{Var}\mu[\varkappa(F_M)+\sum_{|I''_\nu|<\delta}\varkappa(I''_\nu)]-\varepsilon. \tag{3.8}$$

We shall assume for convenience that F_M is a closed set composed of a finite number of closed arcs and that $\varkappa(F_M)$ stands for $\varkappa(\operatorname{int} F_M)$; the parameter M will be assumed to run through a sequence $M_1<M_2<\dots$, $\lim M_n=\infty$. To simplify the notation we shall write F_n for F_{M_n}. Our aim now is to extract a subsequence $\{F_{n_\nu}\}$ which will converge in some sense (to be specified) to a B.–C. set F, and to show using (3.8) that μ_σ cannot be non-negative on F. For that we need

LEMMA 3.1.1. *Let $\{F_n\}$ be a sequence of sets, each one composed of a finite number of closed arcs. Let the following hypotheses hold $(n\to\infty)$:*

(i) $|F_n|\to 0$

(ii) $\varkappa(F^c_n)=O(1)$.

Then there is a subsequence $\{F_{n_\nu}\}$ and a B.–C. set F such that for every $\delta>0$ and some $N=N_\delta$

(a) $F_{n_\nu}\subset F^\delta$,

(b) $F\subset F^\delta_{n_\nu}$

for $\nu>N_\delta$.

Proof. Let $\{I_{kn}\}$ be the complementary arcs of F_n arranged so that $|I_{1n}|\geqslant|I_{2n}|\geqslant\dots$. We show first that $|I_{1n}|$ are bounded away from 0. In fact,

$$\varkappa(F^c_n)=\sum_k \frac{|I_{kn}|}{2\pi}\left(\log\frac{2\pi}{|I_{kn}|}+1\right)\geqslant\frac{|F^c_n|}{2\pi}\left(\log\frac{2\pi}{|I_{1n}|}+1\right)$$

and therefore

$$\log\frac{2\pi}{|I_{1n}|}+1\leqslant\frac{2\pi\varkappa(F^c_n)}{|F^c_n|}. \tag{3.9}$$

Since $|F^c_n|\to 2\pi$ and $\varkappa(F^c_n)=O(1)$ (3.9) shows that $|I_{1n}|$ is bounded away from 0. We can therefore choose a subsequence

$$\{F_{\nu_n}\}=\{F'_n\}$$

such that

$$I_{1n} \to J_1 \quad (n \to \infty), \tag{3.10}$$

where $\{I'_{kn}\}$ are the complementary arcs of F'_n, J_1 is some open arc, $|J_1| > 0$, and (3.10) means that the end points of I'_{1n} tend to the corresponding end points of J_1. If $|J_1| = 2\pi$ then $\{F'_n\}$ is the required subsequence and $F = J_1^c$. If $|J_1| < 2\pi$ then the same argument shows that

$$\log \frac{2\pi}{|I'_{2n}|} + 1 \leqslant \frac{2\pi\varkappa(F'^c_n)}{|F'^c_n| - |I'_{1n}|} \tag{3.11}$$

and since the denominator of the latter faction tends to $2\pi - |J_1| > 0$ the lengths $|I'_{2n}|$ must be bounded away from zero. Therefore a subsequence $\{F''_n\} = \{F'_{\nu_n}\}$ exists such that $I''_{2n} \to J_2$. Continuing this process we shall either arrive after a finite number of steps at a subsequence $\{F^{(s)}_n$ such that

$$I^{(s)}_{kn} \to J_k \quad (n \to \infty;\ k = 1, 2, \dots, s)$$

and $\sum_{k=1}^{s} |J_k| = 2\pi$ in which case $\{F^{(s)}_n\}$ is the required subsequence and $F = (\bigcup_{k=1}^{s} J_k)^c$ is a finite set, or the number of steps is infinite. In the latter case

$$\sum_{k=1}^{\infty} |J_k| = 2\pi. \tag{3.12}$$

In fact, just as (3.9) and (3.11) it is easily seen that

$$\log \frac{2\pi}{|J_s|} + 1 \leqslant \frac{2\pi A}{2\pi - \sum_{k=1}^{s-1} |J_k|},$$

where A is the upper bound for $\varkappa(F^c_n)$, and that proves (3.12) since clearly $|J_s| \to 0 (s \to \infty)$. Taking the diagonal subsequence $\{F^{(n)}_n\}_{n=1}^{\infty}$ we get the required result. Thus our lemma is proved.

Now we can continue the proof of Theorem 3.1. As (3.8) shows, the assumption that μ is not absolutely continuous below implies the existence of a sequence $\{F_n\}_1^\infty$ of sets, each F_n being composed of a finite number of closed arcs, such that

(i) $\varkappa(F_n) \to 0$ and *a fortiori* $|F_n| \to 0$ $(n \to \infty)$;

(ii) $\varkappa(F^c_n) \leqslant A < \infty$ $(n = 1, 2, \dots)$;

(iii) $\mu(F_n) \leqslant -C[\varkappa(F_n) + \sum_{|I_{kn}| < \delta} \varkappa(I_{kn})] - \varepsilon$, (3.13)

where $C = 2\varkappa \operatorname{Var} \mu$, $\{I_{kn}\}$ are the complementary arcs of F_n, and δ and ε are some positive numbers. Using Lemma 3.1.1 we can form a subsequence $\{F_{n_\nu}\}$ converging to a B.–C. set F in the sense that for every $\varrho > 0$ F^ϱ contains all but a finite number of F_{n_ν} and is contained in all but a finite number of $F^{\varrho}_{n_\nu}$. Assume for simplicity that $\{F_n\}$ already is such a subsequence. We claim that the $\varkappa$-singular part μ_σ of the premeasure μ cannot be non-negative on F.

If the contrary is true, then $\mu_\sigma(S)\geqslant 0$ ($\forall S\subseteq F$, $S\in\mathcal{J}$) and in particular $\mu_\sigma(S_n)\geqslant 0$ with $S_n=F_n\cap F$. Using Proposition 2.6 we find $\lim_{\varrho\to 0}\mu(S^{\varrho}_n)\geqslant 0$.

Therefore we can replace in (3.13) F_n by $F_n\diagdown S^{\varrho}n$ and choose ϱ_n so small that (3.13) should still hold though perhaps with a smaller ε and only for sufficiently large n. Thus a sequence of numbers $\varrho_n\downarrow 0$ can be chosen as well as a sequence of sets $\{F_n\}_1^\infty$ (each one composed of a finite number of closed arcs) such that

$$F_n\subset F^{\varrho_n}\diagdown F^{\varrho_{n+1}}$$

and

$$\mu(F_n)\leqslant -C[\varkappa(F_n)+\varkappa(G_n)]-\varepsilon, \tag{3.14}$$

where $G_n=(F^{\varrho_n}\diagdown F^{\varrho_{n+1}})\diagdown F_n$.

Let $\mathcal{I}_n$, $\mathcal{J}_n$ and $\mathcal{K}_n$ denote the systems of arcs I of which F_n, G_n and F^{ϱ_n} are composed respectively; let

$$\mathcal{S}_n=\left(\bigcup_{k=1}^{n}\mathcal{I}_k\right)\cup\left(\bigcup_{k=1}^{n}\mathcal{J}_k\right)\cup\mathcal{K}_{n+1}.$$

Further let $\mathcal{I}_0$ be the system of arcs that form $\partial U\diagdown F^{\varrho_1}$. Summing (3.14) we get

$$\sum_{I\in\mathcal{I}_0}|\mu(I)|+\sum_{I\in\mathcal{S}_n}|\mu(I)|\geqslant\sum_{\nu=1}^{n}|\mu(F_\nu)|\geqslant C\left[\sum_{\nu=1}^{n}\varkappa(F_\nu)+\sum_{\nu=1}^{n}\varkappa(G_\nu)\right]+n\varepsilon$$

$$=C\sum_{I\in\mathcal{S}_n}\varkappa(I)-C\sum_{I\in\mathcal{K}_{n+1}}\varkappa(I)+n\varepsilon=C[\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0}\varkappa(I)-\sum_{I\in\mathcal{K}_{n+1}}\varkappa(I)-\sum_{I\in\mathcal{I}_0}\varkappa(I)]+n\varepsilon.$$

Since

$$\sum_{I\in\mathcal{K}_{n+1}}\varkappa(I)\to 0\quad(n\to\infty),$$

we obtain (for large enough n)

$$\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0}|\mu(I)|\geqslant C\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0}\varkappa(I).$$

We have arrive therefore at a contradiction, because $\mathcal{S}_n\cup\mathcal{I}_0$ is a system of non-overlapping arcs covering ∂U, and $C=2\varkappa\operatorname{Var}\mu$. This contradiction completes the proof of Theorem 3.1.

COROLLARY 3.1.1. *Let $\mu\in\varkappa V$ and $\mu_\sigma\geqslant 0$. Then the function*

$$f(z)=\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu(|d\zeta|)\right\}\quad(z\in U) \tag{3.15}$$

possesses the following properties:

(i) *it is analytic in U and belongs to the class* $\mathcal{N}=A^{-\infty}/A^{-\infty}$ (cf. section 4)*;*

(ii) $f(0)=1$*;*

(iii) *there is a sequence of functions* $\{g_\nu(z)\}_1^\infty$ *belonging to* $A^{-\infty}$ *such that* $h_\nu(z)=f(z)g_\nu(z)$ *belong to* A^{-N} *with some* $N>0$ *and*

$$\|1-h_\nu\|_{-N}\to 0 \quad (\nu\to\infty). \tag{3.16}$$

Proof. μ is $\varkappa$-absolutely continuous below. Taking $\{\mu_\nu\}$ as in Definition 3.1 and defining

$$g_\nu(z)=\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta+z}\mu_\nu(|d\zeta|)\right\} \quad (z\in U) \tag{3.17}$$

we obtain the required sequence. In fact,

$$|g_\nu(z)|=\exp\left\{\int_{\partial U}P(\zeta,z)\,\mu_\nu(|d\zeta|)\right\}\leqslant(1-|z|)^{-\lambda_2\|\mu_\nu\|^+}$$

(see Corollary 2.1.1) so that $g_\nu\in A^{-\infty}$. For the same reason $fg_\nu\in A^{-\lambda_2 C}$ where C is the constant in (3.1). We have further

$$f(z)\,g_\nu(z)=\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}(\mu+\mu_\nu)(|d\zeta|)\right\}=\exp\left\{-\int_0^{2\pi}\left[\frac{d}{d\theta}\,\frac{(e^{i\theta}+z)}{e^{i\theta}-z}\,[\hat{\mu}(\theta)+\hat{\mu}_\nu(\theta)]\,d\theta\right\},$$

and from (3.2) follows easily that $f(z)g_\nu(z)\to 1$ uniformly on compact sets $F\subset U$. Therefore (3.16) holds for any $N>\lambda_2 C$.

§ 4. Proof of Theorem 1.1

Corollary 3.1.1 implies in particular that an element $f\in A^{-\infty}$ possessing representation (3.15) with $\mu\in\varkappa B^+$, $\mu_\sigma=0$, is cyclic, i.e. the closed ideal I_f generated by f is $A^{-\infty}$ itself. Clearly, this covers an important special case of Theorem 1.1, provided that equivalence of the two definitions of $\mathfrak{J}$ can be proved (cf. section 1). For the reader's convenience we shall give here some results from [6] related to the representation of functions of the classes $A^{-\infty}$, $\mathcal{N}$.

PROPOSITION 4.1. [2]. *Every function* $f(z)\in A^{-\infty}$, $f(0)\neq 0$, *possesses a unique representation in the form*

$$f(z)=f(0)\,\tilde{B}_\alpha(z)\,\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu(|d\zeta|)\right\}, \tag{4.1}$$

where $\tilde{B}_\alpha(z)$ *is the "generalized Blaschke product" associated with* $\alpha=\{\alpha_\nu\}=Z_f$*:*

$$\tilde{B}_\alpha(z) = \prod_{\alpha_\nu \in \alpha} \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z} \cdot \frac{|\alpha_\nu|}{\alpha_\nu} \exp\left\{\frac{(\alpha_\nu/|\alpha_\nu|) + z}{(\alpha_\nu/|\alpha_\nu|) - z} \cdot \log \frac{1}{|\alpha_\nu|}\right\} \tag{4.2}$$

and $\mu \in \varkappa B^+$. *Moreover,*

$$\operatorname{Sup} \|\mu\|^+ < \infty \quad (\forall f \in A^{-n}, \ \|f\|_{-n} \leqslant C) \tag{4.3}$$

for any $n > 0$, $C > 0$.

Remark. A corresponding result for the class $\mathcal{N}$ holds as well with $\mu \in \varkappa V$ and the quotient of two generalized Blaschke products instead of $\tilde{B}_\alpha(z)$ in (4.1).

Let $f(z) \in A^{-n}$, $f(0) \neq 0$, $\alpha = \{\alpha_\nu\} = Z_f$. Define

$$\tau_f(F) = -\sum_{(\alpha_\nu/|\alpha_\nu|) \in F} \log \frac{1}{|\alpha_\nu|} \quad (\forall F \in \mathfrak{F}).$$

Then τ_f is a non-positive $\varkappa$-singular measure satisfying condition (1.6) with the constant $C = an$, a being an absolute constant. This result follows immediately from the description of $A^{-\infty}$-zero sets (cf condition (T_n) and (T) in [6]).

Definition 4.1. τ_f will be called the *Blaschke* $\varkappa$*-singular measure* associated with f.

Definition 4.2. [6]. Let $f(z) \in A^{-\infty}$, $f(0) \neq 0$, be represented in the form (4.1). Let μ_σ be the $\varkappa$-singular part of the premeasure μ and τ be the Blaschke $\varkappa$-singular measure. Then

$$\sigma_f = \mu_\sigma - \tau_f \tag{4.4}$$

will be called the $\varkappa$*-singular measure associated with* f. If $f0) = f'(0) = \ldots = f^{(k-1)}(0) = 0$, $f^{(k)}(0) \neq 0$ $(k \geqslant 1)$ and $f_1(z) = z^{-k} f(z)$ then by definition

$$\sigma_f = \sigma_{f_1}.$$

Clearly $\sigma_f \leqslant 0$ for all $f \in A^{-\infty}$.

It will be shown later that Definition 4.2 is equivalent to Definition 1.3.

The notion of a $\varkappa$-singular measure σ_I associated with an ideal $0 \neq I \subset A^{-\infty}$ is reduced to σ_f by means of (1.11).

Definition 4.3. C^∞ is the linear topological space of all infinitely differentiable functions $F(\zeta)$ on ∂U:

$$F(\zeta) = \sum_{-\infty}^{\infty} b_\nu \zeta^\nu \quad (b_\nu = O(|\nu|^{-k}) \ \forall k > 0).$$

Definition 4.4. $C^{-\infty}$ is the linear topological space of all forma series

$$f = \sum_{-\infty}^{\infty} a_\nu \zeta^\nu,$$

where

$$a_\nu = O(|\nu|^k) \quad \text{for some} \quad k = k_f > 0.$$

The spaces A^∞ and $A^{-\infty}$ will be thought of as subspaces of C^∞ and $C^{-\infty}$ respectively.

The multiplication of elements belonging to $C^{-\infty}$ will be understood as formal multiplication of the corresponding series, whenever this leads to meaningful formulas for the coefficients of the product.

PROPOSITION 4.2.

(i) $C^\infty C^{-\infty} \subseteq C^{-\infty}$;

(ii) $C^\infty C^\infty \subseteq C^\infty$;

(iii) $A^{-\infty} A^{-\infty} \subseteq A^{-\infty}$;

(iv) *if* $f \in C^\infty$, $g_\nu \in C^{-\infty}$ $(\nu = 1, 2, \ldots)$ *and* $g_\nu \to g (\nu \to \infty)$ *in the topology of* $C^{-\infty}$, *then* $fg_\nu \to fg$ *in* $C^{-\infty}$;

(v) *if* $g_\nu \to g$ *and* $h_\nu \to h$ $(v \to \infty)$ *in* $A^{-\infty}$, *then* $g_\nu h_\nu \to gh$ *in* $A^{-\infty}$.

The proof is obvious.

Definition 4.5. The *annihilator* of a closed ideal $I \subseteq A^{-\infty}$ is the subspace $\mathcal{A}_I$ of C^∞ whose elements F satisfy

$$Ff \in A^{-\infty} \quad (\forall f \in I). \tag{4.6}$$

Let

$$F_0(\zeta) = \sum_{-\infty}^{\infty} b_\nu \zeta^\nu$$

be some element of $\mathcal{A}_I$; then for any $f \in I$,

$$f(z) = \sum_0^\infty a_\nu z^\nu$$

(4.6) yields

$$\sum_{\nu=0}^{\infty} b_{-k-\nu} a_\nu = 0 \quad (k = 1, 2, \ldots). \tag{4.7}$$

This shows that $F_0 \in \mathcal{A}_I$ implies $F_0 + A^\infty \subseteq \mathcal{A}_I$; in particular, $A^\infty \subseteq \mathcal{A}_I$. Thus what really matters in Definition 4.5 is the non-analytic part of $F(\zeta)$, i.e. the coefficients $\{b_\nu\}_{-1}^{-\infty}$. It is easily seen that the quotient space $\mathcal{A}_I / A^\infty$ is isomorphic to the subspace $\mathcal{A}_I^*$ of A^∞ consisting of those functionals F^* for which $F^*(f) = 0$ $(\forall f \in I)$ (see formula (1.4) for the definition of A^∞ as the dual of $A^{-\infty}$).

PROPOSITION 4.3. *For each closed ideal* $I \subseteq A^{-\infty}$

$$I = \{f \in A^{-\infty}: Ff \in A^{-\infty} \forall F \in \mathcal{A}_I\}. \tag{4.8}$$

This is a direct consequence from Definition 4.5 and from the Hahn-Banach theorem for linear topological spaces (see, e.g., [4], chapter 2).

Now we prove some lemmas which will lead eventually to the proof of Theorem 1.1.

LEMMA 4.1. *Let* $0 \neq F \in C^\infty$ *and* $f_1, f_2, g_1, g_2 \in A^{-\infty}$. *If* $Ff_1 = g_1$, $Ff_2 = g_2$ *then* $f_1 g_2 = f_2 g_1$.

Proof. $(Ff_1)f_2 = F(f_1 f_2) = g_1 f_2$; $(Ff_2)f_1 = F(f_2 f_1) = g_2 f_1$; therefore $g_1 f_2 = g_2 f_1$. All the multiplications and transformations are easily justified.

LEMMA 4.2. *Let* $0 \neq F \in C^\infty$, $f_0 \in A^{-\infty}$ *and* $Ff_0 \in A^{-\infty}$. *Then* $Ff \in A^{-\infty}$ *whenever* $f \in A^{-\infty}$ *is such that* $Z_f \supseteq Z_{f_0}$, $\sigma_f \leqslant \sigma_{f_0}$.

Proof. First take up the case $Z_f = Z_{f_0}$. Using Proposition 4.1 we can represent f_0 and f in the form (4.1); then dividing f_0 by f we obtain

$$\frac{f_0(z)}{f(z)} = \lambda \exp\left\{\int_{\partial U} \frac{\zeta + z}{\zeta - z} \mu(|d\zeta|)\right\}, \tag{4.9}$$

where $\mu \in \varkappa V$ and $\mu_\sigma = \sigma_{f_0} - \sigma_f \geqslant 0$. Applying Corollary 3.1.1 we can find a sequence $\{g_\nu\}_1^\infty$, $g_\nu \in A^{-\infty}$, such that

$$\frac{f_0 g_\nu}{f} \in A^{-\infty}, \ \frac{f_0 g_\nu}{f} \to 1 \quad (\nu \to \infty)$$

in the topology of $A^{-\infty}$. Multiplying by f we get $f_0 g_\nu \to f$ $(\nu \to \infty)$. Since by the hypothesis $Ff_0 \in A^{-\infty}$, we find $(Ff_0)g_\nu = F(f_0 g_\nu) \in A^{-\infty}$ and therefore, using Proposition 4.2, $Ff = \lim_{\nu \to \infty} F(f_0 g_\nu) \in A^{-\infty}$.

If $Z_f \supset Z_{f_0}$ we can construct [6] a function $g \in A^{-\infty}$ such that $Z_g = Z_f \setminus Z_{f_0}$, $\sigma_g = 0$. Then $F(f_0 g) = (Ff_0) g \in A^{-\infty}$, $Z_{f_0 g} = Z_f$, $\sigma_{f_0 g} = \sigma_{f_0} \geqslant \sigma_f$, and the case $Z_f \supset Z_{f_0}$ is thus reduced to that already proved.

LEMMA 4.3. *Let (as in Proposition* 1.1) $F \in \mathfrak{J}$, σ_0 *be a non-negative Borel measure on* F *and* $\Phi(z)$ $(z \in \bar{U})$ *be an outer function belonging to* A^∞ *and vanishing on* F *together with all its derivatives. Define*

$$I(z) = \exp\left\{-\int_{\partial U} \frac{\zeta + z}{\varrho - z} \sigma_0(|d\zeta|)\right\} \quad (z \notin F), \tag{4.10}$$

$$\Psi(\zeta) = \begin{cases} \Phi(\zeta) I^{-1}(\zeta) & (\zeta = \partial U \setminus F) \\ 0 & (\zeta \in F), \end{cases} \tag{4.11}$$

and

$$\Psi_1(z) = \begin{cases} \Phi(z) I(z) & (z \in \bar{U} \setminus F) \\ 0 & (z \in F). \end{cases} \tag{4.12}$$

Then

(i) $\Psi \in C^{\infty}$, $\Psi_1 \in A^{\infty}$;

(ii) $\Psi\Psi_1 = \Phi^2 \in A^{\infty}$;

(iii) *an element* $f \in A^{-\infty}$ *has the property* $\Psi f \in A^{-\infty}$ *if and only if* $\sigma_f \leq -\sigma_0$.

Proof. (i) Since $\Phi^{(n)}(z) = O[d^N(z, F)]$ $(z \in \bar{U})$ for any $n \geq 0$, $N \geq 0$ and $I^{(n)}(z) = O[d^{-2n}(z, F)]$ $(z \in U \setminus F)$, $\Psi(\tau)$ and $\Psi_1(\tau)$ are infinitely differentiable on ∂U. Note that $\Psi(z) = \Phi(z) I^{-1}(z)$ $(z \in U)$ does not belong to $A^{-\infty}$ barring the trivial case $\sigma_0 \neq 0$.

(ii) Obvious.

(iii) Let $f \in A^{-\infty}$ and $\Psi f = g \in C^{-\infty}$. Multiplying by Ψ_1 we get $\Phi^2 f = g\Psi_1$. If $g \in A^{-\infty}$ then equating the $\varkappa$-singular measures on both sides of the equation we find $\sigma_f = \sigma_g - \sigma_0 \leq -\sigma_0$. Conversely, if $\sigma_f \leq -\sigma_0$ then applying Lemma 4.2 we infer from $\Psi\Psi_1 \in A^{-\infty}$ that $\Psi f \in A^{-\infty}$ because Ψ_1 has no zeros and its $\varkappa$-singular measure is $-\sigma_0$.

Incidentally, Lemma 4.3 proves the equivalence of Definition 1.3 and Definition 4.2, as well as Proposition 1.1.

We are now in a position to complete the proof of Theorem 1.1. First prove the second part of the Theorem. Let $\alpha = \{\alpha_\nu\}$ be an $A^{-\infty}$-zero set and let σ_0 be a non-positive $\varkappa$-singular measure. By Theorem 2.2 there is a sequence of B.-C. sets $F_1 \subseteq F_2 \subseteq \ldots$ and a sequence $\{\sigma_\nu\}_1^{\infty}$ of non-positive $\varkappa$-singular measures, σ_ν being in fact the part of σ_0 supported by F_ν, such that for any B.-C. set F $\sigma_0(F) = \lim_{\nu\to\infty} \sigma_\nu(F) = \lim_{\nu\to\infty} \sigma_0(F \cap F_\nu)$. Form as in Lemma 4.3 for all F_ν and the corresponding σ_ν the function

$$\Psi_\nu(\zeta) = \Phi_\nu(\zeta) \exp\left\{-\int_{\partial U} \frac{\zeta + z}{\zeta - z} \sigma_\nu(|d\zeta|)\right\}, \tag{4.13}$$

$\Phi_\nu(z)$ being an outer function of the class A^{∞} with the null set F_ν, and for every zero α_ν let $\pi_\nu(\tau) = (\zeta - \alpha_\nu)^{k_\nu}$, k_ν being the multiplicity of that zero. Then Lemma 4.3 shows that $I(\alpha, \sigma_0) = \{f \in A^{-\infty}: \Psi_\nu f \in A^{-\infty}, \pi_\nu f \in A^{-\infty} \quad \forall \nu \geq 1\}$.

Therefore $I(\alpha, \sigma_0)$ is a closed ideal in $A^{-\infty}$. Its non-triviality follows from the fact that (by Theorem 2.3) there is a premeasure $\mu \in \varkappa B^+$ with $\mu_\sigma = \sigma_0$, and by the results of [6] there is a function $f(z) \in A^{-\infty}$ with $Z_f = \alpha$, $\sigma_f = 0$; therefore

$$g(z) = f(z) \exp\left\{\int_{\partial U} \frac{\zeta + z}{\zeta - z} \mu(|d\zeta|)\right\}$$

meets both conditions $Z_g = \alpha$, $\sigma_g = \sigma_0$.

Take up now the first part of Theorem 1.1. Let $I \neq 0$ be a closed ideal in $A^{-\infty}$ and $\mathcal{A}_I$ be its annihilator. Let $F \neq 0$ be a fixed element of $\mathcal{A}_I$:

$$Ff = g_f \in A^{-\infty} \quad (\forall f \in I). \tag{4.14}$$

By Lemma 4.1 the function

$$h(z)=\frac{g_f(z)}{f(z)}$$

does not depend on the choice of $f\in I$. Since $h(z)$ belongs to the class $\mathcal{N}=A^{-\infty}/A^{-\infty}$ it possesses [6] a unique representation in the form

$$h(z)=\lambda\frac{\tilde{B}_{\alpha'}(z)}{\tilde{B}_{\beta'}(z)}\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu'(|d\zeta|)\right\},\tag{4.15}$$

where $\alpha'=\{\alpha'_\nu\}$ is the zero set and $\beta'=\{\beta'_\nu\}$ is the pole set of $h(z)$ and $\mu'\lambda\in\varkappa V$. Therefore $\beta'\subseteq Z_I$, $\mu'_\sigma\leqslant-\sigma_I$, i.e. each zero of f which is outside Z_I must also be a zero of g_f, and the part of σ_f which goes beyond σ_I must also be a part of σ_{g_f}. Fix now a $f_0\in I$ and assume for simplicity that $f_0(0)\neq 0$. Let $Z_{f_0}\setminus Z_I=\{z_\nu\}$, $\alpha_{f_0}-\sigma_I=\sigma'\leqslant 0$; let further σ' be concentrated on a set $S=\bigcup_\nu F_\nu$, $F_1\subseteq F_2\subseteq\ldots(F_\nu\in\mathcal{F})$ so that $\sigma'=\text{g.l.b.}\{\sigma_\nu\}$, where $\sigma_\nu(F)=\sigma'(F\cap F_\nu)$ and $0\geqslant\sigma_1\geqslant\sigma_2\geqslant\ldots\geqslant\sigma'$. Multiply (4.14) by $[(\zeta-z)(\zeta-z_2)\ldots(\zeta-z_\nu)]^{-1}\Psi_\nu(\zeta)$ where $\Psi_\nu\in C^\infty$ has the form (4.13) and apply Lemma 4.2; then we arrive at the conclusion that $Ff\in A^{-\infty}$ whenever

$$Z_f=Z_I\cup\{z_\nu\}_{n+1}^\infty,\quad \sigma_f=\sigma_I+(\sigma'-\sigma_n),\quad n=1,2,\ldots.$$

By use of Theorem 2.3, Corollary 2.3.1 and the technique developed in [6] for constructing function of the class $A^{-\infty}$ with given zero sets, we can form the following functions:

(a) $g\in A^{-\infty}$ such that $Z_g=Z_I$, $\sigma_g=\sigma_I$;

(b) $p_n\in A^{-\infty}$ such that $Z_{p_n}=\{z_\nu\}_{n+1}^\infty$, $\sigma_{p_n}=0$;

(c) $q_n\in A^{-\infty}$ such that $Z_{q_n}=\varnothing$, $\sigma_{q_n}=\sigma'-\sigma_n$

and ensure that $p_n\to 1$, $q_n\to 1$ in the topology of $A^{-\infty}$. We have for all $n\geqslant 1$

$$Fgp_nq_n\in A^{-\infty};$$

taking the limit when $n\to\infty$ and observing that $p_nq_n\to 1$ we obtain that $Fg\in A^{-\infty}$ and therefore by Lemma 4.2

$$Ff\in A^{-\infty}\quad(\forall f\colon Z_f\supseteq Z_I,\ \sigma_f\leqslant\sigma_I).\tag{4.16}$$

Since F is an arbitrary element of A_I this yields

$$\{f\in A^{-\infty}\colon Ff\in A^{-\infty}\ \forall F\in A_I\}\supseteq I(Z_I,\sigma_I).$$

Using (4.8) we find $I\supseteq I(Z_I,\sigma_I)$. On the other hand, if $f\in I$ then by the definition of Z_I and σ_I we have

$$Z_f\supseteq Z_I,\quad \sigma_f\leqslant\sigma_I$$

and therefore $f\in I(Z_I,\sigma_I)$. Thus

$$I=I(Z_I,\sigma_I)$$

and Theorem 1.1 has been proved.

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Received November 2, 1975.