ON THE COMPUTABILITY OF CONJUGATE POWERS IN FINITELY GENERATED FUCHSIAN GROUPS

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§ 1. Introduction

We define the set of *conjugate powers* of elements U and W in any group G, denoted $CP_G(U, W)$, by

$$CP_G(U, W) = \{(x, y) \in \mathbf{Z}^2;\ U^x \sim_G W^y\}$$

where "$\sim_G$" denotes the conjugacy relation in G. In this paper we show how these sets $CP_G(U, W)$ can be effectively computed for most finitely generated (henceforth f.g.) Fuchsian groups. The Fuchsian groups are the discrete subgroups of the group of all 2×2 real matrices with determinant $+1$.

By a result of Poincaré [11] (see also [8]), the class of f.g. Fuchsian groups consists of free products of cyclic groups, together with the groups

$$G = \langle a_1, b_1, \ldots, a_r, b_r, c_1, \ldots, c_s;\ c_1^{n_1}, \ldots, c_s^{n_s}, R\rangle$$

where R is the word $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_r b_r a_r^{-1} b_r^{-1} c_1 \ldots c_s$, $r, s \geqslant 0$, $n_i > 1$ for each $1 \leqslant i \leqslant s$, and

$$2r - 2 + \sum_{i=1}^{s} (1 - n_i^{-1}) > 0.$$

It therefore follows that the groups

$$G = \langle c_1, c_2;\ c_1^{n_1}, c_2^{n_2}, (c_1 c_2)^{n_3}\rangle \tag{1}$$

are Fuchsian when $n_1^{-1} + n_2^{-1} + n_3^{-1} < 1$. The methods used in this paper do not apply to these groups. They do, however, apply to a more general class than the remaining f.g. Fuchsian groups. Let us denote this new class by Γ_1 and indicate how it is constructed.

To construct Γ_1 we start with a class Γ_0 consisting of free groups and certain tree-products of one-relator groups with torsion. We then get Γ_1 by forming all free products of groups from Γ_0 with a cyclic subgroup amalgamated. Thus, $G \in \Gamma_1$ if and only if

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$$G = G_1 *_C G_2$$

for some $G_1, G_2 \in \Gamma_0$ and (possibly trivial) cyclic subgroup C of G_1 and G_2. We include the trivial group in Γ_0, hence, $\Gamma_0 \subseteq \Gamma_1$. The precise definition of Γ_0 is given in Section 2, and from this it follows that all f.g. Fuchsian groups, except those given by (1), belong to Γ_1.

For the present and later use, let

$$(r, s) + (a, b)\mathbf{Z} = \{(r + ax, s + bx) \in \mathbf{Z}^2;\ x \in \mathbf{Z}\}$$

for any integers r, s, a, and b. If $r, s = 0$, then we write just $(a, b)\mathbf{Z}$ for this set. Also, for any subsets Φ and Φ' of $\mathbf{Z}^2$ we let $\Phi + \Phi'$ be the obvious set of sums.

As our first result we have

THEOREM A. *We can effectively compute the order of elements in any G from Γ_1.*

For the next result, let $|U|$ denote the order of the element U in G.

THEOREM B. *Given U, $W \in G \in \Gamma_1$, we can effectively compute integers a, b, and c such that*

$$CP_G(U, W) = (a, b)\mathbf{Z} \cup (ac, -bc)\mathbf{Z}$$

if $|U|, |W| = \infty$;

$$CP_G(U, W) = (a, b)\mathbf{Z} + (|U|\mathbf{Z}) \times (|W|\mathbf{Z})$$

if $|U|, |W| < \infty$.

Note that the sets $CP_G(U, W)$ are easily described if $|U| < |W| = \infty$, or vice versa.

We give the proof of Theorem B separately for Γ_0 and Γ_1. In the first of these we also show that $0 \leqslant c \leqslant 1$, while in the second $0 \leqslant c \leqslant 2$ or $0 \leqslant c \leqslant |C|$ according as $G = G_1 *_C G_2$ with C infinite or finite.

The following is immediate from Theorems A and B.

COROLLARY C. *All groups in Γ_1 have solvable conjugacy and power-conjugacy problems.*

In Section 7 we indicate how these results can be generalized by iterating the process of forming free products with a cyclic subgroup amalgamated. The class Γ obtained from Γ_0 through this process generalizes a class studied by the author in [5]. From the results in this paper, together with those in [5], it also follows that the HNN groups

$$\langle G_0, p;\ \mathrm{rel}\ G_0,\ pS_1p^{-1} = S_2 \rangle$$

have solvable conjugacy and power-conjugacy problems, where $S_1, S_2 \in G_0 \in \Gamma$, $|S_1| = |S_2|$, and $p \notin G_0$.

This work is the result of considering a question of M. Anshel on the conjugacy problem for free products with cyclic amalgamations of one-relator groups with torsion. M. Anshel and P. Stebe [1] obtained a partial solution of this problem using different techniques.

§ 2. Some basic definitions

We state our definitions with respect to a fixed alphabet $\{a_1, a_2, \dots\}$, but want them to carry over to any alphabet. Thus, let us call U a *word on* $\{a_1, a_2, \dots\}$ if U is a word on $\{a_1, a_2, \dots\} \cup \{a_1^{-1}, a_2^{-1}, \dots\}$ in the usual sence. We use upper case Roman letters in the range $P, \dots, W$, or variations of these such as P', P_i, etc., strictly to denote freely reduced words. If U is such a word, then $l(U)$ denotes its length. As a special symbol, we also use Λ for the empty word. If a_i or a_i^{-1} occurs in U, then we say that U *involves* a_i; and to display the letters in U, we use the notation

$$\text{gen}\,(U) = \{a_i;\ U \text{ involves } a_i\}.$$

Let us call any non-trivial cyclically reduced word *simple* if it is not a proper power. Hence, we can define Γ_0 as the class of all groups

$$G = \langle a_1, a_2, \dots;\ R_1^{n_1}, \dots, R_k^{n_k} \rangle \tag{2}$$

with $k \geqslant 0$ ($k=0$ means G has no relators), where each $n_i > 1$, each R_i is simple, and

$$\text{gen}\,(R_i) \cap \text{gen}\,(R_{i'}) \nsubseteq \text{gen}\,(R_j) \tag{3}$$

for all $i \leqslant j \leqslant i'$ with $i \neq i'$. When $k > 0$, we call $R_1, \dots, R_k$ the *roots* of G, which is meaningful as long as we work with specific presentations for the groups in Γ_0. Recall that we also include the trivial group in Γ_0.

With this definition of Γ_0 it is easily verified that all f.g. Fuchsian groups, except those given by (1), must belong to Γ_1. Recall that Γ_1 consists of all free products of groups from Γ_0 with a cyclic subgroup amalgamated.

Requiring all groups G to be given by specific presentations, we may denote the set of generators in the presentation of G by gen (G). Elements of G can then be represented by freely reduced words on gen (G). We also include 1 as a special symbol for the identity in any group. For any U, $W \in G$, $U = W$ means U and W define the same element in (the abstract group) G, while $U \equiv W$ means they are identical as words.

Let us examine in some detail the groups (2). Those with just one relator form the class of one-relator groups with torsion, a subclass we denote by Γ_*. Most of the problems we need to consider for Γ_0 can be reduced to problems concerning Γ_*.

Suppose now that $G \in \Gamma_0$ is given by (2) with $k \geqslant 2$. For each $2 \leqslant i \leqslant k$ let $G_i \in \Gamma_*$ be the group

$$G_i = \langle \text{gen}\,(R_i);\ R_i^{n_i} \rangle;$$

and similarly, let $G_1' \in \Gamma_*$ be the group

$$G_1' = \langle \text{gen}\,(R_1);\ R_1^{n_1} \rangle.$$

If we then let $G_1 \in \Gamma_*$ be obtained from G_1' by adding the remaining generators of G, we can write G as the tree-product (see A. Karrass and D. Solitar [4])

$$G = G_1 *_{F_1} \dots *_{F_{k-1}} G_k \tag{4}$$

where each F_i is given by

$$F_i = G_i \cap G_{i+1}.$$

From the Freiheitssatz and the condition (3) imposed on the roots R_j, it follows that each

$$F_i = \langle \text{gen}\,(G_i) \cap \text{gen}\,(G_{i+1}); \rangle$$

as a free group.

Note that when $k \geqslant 2$ we can also write G in the form

$$G = G' *_F G_k \tag{5}$$

where $F = F_{k-1}$ and G' is the subgroup generated by $\bigcup_{i=1}^{k-1} \text{gen}\,(G_i)$. Of course, G' belongs to Γ_0 and has $k-1$ relators. Moreover, if $k \geqslant 3$, then G' can also be written as a tree-product (4) of length $k-1$. We utilize these facts to prove results about the class Γ_0 in Section 6.

Before pursuing the various problems in Γ_0, let us consider these for the subclass Γ_*.

§ 3. Conjugate powers in one-relator groups with torsion

Any group $G \in \Gamma_*$ with root R of length $\geqslant 2$ can be presented in the form

$$G = \langle t, a_0, b_0, \dots;\ R^n \rangle \tag{6}$$

where $t, a_0 \in \text{gen}\,(R)$ and R begins with $a_0^{\pm 1}$. If R has exponent sum zero on t, then G can also be realized as an HNN group

$$G = \langle H, t;\ \tilde{R}^n,\ tSt^{-1} = \theta(S)\,(\forall S \in X) \rangle \tag{7}$$

where H belongs to Γ_* and has the root $\tilde{R}$ with $l(\tilde{R}) < l(R)$.

We only sketch here how (7) is obtained from (6), referring the reader to the paper [7] by J. McCool and P. E. Schupp for the details. The first step is to set for each integer i, $a_i = t^i a_0 t^{-i}$, $b_i = t^i b_0 t^{-i}$, etc., and then rewrite R as a word $\tilde{R}$ on these new generators.

If ν is minimal and μ maximal among the subscripts of a-symbols involved in $\tilde{R}$, then let H have generators

$$\text{gen}\,(H) = \{a_\nu, \ldots, a_\mu\} \cup \{b_i;\, i \in \mathbf{Z}\} \cup \ldots$$

and relator $\tilde{R}^n$. By the Freiheitssatz, the subgroups X and Y generated by gen $(H) - \{a_\mu\}$ and gen $(H) - \{a_\nu\}$, respectively, are isomorphic under $\theta: X \to Y$ induced by $a_i \to a_{i+1}$, $b_i \to b_{i+1}$, etc.

Suppose now that G is given by (7). If S is any word on gen (H), let $S^{(x)}$ denote the word obtained from S by shifting each subscript by x. If also gen $(S^{(x)}) \subseteq$ gen (H), then $S^{(x)} = t^x S t^{-x}$.

Let any word S in a group G be called *free* in G if gen (S) generates the free group $\langle$gen $(S);\,\rangle$ in G. By the Freiheitssatz, if $G \in \Gamma_*$ has the root R, then S is free in G if and only if gen $(R) \nsubseteq$ gen (S). From a result of B. B. Newman [10] *it follows that if* S *is free* in G, then gen (S) generates a malnormal subgroup of G. Recall that H is malnormal in G if for any U, $W \in H$, $U = VWV^{-1} \neq 1$ implies $V \in H$.

From B. B. Newman's Lemma 2.1 in [10], we deduce

LEMMA 3.1. *Let $U \in G \in \Gamma_*$ be $\neq 1$. Then, up to cyclic permutation, there exists at most one cyclically reduced free word S in G with $U \sim_G S$. Moreover, if S is such a word, then $U = W^x$ implies $S \equiv S_0^x$ with $W \sim_G S_0$.*

Proof. Suppose that S_1 and S_2 both satisfy the lemma. If R is the root of G, then gen$(R) \nsubseteq$ gen (S_i) for $i = 1, 2$. Thus, for some a, $b \in$ gen (R), $a \notin$ gen (S_1) and $b \notin$ gen (S_2). If gen $(R) \subseteq$ gen $(S_1) \cup$ gen (S_2), then $b \in$ gen (S_1) and $a \in$ gen (S_2). Since $S_1 \sim_G S_2$, this violates Lemma 2.1 in [10], hence, gen $(R) \nsubseteq$ gen $(S_1) \cup$ gen (S_2). But then S_1 and S_2 belong to the malnormal subgroup F of G generated by gen $(S_1) \cup$ gen (S_2), and therefore $S_1 \sim_F S_2$. This proves the first half.

To complete the proof, suppose that S is free in G with $U = W^x = VSV^{-1}$. If we set $W_0 = V^{-1}WV$, then $W_0^x = S$, and therefore $W_0 S W_0^{-1} = S \neq 1$. If F is the malnormal subgroup of G generated by gen (S), then $W_0 \in F$ and the result follows. □

The proof of the next lemma is given in Section 5 where the necessary techniques are developed.

LEMMA 3.2. *If $U \in G \in \Gamma_*$, then we can effectively decide if there exists any free word S in G conjugate to U. Moreover, we can effectively compute such an S, provided any exists.*

Let us establish some terminology concerning elements of the HNN groups (7). Words on gen (G) without any t's are called *t-free*. If U involves t, then U can be written in the form

$$U \equiv U_0 t^{\varepsilon_1} U_1 \dots t^{\varepsilon_k} U_k$$

with each U_i t-free. (Lower case Greek letters denote ± 1.) The number of t's occurring in U is called the *t-length* of U, denoted $l_t(U)$. If U contains no subword $t^{\varepsilon_i} U_i t^{-\varepsilon_i}$ with $\varepsilon_i = 1$ and $U_i \in X$ or $\varepsilon_i = -1$ with $U_i \in Y$, then U is called *t-reduced*. If all cyclic premutations of U are t-reduced, and U is either t-free or begins with $t^{\pm 1}$, then U is called *cyclically t-reduced*. Let us also call U and W *t-parallel* if $l_t(U) = l_t(W) = k$, and they contain identical k-tuples of $t^{\pm 1}$.

To study the sets $CP_G(U, W)$ in Γ_* we need

LEMMA 3.3. *Let G be presented by* (7). *If U, $W \in H$, then* $U \sim_G W$ implies

$$U = V_0 t^x V_1 W V_1^{-1} t^{-x} V_0^{-1}$$

for some t-free V_0 and V_1.

Proof. By Britton's Lemma [2], if V is t-reduced and involves t, then $U = VWV^{-1}$ implies $U \sim_H S_1$ and $W \sim_H S_2$ for some cyclically reduced free words S_1 and S_2 in H. It thus suffices to prove the lemma for $U \equiv S_1$ and $W \equiv S_2$. Now, if $V \equiv V_0 t^{\varepsilon_1} V_1 \dots t^{\varepsilon_k} V_k$, then we may assume the words $t^{\varepsilon_i} V_i t^{-\varepsilon_i}$ to be t-reduced for each $1 \leqslant i \leqslant k$ with $V_i \not\equiv \Lambda$. It remains to show that $V_i \equiv \Lambda$ for each $1 \leqslant i \leqslant k$. To this end, let i be maximal with $V_i \not\equiv \Lambda$. But then

$$t^{\varepsilon_i} V_i S_2^{(\varepsilon_k (k-i))} V_i^{-1} t^{-\varepsilon_i} = t^{\varepsilon_i} T t^{-\varepsilon_i} = T^{(\varepsilon_i)}$$

for some free word T. Lemma 3.1 now implies that $T \equiv P \bar{T} P^{-1}$ with $\bar{T}$ a cyclic premutation of $S_2^{(\varepsilon_k(k-i))}$. By malnormality of the subgroup generated by gen (T), we easily see that $t^{\varepsilon_i} V_i t^{-\varepsilon_i}$ cannot be t-reduced, a contradiction. □

LEMMA 3.4. *Let $U \in G \in \Gamma_*$ be $\neq 1$, and suppose that $(x, y) \in CP_G(U, U)$.* Then

(i) $|U| = \infty$ *implies* $|x| = |y|$;

(ii) $|U| < \infty$ *implies* $U^x = U^y$.

Proof. We use induction on the length of the root R of G. If this length is 1, then G is a free product of a finite cyclic group and a free group. Both (i) and (ii) are easily established in this case.

Suppose now that $l(R) \geqslant 2$ with the result established for all $G' \in \Gamma_*$ having roots R' satisfying $l(R') < l(R)$. Assume first that G can be realized as the HNN group (7). If U is cyclically t-reduced and involves t, then $|U| = \infty$ and the result follows from Collins' Lemma (p. 123 in [3]). If $U \in H$, then Lemmas 3.1 and 3.3 imply $U^x \sim_G U^y$ only if $U^x \sim_H U^y$. Since $l(\tilde{R}) < l(R)$, it remains to consider the case where no generator in gen (R) has exponent sum zero in R. But J. McCool and P. E. Schupp showed in [7] that G can then be imbedded in an HNN group of the type (7) with the root $\tilde{R}$ of H satisfying $l(\tilde{R})<l(R)$. The case just considered can therefore be applied. $\square$

Note that $U\equiv abab^{-1}$ is of infinite order in $G=\langle a, b;\, a^2\rangle\in\Gamma_*$ and satisfies $U\sim_G U^{-1}$.

B. B. Newman proved in [10] that the centralizer of any nontrivial $U\in G\in\Gamma_*$ is cyclic. Hence, if we denote the centralizer of U in G by $C_G(U)$, then for any $U\in G\in\Gamma_*\neq 1$, $C_G(U)=\langle T\rangle$ for some T. Here $\langle T\rangle$ denotes the subgroup generated by T.

Pending a proof of Lemma 3.2, we can now establish

PROPOSITION 3.5. *Let U, $W\in G\in\Gamma_*$ be of infinite order. Then we can effectively compute integers a, b, and c with $0\leqslant c\leqslant 1$ such that*

$$CP_G(U, W)=(a, b)\mathbf{Z}\cup(ac, -bc)\mathbf{Z}.$$

Moreover, a and b are relatively prime if $a\neq 0$.

Proof. Note that we can determine the order of elements of G. Let us show first that $CP_G(U, W)$ has the asserted form. Because of Lemma 3.4 it suffices to show: If $(xz, yz)\in CP_G(U, W)$ for some x, y, $z\in\mathbf{Z}$ with $z\neq 0$, then $(x, y)\in CP_G(U, W)$. This is trivial if $x=0$. If $x\neq 0$, then

$$U^{xz}=VW^{yz}V^{-1}=(VWV^{-1})^{yz}$$

for some V. Let $C_G(U^{xz})=\langle T\rangle$, and note that U and VWV^{-1} must therefore belong to $\langle T\rangle$. Now, if $U=T^p$ and $VWV^{-1}=T^q$, then $T^{pxz}=T^{qyz}$. Since $|T|=\infty$, we must have

$$U^x=T^{px}=T^{qy}=(VWV^{-1})^y=VW^yV^{-1}.$$

By a result of B. B. Newman [9], G has solvable conjugacy problem. Hence, it suffices to determine a', $b'>0$ such that a, $b\neq 0$ implies $|a|=a'$ and $|b|=b'$. For this, let us proceed by induction on the length of the root R of G. The case with $l(R)=1$ is trivial, so suppose that $l(R)\geqslant 2$ with the result established for all $G'\in\Gamma_*$ having roots R' with $l(R')<l(R)$.

Suppose first that G can be realized as the HNN group (7). J. McCool and P. E. Schupp [7] proved that H has solvable generalized word problem with respect to X and Y, hence, we can effectively cyclically t-reduce words in G. Suppose therefore that U and W are cyclically t-reduced. If only one of them is t-free, let a', $b'=0$. If both U and W involve t, let a', $b'>0$ be minimal such that $l_t(U^{a'})=l_t(W^{b'})$. If U, $W\in H$, we use the inductive hypothesis and compute $CP_H(U, W)$. From Lemmas 3.1 and 3.3 it follows that $CP_H(U, W)\neq\{(0, 0)\}$ implies $CP_H(U, W)=CP_G(U, W)$. Suppose therefore that $CP_H(U, W)=\{(0, 0)\}$, and consider $U^x\sim_G W^y$. If x, $y\neq 0$, then $U\sim_H S_1$ and $W\sim_H S_2$ for some cyclically reduced free words S_1 and S_2 from X or Y (see Lemma 3.1). But then $S_1^x\sim_G S_2^y$, and so by Lemma 3.3, S_1^x must be a cyclic permutation of $(S_1^{(z)})^y$ for some z. By Lemma 3.2 we can compute S_1 and S_2 and therefore also determine z. It is now elementary to compute a' and b'.

Finally, suppose that G cannot be realized as the HNN group (7). By the remark at the end of the last proof, G can be imbedded in such an HNN group with the inductive hypothesis applying to the base group. Since this imbedding is clearly effective, and we only need to compute the integers a' and b', the above case applies. □

The sets $CP_G(U, W)$ with $|U|, |W| < \infty$ are considered in Section 6.

§ 4. Some more definitions

In this paper we use two constructions of generalized free products

$$G = G_1 *_H G_2$$

of groups G_1 and G_2 from Γ_0. In the first of these we amalgamate a cyclic subgroup H and get G in Γ_1; in the second (see (5) in Section 2) the amalgamated subgroup H is free on gen $(G_1) \cap$ gen (G_2) and G belongs to Γ_0.

For both of the above constructions we choose the natural presentation for G, hence,

$$\text{gen}(G) = \text{gen}(G_1) \cup \text{gen}(G_2).$$

Now, any word U on gen (G) can be written in the (not necessarily unique) form

$$U \equiv U_1 \dots U_k \tag{8}$$

where gen (U_i) is contained in gen (G_1) or gen (G_2) for each i. If $U \not\equiv \Lambda$ and gen $(U_i U_{i+1})$ is not contained in gen (G_1) or gen (G_2) for any $1 \leqslant i < k$, then we call each U_i a *syllable of* U. The number of syllables in U is called the *s-length* of U, denoted $l_s(U)$.

Words $U \in G$ are called *s-reduced* if either $l_s(U) \leqslant 1$ or no syllable of U belongs to H; that is, for no decomposition (8) of U into syllables $U'_1 \dots U'_k$ does $U'_i \in H$ for any $1 \leqslant i \leqslant k$. If also all cyclic syllable-permutations (henceforth s-permutations) of U are s-reduced, then we call U *cyclically s-reduced.*

Both for generalized free products G in Γ_0 and Γ_1 we need to determine for given cyclically s-reduced words whether or not these are conjugate in G. The main tool to deal with such problems is Solitar's Theorem (Thm. 4.6 in [6]). In part, this theorem asserts for cyclically s-reduced words U and W of the same *s-length* $\geqslant 2$, that $U \sim_G W$ if and only if $U = SW_\pi S^{-1}$ for some cyclic s-permutation W_π of W and $S \in H$. If we write $U \equiv U_1 \dots U_k$ and $W_\pi \equiv W_1 \dots W_k$ in terms of syllables, and examine the identity

$$U_1 \dots U_k S W_k^{-1} \dots W_1^{-1} = S,$$

then we note that $U_k S W_k^{-1} = S_1 \in H$, $U_{k-1} S_1 W_{k-1}^{-1} = S_2 \in H$, etc. Thus, we are led to consider the sets

$$\{(S, S') \in H \times H;\ U_i S W_i^{-1} = S'\}$$

in groups from Γ_0. However, for some of our applications we need to consider a more general situation.

Let G be a given group with subgroups H and K. For any $U, W \in G$, consider the following subset of the direct product $H \times K$:

$$\text{gph}\,(U, W; H, K) = \{(S, T) \in H \times K;\ USW = T (\text{in } G)\}.$$

This is the graph of the function $G \to G$, given by $V \to UVW$, restricted to the (possibly empty) subset of H mapped into K.

For any subgroup N of $H \times K$ and element $(S_0, T_0) \in H \times K$, let $N(S_0, T_0)$ and $(S_0, T_0)N$ denote the right and left translates of N by (S_0, T_0). For the next lemma, note that gph $(U, W; H, K)$ is a subgroup of $H \times K$ if and only if $W = U^{-1}$.

LEMMA 4.1. *If* $(S_0, T_0) \in$ gph $(U, W; H, K)$, *then*

$$\text{gph}\,(U, W; H, K) = [\text{gph}\,(U, U^{-1}; H, K)](S_0, T_0) = (S_0, T_0)[\text{gph}\,(W^{-1}, W; H, K)].$$

Proof. If $US_0W = T_0$ then $USW = T$ if and only if $USW(US_0W)^{-1} = USS_0^{-1}U^{-1} = TT_0^{-1}$. Hence $(S, T) \in$ gph $(U, W; H, K)$ if and only if $(SS_0^{-1}, TT_0^{-1}) \in$ gph $(U, U^{-1}; H, K)$. The other half of the proof is similar. □

Much of the remaining work in this paper concerns the sets gph $(U, W; H, K)$ for free and cyclic subgroups H and K of groups in the subclass Γ_* of Γ_0. In the special case when $H = \langle S \rangle$ and $K = \langle T \rangle$ as infinite cyclic subgroups of G, then we identify H and K with $\mathbf{Z}$, and set

$$\text{gph}\,(U, W; H, K) = \{(x, y) \in \mathbf{Z}^2;\ US^x W = T^y\}.$$

If this set is non-empty, then Lemma 4.1 implies that

$$\text{gph}\,(U, W; H, K) = (r, s) + (a, b)\mathbf{Z}$$

for any r, s with $US^rW = T^s$, provided $(a, b)\mathbf{Z} =$ gph $(U, U^{-1}; H, K)$. In the next two sections we show how we can effectively compute such integers r, s, a, and b.

Let us also consider the graphs for free subgroups H and K of $G \in \Gamma_*$ generated by subsets of gen (G).

LEMMA 4.2. *Let H and K be free subgroups of $G \in \Gamma_*$ generated by subsets of* gen (G), *and set $N = H \cap K$. Then, if* gph $(U, U^{-1}; H, K)$ *is non-trivial, there exists at least one pair $(P, Q) \in H \times K$ with $U = QP^{-1}$ and*

$$\text{gph}\,(U, U^{-1}; H, K) = (P, Q)[\text{gph}\,(1, 1; N, N)](P, Q)^{-1}.$$

Proof. Suppose first that $U = QP^{-1}$, and note that in this case, $(S,T) \in \text{gph}\,(U, U^{-1}; H, K)$ if and only if $P^{-1}SP = Q^{-1}TQ = S' \in N = H \cap K$. Moreover, if $USU^{-1} = T \neq 1$ for some $(S, T) \in H \times K$, then Lemma 3.1 implies that $S \equiv S_1 S' S_1^{-1}$ and $T \equiv T_1 T' T_1^{-1}$ with S', $T' \in N$. But then, since N is malnormal in G and $(T_1^{-1} U S_1) S' (T_1^{-1} U S_1)^{-1} = T' \neq 1$, we must have $T_1^{-1} U S_1 = S_0 \in N$. This in turn shows that $U = QP^{-1}$ for $Q = T_1 S_0 \in K$ and $P = S_1 \in H$. □

In view of Lemmas 4.1 and 4.2, we may say that gph $(U, W; H, K)$ has been *computed* whenever elements (S_0, T_0), $(P, Q) \in H \times K$ have been effectively determined for which gph $(U, W; H, K) \neq \emptyset$ if and only if $US_0 W = T_0$, and gph $(U, U^{-1}; H, K)$ is non-trivial if and only if $U = QP^{-1}$ and $H \cap K \neq (1)$. We assume here that H, K, and G satisfy the hypotheses of Lemma 4.2. The problems involved in actually computing these sets are considered in Section 5.

Conjugacy between elements of $G_1 *_H G_2$ belonging to the factors G_1 and G_2, will be considered in Section 6.

§ 5. Graphs in one-relator groups with torsion

Consider the graphs gph $(U, W; C_1, C_2)$ for cyclic subgroups $C_1 = \langle S \rangle$ and $C_2 = \langle T \rangle$ of $G \in \Gamma_*$. To obtain results about such sets we proceed by induction on the length of the root of G. The HNN construction (7) allows us to apply the inductive hypothesis, but this construction also introduces new problems. To illustrate this, suppose S and T to be t-free while $U \equiv U_0 t^{e_1} U_1 \dots t^{e_k} U_k$ and $W^{-1} \equiv W_0^{-1} t^{e_1} W_1^{-1} \dots t^{e_k} W_k^{-1}$ with $k \geqslant 2$. If now U and W are t-reduced and satisfy $US^x W = T^y$, that is, if

$$U_0 t^{e_1} U_1 \dots t^{e_k} U_k S^x W_k t^{-e_k} \dots W_1 t^{-e_1} W_0 = Y^y,$$

then $U_k S^x W_k = S_1$ in X or Y, $U_{k-1} S_1^{(e_k)} W_{k-1} = S_2$ in X or Y, etc. Thus, we need to consider graphs gph $(U, W; H, K)$ in groups $G \in \Gamma_*$ for the following combinations: H and K cyclic, H cyclic and K free on a subset of gen (G) (or vice versa), and finally, both H and K free on subsets of gen (G).

For the remainder of this section, let us use the following convention: All subgroups denoted by F, F', F_1, etc. of given groups $G \in \Gamma_*$ are assumed to be freely generated by recursive subsets of gen (G). Cyclic subgroups are denoted by C, C_1, etc.

Before turning to the various problems involved in computing the relevant graphs, we need a definition and a lemma. But first, let us recall the "Spelling Theorem" of B. B. Newman [9] concerning groups $G \in \Gamma_*$ with relator R^n (R simple). This theorem asserts for any U, $S \in G$ with S free in G and U not free, that $U = S$ implies $U \equiv U_1 U' U_2$ with $U'V$ a cyclic permutation of $R^{\pm n}$ for some V with $l(V) < l(R)$. Let us therefore call the process of replacing U' by V^{-1} an *R-reduction* of U where we assume U' to be maximal so that $U_1 V^{-1} U_2$ is freely reduced. Further R-reductions of $U_1 V^{-1} U_2$ are also called R-reductions of U, etc.

LEMMA 5.1. *Let S and T be free in $G \in \Gamma_*$ where G has the relator R^n. Then at most one R-reduction is possible in ST (none if $n > 2$).*

Proof. The remark about n is obvious, so suppose that $n = 2$. If an R-reduction is possible in ST, then $S \equiv S_1 S'$, $T \equiv T' T_1$, and $S'T'V \equiv R_*^{2\varepsilon}$ for some cyclic permutation R_* of R where $l(V) < l(R)$. But then $\operatorname{gen}(R) = \operatorname{gen}(S') \cup \operatorname{gen}(T')$, and therefore $R_\varepsilon^* \equiv UV$ with $V \equiv V_1 V_2$, $S' \equiv UV_1$, and $T' \equiv V_2 U$. Moreover, for some minimal subwords W_1 of V_1 and W_2 of V_2, we must have $\varnothing \neq \operatorname{gen}(V_i) - \operatorname{gen}(V_j) \subseteq \operatorname{gen}(W_i)$ for $1 \leqslant i \neq j \leqslant 2$. Clearly, if $W_i W$ and $W_i W'$ are cyclic permutations of R^τ and $R^{\tau'}$ respectively, then $\tau = \tau' = \varepsilon$ and $W \equiv W'$. This follows since $\operatorname{gen}(W_i) \subseteq \operatorname{gen}(W_j)$ only if $i = j$. Suppose now that $S_1 V_2^{-1} V_1^{-1} T_1$ is the result of an R-reduction. Any new R-reduction of this word must involve all of W_1^{-1} or W_2^{-1}, hence, must be with respect to a cyclic permutation of $R^{-2\varepsilon}$. By the uniqueness of W_1 and W_2, we then get a contradiction to the necessary fact that $S_1 U V_1 V_2 U T_1$ was freely reduced. □

Most of the combinatorial difficulties involved in computing the sets $\operatorname{gph}(U, W; H, K)$ in $G \in \Gamma_*$ for $H = F_1$ or C_1 and $K = F_2$ or C_2, are handled by the next three lemmas.

LEMMA 5.2. *Let $G \in \Gamma_*$ have the root R with exponent sum zero on t where R involves t. Then, if we can effectively compute the sets* $\operatorname{gph}(U', W'; F_1', F_2')$ *in any $G' \in \Gamma_*$ having root R' with $l(R') < l(R)$, we can also effectively compute the sets* $\operatorname{gph}(U, W; F_1, F_2)$ *in G.*

Proof. We must show that given any $U, W \in G$, we can effectively determine at least one pair $(S, T) \in F_1 \times F_2$ such that $\operatorname{gph}(U, W; F_1, F_2) \neq \varnothing$ if and only if $USW = T$.

Let G be realized as the HNN group (7), and note that the hypothesis of the lemma applies to H.

Case 1. $t \notin F_1 \cup F_2$. It follows that $\operatorname{gen}(F_i) \subseteq \operatorname{gen}(H)$ for $i = 1, 2$. We may restrict ourselves to t-reduced words U and W with U and W^{-1} t-parallel. Consider $U \equiv U_0 t^{\varepsilon_1} U_1 \dots t^{\varepsilon_k} U_k$ and $W^{-1} \equiv W_0^{-1} t^{\varepsilon_1} W_1^{-1} \dots t^{\varepsilon_k} W_k^{-1}$ with $k > 0$, and suppose that $(S, T) \in F_1 \times F_2$ satisfies $USW = T$. It then follows from

$$U_0 t^{\varepsilon_1} U_1 \dots t^{\varepsilon_k} U_k S W_k t^{-\varepsilon_k} \dots W_1 t^{-\varepsilon} W_0 = T \tag{9}$$

that $U_k S W_k = T' \in F_2'$, where $F_2' = X$ if $\varepsilon_k = 1$, and $F_2' = Y$ if $\varepsilon_k = -1$. By assumption, we can effectively compute $\operatorname{gph}(U_k, W_k; F_1, F_2')$ in H. If this set is finite, then S is uniquely determined, so suppose that

$$\operatorname{gph}(U_k, W_k; F_1\ F_2') = (P, Q)[\operatorname{gph}(1, 1; F, F)](P^{-1}S_0, Q^{-1}T_0)$$

as an infinite set, where $U_k = QP^{-1}$ and $F = F_1 \cap F_2'$ as guaranteed by Lemma 4.2. Since we now have $U_k S W_k = T' = Q\bar{T}Q^{-1}T_0$ for some $\bar{T} \in F$, let us replace $t^{\varepsilon_k} U_k S W_k t^{-\varepsilon_k}$ by $Q^{(\varepsilon_k)}\bar{T}^{(\varepsilon_k)}(Q^{(\varepsilon_k)})^{-1}T_0^{(\varepsilon_k)}$ in (9). This produces a new equation $U'S'W' = T$ where U' and $(W')^{-1}$ are t-parallel, $S' \equiv \bar{T}^{(\varepsilon_k)} \in F_1' = t^{\varepsilon_k} F t^{-\varepsilon_k}$ and $l_t(U') < l_t(U)$.

The above discussion shows how we can compute the sets gph $(U, W; F_1, F_2)$ in G for any free subgroups F_1 and F_2 of H with gen (F_i) recursive subsets of gen (H) for $i = 1, 2$, using induction of $l_t(U)$.

Case 2. $t \in F_1$, $t \notin F_2$. (The case with $t \notin F_1$, $t \in F_2$ is similar.) By relabelling the generators if necessary, we may assume that $a_0 \notin F_1$. The elements of F_1 can then be written as t-reduced words St^x with $S \in F_1' = F_1 \cap H$. Since $USt^xW = T \in F_2$ implies that $|x| \leqslant l_t(U) + l_t(W)$, it suffices to consider gph $(U, t^xW; F_1', F_2)$ for each such x, using Case 1.

Case 3. $t \in F_1 \cap F_2$. Let $F_i' = F_i \cap H$ for $i = 1, 2$, and note that we can effectively compute gph $(V, 1; F_1', F)$ and gph $(1, V; F_2', F)$ in H for any $V \in H$ and $F = X, Y$. But then, if U and W are t-reduced, we may assume for any $(S, T) \in F_1 \times F_2$ that $T^{-1}US$ is t-reduced if $U \notin H$, and SWT^{-1} is t-reduced if $W \notin H$. This in turn implies $T^{-1}USW \neq 1$ or $WT^{-1}US \neq 1$ if $T^{-1}US$ is t-reduced and S and T are not both t-free. A similar statement holds when SWT^{-1} is t-reduced. Since we can decide by Case 1 if $USW = T$ is possible for any $(S, T) \in F_1' \times F_2'$, we need only consider U, $W \in H$.

Suppose now that U, $W \in H$. As in Case 2, we may assume that $a_0 \notin F_1$ and write elements of F_1 as t-reduced words St^x with $S \in F_1'$. Elements of F_2 can be written as t-reduced words $T_0 t^{\tau_1} T_1 \ldots t^{\tau_r} T_r$ with $T_i \in F_2'$ for each i. If now

$$USt^xW = T_0 t^{\tau_1} T_1 \ldots t^{\tau_r} T_r,$$

then $|x| = r$ and $x = \tau_i |x|$ for each i. By symmetry, it suffices to treat the case with $x = r > 0$ and thus each $\tau_i = 1$. Note that we must have $WT' = P \in X$ for some $T' \in F_2'$. Since gph $(W, 1; F_2', X)$ can be effectively computed in H, we may as well assume that $W \equiv P \not\equiv \Lambda$. Moreover, let PT' be freely reduced for all $T' \in F_2'$. The words $T_0 t T_1 \ldots t T_r$ may be written such that each T_i is either empty or begins with $a_\mu^{\pm 1}$ for $1 \leqslant i \leqslant r$. Suppose now that $T_q \not\equiv \Lambda$ and $T_i \equiv \Lambda$ for each $i > q$ in the above. We then get

$$USt^q P^{(r-q)} = T_0 t T_1 \ldots t T_q. \tag{10}$$

If $q \geqslant 1$, then we must have $P^{(r-q)} T_q^{-1} = P_q \in X$. From the restrictions above (on PT'), it follows that P_q must result after one $\tilde{R}$-reduction of $P^{(r-q)} T_q^{-1}$. By the proof of Lemma 5.1, $P^{(r-q)}$ must contain a certain unique subword W_1 which then uniquely determines $r - q$. Due to the shifting of subscripts in $P_q^{(1)}$, no $\tilde{R}$-reduction is possible in $P_q^{(1)} T_{q-1}^{-1}$, therefore $T_{q-1} \equiv \Lambda$. Similarly, we must have $T_i \equiv \Lambda$ for each $1 \leq i < q$. From the above equation (10) we now get

$$USP_q^{(q)} = T_0.$$

Since P_q must involve a-symbols (from the $\tilde{R}$-reduction), we get $q \leq \mu - \nu$. For each such q we can decide if $USP_q^{(q)} = T_0$ is possible in H. It remains to consider the case with $q = 0$ in (10). Then $USP^{(r)} = T_0$, and if P involves a-symbols, then the remark about P_q applies. If P involves no a-symbols, then the words $SP^{(r)}$ and T_0 are both free in H. Since U^{-1} must result after free reductions and possibly one $\tilde{R}$-reduction of the word $SP^{(r)}T_0^{-1}$, it follows that r is uniquely determined, hence, we can solve $USP^{(r)} = T_0$ in H. □

LEMMA 5.3. *Exactly like Lemma* 5.2 *with F_1' and F_1 replaced by C_1' and C_1. Also include the assertion from Lemma* 5.2 *about* gph $(U', W'; F_1', F_2')$.

Proof. The case with C_1 finite is trivial, so suppose that $C_1 = \langle S \rangle$ with $|S| = \infty$. Lemma 3.1 together with Lemma 4.1 show that if gph $(U, W; C_1, F_2)$ is infinite, then $USU^{-1} \in F_2$ and therefore $UW \in F_2$. It suffices therefore to find just one x such that $US^xW \in F_2$ if and only if gph $(U, W; C_1, F_2) \neq \emptyset$.

Let G be realized as the HNN group (7), and note that the hypotheses of the lemma apply to H.

By changing U and W if necessary, assume that S is cyclically t-reduced.

Case 1. $t \notin F_2$. If $S \notin H$, then $US^xW \in F_2$ forces a bound on $|x|$, so we need only consider the case with $S \in H$. It then suffices to consider t-reduced U and W with U and W^{-1} t-parallel. The result is now easily obtained by induction on $l_t(U)$ (see Case 1 in the proof of Lemma 5.2), using the remark above concerning gph $(U', W'; C_1, F_2')$ when this is infinite.

Case 2. $t \in F_2$. By relabelling the generators if necessary, we may assume that $a_0 \notin F_2$. Elements of F_2 may therefore be written in the form Tt^y with $T \in F_2' = F_2 \cap H$.

If $S \in H$, then $US^xW = Tt^y$ implies $|y| \leq l_t(U) + l_t(W)$. For each such y we can use induction on $l_t(U)$, just like we indicated for Case 1, and consider $US^x(Wt^{-y}) = T$.

For the remainder of the proof, assume that $S \equiv t^{\varepsilon_1}S_1 \ldots t^{\varepsilon_r}S_r$ with $r \geq 1$. It suffices to obtain a bound on $|x|$, so by symmetry, we need only treat the case with $x > 0$. We may assume x to be large enough for us to t-reduce US^xW and obtain a t-reduced word $U'S^{x'}W'$ with $x' > 0$. But then we may also assume all ε_i's in S to be equal. By symmetry, let us only consider the case with each $\varepsilon_i = 1$. If now $U'S^{x'}W' = Tt^y = t^yT^{(-y)}$ with $x', y > 0$, then we must have $SW't^{-z} = Q \in Y$ and $t^{-z'}U' = Q_0 \in Y$ for some z and z'. All the above reductions are effective, so we may as well assume that $U \equiv Q_0 \in Y$, $W \equiv Q \in Y$, and $x' = x$. We now have

$$Q_0(tS_1 \ldots tS_r)^xQ = Tt^{xr}.$$

If $x>0$, then we must have $S_rQ=P_r\in X$, $S_{r-1}P_r^{(1)}=P_{r-1}\in X$, etc. By changing Q_0 and Q if necessary, we may therefore assume S_i to be a word on gen (Y) for each i. Note that the P_i's are unique.

Let us now write

$$tS_1 \dots tS_r = S_0 tS_1' \dots tS_r' \equiv S'$$

where each S_i' is either empty or begins with $a_\mu^{\pm 1}$ for $i>0$. Next we write

$$tS_1' \dots tS_r' S_0 = S_0' tS_1'' \dots tS_r'' \equiv S''$$

with S'' satisfying the same conditions as S'. Note that if $S_i' \not\equiv \Lambda$ for some $i>0$, then we either get $S_0' \equiv \Lambda$ or $l(S'')<l(S')$. Thus, after a finite number of steps, that we can keep track of, we must arrive at either $\bar{S}_0 t^r$ with $\bar{S}_0$ involving no a_i's, or $t\bar{S}_1 \dots t\bar{S}_r$ with each $\bar{S}_i$ empty or beginning with $a_\mu^{\pm 1}$. By changing Q_0 and Q if necessary, we may assume that $S\equiv S_0 t^r \equiv \bar{S}_0 t^r$ or $S\equiv tS_1 \dots tS_r \equiv t\bar{S}_1 \dots t\bar{S}_r$ with the conditions just mentioned satisfied. We may also assume that $S_0 S_0^{(r)}$ is freely reduced, otherwise we can replace S_0 with a shorter word S_0'.

Suppose now that $Q_0(S_0t^r)^xQ=Tt^{xr}$ for some $x>0$. We then arrive at

$$Q_0 S_0 S_0^{(r)} \dots S_0^{(xr-r)} Q^{(xr)} = T \in F_2'.$$

If Q involves some a_i, then $xr \leqslant \mu - \nu$. Freely reducing both sides gives an equation in $Y \cap F_2'$ that can be solved by inspection. Finally assume that $S\equiv tS_1 \dots tS_r$ with at least one $S_i \not\equiv \Lambda$. Let q be maximal with $S_q \not\equiv \Lambda$, and note that $S_q Q^{(r-q)} = P \in X \cap Y$. Since S_q begins with $a_\mu^{\pm 1}$, it must be completely absorbed in $Q^{(r-q)}$, hence $l(P)<l(Q)$. Repeating this argument, note that if $S^kQt^{-kr}=P_k\in X$ for each $k>0$, then $l(P_{k+1})<l(P_k)$ for each such k. From this it is easy to bound x. □

LEMMA 5.4. *Exactly like Lemma 5.2 with F_i' and F_i replaced by C_i' and C_i for $i=1, 2$. Also include the assertions from the previous two lemmas about* gph $(U', W'; F_1', F_2')$ *and* gph $(U', W'; C_1', F_2')$.

Proof. Let $C_1=\langle S\rangle$ and $S_2=\langle T\rangle$. If S or T is of finite order, then we can easily list all pairs (S^x, T^y) with $US^xW=T^y$. Suppose therefore that $|S|=|T|=\infty$.

Let G be realized as the HNN group (7), and assume S and T to be cyclically t-reduced. If now $T\in H$ with $S\notin H$, then $US^xW=T^y$ forces a bound on $|x|$. For each such x we can decide if $US^xW=V_x\in H$, and then if $V_x\in\langle T\rangle$. Just let $C_0=\langle 1\rangle$ in H, and consider gph $(V_x, 1; C_0, C_2)$. The case with $S\in H$ and $T\notin H$ is similar. Two cases remain.

Case 1. $S, T\in H$. It suffices to consider t-reduced U and W with U and W^{-1} t-parallel. Since we can compute gph $(U', W'; C_1, F_2')$ in H, the result is easily obtained by induction on $l_t(U)$.

Case 2. $S,\ T \notin H$. Note that $|y|\, l_t(T) \leqslant l_t(U) + l_t(W) + |x|\, l_t(S)$ whenever $US^xW = T^y$, hence, it suffices to bound $|x|$. By symmetry we need only treat the case with $x, y > 0$. Let $a, b > 0$ be minimal such that S^a and T^b have the same t-length. Then $US^xW = T^y$ if and only if $U_r S^{ax'} W_s = T^{by'}$ where $x = ax' + r$, $y = by' + s$, $0 \leqslant r < a$, $0 \leqslant s < b$, $U_r = US^r$, and $W_s = WT^{-s}$. Since there are only finitely many such pairs (r, s) to consider, it suffices to treat the case with $a = b = 1$.

Following a remark in Case 2 of the last proof, we may assume that US^xW is t-reduced for all $x > 0$. Suppose now that $S \equiv t^{\varepsilon_1} S_1 \dots t^{\varepsilon_r} S_r$ and $T \equiv t^{\tau_1} T_1 \dots t^{\tau_r} T_r$ satisfy $US^xW = T^y$ for some $x > 0$. This implies

$$SWT^{-z}(T')^{-1} = Q \in F_2'$$

for some $z \geqslant 0$ and terminal segment T' of T beginning with $t^{\pm 1}$; and

$$(T'')^{-1} T^{-z'} U = Q_0 \in F_2'$$

for some $z' \geqslant 0$ and initial segment T'' of T for which $T \equiv T'' \bar{T}$ and $\bar{T}$ begins with $t^{\pm 1}$. Here $F_2' = Y$ if $\varepsilon_1 = 1$, $F_2' = X$ if $\varepsilon_1 = -1$. Without loss of generality, we may assume that $U \equiv Q_0$, $W \equiv Q$, and then consider

$$Q_0 (t^{\varepsilon_1} S_1 \dots t^{\varepsilon_r} S_r)^x Q = (t^{\varepsilon_1} T_1 \dots t^{\varepsilon_r} T_r)^x.$$

Note that we must have $y = x$ and S and T t-parallel. Suppose now that $\varepsilon_{i+1} = -\varepsilon_i$ for some $1 \leqslant i < r$, and let S' and T' be the terminal segments of S and T beginning with $t^{\varepsilon_{i+1}}$. If we set

$$S' S^z Q T^{-z} (T')^{-1} = P_z \in F'$$

for $z = 0, 1$ where F' depends on ε_{i+1}, then we must have $S_i P_1 P_0^{-1} S_i^{-1} \in F'$ as well. Hence, by malnormality of F' we get $S_i \in F'$ if $P_1 \neq P_0$. Since $S_i \in F'$ would violate the assumption that $t^{\varepsilon_i} S_i t^{-\varepsilon_i}$ is t-reduced, we must have either all ε_i's equal or $P_1 = P_0$. The latter implies $S^x Q T^{-x} = Q$ for all x, thus bounding x by 0.

By symmetry, it suffices now to treat the case with $S \equiv tS_1 \dots tS_r$ and $T \equiv tT_1 \dots tT_r$. If $SQT^{-1} \neq Q$, then for each i we must have gph $(S_i, T_i^{-1};\ Y, X)$ infinite; otherwise no $x > 1$ can satisfy $Q_0 S^x Q = T^x$. But then, by Lemma 4.2 we must have each $S_i = P_i Q_i^{-1}$ for some $(P_i, Q_i) \in X \times Y$. By assumption we can effectively compute such pairs, and by changing Q_0 and Q if necessary, we may assume each S_i to be a word on gen (Y). By a similar argument we may also assume each T_i to be a word on gen (Y).

We can now assume that we have applied the cyclic reduction process from the last proof to S and T, and consider the following subcases:

Subcase 1. $S \equiv S_0 t^r$ and $T \equiv T_0 t^r$, with S_0 and T_0 involving no a_i's, and $S_0 S_0^{(r)}$ and $T_0 T_0^{(r)}$ freely reduced. From $Q_0 S^x Q = T^x$ we then get the equation

$$Q_0 S_0 S_0^{(r)} \ldots S_0^{(xr-r)} Q^{(xr)} = T_0 T_0^{(r)} \ldots T_0^{(xr-r)}$$

in Y. It is easy to bound x if $l(S_0) \neq l(T_0)$. Also, if $l(S_0) = l(T_0)$, then we can decide by inspection if the equation can hold for any x with $(x-2)\,l(S_0) > l(Q_0) + l(Q)$.

Subcase 2. $S \equiv S_0 t^r$ as in Subcase 1 and $T \equiv tT_1 \ldots tT_r \not\equiv t^r$ with each T_i empty or beginning with $a_\mu^{\pm 1}$. (The case with S and T interchanged is similar.) From $Q_0 S^x Q = T^x$ with $x > 0$ we then get

$$Q_0 S_0 S_0^{(r)} \ldots S_0^{(xr-r)} t^{xr} Q = (tT_1 \ldots tT_r)^x.$$

But then $QT_r^{-1} = P \in X \cap Y$, and therefore all of T_r^{-1} must be absorbed in Q. If $T_r \not\equiv \Lambda$, then $l(P) < l(Q)$. Since $T_j \not\equiv \Lambda$ for at least one j, we must have $t^r Q T^{-1} = Q_1 \in Y$ with $l(Q_1) < l(Q)$. This forces a bound on x.

Subcase 3. $S \equiv tS_1 \ldots tS_r \not\equiv t^r$ and $T \equiv tT_1 \ldots tT_r \not\equiv t^r$ where each S_i and T_j is either empty or begins with $a_\mu^{\pm 1}$. If now

$$Q_0 (tS_1 \ldots tS_r)^x Q = (tT_1 \ldots tT_r)^x$$

for some $x > 0$, then we must have $S_r Q T_r^{-1} = P_r \in X \cap Y$, $S_{r-1} P_r^{(1)} T_{r-1}^{-1} = P_{r-1} \in X \cap Y$, etc. All of S_r and T_r^{-1} must be absorbed in the free reductions of $S_r Q T_r^{-1}$. Hence, unless S_r and T_r are empty, we must have $l(P_r) < l(Q)$. Since at least one S_i is non-empty, it follows that $SQT^{-1} = Q_1 \in Y$ with $l(Q_1) < l(Q)$. This forces a bound on x, and completes the proof of the lemma. □

We can now establish all the needed results about computability of graphs in any $G \in \Gamma_*$. Recall our assertions about free groups F_1 and F_2 in G.

PROPOSITION 5.5. *We can effectively compute the sets* gph $(U, W; H, K)$ *for any* $U, W \in G \in \Gamma_*$ *where* $H = F_1$ *or* C_1 *and* $K = F_2$ *or* C_2 *as subgroups of* G.

Proof. We use induction on the length of the root R of G. If $l(R) = 1$, then $G = C * F$ for some finite cyclic group C and free group F. F_1 and F_2 must be subgroups of F, which we may assume to be nonempty. All three types of graphs can be effectively computed in C and F. But then, by modifying the techniques in the three last lemmas, we can also compute these sets in G. It is of course considerably easier to work with free products than with HNN groups.

Suppose now that $l(R) \geq 2$, and that the proposition holds for all $G' \in \Gamma_*$ having roots R' with $l(R') < l(R)$. If R has exponent sum zero on one of its generators, then the above lemmas apply. Finally assume that the exponent sum is non-zero on all generators in R, and in particular, assume that $x \neq 0$ and $y \neq 0$ are the exponent sums of t and a_0 respectively in R. Then let $\hat{G} \in \Gamma_*$ be obtained from G by replacing the generators t and a_0 by $\hat{t}$ and $\hat{a}_0$, and then replacing the root R by the cyclic reduction of $\hat{R}$, where $\hat{R}$ is obtained from R by replacing each t by $\hat{a}_0\hat{t}^{-y}$ and each a_0 by $\hat{t}^x$. This construction also defines an imbedding

$$\Psi: G \to \hat{G}.$$

Now, $\hat{G}$ can be realized as an HNN group with stable letter $\hat{t}$ and base $\hat{H} \in \Gamma_*$ having root of length less that $l(R)$ (see [7]). For any $V \in G$, let $\hat{V}$ be the $\hat{t}$-reduced form of $\Psi(V) \in \hat{G}$. Similarly, if H is a subgroup of G, let $\hat{H}$ be the image in $\hat{G}$ of H under Ψ.

The case with C_1 or C_2 finite is trivial, so suppose that both are infinite. Then

$$\text{gph}\,(U, W; C_1, C_2) = \text{gph}\,(\hat{U}, \hat{W}; \hat{C}_1, \hat{C}_2)$$

in $\mathbf{Z}^2$, so this set can be effectively computed since Ψ is clearly effective. Next, consider gph $(U, W; C_1, F_2)$ with $C_1 = \langle S \rangle$. By relabelling the generators of G if necessary, we may assume that $t \notin F_2$. If also $a_0 \notin F_2$, let $\bar{F}_2 = \hat{F}_2$ in $\hat{G}$; otherwise let $\bar{F}_2$ be generated by gen $(F_2) - \{a_0\}$ together with $\hat{t}$. Now compute gph $(\hat{U}, \hat{W}; \hat{C}_1, \bar{F}_2)$ in $\hat{G}$. The case with this set finite is easy, so suppose it to be infinite. By an earlier remark, we must have $\hat{U}\hat{S}\hat{U}^{-1} = \bar{S}_0 \in \bar{F}_2$ and $\hat{U}\hat{W} = \bar{T}_0 \in \bar{F}_2$. But then, $US^xW = T \in F_2$ if and only if $(\hat{U}\hat{S}\hat{U}^{-1})^x\hat{U}\hat{W} = \bar{S}_0^x\,\bar{T}_0 = \hat{T}$. We can now decide if any $x \in \mathbf{Z}$ and $\hat{T} \in \hat{F}_2$ can satisfy this equation in $\bar{F}_2$.

It remains to consider gph $(U, W; F_1, F_2)$. Up to relabelling of the generators, the following three cases exhaust all possibilities.

Case 1. $t, a_0 \in F_1$, $t \notin F_2$. Let $\bar{F}_1$ be generated by $(\text{gen}\,(F_1) - \{t, a_0\}) \cup \{\hat{t}, \hat{a}_0\}$ in $\hat{G}$, while $\bar{F}_2$ is the group just considered above. Now compute gph $(\hat{U}, \hat{W}; \bar{F}_1, \bar{F}_2)$ in $\hat{G}$. The case with this set finite is easy, so suppose that

$$\text{gph}\,(\hat{U}, \hat{W}; \bar{F}_1, \bar{F}_2) = (P, Q)[\text{gph}\,(1, 1; \bar{F}, \bar{F})](P^{-1}S_0, Q^{-1}T_0)$$

as an infinite set where $\hat{U} = QP^{-1}$, $(P, Q) \in \bar{F}_1 \times \bar{F}_2$, and $\bar{F} = \bar{F}_1 \cap \bar{F}_2$ (see Lemma 4.2). If now $USW = T$ for some $(S, T) \in F_1 \times F_2$, then

$$\hat{S} = P\bar{S}P^{-1}S_0 \quad \text{and} \quad \hat{T} = Q\bar{S}Q^{-1}T_0$$

for some $\bar{S} \in \bar{F}$. Since $\hat{a}_0 \notin \bar{F}$, it is easy to decide if these equations have a solution in $\bar{F}_1$ and $\bar{F}_2$.

Case 2. $t \notin F_1 \cup F_2$. This is essentially like Case 1, only easier.

Case 3. gen $(R) = \{t, a_0\}$, $t \in F_1$, $a_0 \in F_2$. The cases with F_1 or F_2 cyclic have been considered, so suppose that $F_1 = \langle t; \rangle * F_1'$ and $F_2 = \langle a_0; \rangle * F_2'$ with F_1' and F_2' non-trivial. Hence, we must also have $G = G_0 * F$ for some non-trivial free group F where $G_0 = \langle t, a_0; R^n \rangle$. Let us assume that no terminal segment of U or initial segment of W belongs to F_1, and similarly for U^{-1} and W^{-1} with respect to F_2. It is now easy to decide if $USW = T$ is possible for any $(S, T) \in F_1 \times F_2$ with U, W, S, and T s-reduced. □

Note that we have also shown: All groups in Γ_* have solvable generalized word problem with respect to cyclic subgroups. This because $U \in C_2$ if and only if gph $(U, 1; F_1, C_2) \neq \emptyset$ where F_1 is the trivial group.

We can now prove Lemma 3.2 as well.

Proof of Lemma 3.2. Let $U \in G \in \Gamma_*$ where G has the root R. We must decide if $U \sim_G S$ is possible for any free word S in G; then we must show how such an S can be effectively constructed whenever solutions exist. We proceed by induction on $l(R)$, observing that the problem is trivial when $l(R) = 1$. Suppose now that the result has been established for all $G' \in \Gamma_*$ having roots R' of length less than $l(R)$, where $l(R) \geqslant 2$.

First we consider the case when G can be realized as the HNN group (7). Let U be cyclically t-reduced. Note that if S is free in $G \in \Gamma_*$ and the t-reduced form S' of S belongs to H, then S' is also free in $H \in \Gamma_*$. Suppose now first that U is t-free. By the inductive hypothesis, suppose that we have determined a free word S in H with $U \sim_H S$. Moreover, assume this S to be cyclically reduced. Suppose also that there exists a free word T in $G \in \Gamma_*$ with $U \sim_G T$. The cyclically t-reduced form T' of T must belong to H, moreover, for some $V \equiv V_0 t^x V_1$ we must have $T' = VSV^{-1}$ (see Lemma 3.3). It now suffices to set V_0, $V_1 \equiv \Lambda$, and check if $T' \equiv S^{(x)}$ for any $x \in \mathbf{Z}$ and free word T in $G \in \Gamma_*$. Finally, suppose that $U \equiv t^{\varepsilon_1} U_1 \dots t^{\varepsilon_k} U_k$ with $k \geqslant 1$. By Collins' Lemma (p. 123 in [3]), if $U \sim_G S$ with S cyclically t-reduced, then we must also have $l_t(S) = k$. Replacing a_0 successively by the elements from gen $(R) - \{t\}$ in the HNN construction of G, we may assume that S involves no a-symbols. Hence, S can be written $S \equiv S_0 t^x$ with $|x| = k$. It is now enough to consider the case with $U \equiv tU_1 \dots tU_k$ and $S \equiv S_0 t^k$. By Collins' Lemma, if $U \sim_G S$, then we may assume that

$$tU_1 \dots tU_k = QS_0 t^k Q^{-1} \tag{11}$$

for some $Q \in Y$. But then $U_k Q = P \in X$, $U_{k-1} P^{(1)} = P' \in X$, etc. By Proposition 5.5 we can effectively compute pairs $(P_i, Q_i) \in X \times Y$ with $U_i = P_i Q_i^{-1}$ for each $1 \leqslant i \leqslant k$ (if (11) can be satisfied), hence, we may as well assume each U_i to be a word on gen (Y). We can now use the cyclic reduction process on U that we applied to the generator of C_1 in the proof of Lemma 5.3. If we arrive at a word without a-symbols, then we are done. Suppose therefore that we arrive at the word $V \equiv tV_1 \dots tV_k$ with each V_i empty or beginning with $a_\mu^{\pm 1}$ and $V_i \not\equiv \Lambda$ for at least one i. If now

$$tV_1 \dots tV_k = QS_0 t^k Q^{-1}$$

then $V_k Q = P_k \in X$, $V_{k-1} P_k^{(1)} = P_{k-1} \in X$, etc. Thus, $Q \equiv V_k^{-1} P_k$, $P_k^{(1)} \equiv V_{k-1}^{-1} P_{k-1}, \dots, P_2^{(1)} \equiv V_1^{-1} P_1$. After these t-reductions we arrive at

$$P_1^{(1)} = QS_0.$$

But this equation cannot be satisfied for any S_0 without a-symbols. To see this, note that $P_1^{(1)} = QS_0$ is an equation in Y, and

$$Q \equiv V_\kappa^{-1}(V_{k-1}^{-1})^{(-1)} \dots (V_1^{-1})^{(-k+1)} P_1^{(-k+1)}$$

contains strictly more a_i's than $P_1^{(1)}$.

The case remains where G cannot be realized directly as the HNN group (7). We then use the imbedding

$$\Psi': G \to \hat{G}$$

from the proof of Proposition 5.5. Suppose now that there exists a free word $S \in G$ and a free word $T \in \hat{G}$, such that

$$U \sim_G S \quad \text{and} \quad \hat{U} \sim_{\hat{G}} T.$$

If $t \notin \text{gen}(S)$, where $\Psi'(t) = \hat{a}_0 \hat{t}^{-y}$, then $\hat{S}$ is also a free word in $\hat{G}$, hence, by Lemma 3.1 $\hat{S}$ must be a cyclic permutation of T. We assume here that S and T are cyclically reduced. Since the imbedding Ψ' depends on the particular choice of t, $a_0 \in \text{gen}(R)$, we must repeat this imbedding for each such choice, and then check if $T \in \Psi''(G)$ for some such Ψ''. □

§ 6. The main results for the class Γ_0

In this section we complete our study of the class Γ_0. To this end, let $G \in \Gamma_0$ be given by

$$G = \langle a_1, a_2, \dots; R_1^{n_1}, \dots, R_k^{n_k} \rangle \tag{12}$$

subject to the conditions on (2) in Section 2. In that section we also showed that if $k \geq 2$, then G can be realized as a tree-product

$$G = G_1 *_{F_1} \dots *_{F_{k-1}} G_k \tag{13}$$

where each $G_i \in \Gamma_*$ and each F_i is free on gen $(G_i) \cap$ gen (G_{i+1}). Moreover, if $G' \in \Gamma_0$ is the subgroup of G generated by $\bigcup_{i=1}^{k-1}$ gen (G_i), then

$$G = G' *_F G_k \tag{14}$$

where $F = F_{k-1}$. Note that G' has $k-1$ relators, hence, this gives us a means of proving results about Γ_0 by induction on k in (12).

B. B. Newman proved in [10] that if J is malnormal in H_1 and H_2, then H_1 and H_2 are malnormal in $H_1 *_J H_2$. Using this result together with transitivity of malnormality, it is easy to prove by induction on k:

Lemma 6.1. *Let $G \in \Gamma_0$ be given by* (12) *with $k \geq 2$. Then the subgroups G_i in* (13) *and G' in* (14) *are malnormal in G.*

Suppose now that G is given by (12) with $k \geq 1$. If $k = 1$, let $G_1 = G$; otherwise let the subgroups G_i be defined by (13). Now, if U is a word on gen (G) with gen $(U') \subseteq$ gen (G_i) for some subword U' of U and $1 \leqslant i \leqslant k$, then we call any R_i-reduction (see Section 5) of U' in G_i an *R-reduction of U*. If no R-reductions are possible in U, then we call U *R-reduced*. If also all cyclic permutations of U are R-reduced, then we call U *cyclically R-reduced*. Note-that these reductions are effective.

Let $k \geqslant 2$ and consider G as the generalized free product (14). Then, if U is an R-reduced word on gen (G), we claim that U is also s-reduced (see Section 4) as an element of $G' *_F G_k$. This is so because any s-reduction of U must involve an R-reduction. Similarly, if U is cyclically R-reduced, then it must also be cyclically s-reduced.

Using the above ideas we can prove Theorem A for the class Γ_0. We state this as a lemma.

LEMMA 6.2. *We can effectively compute the order of elements in any G from Γ_0.*

Proof. We use induction on k where G is given by (12). The result is well-known for $k \leqslant 1$, so suppose that $k \geqslant 2$ with the lemma established for all $G' \in \Gamma_0$ having less that k relators. Write $G = G' *_F G_k$ as in (14), and consider $U \in G$. By the remarks above, we may assume that U is cyclically R-reduced. Now, if U belongs to G' or G_k, then the inductive hypothesis applies, while otherwise, $|U| = \infty$. □

Our next result generalizes Proposition 3.5 and establishes Theorem B for the class Γ_0.

THEOREM 6.3. *Given $U, W \in G \in \Gamma_0$, we can effectively compute integers a, b, and c with $0 \leqslant c \leqslant 1$ such that*

$$CP_G(U, W) = (a, b)\mathbf{Z} \cup (ac, -bc)\mathbf{Z}$$

if $|U|, |W| = \infty$;

$$GP_G(U, W) = (a, b)\mathbf{Z} + (|U|\mathbf{Z}) \times (|W|\mathbf{Z})$$

if $|U|, |W| < \infty$.

Proof. We use induction on the number k of roots in the presentation (12) of G. The case with $k=0$ is trivial. Also, $k=1$ with $|U|, |W| = \infty$ is covered by Proposition 3.5.

To complete the case with $k=1$, let U and W be of finite order in $G \in \Gamma_*$ where G has the relator R^n. Now, if U or W equals 1, then we can clearly take $a, b = 0$. Suppose therefore that $U, W \neq 1$. Hence, $U \sim_G R^p$ and $W \sim_G R^q$ for some $0 < p, q < n$. From Lemma 3.4 it follows that $U^x \sim_G W^y$ if and only if $R^{px} = R^{qy}$. The integers p and q can be effectively computed, hence, we can decide if U and W are power-conjugate, that is, if $U^x \sim_G W^y \neq 1$ for some $x, y \in \mathbf{Z}$. If U and W are not power-conjugate, let $a, b = 0$; otherwise determine the minimal $a > 0$ in $\mathbf{Z}$ with $R^{pa} = R^{qb'} \neq 1$ for some $b' \in \mathbf{Z}$. Then determine the minimal $b > 0$ for which $R^{pa} = R^{qb}$. It remains to show that $U^x \sim_G W^y$ implies $U^x = U^{az}$ and $W^y = W^{bz}$ for some $z \in \mathbf{Z}$. But this is easy enough, just use the Euclidean algorithm and write $x = az + r$ with $0 \leqslant r < a$. Then observe that

$$R^{pr} = R^{p(x-az)} = R^{px}R^{-paz} = R^{qy}R^{-qbz} = R^{q(y-bz)}.$$

By minimality of a, it follows that $R^{pr} = 1$, and hence,

$$U^x \sim_G R^{px} = R^{paz} \sim_G U^{az}$$

and

$$W^y \sim_G R^{qy} = R^{qbz} \sim_G W^{bz}.$$

The result now follows from Lemma 3.4.

Suppose next that G has $k \geqslant 2$ roots, and that the theorem is valid for all $G' \in \Gamma_0$ with less than k roots. We can then write $G = G' *_F G_k$ as in (14) and apply the inductive hypothesis to both G' and G_k. Let U and W be cyclically R-reduced. By Lemma 6.1 it is clear that

$$CP_G(U, W) = CP_{G''}(U, W)$$

if $U, W \in G''$ for $G'' = G'$ or G_k. Moreover, if U and W belong to distinct factors, then $U^x \sim_G W^y$ implies U^x and W^y must both be conjugate in their factor to some $T \in F$. By symmetry we may assume that $W \in G_k$. If $W^y \neq 1$, then W must be conjugate in G_k to some $T_0 \in F$ with $T = T_0^y$. By Lemma 3.2 we can effectively determine such a T_0 if it exists, so because we then get

$$CP_G(U, W) = CP_{G'}(U, T_0),$$

it remains to consider $W^y = 1$. But in this case (i.e. $W^y \sim_G T \in F$ implies $T = 1$) we may set $a, b, c = 0$. The case with $l_s(U)$, $l_s(W) \geqslant 2$ remains. Since we can easily determine minimal integers $a', b' > 0$ with $l_s(U^{a'}) = l_s(W^{b'})$, we may as well assume that $l_s(U) = l_s(W)$. Now, by Solitar's Theorem we know that $U^x \sim_G W^x$ for some $x \neq 0$ if and only if $U^x = S(W^x)_\pi S^{-1}$ for some s-permutation $(W^x)_\pi$ of W^x and $S \in F$. Since $(W^x)_\pi = W_\pi^x$ for some s-permutation W_π of W, note that if $x > 1$, then

$$U^i S W_\pi^{-i} = S_i \in F$$

for each $1 \leqslant i \leqslant x$. But then we must have $S = S_1$; otherwise $USS_1^{-1}U^{-1} = S_1 S_2^{-1} \neq 1$, and by malnormality of F in G, we then get $U \in F$. The case with $x < -1$ is similar, so we can conclude that $U^x \sim_G W^x$ for some $x \neq 0$ if and only if $U \sim_G W$. Thus, it suffices to determine whether or not $U \sim_G W^\varepsilon$ for $\varepsilon = \pm 1$. Let us just consider $\varepsilon = 1$. Now, if $U \equiv U_1 \ldots U_r$ and $W \equiv W_1 \ldots W_r$ in terms of syllables and $U \sim_G W$, then

$$U_1 \dots U_r S(W_1 \dots W_r)_\pi^{-1} = S$$

for some s-permutation $(W_1 \dots W_r)_\pi$ of W. There are only finitely many such s-permutations, so let us just consider the trivial one. By considering $S = U^{-1}SW$ if necessary, we may assume that $U_r, W_r \in G_k$. In this factor we can effectively compute gph $(U_r, W_r^{-1}; F, F)$ by Proposition 5.5. Moreover, this set can contain at most one pair (S, T), otherwise $U_r, W_r \in F$. Now all we need to do is to check if $USW^{-1}S^{-1} = 1$ for this S. □

The final result we need for the class Γ_0 is the following generalization of part of Proposition 5.5.

LEMMA 6.4. *For any cyclic subgroups C_1 and C_2 of $G \in \Gamma_0$, and elements $U, W \in G$, we can effectively compute* gph $(U, W; C_1, C_2)$.

Proof. The case with C_1 or C_2 finite is trivial, so suppose that $C_1 = \langle S \rangle$ and $C_2 = \langle T \rangle$ with $|S|, |T| = \infty$. In this case we identify C_1 and C_2 with $\mathbf{Z}$ and set

$$\text{gph}\,(U, W; C_1, C_2) = \{(x, y) \in \mathbf{Z}^2;\ US^xW = T^y\}.$$

As a consequence of Lemma 4.1, we know that this set is either empty or takes the form

$$\text{gph}\,(U, W; C_1, C_2) = (r, s) + (a, b)\mathbf{Z}.$$

Let us proceed by induction on the number k of roots in G. The case with $k = 0$ was treated in [5], while $k = 1$ is covered by Proposition 5.5. Suppose now that $k \geq 2$ with the lemma established for all $G' \in \Gamma_0$ having less than k roots. Then write $G = G' *_F G_k$ as in (14), noting that the inductive hypothesis applies to both factors. By standard arguments, we may assume that S and T are cyclically R-reduced, U and W R-reduced. The case with $l_s(S) = 1 < l_s(T)$, or vice versa, is trivial since we can then bound $|x|$ or $|y|$. Let us now consider

Case 1. $l_s(S), l_s(T) \geq 2$. By considering a finite number of cases, we may assume that S and T have the same s-length, and that gph $(U, W; C_1, C_2) \neq \varnothing$ if and only if $US^xW = T^y$ for some $x, y \geq 0$. It suffices to bound x since this also yields a bound on y. Assume therefore that x is large enough so that we can s-reduce US^xW and obtain an s-reduced word $U'S^{x'}W'$ with $x' \geq 2$. (We accomplish this by R-reductions.) If now $U'S^{x'}W' = T^y$, then

$$SW'T^{-y_1}T_1^{-1} = P \in F$$

and

$$T_2^{-1}T^{-y_2}U' = Q \in F$$

for some syllable-segments T_1 and T_2 of T for which $T'T_1 \equiv T \equiv T_2T''$, where $y_1, y_2 \geq 0$. Since we now have

$$QS^{x'-1}P = T''T^{y'}T' = T_0^{x'-1}$$

where $T_0 \equiv T'' T_2 \equiv T_1 T'$ (comparing s-lengths), it suffices to show that we can take $x'-1=1$. But clearly, if $x'-1>1$, then

$$S^i P T_0^{-i} = P_i \in F$$

for each $1 \leqslant i \leqslant x'-1$, hence, malnormality of F in G implies $P=P_1$.

Case 2. $l_s(S)$, $l_s(T)=1$. Suppose that $U \equiv U_1 \ldots U_p$ and $W \equiv W_1 \ldots W_q$ as s-reduced decompositions into syllables where we allow $U \equiv U_1 \equiv \Lambda$ and $W \equiv W_1 \equiv \Lambda$. Suppose further that

$$U_1 \ldots U_p S^x W_1 \ldots W_q = T^y$$

for some x and y. If U, S, W, and T belong to one and the same factor, then the inductive hypothesis applies, and otherwise we must have $U_p S^x$, $S^x W_1$, or $U_p S^x W_1$ in F. If this is an equation in G_k, then by Proposition 5.5 we can compute the corresponding graph. Moreover, unless the third possibility occurs with $U_p S U_p^{-1} = S_0 \in F$ and $U_p W_1 \in F$, the x is unique. Also, in this case with x not unique we can shorten U and W by a syllable, and repeat the argument with S replaced by $S_0 \in F$. If $S \notin G_k$ and U_p or W_1, as the case may be, belongs to G', then let U_* and W_* be the remaining segments of U and W. We now get $U_*^{-1} T^y W_*^{-1} \in F$, and hence, if U_*, $W_* \notin G_k$, then we arrive at the graph gph $(U_i, W_j; F, F)$ in G_k, where $U_i \equiv U_p$ or U_{p-1}(Λ if $p=1$) and $W_j \equiv W_1$ or W_2 (Λ if $q=1$). By Lemma 4.1, at most one pair S_0, $T_0 \in F$ can satisfy $U_i S_0 W_j = T_0$, otherwise U_i, $W_j \in F$. By Proposition 5.5 we can compute this pair (S_0, T_0), and hence, also determine x and y. It remains to consider the case with U_*, $W_* \in G_k$. But this is just like the first part. □

The following corollary is immediate.

COROLLARY 6.5. *The groups in* Γ_0 *have solvable generalized word problem with respect to cyclic subgroups.*

With these results for Γ_0 we can turn to the main theorems for Γ_1.

§ 7. Proofs of the main theorems

With the results thus far established in this paper and techniques used in [5], the following is easy to prove, hence we omit the proof here.

LEMMA 7.1. *Let* $G = G_1 *_C G_2 \in \Gamma_1$. *Then for any* U, $W \in G$ *we can effectively compute* gph $(U, W; C, C)$ *in* G.

Note that by Corollary 6.5 we can effectively s-reduce and cyclically s-reduce elements in any $G = G_1 *_C G_2 \in \Gamma_1$.

Proof of Theorem A. Let $U \in G = G_1 *_C G_2 \in \Gamma_1$. To compute $|U|$, let us first cyclically s-reduce U and obtain U'. If now $l_s(U') > 1$, then $|U| = \infty$. If instead U' belongs to G_1 or G_2, then we can apply Lemma 6.2. □

We can also give the

Proof of Theorem B. We have already proved this for all groups in Γ_0, so assume that $G = G_1 *_C G_2 \in \Gamma_1$ with C non-trivial. Let $U, W \in G$ be cyclically s-reduced, and consider first

Case 1. $l_s(U), l_s(W) > 1$. As usual, we only treat the case with U and W of the same s-length. Moreover, by Lemma 7.1 and Solitar's Theorem, it suffices to obtain a bound on $x > 0$ for which $U^x \sim_G W^{\varepsilon x}$ is possible for $\varepsilon = \pm 1$. By a result in [5], we may take $x \leqslant 2$ if C is infinite. Also, if $C = \langle S \rangle$ in G with $|S| < \infty$, then $U^x \sim_G W^{\varepsilon x}$ implies

$$U^i S^z W_\pi^{-\varepsilon i} = S^{z_i}$$

for each $1 \leqslant i \leqslant x$, where

$$U^x = S^z W_\pi^{\varepsilon x} S^{-z}$$

Here W_π is some cyclic s-permutation of W. But then, if $x > |S|$, we get $S^{z_i} = S^{z_j}$ for some $j < i$, hence,

$$U^i S^z W_\pi^{-\varepsilon i} = U^j S^z W_\pi^{-\varepsilon j}$$

implies

$$U^{i-j} = S^z W_\pi^{(i-j)\varepsilon} S^{-z}$$

with $i - j < x$.

Case 2. $l_s(U), l_s(W) = 1$. If $U, W \in G_i$ ($i = 1$ or 2) and some $(x, y) \in CP_G(U, W)$ does not belong to $CP_{G_i}(U, W)$, then

$$U^x \sim_{G_i} S^z = T^z \sim_{G_j} T^{z'} = S^{z'} \sim_{G_i} W^y$$

where $C = \langle S \rangle$ in G_i and $C = \langle T \rangle$ in G_j ($j \neq i$). If $|T| < \infty$, then $T^z = T^{z'}$ and therefore $S^z = S^{z'}$. Hence, we must have $|T| = \infty$. But then $z' = -z$ and thus $(x, -y) \in CP_{G_i}(U, W)$. Since by Theorem 6.3,

$$CP_i(U, W) = (a', b')\mathbf{Z} \cup (a'c', -b'c')\mathbf{Z}$$

with $0 \leqslant c' \leqslant 1$, we must have $c' = 0$. Therefore, if $T \sim_{G_j} T^{-1}$, then we get

$$CP_G(U, W) = (a, b)\mathbf{Z} \cup (a, -b)\mathbf{Z}$$

with $a' = a$, $b = b'$; if $T \nsim_{G_j} T^{-1}$, then $CP_G(U, W) = CP_{G_i}(U, W)$.

Finally consider $U \in G_i$ and $W \in G_j$ with $i \neq j$. Since $U^x \sim_G W^y$ if and only if

$$U^x \sim_{G_i} S^z = T^z \sim_{G_j} W^y,$$

we can construct $CP_G(U, W)$ from $CP_{G_i}(U, S)$ and $CP_{G_j}(T, W)$. □

As for the generalizations alluded to in the introduction, let Γ be the smallest class containing Γ_0 which is closed under the formation of free products with cyclic subgroups amalgamated.

Using techniques from this paper and from [5], we can generalize Lemmas 6.4 and 7.1 to groups in Γ. Also, if $G = G_1 *_C G_2 \in \Gamma$, and $|U| < \infty$ in G, then we must have U conjugate to an element of a factor. Continuing, we note that U must be conjugate to an element of a subgroup G' of G with $G' \in \Gamma_0$.

It follows from the above that we must get

THEOREM B'. *Given $U, W \in G \in \Gamma$, we can effectively compute integers $a, b, c_1, \ldots, c_n$ such that*

$$CP_G(U, W) = (a, b)\,\mathbf{Z} \cup \left[\bigcup_{i=1}^{n} (ac_i, -bc_i)\,\mathbf{Z}\right]$$

if $|U|, |W| = \infty$;

$$CP_G(U, W) = (a, b)\,\mathbf{Z} + (|U|\mathbf{Z}) \times (|W|\mathbf{Z})$$

if $|U|, |W| < \infty$.

Note that the generalized version of Lemma 6.4 implies we can cyclically s-reduce elements of any $G \in \Gamma$, hence we can compute $|U|$ and $|W|$ in Theorem B'.

The corresponding generalization of the results in [5] about HNN groups are straightforward.

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