IMMERSION AND EMBEDDING OF PROJECTIVE VARIETIES

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§ 0. Introduction

Let $f: X \to Y$ be a morphism of algebraic varieties defined over an algebraically closed field. If X and Y are nonsingular, and the induced map $df: T(X) \to T(Y)$ of tangent bundles is a monomorphism, then f is called an *immersion*. A one-to-one immersion is

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called an *embedding*. If X or Y is singular, there are several possible definitions of immersion and embeding, corresponding to the various tangent cones which can serve as the tangent space at a singular point.

In any category where one has notions of immersion and embedding, a natural question is the following: Given objects X and Y, and some equivalence class $[f]$ of mappings $X \to Y$, does $[f]$ contain an immersion or an embedding? If f is a generic member of $[f]$, one might hope that the *ramification locus* of f (that is, the subset of X where f fails to be an immersion) and the *double locus* of f (that is, the preimage in X of the self-intersection of $f(X)$ in Y) determine classes in the homology of X which could be calculated in terms of invariants of X, Y and $[f]$. The question could then be rephrased in terms of the vanishing of these homology classes.

Our main accomplishment in this paper is to carry out this program in the case when X is a (possibly singular) subvariety of some projective space, $Y = \mathbf{P}^m$ is a smaller projective space, and $[f]$ is the set of all projections of X to $\mathbf{P}^m$. Our definitions of immersion and embedding for singular varieties guarentee that the ramification and double homology classes we calculate have uniform dimension. In fact, these classes can be represented by positive cycles whose supports are, respectively, the ramification and double loci of a generic projection of X to $\mathbf{P}^m$ (see § 5.2 for a precise statement of the theorem).

On manifolds it is possible to have positive cycles which are homologous to 0. So, for instance, it is possible for the ramification homology class determined by maps in a homotopy equivalence class $[f]$ to be 0 without having the ramification locus of any of the maps in $[f]$ actually be empty. This cannot happen in our case, since the degree of a homology class on a projective variety is equal to the degree of an algebraic cycle which represents it, and a positive algebraic cycle can have degree 0 only if its support is empty. Hence the theorem we obtain is much stronger than is possible in the topological case.

If X is a nonsingular n-dimensional subvariety of k-dimensional projective space, $k \leqslant 2n$, and m is less than k, we have a simple proof of the following result:

(*) *If X can be immersed in $\mathbf{P}^m$ by projection, then it can be so embedded.*

Our definitions of immersion and embedding for singular varieties preserve both this result and our proof of it (§ 5.2, Corollary 1). We are able to deduce that (*) is true also if one defines immersion and embedding using the Zariski tangent space (which is not the definition we make, although it is perhaps the most obvious one) (§ 5.2, Corollary 2).

We note that there is no theorem like (*) in the category of smooth manifolds. For example, the Klein bottle K can be realized as a submanifold of $\mathbf{R}^4$ which can be immersed in $\mathbf{R}^3$ by projection; but K cannot be embeded in either $\mathbf{R}^3$ or real projective 3-space.

Our formulas for the ramification and double classes are in terms of homology invariants of X which are Poincaré dual to the Segre, or inverse Chern, classes of X when X is nonsingular (§ 4.3). The definition of these classes is similar to a definition of canonical classes given by B. Segre [1, p. 113]. Since Chern classes and Segre classes of nonsingular varieties are related multiplicatively (see § 4.2), and multiplication is lost in homology, one should not expect any obvious relation between these classes and the various homology Chern classes for singular varieties which have been defined in [3] and [14].

The literature on immersions and embeddings is much more extensive in topology than in geometry.

A survey of results in the topological case can be found in [4]. The homology class determined by the ramification locus of the projection of a manifold to euclidean space has been studied by many people, including Pontrjagin [20] and Thom [25] in the smooth case, and McCrory [15] in the simplicial case. McCrory [16] has also considered the homology class determined by the double locus of such a map.

The ramification class of a morphism between nonsingular algebraic varieties is given by Porteous's formula [22], which is proved in great generality by Kempf and Laksov [10].

In the paper which first made us interested in the subject, Holme [8] finds a sequence of numbers which he shows must be 0 in order for a nonsingular variety $X^n \subset \mathbf{P}^N$, $N=2n+1$, to be embeddable in $\mathbf{P}^m$ by projection. His idea is to determine the dimension of $\operatorname{Sec} X \subset \mathbf{P}^N$, the variety of secants of X. (If the dimension of $\operatorname{Sec} X$ is less than or equal to m, then X can be embedded in $\mathbf{P}^m$ by projection.) To determine this dimension he calculates, in $A(\mathbf{P}^N \times \mathbf{P}^N)$, the class of the subvariety

$$\{(x, y) \mid x \in X, \text{ and the line through } x \text{ and } y \text{ is secant to } X\}$$

of $\mathbf{P}^N \times \mathbf{P}^N$. (This variety is N-dimensional, and maps onto $\operatorname{Sec} X$ by projection on the second factor.) Writing this class in terms of the natural basis

$$\{[\mathbf{P}^N \times \mathbf{P}^0], [\mathbf{P}^{N-1} \times \mathbf{P}^1], \ldots, [\mathbf{P}^0 \times \mathbf{P}^N]\}$$

of $A_N(\mathbf{P}^N \times \mathbf{P}^N)$, he observes that if the coefficients of $[\mathbf{P}^{N-m-1} \times \mathbf{P}^{m+1}], \ldots, [\mathbf{P}^0 \times \mathbf{P}^N]$ are all 0, then the dimension of $\operatorname{Sec} X$ is less than or equal to m, so X is embeddable in $\mathbf{P}^m$. The coefficient of $[\mathbf{P}^{N-m-1} \times \mathbf{P}^{m+1}]$ turns out to be the degree of the double cycle of a generic projection $X \to \mathbf{P}^m$.

Peters and Simonis [19] calculate the number of secants to $X^n \subset \mathbf{P}^{2n+1}$ through a general point in $\mathbf{P}^{2n+1}$. This number is just half of the degree of the double cycle of a generic projection $X \to \mathbf{P}^{2n}$. Their approach is similar to ours, although we did not become aware of their work until ours was substantially complete.

The more general problem of calculating the degree of the double cycle of morphism $X \to Y$, where X and Y are both nonsingular, was dealt with by Severi as early as 1902 [24]. In 1940, Todd [26] calculated the class of this cycle, in the ring of equivalence which had been developed by Severi, in terms of his "invariant systems" (which are in fact Chern classes and Segre classes).

In two recent, very important papers, Laksov has given modern proofs of Todd's results. In the first [12] he defines a double cycle for a generic morphism $X \to \mathbf{P}^m$, where X is nonsingular, and calculates its rational equivalence class in terms of the Segre classes of X. He does not assume X to be a subvariety of $\mathbf{P}^N$ and take the morphism to be a projection, so his theorem is stronger than ours in that respect. He constructs the double cycle by defining a homomorphism $O_B^{m+1} \to F$, where F is a rank 2 bundle on B = the blow-up of $X \times X$ along its diagonal; the double cycle is then the push-down to X of the support of the cokernel of this homomorphism. The bundle F is obtained from a secant construction due to Schwarzenberger.

The relation between Laksov's approach and ours is as follows: The sheaf homomorphism $O_B^{m+1} \to F$ gives a correspondence from B to $G_1(\mathbf{P}^m)$, the Grassmannian of lines in $\mathbf{P}^m$. If X is a nonsingular subvariety of $\mathbf{P}^N$, and $X \to \mathbf{P}^m$ is a generic projection, then this correspondence can be factored as

$$B \xrightarrow{\gamma} G_1(\mathbf{P}^N) \dashrightarrow G_1(\mathbf{P}^m),$$

where γ is the morphism we define in § 2.2. The set where $G_1(\mathbf{P}^N) \dashrightarrow G_1(\mathbf{P}^m)$ is not defined is precisely the Schubert variety $W(H)$ of lines in $\mathbf{P}^N$ which intersect the space H defining the projection of X to $\mathbf{P}^m$; so the support of the cokernel of $O_B^{m+1} \to F$ (that is, the set where $B \dashrightarrow G_1(\mathbf{P}^m)$ is not defined) is the pullback of $W(H)$ to B. Therefore our definition of the double cycle and his agree in this case (cf. § 3.2, Theorem (ii)).

In the second paper [13] he calculates the rational equivalence class of a double cycle for a map $X \to Y$ between nonsingular varieties. His argument can easily be modified to allow X to be singular, and so that the Segre classes that we have defined for singular varieties (§ 4.3) come into the formula [3].

Another recent paper that is closely related to ours is another one by Holme [9], in which he extends his earlier result to include the case when X is singular. His whole approach is different from ours (cf. our discussion of his work above), but for comparison purpoees what he does is to define ramification using the Zariski tangent space of X, rather than the tangent star (see § 1.4 below). This has the advantage that the numbers he obtains are obstructions to being able to project X isomorphically. Our definition of embedding is not as strong as this.

I would like to thank my advisor, William Fulton, for his help in preparing this work. I am also grateful to Alan Landman and Clint McCrory for suggesting points of view that have proved fruitful.

§ 1. Basic definitions

In this section we define notions of immersion and embedding for singular algebraic varieties.

The terms "immersion" and "embedding" are borrowed from differential topology. In the category of smooth manifolds, an *immersion* is a mapping which does not collapse tangent spaces (i.e., the induced differential homomorphisms between tangent spaces are all one-to-one), and an *embedding* is a one-to-one immersion.

There are obvious analogues of these notions for nonsingular varieties. But in the singular case, "collapsing tangent spaces" can be defined in various ways, since there are several tangent cones that are reasonable replacements for the tangent space, and, if a cone is not linear, "finite-to-one" might replace "one-to-one" as a criterion for non-collapsing.

The tangent cone we use in defining immersion and embedding (which we call the tangent star, for lack of a better name) is not an obvious choice. In § 1.3 we describe some good properties that our definition has, and in § 1.4 we give examples to show that more obvious definitions do not have these properties.

A *variety* is a projective and reduced scheme defined over an algebraically closed field. A nonsingular variety is always assumed to be irreducible.

1.1. The tangent star

Let X be a variety, and let $\mathfrak{J}$ be the ideal sheaf of the diagona lin $X \times X$; then $\oplus_{j=0}^{\infty} \mathfrak{J}^j/\mathfrak{J}^{j+1}$ is a sheaf of algebras on $X \approx$ diagonal in $X \times X$. We define

$$T(X) = \operatorname{Spec}\left(\bigoplus_{j=0}^{\infty} \mathfrak{J}^j/\mathfrak{J}^{j+1}\right).$$

If x is a closed point in X/k, the scheme

$$T(X)_x = T(X) \times_X \operatorname{Spec}(k(x))$$

is called the *tangent star* to X at x.

If $\Theta(X) = \mathrm{Spec}\,(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2))$, then $\Theta(X)_x$ is the Zariski tangent space to X at x. The surjection $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2) \to \oplus_{j=0}^{\infty} \mathcal{J}^j/\mathcal{J}^{j+1}$ [6, IV 16.1.2.2] induces an inclusion $T(X) \hookrightarrow \Theta(X)$, so the tangent star is a subscheme of the Zariski tangent space.

If X is a subvariety of $\mathbf{C}^N$, then $T(X)_x$ is the union of all lines l through x for which there are sequences $\{y_i\}$, $\{y_i'\}$ of points in X converging to x such that the sequence of lines $\{\mathrm{span}\,(y_i, y_i')\}$ converges to l. (See Appendix C.)

The tangent star is different, in general, from both the usual tangent cone and the Zariski tangent space. For example, let $X \subset \mathbf{C}^3$ be the union of three lines not lying in a plane which intersect in a point x. Then the tangent cone of X at x is X itself, $T(X)_x$ is the union of the three planes spanned by the three pairs of lines of X, and $\Theta(X)_x$ is $\mathbf{C}^3$ [28, p. 212].

At a nonsingular point, the tangent star and Zariski tangent space are the same.

1.2. Ramification, immersion, and embedding

Let X and Y be varieties, and let $f: X \to Y$ be a morphism. Then f induces a linear map df_x: $\Theta(X)_x \to \Theta(Y)_{f(x)}$ for each x in X. We say that f *ramifies* at x if the map $T(X)_x \to \Theta(Y)_{f(x)}$ induced by df_x is not quasifinite.(1) The set of all points in X where f ramifies is called the *ramification locus*. If the ramification locus is empty, then we call f an *immersion*.

The *double locus* of f is the union of the ramification locus with the set of points $x \in X$ for which there is a $y \in X$, $y \neq x$, such that $f(y) = f(x)$. If the double locus is empty, then f is called an *embedding*. In other words, an embedding is an immersion which is globally one-to-one.

We note that an embedding is an isomorphism when X is non-singular, but that it is not an isomorphism in general.

If X and Y are nonsingular complex varieties, these definitions coincide with the usual definitions of immersion and embedding for maps between complex manifolds.

1.3. Desirable properties of immersion and embedding

In differential topology, immersions and embeddings of manifolds have been studied extensively [4]. It is known, for example, that if M is a simplicial n-dimensional

(1) We use quasifinite (that is, finite-to-one) rather than one-to-one, because one-to-one leads to ramification loci that are too big: let X be the union of three planes in $\mathbf{P}^4$ having a line l in common, but otherwise in general position. Then at any point y in l, $T(X)_y$ can be identified with a dense open subset of the union of the three 3-spaces spanned by the three pairs of planes in X; so a generic projection of $T(X)_y$ to $\mathbf{P}^3$ is quasifinite, but not one-to-one. The ramification locus for a mapping from a surface to a 3-dimensional variety should have dimension 0, not dimension 1 (cf. § 1.3).

submanifold of some euclidean space, and $f: M \to \mathbf{R}^m$ is a suitable projection to a smaller euclidean space, then the ramification and double loci of f determine $(2n-m-1)$- and $(2n-m)$-dimensional classes in the homology of M which are Poincaré dual to characterisic classes of M [15], [16].

Our definitions of immersion and embedding for varieties lead to similar results in the geometric case. That is, if X is a subvariety of some projective space, and $f: X \to \mathbf{P}^m$ is a suitable projection, we are able to find positive cycles on X whose supports are the ramification and double loci of f, and to calculate the rational equivalence classes of these cycles in terms of characteristic classes of X. The formulas for these equivalence classes are more complicated than their topological counterparts (§ 5.2), reflecting the fact that the tangent bundle to projective space is not trivial.

There are two important features from the case when X is non-singular that are preserved by our definitions of immersion and embedding: the first is that the ramification and double cycles are uniformly of diemensions $2n-m-1$ and $2n-m$; the second is a simple relation between the ramification and double classes which leads to the following rather striking result (§ 5.2, Corollary 1):

(*) *Any variety sitting in a projective space of twice its dimension which can be immersed in a lower dimensional space by projection can be so embedded.*

In the next section we discuss alternate, more obvious, definitions of immersion and embedding which do not enjoy one or the other of these properties.

1.4. Other possible definitions

There are several other tangent cones one might use to define ramification, and hence immersion and embedding. The most obvious are (cf. [28])

T_1: The usual tangent cone;

T_2: The cone obtained by closing up the bundle of tangent spaces over nearby non-singular points;

Θ: The Zariski tangent space.

The following example shows that (*) cannot hold if one defines ramification using either T_1 or T_2: If X is the union of two lines in $\mathbf{P}^2$ intersecting at x, then $T_1(X)_x = T_2(X)_x$ can be identified with an affine subset of X. So projecting X to $\mathbf{P}^1$ from a point off X induces a quasifinite map on all tangent cones, and so is an immersion. But it is not an embedding, even though X is a 1-dimensional subvariety of $\mathbf{P}^2$. (This example also shows that the double locus need not be closed when either of these tangent cones is used.)

Defining immersion and embedding with Θ has the advantage that an embedding is always an isomorphism. However, using $\Theta(X)$ in our subsequent argument rather than $T(X)$ leads to ramification cycles which may not have uniform dimension: for example, let X be a union of planes in $\mathbf{P}^4$, $X=\bigcup_{i=1}^4 P_i$, where P_1, P_2, and P_3 have a line l in common, but are otherwise in general position, and P_4 intersects P_i in a single point $x_i \notin l$, $i=1, 2, 3$. Then $\dim \Theta(X)_{x_i}=4$, $i=1, 2, 3$, and $\dim \Theta(X)_y=4$ for any point y in l. So any projection of X to $\mathbf{P}^3$ ramifies at x_1, x_2, x_3, and along l. The ramification cycle in this case is $[x_1]+[x_2]+[x_3]+[l]$ (see § 3.1 for notation).

Remarks. 1. It follows from our main result that (*) is true even if immersion and embedding are defined using Θ (§ 5.2, Corollary 2).

2. Another definition of ramification that at first glance may seem reasonable is: *$f: X\to Y$ ramifies at $x\in X$ if the induced map $T_2(X)_x \to \Theta(Y)_{f(x)}$ is not one-to-one.* However, the following example shows that this definition is not a good one: let X be the curve in $\mathbf{C}^3$ defined by z^2-x^3 and $z-y^3$ (where x, y and z are the coordinates in $\mathbf{C}^3$), and let p be the singular point $(0, 0, 0)$ of X. Then $T_2(X)_p$ is the x-axis with multiplicity 2, so no projection of X to $\mathbf{C}^2$ induces a one-to-one map $T_2(X)_p\to \mathbf{C}^2$; however, most projections of X are isomorphisms at p.

A better definition utilizing T_2 is: *$f: X\to Y$ ramifies at $x\in X$ if the induced map $T_2(X)_x\to T_2(f(X))_{f(x)}$ is not one-to-one.* We do not know any example where the resulting ramification locus is different from the one we have defined in § 1.2.

§ 2. The ramification and double loci of a projection

In this section we give a geometric description of the ramification and double loci for a projection of $X\subset \mathbf{P}^N$ to $\mathbf{P}^m$ which provides the key to obtaining cycle structures on these sets in § 3.

2.1. Projection from a linear space

If H is a subspace of $\mathbf{P}^N$ cut out by $m+1$ independent linear forms $L_0, \ldots, L_m$, then there is a morphism $\mathbf{P}^N\setminus H\to \mathbf{P}^m$ defined by $x\mapsto (L_0(x), \ldots, L_m(x))$. The induced morphism on any quasiprojective subset of $\mathbf{P}^N\setminus H$ is called the *projection from H*.

LEMMA. *Let $C\subset \mathbf{A}^N\subset \mathbf{P}^N$ be an affine cone, and let H be a linear subspace of $\mathbf{P}^N$ disjoint from C. The projection of C from H is quasifinite if and only if $H\cap \bar{C}=\emptyset$, where $\bar{C}$ the closure of C in $\mathbf{P}^N$.*

Proof. If $H \cap \bar{C} = \emptyset$, then the projection of $\bar{C}$ from H is finite [18, p. 246] and so has finite fibres [18, p. 243]. Hence *a fortiori* the projection of C from H has finite fibres, and so is quasifinite.

If $H \cap \bar{C} \neq \emptyset$, then since C is a cone there is a line in C (through the vertex of C) whose projective closure intersects H. This line is projected from H to a point. So the projection of C from H has at least one non-finite fibre, and hence cannot be quasifinite.

2.2. A geometric description of the ramification and double loci of a projection

If x is a point in $\mathbf{P}^N$, and L is a hyperplane disjoint from x, then there is a canonical identification of $\Theta(\mathbf{P}^N)_x$ with $\mathbf{P}^N \setminus L$, [18, p. 327]. If H is a linear subspace of L, and f: $\mathbf{P}^N \setminus H \to \mathbf{P}^m$ is the projection from H, then one easily sees that the following diagram commutes:

$$\begin{array}{ccc} \Theta(\mathbf{P}^N)_x & \xrightarrow{df_x} & \Theta(\mathbf{P}^m)_{f(x)} \\ \cap & & \cap \\ \mathbf{P}^N / H & \xrightarrow{f} & \mathbf{P}^m \end{array}$$

(here $\Theta(\mathbf{P}^m)_{f(x)}$ is canonically identified with $\mathbf{P}^m \setminus (\mathbf{P}^m \cap L)$, as above).

If $X \subset \mathbf{P}^N \setminus H$ is a variety, and $x \in X$, then the inclusions $T(X)_x \subset \Theta(X)_x \subset \Theta(\mathbf{P}^N)_x \subset \mathbf{P}^N$ identify $T(X)_x$ with an affine cone in $\mathbf{P}^N$. Since the above diagram commutes, the projection of X from H ramifies at x if and only if the projection of $T(X)_x$ from H is not quasifinite. But by the lemma in § 2.1, this happens if and only if $\overline{T(X)_x} \cap H \neq \emptyset$, where $\overline{T(X)_x}$ is the closure of $T(X)_x$ in $\mathbf{P}^N$. Therefore the set

$$R(H) = \{x \in X \mid \overline{T(X)_x} \cap H \neq \emptyset\}$$

is the ramification locus of the projection of X from H.

Two different points in X have the same image under projection from H if and only if the line through the two points intersects H. Therefore

$$D(H) = \{x \in X \mid \exists\, y \in X,\ y \neq x, \text{ such that the line through } x \text{ and } y \text{ intersects } H\}$$

is the double locus of the projection of X from H.

One sees that $R(H)$ and $D(H)$ are closed subsets of X as follows: let G be the Grassmannian of lines in $\mathbf{P}^N$, let $\widetilde{X \times X}$ be the closure in $X \times X \times G$ of the graph of the mapping

$$X \times X \setminus \text{diagonal} \to G,\ (x, y) \to \text{the line through } x \text{ and } y,$$

and let $P(X)$ be the part of $\widetilde{X \times X}$ lying over the diagonal in $X \times X$, i.e.

$$P(X) = \widetilde{X \times X} \times_{X \times X} \text{diagonal}.$$

Then the fibre of $P(X)$ over (x, x) consists, as a set, of all lines through x in $\overline{T(X)}_x$ (for proof, see Appendix C).

The projection

$$\gamma: \widetilde{X \times X} \to G$$

takes a point in $P(X)$ to the corresponding line in G, and takes the point lying over $(x, y) \in X \times X \setminus \text{diagonal}$ to the line through x and y. So if

$$W(H) \subset G$$

is the Schubert variety of lines which intersect H, then $\gamma^{-1}W(H)_{(x,x)}$ is non-empty provided there is a line in $\overline{T(X)}_X$ which intersects H, and $\gamma^{-1}W(H)_{(x,y)}$ is non-empty provided the line through x and y intersects H. That is,

$$x \in R(H) \Leftrightarrow \gamma^{-1}W(H)_{(x,x)} \neq \emptyset, \text{ and } x \in D(H) \Leftrightarrow \gamma^{-1}W(H)_{(x,y)} \neq \emptyset$$

for some y.

We summarize these observations:

PROPOSITION. *Let $\gamma: \widetilde{X \times X} \to G$ and $\pi: \widetilde{X \times X} \to X \times X \xrightarrow{p_1} X$ be the projections, and let $g = \gamma|_{P(X)}$, $p = \pi|_{P(X)}$. Then*

$$\text{(i)}\ \ R(H) = pg^{-1}W(H)$$

and

$$\text{(ii)}\ \ D(H) = \pi\gamma^{-1}W(H)$$

where $R(H)$ and $D(H)$ are the ramification and double loci of the projection of X from H. It follows that $R(H)$ and $D(H)$ are closed subsets of X.

In the next section we will construct ramification and double cycles for the projection of X from H by means of cycle-theoretic analogues of (i) and (ii).

§ 3. Ramification and double cycles

In this section we find positive cycles whose supports are the ramification and double loci discussed in § 2.

3.1. Algebraic cycles

We recall some definitions from [2], [23].

Let X be a projective scheme. The free abelian group on the set of irreducible subvarieties of X is called the group of *algebraic cycles* of X. If V is an irreducible subvariety of X, we denote the corresponding cycle by $[V]$. The subgroup generated by the p-dimensional subvarieties of X is called the group of *p-cycles*.

Let X and Y be projective schemes defined over a field k, and let $f: X \to Y$ be a proper morphism. There is a corresponding homomorphism f_* of cycle groups, which is defined as follows: If V is an irreducible subvariety of X, then

$$f_*[V] = d[fV],$$

where $d = 0$ if $\dim dV < \dim V$, and $d = [k(V): k(fV)]$, the degree of the function field extension, if $\dim fV = \dim V$.

A *positive cycle* is a cycle $\sum m_i[V_i]$ such that $m_i > 0$ for all i. The set $\cup V_i$ is called the *support* of the cycle.

If Y is non-singular, and W is an irreducible subvariety of Y such that all components of $f^{-1}W$ have dimension $\dim X - \operatorname{codim}_y W$, then Serre has defined a positive cycle

$$[W]._f[X]$$

whose support is $f^{-1}W$ [23, V § 7], [2, § 2.1]. This cycle is called the *pull-back* of $[W]$ by f.

3.2. Main theorem

The following theorem says that the set-theoretic identities $R(H) = pg^{-1}W(H)$ and $D(H) = \pi\gamma^{-1}W(H)$ from § 2.2 have reasonable cycle-theoretic analogues.

THEOREM. *Let X be a purely n-dimensional subvariety of $\mathbf{P}^N$. For a generic linear subspace H of $\mathbf{P}^N$ having dimension $N-m-1$* (1)

(i) $\mathbf{R}(H) = p_*([W(H)]._g[P(X)])$ *is a positive cycle whose support is the union of the components of $R(H)$ having dimension $2n-m-1$. (These are the largest components of $R(H)$, and include all components not contained in the singular locus of X.)*

(ii) $\mathbf{D}(H) = \pi_*([W(H)]._\gamma[\widetilde{X \times X}])$ *is a positive $(2n-m)$-cycle whose support is all of $D(H)$.*

Proof. We note first that the lemma proved in Appendix A implies that all components of $g^{-1}W(H)$ have dimension $2n-m-1$, and all components of $\gamma^{-1}W(H)$ have dimension $2n-m$. It follows that $[W(H)]._g[P(X)]$ and $[W(H)]._g[\widetilde{X \times X}]$ are positive cycles whose supports are $g^{-1}W(H)$ and $\gamma^{-1}W(H)$ respectively (cf. § 3.1). Therefore $\mathbf{R}(H)$ and $\mathbf{D}(H)$ are positive cycles of the stated dimensions.

To complete the proof of (ii), it suffices to show that for any component V of $\gamma^{-1}W(H)$, $\pi(V)$ has the same dimension as V. V is not contained in $P(X)$, since if it were it would be

(1) When we say that a statement is true for a *generic* (i-dimensional) H, we mean that there is a dense open subset U of the Grassmannian of i-dimensional subspaces of $\mathbf{P}^N$ such that the statement is true for all H corresponding to points in U.

a component of $g^{-1}W(H)$, which its dimension does not allow. Letting V' be the open subset of V disjoint from $P(X)$, we show that the fibres of $V' \to \pi(V')$ are finite, which implies that the dimensions of V' and $\pi(V')$, and hence of V and $\pi(V)$, are the same: in fact, the (set-theoretic) fibre of $\pi|_{V'}$ over a point x in X is in one-to-one correspondence with the set of points y in X different from x such that the line through x and y intersects H. This set is contained in $L \cap X$, where L is the smallest linear subspace of $\mathbf{P}^N$ containing both H and x. Since H has codimension l in L, and we are working over an algebraically closed field, the dimension of $L \cap X$ must be 0 in order for X and H to be disjoint. So $L \cap X$ is finite, and hence the fibre of $\pi|_{V'}$ over x is also finite.

To complete the proof of (i), we consider the variety $\tilde{X} \subset X \times G_n(\mathbf{P}^N)$ which is the closure of the graph of the "Gauss map"

$$X \setminus \text{singular locus} \to G_n(\mathbf{P}^N),$$

$$x \mapsto \text{the projective closure of } \Theta(X)_x.$$

($G_n(\mathbf{P}^N)$ is the Grassmannian of n-planes in $\mathbf{P}^N$). If p_1 and p_2 are the projections of $\tilde{X}$ to X and to $G_n(\mathbf{P}^N)$, then $R(H)$ and $p_1 p_2^{-1} W'(H)$ coincide on the nonsingular part of X (where $W'(H)$ is the Schubert variety of n-planes which intersect H). By the lemma of Appendix A, all component of $p_2^{-1} W'(H)$ have dimension $2n-m-1$. Since p_1 is an isomorphism on the nonsingular part of X, all components of $R(H)$ which intersect the nonsingular part must also have dimension $2n-m-1$.

Remarks. 1. If ramification is defined using the tangent cone T_2 (cf. § 1.4), then the ramification locus of the projection of X from H is $p_1 p_2^{-1} W'(H)$, which has the advantage over $R(H)$ of being of uniform dimension when H is generic. The disadvantages of this definition were discussed in § 1.4.

2. Components of $R(H)$ having dimension less than $2n-m-1$ are possible: for example, let X be the union of two planes in $\mathbf{P}^4$ which intersect in a point x, and let H be a line which is disjoint from X. Then $R(H) = \{x\}$, but $2n-m-1=1$.

§ 4. Chow theory and Chern classes

In this section we state the facts about Chow homology and cohomology, and about Chern classes, which we use subsequently. The reader is referred to [2] and [5] for details. We define Segre classes for singular varieties in § 4.3.

Our argument does not require Chow theory specifically. Any homology-cohomology theory that has a projection formula and that carries Chern classes in cohomology, such as ordinary singular theory, would work just as well.

4.1. Chow homology and cohomology

If X is a projective scheme, the *Chow homology group* of X is the graded group $A_. X = \oplus A_p X$, where $A_p X$ is the group of p-cycles modulo rational equivalence. If $f: X \to Y$ is a proper morphism of projective schemes, then the homomorphism f_* defined on cycles induces a homomorphism $A_. X \to A_. Y$, which is also denoted f_*. If V is an irreducible subvariety of X, then $[V]$ denotes either the cycle determined by V or the equivalence class of that cycle in $A_. X$. The *fundamental class* of X is the class

$$[X] = \sum m_i [V_i],$$

where $V_1, V_2, \ldots$ are the reduced, irreducible components of X, and m_i is the multiplicity of V_i in X.

One can also define a graded ring $A^{\cdot} X$, the *Chow cohomology ring*, which carries Chern classes for any locally free sheaf on X. (If X is nonsingular of dimension n, then $A^q X = A_{n-q} X$.) A proper morphism $f: X \to Y$ induces a ring homomorphism $f^*: A^{\cdot} Y \to A^{\cdot} X$.

There is a *cap product* $\frown: A^{\cdot} X \otimes A_. X \to A_. X$ which makes $A_. X$ into an $A^{\cdot} X$-module, and which also satisfies a *projection formula*: If $f: X \to Y$ is proper, and $x \in A_. X$, and $y \in A^{\cdot} Y$, then

$$f_*(f^* y \frown x) = y \frown f_* x.$$

4.1.1. *If Y is a nonsingular variety, $f: X \to Y$ is a morphism, $V \subset Y$, and $[V]._f [X]$ is defined, then the class of $[V]._f [X]$ in $A_. X$ is $f^*[V] \frown [X]$.*

4.2. Chern classes and Segre classes

Let X be a nonsingular variety, let $\mathfrak{F}$ be a locally free sheaf of rank r on X, and let $P(\mathfrak{F}) = \operatorname{Proj}(\operatorname{Sym}_{O_X} \mathfrak{F})$ [6, II]. If $\xi \in A^1 P(\mathfrak{F})$ is the Chern class of the Serre line bundle $O_{P(\mathfrak{F})}(1)$ on $P(\mathfrak{F})$, then

$$A^{\cdot} P(\mathfrak{F}) \approx A^{\cdot} X[\xi] \Big/ \sum_{i=0}^{r} (-1)^i c_i \xi^{r-i}$$

where $c_0 = 1 = [X] \in A^0 X$, and $c_i = c_i(\mathfrak{F}) \in A^i X$ is the ith Chern class of $\mathfrak{F}$.

If $p: P(\mathfrak{F}) \to X$ is the natural projection, then

$$s_i(\mathfrak{F}) = p_*(\xi^{d-1+i} \frown [P(\mathfrak{F})]) \in A^i X$$

is the ith *Segre class* of $\mathfrak{F}$. The Chern classes and Segre classes are related by the equation

$$\left(\sum_{i=0}^{d}(-1)^i c_i\right)\left(\sum_{i=0}^{n} s_i\right)=1, \quad n=\dim X$$

(so $s_1=c_1$, $s_2=c_1^2-c_2$, ...).

The ith *Segre class of* X, $s_i(X)$, is defined to be $s_i(\Omega_X)$, where Ω_X is the cotangent sheaf on X.

4.3. Segre classes for singular varieties

Let X be a purely n-dimensional variety, let $\mathfrak{J}$ be the ideal sheaf of the diagonal in $X\times X$, let $P(X)=\operatorname{Proj}(\oplus_{j=0}^{\infty}\mathfrak{J}^j/\mathfrak{J}^{j+1})$ (1), let $p: P(X)\to X$ be the natural projection, let $O_{P(X)}(1)$ be the Serre line bundle on $P(X)$, and let $\xi=c_1(O_{P(X)}(1))\in A^1P(X)$. We define

$$s_i(X)=p_*(\xi^{n-1+i}\frown[P(X)])\in A_{n-i}X$$

to be the ith *Segre class of* X. When X is nonsingular this definition agrees with the one given in § 4.2 above, since in this case $\Omega_X=\mathfrak{J}/\mathfrak{J}^2$ and $\operatorname{Sym}_{O_X}(\mathfrak{J}/\mathfrak{J}^2)=\oplus_{j=0}^{\infty}\mathfrak{J}^j/\mathfrak{J}^{j+1}$ [6, IV 16].

We note that $s_i(X)=0$ if $i<0$ or $i>n$.

§ 5. Homology classes of the ramification and double cycles

In this section we calculate the classes in $A_{.}X$ of the cycles we found in § 3.2. Our formulas are in terms of the Segre classes defined in § 4.3. In § 5.2 we state our main results.

5.1. Calculation of the classes

Let X be an n-dimensional subvariety of $\mathbf{P}^N$, and let H be an $(N-m-1)$-dimensional subspace of $\mathbf{P}^N$ which is generic in the sense of § 3.2. According to 4.1.1, the homology classes of the ramification and double cycles of the projection of X from H (cf. § 3.2) are, respectively, $p_*(g^*[W(H)]\frown[P(X)])$ and $\pi_*(\gamma^*[W(H)]\frown[\widetilde{X\times X}])$ (Recall that $\gamma: \widetilde{X\times X}\to G=G_1(\mathbf{P}^N)$ and $\pi: \widetilde{X\times X}\to X\times X\xrightarrow{p_1}X$ are projections (§ 2.2), $g=\gamma|_{P(X)}$, $p=\pi|_{P(X)}$, and $W(H)\subset G_1(\mathbf{P}^N)$ is the Schubert variety of lines which intersect H.)

In Appendix B we derive the following formulas:

$$g^*[W(H)]=(2n-m-1)(p^*u)^m+\sum_{j=0}^{m-1}\binom{m+1}{j+2}(p^*u)^{m-1-j}\xi^{j+1}\in A^{\cdot}P(X) \tag{1}$$

$$\gamma^*[W(H)]=\sum_{k=0}^{n}(\pi^*u)^k t^{m-k}+\sum_{j=0}^{m-1}(-1)^{j+1}\binom{m+1}{j+2}(\pi^*u)^{m-1-j}\eta^{j+1}\in A^{\cdot}\widetilde{X\times X} \tag{2}$$

(1) If X is a subvariety of $\mathbf{P}^N$, then this is the same as the space defined in § 2.2 (Appendix C).

where $u \in A^{\cdot}X$ is the class of a hyperplane section, ξ is the Chern class of the Serre line bundle $O_{P(X)}(1)$, t is the pullback of u by the projection $\widetilde{X \times X} \to X \times X \xrightarrow{p_2} X$, and η is the pull-back of the class of the exceptional divisor in $\widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ (1) by the inclusion $\widetilde{X \times X} \hookrightarrow \widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ (cf. Appendix C).

From (1) we may calculate directly that

$$\begin{aligned}
&p_*(g^*[W(H)] \frown [P(X)]) \\
&\quad = (2n-m-1)\,((p^*u)^m \frown [P(X)]) + \sum_{j=0}^{m-1} \binom{m+1}{j+2} p_*((p^*u)^{m-1-j}\xi^{j+1} \frown [P(X)]) \\
&\quad = (2n-m-1)\, u^m \frown p_*[P(X)] + \sum_{j=0}^{m-1} \binom{m+1}{j+2} u^{m-1-j} \frown p_*(\xi^{j+1} \frown [P(X)]) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(projection formula)} \\
&\quad = \sum_{j=n-2}^{m-1} \binom{m+1}{j+2} u^{m-1-j} \frown s_{j-n+2}(X)
\end{aligned}$$

since $P(X)$ has dimension $2n-1>n$, so $p_*[P(X)]=0$. (If $n=1$, then $2n-m-1 \leqslant 0$). Changing indices, this becomes

$$p_*(g^*[W(H)] \frown [P(X)]) = \sum_{i=0}^{m-n+1} \binom{m+1}{i+n} u^{m-n-i+1} \frown s_i(X) \in A_{.}X. \tag{3}$$

In order to calculate $\pi_*(\gamma^*[W(H)] \frown [\widetilde{X \times X}])$, we need the following lemma:

LEMMA. $\pi_*(\eta^{n+i} \frown [\widetilde{X \times X}]) = (-1)^{n-1+i} s_i(X)$.

Proof. Let $k\colon P(X) \hookrightarrow \widetilde{X \times X}$ be the inclusion. Then $k^*\eta = -\xi$, and $k_*[P(X)] = \eta \frown [\widetilde{X \times X}]$ (see Appendix B, § 4). So

$$\begin{aligned}
\pi_*(\eta^{n+i} \frown [\widetilde{X \times X}]) &= \pi_*(\eta^{n-1+i} \frown (\eta \frown [\widetilde{X \times X}])) \\
&= \pi_*(\eta^{n-1+i} \frown k_*[P(X)]) \\
&= \pi_* k_*(k^*(\eta^{n-1+i}) \frown [P(X)]) \\
&\qquad \text{(projection formula)} \\
&= p_*((-\xi)^{n-1+i} \frown [P(X)]) \\
&= (-1)^{n-1+i} s_i(X).
\end{aligned}$$

(1) $\widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ is the blow-up of $\mathbf{P}^N \times \mathbf{P}^N$ along its diagonal; or it can be defined as in § 2.2, in which case the exceptional divisor is $P(\mathbf{P}^N)$. See Appendix C.

Then, using (2), we have

$$\pi_*(\gamma^*[W(H)]\frown\widetilde{[X\times X]})$$

$$= \sum_{k=0}^{m} \pi_*((\pi^*u)^k t^{m-k}\frown\widetilde{[X\times X]}) + \sum_{j=0}^{m-1}(-1)^{j+1}\binom{m+1}{j+2}\pi_*((\pi^*u)^{m-1-j}\eta^{j+1}\frown\widetilde{[X\times X]})$$

$$= \sum_{k=0}^{m} u^k\frown\pi_*(t^{m-k}\frown\widetilde{[X\times X]}) + \sum_{j=0}^{m-1}(-1)^{j+1}\binom{m+1}{j+2}u^{m-1-j}\frown\pi_*(\eta^{j+1}\frown\widetilde{[X\times X]})$$

(projection formula)

$$= u^{m-n}\frown\pi_*(t^n\frown\widetilde{[X\times X]}) - \sum_{j=n-1}^{m-1}\binom{m+1}{j+2}u^{m-1-j}\frown s_{j-n+1}(X).$$

($t^l\frown\widetilde{[X\times X]}$ has dimension $>n$ for $l<n$, and t^l is the pullback of $u^l\in A^{\cdot}X$, which is 0 for $l>n$). Letting d be the degree of X, and changing indices, this becomes

$$\pi_*(\gamma^*[W(H)]\frown\widetilde{[X\times X]}) = u^{m-n}\frown d[X] - \sum_{i=0}^{m-n}\binom{m+1}{i+n+1}u^{m-n-i}\frown s_i(X)\in A_{\cdot}X. \tag{4}$$

5.2. Main results

The following theorem is an immediate consequence of the theorem in § 3.2, and of formulas (3) and (4) above.

THEOREM. *Let X be a purely n-dimensional subvariety of $\mathbf{P}^N$, let $\mathbf{R}(H)$ and $\mathbf{D}(H)$ be the ramification and double cycles of the projection of X from a generic $(N-m-1)$-dimensional linear space H as defined in* § 3.2, *and let R_m and D_m be the Chow homology classes of $\mathbf{R}(H)$ and $\mathbf{D}(H)$ respectively. Then*

$$R_m = \sum_{i=0}^{m-n+1}\binom{m+1}{i+n}u^{m-n-i+1}\frown s_i(X)$$

and

$$D_m = u^{m-n}\frown d[X] - \sum_{i=0}^{m-n}\binom{m+1}{i+n+1}u^{m-n-i}\frown s_i(X)\ (^1)$$

where $u\in A^1X$ is the class of a hyperplane section, $s_i(X)\in A_{n-i}X$ is the ith *Segre class of X* (§ 4.3), *$[X]\in A_nX$ is the fundamental class of X, and d is the degree of X. It follows that*

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(1) Laksov [13] defines the rth *Todd class* $t_r(f)$ of a morphism $f\colon X\to Y$ to be the class

$$t_r(f) = \sum_{i=0}^{r} f^*c_{r-i}(Y)\frown s_i(X).$$

Using this notation we can rewrite these formulas more compactly as $R_m = t_{m-n+1}(f)$, $D_m = f^*f_*[X] - t_{m-n}(f)$, where $f\colon X\to\mathbf{P}^N$ is a generic projection.

(i) *Both* R_m *and* D_m *have non-negative degree,*

(ii) *If* X *can be immersed in* $\mathbf{P}^m$ *by projection, then* $R_m=0$, *and*

(iii) X *can be embedded in* $\mathbf{P}^m$ *by projection if and only if* $D_m=0$.

If X *is nonsingular then the converse of* (ii) *also holds.*

Using the identity

$$\binom{m+2}{i+n+1}-\binom{m+1}{i+n+1}=\binom{m+1}{i+n},$$

one can see that the classes R_m and D_m are related as follows:

$$uD_m-D_{m+1}=R_m.$$

This relation leads to a rather surprising corollary:

COROLLARY 1. *Let* X *be an n-dimensional subvariety of* $\mathbf{P}^N$, $N\leqslant 2n$. *If* X *can be immersed in a lower dimensional projective space by projection, then it can be so embedded.*

Proof. If X can be immersed in $\mathbf{P}^m$ by projection, then certainly X can be immersed in $\mathbf{P}^{m+1}$, $\mathbf{P}^{m+2}$, ... by projection, so $R_m=R_{m+1}=\ldots=R_{2n-1}=0$. Hence

$$\begin{aligned} uD_m-D_{m+1}&=R_m=0\\ uD_{m+1}-D_{m+2}&=0\\ &\vdots\\ uD_{2n-1}-D_{2n}&=0\end{aligned}$$

and therefore $u^{2n-m}D_m=D_{2n}$. But $D_{2n}=0$ (since $X\subset\mathbf{P}^{2n}$), so $D_m=0$, and X can be embedded in $\mathbf{P}^m$ by projection.

Note. This proof does not work when $N=2n+1$ because all terms in the equation $uD_{2n}-D_{2n+1}=R_{2n}$ are 0, all having dimension -1.

There is a similar result for the stronger notions of immersion and embedding defined using the Zariski tangent space (cf. § 1.4):

COROLLARY 2. *Let* X *be an n-dimensional subvariety of* $\mathbf{P}^N$, $N\leqslant 2n$. *If* X *can be projected to* $\mathbf{P}^m$ *so that the induced maps of Zariski tangent spaces are all one-to-one, then* X *can be projected isomorphically to* $\mathbf{P}^m$.

Proof. One need only show that a generic projection of X to $\mathbf{P}^m$ is one-to-one. Since such a projection induces one-to-one maps on Zariski tangent spaces by hypothesis, it is *a fortiori* an immersion. By Corollary 1, it is therefore an embedding; in particular, it is one-to-one.

Appendix A

In this appendix we prove that the dimension of the inverse image of a generic Schubert variety is always what it should be.

Let G be the Grassmannian parameterizing r-dimensional subspaces of $\mathbf{P}^N$. For integers $0 \leqslant a_0 \leqslant a_1 \leqslant \ldots \leqslant a_r \leqslant N$, let F be the flag variety of all sequences $A_0 \subset A_1 \subset \ldots \subset A_r$ of subspaces of $\mathbf{P}^N$ such that $\dim A_i = a_i$. The points of F can be identified with Schubert subvarieties of G,

$$(A_0, A_1, \ldots, A_r) \leftrightarrow \{L \in G \mid \dim (L \cap A_i) \geqslant i,\ i=0, \ldots, r\}.$$

LEMMA. *Let X be a variety, and let $f: X \to G$ be a proper morphism. Then for Schubert varieties Ω in a dense open subset of F, the components of $f^{-1}\Omega$ have codimension in X equal to the codimension of Ω in G.*

Proof. Let Y be a nonempty irreducible subset of fX over which the fibres of f have constant dimension; then

$$\operatorname{codim}_{f^{-1}Y} f^{-1}(Y \cap \Omega) = \operatorname{codim}_Y Y \cap \Omega = \operatorname{codim}_{\overline{Y}} \overline{Y} \cap \Omega$$

where $\overline{Y}$ is the closure of Y in G. If the lemma holds for subvarieties of G, then $\operatorname{codim}_{\overline{Y}} \overline{Y} \cap \Omega = \operatorname{codim}_G \Omega$ for all Ω in a nonempty open subset of F. Since $X = \cup f^{-1} Y$ for finitely many such Y, the lemma then holds for X.

So we are reduced to the case when X is a subvariety of G, and we must show that for all Ω in an open subset of F, Ω and X intersect in the right dimension. Since it is not possible for the dimension of a component of $\Omega \cap X$ to be smaller than it should be [27, p. 146, Cor. 1], it suffices to show that

$$\operatorname{codim}_X \Omega \cap X \geqslant \operatorname{codim}_G \Omega.$$

Let p be the projection of

$$\bigcup_{\Omega \in F} \Omega \times \{\Omega\} \subset G \times F$$

on G; then $\Omega \cap X = p(p^{-1}X \times_F \Omega)$, so

$$\dim (\Omega \cap X) \leqslant \dim (p^{-1}X \times_F \Omega).$$

A standard fact about the dimension of fibres [18, p. 96, Cor. 1] implies that, for all Ω in a nonempty open subset of F,

$$\begin{aligned}\dim (p^{-1}X \times_F \Omega) &\leqslant \dim p^{-1}X - \dim F \\ &= \dim X - \operatorname{codim}_G \Omega.\end{aligned}$$

Therefore $\operatorname{codim}_X \Omega \cap X \geqslant \operatorname{codim}_G \Omega$, as required.

Appendix B

The purpose of this appendix is to derive formulas (1) and (2) of § 5.1.

§ 1. Let G be the Grassmannian parameterizing lines in $\mathbf{P}^N$. There is an exact sequence of bundles on G,

$$0 \to \check{K} \to \mathbf{E}^{N+1},$$

where $\mathbf{E}^{N+1} = \mathbf{A}^{N+1} \times G$ is the trivial rank $N+1$ bundle, and $\check{K}$ is the "tautological" rank 2 bundle (which has as fibre over a point in G the plane in $\mathbf{A}^{N+1}$ which determines the line in $\mathbf{P}^N$ corresponding to that point).

Let $P = \operatorname{Proj}(\operatorname{Sym}_{O_G} \mathcal{K})$, where $\mathcal{K}$ is the sheaf of sections of K. Let $c_i = c_i(\mathcal{K}) \in A^1G$, $i = 1, 2$, let $\zeta = c_1(O_P(1)) \in A^1P$, and let $\bar{c}_i$, ζ_1, $\zeta_2 \in A^1P \times_G P$ be the pullbacks of these classes by the projections $P \times_G P \to G$, $P \times_G P \xrightarrow{p_i} P$, $i = 1, 2$. Finally, let $\delta \in A^1P \times_G P$ be the class of the diagonal.

LEMMA 1. (i) $\bar{c}_1 = \zeta_1 + \zeta_2 - \delta$, *and* (ii) $\bar{c}_2 = \zeta_2(\zeta_1 - \delta)$.

Proof. In $A^1P \times_G P$ the equations $\zeta_1^2 - \bar{c}_1\zeta_1 + \bar{c}_2 = 0$ and $\zeta_2^2 - \bar{c}_1\zeta_2 + \bar{c}_2 = 0$ both hold (§ 4.2). The second of these, together with (i), implies (ii):

$$\begin{aligned}\bar{c}_2 &= \zeta_2(\bar{c}_1 - \zeta_2)\\ &= \zeta_2((\zeta_1 + \zeta_2 - \delta) - \zeta_2)\\ &= \zeta_2(\zeta_1 - \delta).\end{aligned}$$

To prove (i), we argue as follows: since c_1 generates A^1G (this follows from the Basis theorem, [7, p. 350]), the classes ζ_1, ζ_2 and $\bar{c}_1$ generate $A^1P \times_G P$. So we may write

$$\delta = a(\zeta_1 + \zeta_2) + b\bar{c}_1,$$

where a and b are integers.

In determining a and b we will use the fact that the composition

$$P \xhookrightarrow{d} P \times_G P \xrightarrow{p_i} P,$$

where d is the diagonal embedding and p_i is either of the two projections, is the identity.

Claim 1. $a = 1$, and $p_{1*}\zeta_2 = [P]$.

Proof. $p_{1*}\delta = p_{1*}d_*[P] = id_*[P] = [P]$, so

$$\begin{aligned}[P] &= p_{1*}(a(\zeta_1 + \zeta_2) + b\bar{c}_1)\\ &= ap_{1*}\zeta_1 + ap_{1*}\zeta_2 + bp_{1*}\bar{c}_1\\ &= ap_{1*}\zeta_2\end{aligned}$$

(both $p_{1*}\zeta_1$ and $p_{1*}\bar{c}_1$ are 0, by the projection formula). Since $p_{1*}\zeta_2$ is positive (because ζ_2 is positive), the claim follows.

Claim 2. $b=-1$.

Proof. Since $d^*\zeta_i = d^*p_i^*\zeta = \zeta$ for $i=1, 2$, we have

$$d_*\zeta = \zeta_1 d_*[P] = \zeta_2 d_*[P]$$

by the projection formula, and therefore

$$\delta(\zeta_1-\zeta_2) = d_*[P](\zeta_1-\zeta_2) = 0.$$

Substituting $\delta=\zeta_1+\zeta_2+b\bar{c}_1$ in this equation gives

$$\zeta_1^2-\zeta_2^2+b\bar{c}_1(\zeta_1-\zeta_2) = 0. \tag{1}$$

From $\zeta_i^2=\bar{c}_1\zeta_i-\bar{c}_2$ for $i=1, 2$, we have

$$\zeta_1^2-\zeta_2^2 = \bar{c}_1(\zeta_1-\zeta_2),$$

which combines with (1) to give

$$(1+b)\bar{c}_1(\zeta_1-\zeta_2) = 0.$$

So we may conclude that either $\bar{c}_1(\zeta_1-\zeta_2)=0$ or $b=-1$.

Letting $p: P\to G$ be the projection, we have

$$\begin{aligned} p_{1*}(\bar{c}_1(\zeta_1-\zeta_2)) &= p_{1*}(\bar{c}_1\zeta_1) - p_{1*}(\bar{c}_1\zeta_2) \\ &= (p^*c_1)p_{1*}\zeta_1 - (p^*c_1)p_{1*}\zeta_2 \\ &\qquad \text{(projection formula)} \\ &= -p^*c_1 \end{aligned}$$

since $p_{1*}\zeta_1=0$ and $p_{1*}\zeta_2=[P]$). $p^*c_1 \neq 0$, because $c_1\neq 0$ and p^* is an injection. So $\bar{c}_1(\zeta_1-\zeta_2)\neq 0$, and hence $b=-1$.

§ 2. Let Δ be the diagonal in $\mathbf{P}^N\times\mathbf{P}^N$. As in § 2.2, let $\widetilde{\mathbf{P}^N\times\mathbf{P}^N}$ be the closure in $\mathbf{P}^N\times\mathbf{P}^N\times G$ of the graph of the mapping

$$\mathbf{P}^N\times\mathbf{P}^N\setminus\Delta\to G$$

$$(x, y)\mapsto \text{the line through } x \text{ and } y.$$

Let $E=\widetilde{\mathbf{P}^N\times\mathbf{P}^N}\times_{\mathbf{P}^N\times\mathbf{P}^N}\Delta$, and let $p: P\to\mathbf{P}^N$ be the projection of the diagonal in $P\times_G P$ on Δ.

LEMMA 2. *There is a morphism* $\alpha\colon \widetilde{\mathbf{P}^N\times\mathbf{P}^N}\hookrightarrow P\times_G P$ *making the following square commute for* $i=1, 2$*:*

$$\begin{tikzcd} \widetilde{\mathbf{P}^N\times\mathbf{P}^N} \arrow[r, "\alpha"] \arrow[d, "p_i"] & P\times_G P \arrow[d, "p_i'"] \\ \mathbf{P}^N & P \arrow[l, "p"'] \end{tikzcd}$$

(*the vertical arrows are the obvious projections*). *Moreover* $\alpha^*\delta=[E]$.

Proof. The exact sequence $O_G^{N+1}\to\mathcal{K}\to 0$ of sheaves on G gives an inclusion $P\hookrightarrow\mathbf{P}^N\times G$, and hence $P\times_G P$ is contained in $\mathbf{P}^N\times\mathbf{P}^N\times G$. $P\times_G P$ coincides with $\widetilde{\mathbf{P}^N\times\mathbf{P}^N}$ over $\mathbf{P}^N\times\mathbf{P}^N\setminus\Delta$, and so, since $P\times_G P$ is closed, and $\widetilde{\mathbf{P}^N\times\mathbf{P}^N}$ is reduced, we have $\widetilde{\mathbf{P}^N\times\mathbf{P}^N}\xhookrightarrow{\alpha} P\times_G P$.

Since $E=\widetilde{\mathbf{P}^N\times\mathbf{P}^N}\times_{P\times_G P}D$, where D is the diagonal in $P\times_G P$, it follows from [2, § 2.2, Lemma (4)] that $\alpha^*\delta=[E]$.

Remark. α is actually an isomorphism, since $P\times_G P$ is reduced and irreducible.

§ 3. Let $i\colon \widetilde{X\times X}\hookrightarrow\widetilde{\mathbf{P}^N\times\mathbf{P}^N}$ be the inclusion induced by $X\hookrightarrow\mathbf{P}^N$. Then $\gamma\colon\widetilde{X\times X}\to G$ is the composition

$$\widetilde{X\times X}\xrightarrow{\;i\;}\widetilde{\mathbf{P}^N\times\mathbf{P}^N}\xrightarrow{\;\alpha\;}P\times_G P\longrightarrow G.$$

Let $\eta=i^*[E]\in A^1\widetilde{X\times X}$, let $u\in A^1X$ be the class of a hyperplane section, and let p_1, p_2: $X\times X\rightrightarrows X$ be the projections. Recall that $c_i=c_i(\mathcal{K})\in A^{\cdot}G$.

LEMMA 3. (i) $\gamma^*c_1=p_1^*u+p_2^*u-\eta$, and (ii) $\gamma^*c_2=p_2^*u(p_1^*u-\eta)$.

Proof. This is an immediate consequence of Lemma 1, once we have established the following claims:

Claim 1. $i^*\alpha^*\zeta_1=p_1^*u$, *and* $i^*\alpha^*\zeta_2=p_2^*u$.

Proof. We know from Lemma 2 that the following diagram is commutative for $i=1, 2$:

$$\begin{tikzcd} \widetilde{X\times X} \arrow[rr, "\alpha\circ i"] \arrow[d, "p_i"] & & P\times_G P \arrow[d, "p_i'"] \\ X \arrow[r, "j"] & \mathbf{P}^N & P \arrow[l, "p"'] \end{tikzcd} .$$

Hence $p_i^*j^*O_{\mathbf{P}^N}(1)=(\alpha\circ i)^*p_i'^*O_{\mathbf{P}^N}(1)$, $i=1, 2$. The Chern class of the former is p_i^*u, and then Chern class of the latter is $i^*\alpha^*\zeta_i$ (since $p^*O_{\mathbf{P}^N}(1)=O_P(1)$, $\zeta=c_1O_P(1)$, and $\zeta_i=p_i'^*\zeta$).

Claim 2. $i^*\alpha^*\delta = \eta$.

Proof. $\alpha^*\delta = [E]$ (Lemma 2), so $i^*\alpha^*\delta = i^*[E] = \eta$ by definition.

THEOREM. *If H is an $(N-m-1)$-dimensional subspace of $\mathbf{P}^N$, and $W(H)$ is the Schubert variety in G of lines which intersect H, then*

$$\gamma_*[W(H)] = \sum_{k=0}^{m} (p_1^* u)^k (p_2^* u)^{m-k} + \sum_{j=0}^{m-1} (-1)^{j+1} \binom{m+1}{j+2} (p_1^* u)^{m-1-j} \eta^{j+1}.$$

Proof. Let Q be the universal quotient on G, that is, the rank $N-1$ bundle which makes the sequence

$$0 \to \check{K} \to \mathbf{E}^{N+1} \to Q \to 0$$

exact. It is well-known [11, Prop. 5.6] that $[W(H)] = c_m(Q)$. The exactness of the above sequence implies that

$$(1 + c_1(\check{K}) + c_2(\check{K}))(1 + c_1(Q) + c_2(Q) + \dots + c_{N-1}(Q)) = 1,$$

since $\mathbf{E}^{N+1}$ is trivial. Writing $c_1 = c_1(K) = -c_1(\check{K})$, $c_2 = c_2(K) = c_2(\check{K})$ as before, we may solve for the Chern classes of Q to obtain equations

$$c_m(Q) = c_1 c_{m-1}(Q) - c_2 c_{m-2}(Q), \quad m = 1, \dots, N-1.$$

Hence

$$\begin{aligned} \gamma^* c_m(Q) &= (\gamma^* c_1)\gamma^* c_{m-1}(Q) - (\gamma^* c_2)\gamma^* c_{m-2}(Q) \\ &= (p_1^* u + p_2^* u - \eta)\gamma^* c_{m-1}(Q) - p_2^* u (p_1^* u - \eta)\gamma^* c_{m-2}(Q) \end{aligned}$$

(Lemma 3). Solving these equations inductively (1) gives the stated formula for $\gamma^* c_m(Q) = \gamma^*[W(H)]$.

Let $p: P(X) \to X$ be the natural projection, let $g: P(X) \to G$ be the restriction of γ, and let $\xi = c_1 O_{P(X)}(1) \in A^1 X$.

COROLLARY. *If H and $W(H)$ are as in the theorem above, then*

$$g^*[W(H)[= (2n - m - 1)(p^* u)^m + \sum_{j=0}^{m-1} \binom{m+1}{j+2} (p^* u)^{m-1-j} \xi^{j+1}$$

Proof. Let $k: P(X) \hookrightarrow \widetilde{X \times X}$ be the inclusion. Then $g = \gamma \circ k$, so $g^* = k^* \gamma^*$, and the

(1) In solving one can make use of the identity $(p_1^* u - p_2^* u)\eta = 0$ (which follows from the corresponding identity $(c_1 - c_2)\delta = 0$ found in the course of the proof of Lemma 1). This identity also implies that the formula for $\gamma^*[W(H)]$ is symmetric in $p_1^* u$ and $p_2^* u$.

formula for $g^*[W(H)]$ is an immediate consequence of the theorem above and the following two claims:

Claim 1. $k^*p_i^*u = p^*u$ *for* $i = 1, 2$.

This is because $p = p_1 \circ k = p_2 \circ k$ (recall that $p_1, p_2 \colon \widetilde{X \times X} \rightrightarrows X$ are the projections).

Claim 2. $k^*\eta = -\xi$.

This is one of the statements in the lemma below.

§ 4. The following lemma is used in the proof of the lemma in § 5.1:

LEMMA. *Let* $k \colon P(X) \to \widetilde{X \times X}$ *and* $i \colon \widetilde{X \times X} \hookrightarrow \widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ *be the inclusions, let* E *be the exceptional divisor in* $\widetilde{\mathbf{P}^N \times \mathbf{P}^N}$, *let* $\eta = i^*[E]$, *and let* $\xi = c_1 O_{P(X)}(1)$. *Then* (i) $k^*\eta = -\xi$, *and* (ii) $\eta \frown [\widetilde{X \times X}] = k_*[P(X)]$.

Proof. The diagram

$$\begin{tikzcd} P(X) \arrow[r, hook, "k"] \arrow[d, hook, "i'"] & \widetilde{X \times X} \arrow[d, hook, "i"] \\ E \arrow[r, hook, "k'"] & \widetilde{\mathbf{P}^N \times \mathbf{P}^N} \end{tikzcd}$$

is commutative, and it is known that $k'^*[E] = -\xi'$, where $\xi' = c_1 O_E(1)$ [21, p. 123]. Since $i'^*\xi' = \xi$, it follows that

$$k^*\eta = k^*i^*[E] = i'^*k'^*[E] = i'^*(-\xi') = -\xi.$$

To prove (ii), we note that the above diagram is a fibre square. Since $[P(X)] = [E]\cdot_{i'}[P(X)]$, we may use [2, § 2.2, Lemma (4)] to conclude that

$$k_*[P(X)] = k_*([E]\cdot_{i'}[P(X)]) = [E]\cdot_i[\widetilde{X \times X}].$$

In Chow groups, this equation becomes $k_*[P(X)] = \eta \frown [\widetilde{X \times X}]$, as required (cf. 4.1.1).

Appendix C

In this appendix we show that the two definitions of $P(X)$ that we have given (§ 2.2 and § 4.3) agree. An immediate corollary is the set-theoretic identity of $P(X)_x$ with the set of lines through x in $T(X)_x$, which was used in § 2.2. We also justify a remark we made in § 1.1.

§ **1.** Let Ω be the cotangent sheaf on $\mathbf{P}^N$, and let

$$P = \mathrm{Proj}\,(\mathrm{Sym}_{O_{\mathbf{P}^N}} \Omega(1)).$$

The underlying set of P_x, for x in $\mathbf{P}^N$, is the set of all lines in $\mathbf{P}^N$ through x.

Let G be the Grassmannian parameterizing lines in $\mathbf{P}^N$. Our first goal is to define a morphism $P \to G$ whose underlying set-theoretic map is the obvious one. Recall that a morphism $P \to G$ is just an exact sequence $O_P^{N+1} \to M \to 0$ of sheaves on P, where M is locally free of rank 2 [17, p. 32].

The exact sequence

$$0 \to \Omega(1) \to O_{\mathbf{P}^N}^{N+1} \to O_{\mathbf{P}^N}(1) \to 0$$

of sheaves on $\mathbf{P}^N$ lifts by the natural projection $r: P \to \mathbf{P}^N$ to an exact sequence on P, which we incorporate into the following diagram of sheaves on P:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Z \arrow[r, equal] \arrow[d] & Z \arrow[d] & & \\
0 \arrow[r] & r^*\Omega(1) \arrow[r] \arrow[d] & O_P^{N+1} \arrow[r] \arrow[d] & r^*O_{\mathbf{P}^N}(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & O_P(1) \arrow[r, dashed] \arrow[d] & M \arrow[r, dashed] \arrow[d] & r^*O_{\mathbf{P}^N}(1) \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

Here $O_P(1)$ is the Serre line bundle on P, and Z is defined to make the left column exact; then M is defined to make the center column exact. Maps indicated by the broken arrows can then be defined so that the bottom row is exact. Since $O_P(1)$ and $r^*O_{\mathbf{P}^N}(1)$ are both line bundles, M must be locally free of rank 2, thus defining the morphism $P \to G$.

§ **2.** Let $\widetilde{\mathbf{P}^N \times \mathbf{P}^N} \subset P^N \times \mathbf{P}^N \times G$ be the closure of the graph of $\mathbf{P}^N \times \mathbf{P}^N \setminus \text{diagonal} \to G$, $(x, y) \mapsto$ the line through x and y. Let I be the ideal sheaf of the diagonal in $\mathbf{P}^N \times \mathbf{P}^N$, and let

$$B = \mathrm{Proj}\left(\bigoplus_{j=0}^{\infty} I^j\right)$$

(this is the blow-up of $\mathbf{P}^N \times \mathbf{P}^N$ along its diagonal [6, II 8]).

PROPOSITION. *$\widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ and B are isomorphic as $\mathbf{P}^N \times \mathbf{P}^N$-schemes.*

Proof. We define an injection $B \hookrightarrow \mathbf{P}^N \times \mathbf{P}^N \times G$ whose set-theoretic image coincides with $\widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ off the diagonal in $\mathbf{P}^N \times \mathbf{P}^N$. Since both $\widetilde{\mathbf{P}^N \times \mathbf{P}^N}$ and B are complete, reduced and irreducible, this suffices to show that they are scheme-theoretically isomorphic. In fact there is a natural projection $B \to \mathbf{P}^N \times \mathbf{P}^N$, so it is enough to give an appropriate morphism $B \to G$.

Holme [8] defines a morphism $\lambda: B \to P$, which can be composed with the morphism $P \to G$ defined in § 1 above to give a morphism $B \to G$. The underlying set-theoretic map of λ takes the point lying over (x, y), $x \neq y$, to the point in P_y corresponding to the line through x and y. Taken together with the set-theoretic description of $P \to G$ given in § 1, this shows that the morphism $B \to G$ acts as desired on sets.

§ 3. Let X be a subvariety of $\mathbf{P}^N$, and let $\mathfrak{J}$ be the ideal sheaf of the diagonal in $X \times X$. Let

$$\widetilde{X \times X} = \operatorname{Proj}\left(\bigoplus_{j=0}^{\infty} \mathfrak{J}^j\right)$$

be the blow-up of $X \times X$ along its digonal. This is the proper transform of $X \times X$ in B, so by the above proposition it agrees with the definition given in § 2.2. From this it follows that

$$\widetilde{X \times X} \times_{X \times X} \Delta = \operatorname{Proj}\left(\bigoplus_{j=0}^{\infty} \mathfrak{J}^j/\mathfrak{J}^{j+1}\right),$$

that is, the definitions of $P(X)$ given in §§ 2.2 and 4.3 agree. Since $T(X) = \operatorname{Spec}\,(\oplus_{j=0}^{\infty} \mathfrak{J}^j/\mathfrak{J}^{j+1})$ by definition, the set $P(X)_x$ consists of all lines through x in $T(X)_x$, as claimed in § 2.2.

The following remark was made in § 1.1:

If X is a subvariety of $\mathbf{P}^N$ over $\mathbf{C}$, then the closure of $T(X)_x$ in $\mathbf{P}^N$ is the union of all lines l through x for which there are sequences $\{y_i\}$, $\{y_i'\}$ of points in X converging to x such that the sequence of lines $\{\operatorname{span}(y_i, y_i')\}$ converges to l.

Proof. The points in $P(X)_x$ are exactly the limits of the sequences of points in $\widetilde{X \times X}$ which are lifted from sequences $\{(y_i, y_i')\}$ of points in $X \times X \setminus$ diagonal converging to (x, x). So the lines in $\overline{T(X)}_x$ through x correspond precisely to limits of the sequences $\{\operatorname{span}(y_i, y_i')\}$.

References

[1]. Baldassarri, M., *Algebraic varieties*. Springer-Verlag, 1956.

[2]. Fulton, W., Rational equivalence on singular varieties. *Publ. Math. I.H.E.S.*, 45 (1975), 147–167.

[3]. Fulton, W. & Johnson, K., Canonical classes of singular varieties. To appear.

[4]. GITLER, S., Immersion and embedding of manifolds. *Amer. Math. Soc. Proc. Sym.* XXII (1971), 87–96.

[5]. GROTHENDIECK, A., La theorie des classes de Chern. *Bull. Soc. Math. France*, 86 (1958), 137–154.

[6]. GROTHENDIECK, A. & DIEUDONNE, J., Elements de geometrie algebrique. *Publ. Math. I.H.E.S.* 8, 32.

[7]. HODGE, W. V. D. & PEDOE, D., *Methods of algebraic geometry*, vol. II. Cambridge University Press, 1952.

[8]. HOLME, A., Embedding-obstruction for algebraic varieties I. Preprint, University of Bergen, Norway (1974). To appear in *Advances in Math.*

[9]. —— Embedding-obstruction for singular algebraic varieties in $\mathbf{P}^N$. *Acta Math.*, 135 (1976), 155–185.

[10]. KEMPF, G. & LAKSOV, D., The determinantal formula of Schubert calculus. *Acta Math.*, 132 (1974), 153–162.

[11]. KLEIMAN, S., Geometry on Grassmannians and applications to splitting bundles and smoothing cycles. *Publ. Math. I.H.E.S.*, 36 (1969), 281–298.

[12]. LAKSOV, D., Secant bundles and Todd's formula for the double points of maps into $\mathbf{P}^N$. To appear.

[13]. —— Residual intersections and Todd's formula for the double locus of a morphism. *Acta Math.*, 140 (1978), 74–91.

[14]. MACPHERSON, R., Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100 (1974), 423–432.

[15]. MCCRORY, C., Geometric homology operations. To appear in *Advances in Math.*

[16]. —— Lectures, Brown University, Spring 1976.

[17]. MUMFORD, D., *Lectures on curves on an algebraic surface*. Ann. of Math. Studies, no. 59, Princeton Univ. Press, 1966.

[18]. —— *Introduction to algebraic geometry*. Lecture notes, Harvard University, 1966.

[19]. PETERS, C. A. M. & SIMONIS, J., A secant formula. *Quart. J. Math. Oxford (2)*, 27 (1976), 181–189.

[20]. PONTRJAGIN, L. S., Vector fields on manifolds. *Mat. Sb.*, 24 (66) (1949), 129–162; English transl., *Amer. Math. Soc. Transl.* (1) 7 (1962), 220–278.

[21]. PORTEOUS, I. R., Blowing up chern classes. *Proc. Cambridge Phil. Soc.*, 56 (1960), 118–124.

[22]. —— Simple singularities of maps, *Proc. Liverpool Singularities Sym.* I, Springer Lecture Notes in Mathematics 192 (1965).

[23]. SERRE, J.-P., *Algebre locale—multiplicites*. Springer Lecture Notes in Mathematics 11 (1965).

[24]. SEVERI, F., Sulle intersezioni delle varieta algebriche e sopra i loro caratteri e singolarita proiettive. *Mem. Accad. Sci. Torino S. II*, 52 (1902), 61–118.

[25]. THOM, R., Les singularites des applications differentiables. *Ann. Inst. Fourier*, 6 (1956), 43–87.

[26]. TODD, J. A., Invariant and covariant systems on an algebraic variety. *Proc. London Math. Soc.*, 46 (1940), 199–230.

[27]. WEIL, A., *Foundations of algebraic geometry*, 2nd edition. Amer. Math. Soc. Coll. Publ., 29, Providence, 1962.

[28]. WHITNEY, H., Local properties of analytic varieties. In *Differential and Combinatorial Topology*, ed. S. Cairns, Princeton Univ. Press, 1965.

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