A FIXED POINT THEOREM IN SYMPLECTIC GEOMETRY

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There are a number of fixed point theorems peculiar to symplectic geometry. A particularly simple example is the theorem that any area-preserving mapping ψ of the two-dimensional sphere into itself possesses at least two distinct fixed points (see [6, 8]) although an arbitrary orientation-preserving mapping may have only one single fixed point. In higher dimensions such global theorems are not available, but it is known (see [11]) that any symplectic map ψ which is C^1-close to the identity map of a simply connected, compact symplectic manifold into itself has at least two fixed points. These fixed points are found as critical points of appropriate functions on the manifold. In this note we will derive a generalization of such a perturbation theorem which has various applications in mechanics.

To formulate our result we need some concepts of symplectic geometry: A smooth manifold Σ is called symplectic if there exists a non-degenerate closed 2-form ω on Σ; the symplectic manifold consists in fact of the pair (Σ, ω). If ω is even exact and given by $\omega = d\alpha$, α being a 1-form we call (Σ, α) an exact symplectic manifold. The most familiar example of an exact symplectic manifold is the cotangent bundle of any manifold with its natural 1-form.

A differentiable mapping ψ of Σ into itself is called symplectic if it preserves the two-form ω, i.e. if $\psi^*\omega = \omega$. Similarly, we call a mapping ψ exact symplectic if (Σ, α) is exact and $\psi^*\alpha - \alpha$ is exact, i.e. $= dF$ where F is a function of Σ. We apply the same terminology for mappings ψ of an open set $D_1 \subset \Sigma$ into another $D_2 \subset \Sigma$.

Of course, every exact symplectic mapping is also symplectic since $\psi^*\alpha = \alpha + dF$ implies

$$\psi^*\omega = d(\psi^*\alpha) = d\alpha = \omega.$$

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The converse, however, is not true in general. Indeed from $\psi^*\omega=\omega$ we conclude that

$$d(\psi^*\alpha-\alpha)=0$$

i.e. $\psi^*\alpha-\alpha$ is a closed 1-form which need not be exact. But for simply connected, exact symplectic manifolds the two concepts coincide.

We illustrate the difference of the two concepts with a simple example in the plane. The x–y-plane with 1-form $ydx=\alpha$ is exact symplectic and the corresponding two-form $\omega=d\alpha=dy\wedge dx$ is the area element. Any mapping whose Jacobian is identically $=1$ is symplectic. But in the non simply connected domain $\mathbf{R}^2\setminus\{0\}$ the mapping ψ: $(x, y)\to(X, Y)$ given by

$$x=x\left(1+\frac{\varepsilon^2}{x^2+y^2}\right)^{1/2}, \quad Y=y\left(1+\frac{\varepsilon^2}{x^2+y^2}\right)^{1/2}$$

is symplectic, but not exact symplectic for any $\varepsilon\neq 0$, as one easily verifies.

In geometrical terms an exact symplectic mapping in the plane does not only preserve the area elment ω but also the line integral $\oint\alpha$ over any closed curve.

The next concept we need is that of a coisotropic submanifold of Σ. For this purpose we first define the concept of a coisotropic subspace V of a symplectic vector space (S, ω), where ω defines an alternating non-degenerate bilinear form. We denote by V^ω the subspace of all $w\in S$ for which $\omega(w, v)=0$ for all $v\in V$. In other words, V^ω is "orthogonal with respect to ω" to V.

One calls a subspace $V\subset S$ "isotropic" if $V\subset V^\omega$, i.e. if ω vanishes in V. Similarly, a space V is called "coisotropic" if $V^\omega\subset V$, which is the same as saying that its ω-orthogonal complement V^ω is isotropic. (See [11].) Since ω is nondegenerate $\dim S=2n$ is even and

$$\dim V+\dim V^\omega=\dim S=2n.$$

Hence for a coisotropic subspace we have $2\dim V^\omega\leqslant 2n$, i.e. $\dim V\geqslant n=\frac{1}{2}\dim S$. Incidentally, every subspace V with $\dim V=2n-1$ is clearly coisotropic.

Let M be a smooth manifold and j: $M\to\Sigma$ an embedding of M in Σ. Then $dj(T_pM)$ is a subspace of $T_{j(p)}\Sigma$ and we call M coisotropic if $dj(T_pM)$ is a coisotropic subspace of $T_{j(p)}\Sigma$ for every $p\in M$. We will denote the dimensions of Σ, M by $2n$, $2n-r$ so that $0\leqslant r\leqslant n$.

For a coisotropic manifold M the space $(dj(T_pM))^\omega$ is, by definition, an r-dimensional subspace of $T_{j(p)}\Sigma$ and therefore has a preimage in TM under j. We denote this preimage simply by $(TM)^\omega$, defining an r-dimensional distribution in M. It turns out that this distribution is integrable so that, by Frobenius' theorem, one has an r-dimensional foliation of M. We denote the r-dimensional leaf through $p \in M$ by L_p; so that L_p is tangential to the given distribution.

We illustrate these concepts with a simple example: Let $\Sigma = \mathbf{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ and $\alpha = \sum_{k=1}^{n} y_k dx_k$ which makes $\mathbf{R}^{2n}$ into an exact symplectic manifold. We consider a submanifold of codimension 1 which is always coisotropic. We describe this submanifold M by a function $H = H(x, y)$ as

$$H = 0,$$

where we assume that $dH \neq 0$ on M. The one-dimensional space $(T_p M)_\omega$ is, in this case, spanned by the tangent vector

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}, \tag{1}$$

which is clearly tangent to the "energy surface" M. The leaves L_p of M are in this case simply the orbits of the above systems on $H = 0$. Clearly these leaves need not be compact, in general, and may even be dense on M.

In the following we frequently will identify $j(M)$ and M as well as $j(L_p)$ and L_p, and set $\psi|_M = \psi \circ j$; $j = \mathrm{id}_M$. This is, of course justified for embeddings j, but we point out that the result holds for immersions j also.

THEOREM. *Let (Σ, α) be a simply connected exact symplectic manifold and let*

$$j: M \to \Sigma$$

define a smooth embedding of a s smooth, compact coisotropic manifold into Σ. Finally, let ψ be a differentiable, exact symplectic mapping of a neighborhood $U(j(M))$ of $j(M)$ into Σ such that

$$|\psi \circ j - j|_{C^1}$$

is sufficiently small.

Then there exist at least two points $p \in M$ such that

$$\psi j(p) \subset j(L_p),$$

i.e. p and $\psi j(p)$ lie on the same leaf in Σ.

We discuss some consequences of this theorem:

1. For $r = 0$ we can take $M = \Sigma$, $j =$ identity if Σ is also compact. In this case the leaves $L_p = (p)$ are points and the above theorem asserts: Every symplectic mapping ψ, C^1-close to the identity, on a simply connected compact, symplectic manifold has at least two

fixed points. Here one can drop the requirement that ψ or M be exact symplectic, since ψ is defined on a simply connected manifold; see [11], p. 29.

2. For $r=n$ one has $L_p=M$, if M is connected. In this case M is called a Lagrange manifold. In this case the theorem asserts that the image $\psi\circ j(M)$ of any Lagrange manifold $j(M)$ intersects $j(M)$. This follows from Weinstein's results [10] on intersection of nearby Lagrange manifold and our case can be viewed as an extension of this statement.

3. For the intermediate case, $r=1$, we have a submanifold of codimension 1 with a 1-dimensional foliation. For example, if $\Sigma=\mathbf{R}^{2n}$, $\alpha=\sum_{k=1}^{n} y_k dx_k$ (notation as above) and M is given by

$$H=0$$

where $H=H(x, y)$ is a C^2-function with $dH\neq 0$ on M. Then the foliation on M is given by

$$\dot{x}_k=\frac{\partial H}{\partial y_k}, \quad \dot{y}_k=-\frac{\partial H}{\partial x_k}.$$

To apply our theorem to this situation we assume M to be compact, and assume that ψ is any exact symplectic mapping in $U(M)$ such that $\psi|_M$ is close to $\mathrm{id}|_M$. In particular, ψ need not map M into itself. In this case the theorem asserts the existence of a point $p\in M$ such that $\psi(p)$ lies on the orbit of the above systems through p; for *this* point $\psi(p)\in M$, thus M and $\psi(M)$ intersect at $\psi(p)$.

For $n=1$ this result just states that a closed curve M in $\mathbf{R}^2$ and its image under ψ intersect, which follows simply from the preservation of the area $\oint \alpha$ enclosed by such a curve.

For the higher dimensional case consider the example

$$H=\tfrac{1}{2}\left(\sum_{k=1}^{n}(x_k^2+y_k^2)-1\right)$$

where M is the unit sphere and the leaves are circles $x_k+iy_k=c_k e^{-it}$, $\sum_{k=1}^{n}|c_k|^2=1$. If ψ is the translation, say, $x_k\to x_k+\varepsilon a_k$; $y_k\to y_k$ then there is at least one circle intersecting its image circle. Incidentally, this example shows also the necessity for the smallness condition: If ε is sufficiently large no such intersection exists.

4. Poincaré's perturbation theory of periodic orbits. We consider a system (1) in $\mathbf{R}^{2n}$ and assume that the energy surface $M=\{(x, y)\,|\,H=0\}$ is a regular compact manifold such that all orbits on M are periodic of a constant period $=T$. We claim that for any system with a Hamiltonian $\tilde{H}$ for which $|\tilde{H}-H|_{C^2}$ is small, there exists at least one periodic orbit on $\tilde{H}=0$ whose period is close to T.

To show how this result follows from the theorem we denote by φ^t the flow associated with the system (1) and $\tilde{\varphi}^t$ that for the perturbed system, so that $|\tilde{\varphi}^t - \varphi^t|_{C^1}$ small for $0 \leq t \leq T$. Moreover, let j be a mapping taking $M = \{(x, y) \mid H = 0\}$ into $\tilde{M} = \{\tilde{H} = 0\}$, e.g. along normals of M. Since $\varphi^T|_M = \text{id}$ we set $\psi = \tilde{\varphi}^T$ so that $\psi \circ j$ is close to j. Moreover, ψ is exact symplectic as is well known. The foliation $j(L_p)$ on $\tilde{M} = j(M)$ is given by the flow $\tilde{\varphi}^t$. Thus the theorem asserts the existence of a point $j(p) = q \in \tilde{M}$ such that

$$\psi(q) = \psi \circ j(p) \in j(L_p)$$

or

$$\psi(q) = \tilde{\psi}^s(q)$$

for some small s. Hence

$$\tilde{\varphi}^{T-s}(q) = q$$

and the orbit through q has period $T - s \sim T$. We did not assert the existence of two orbits since the two points of the theorem may lie on the same periodic orbit.

Arguments proving similar results go back to Poincaré [7] and our proof can be viewed as a generalization of his. The basic idea is to construct an auxiliary function on M whose critical points are the desired points which are mapped along the foliation.

We recall Poincaré's idea for a mapping

$$X = f(x, y), \quad Y = g(x, y)$$

in $\mathbf{R}^{2n}$ which is assumed to be exact symplectic, hence

$$\sum_{k=1}^{n} (Y_k dx_k - y_k dx_k) = df$$

is exact. Poincaré constructed in place of this the differential

$$\begin{aligned}\beta &= \sum_{k=1}^{n} \{(Y_k - y_k) dX_k - (X_k - x_k) dy_k\} \\ &= \sum_{k=1}^{n} (Y_k dX_k + x_k dy_k) - d\left(\sum_{k=1}^{n} y_k X_k\right) \qquad (2) \\ &= d\left(f - \sum_{k=1}^{n} y_k (X_k - x_k)\right) = dg\end{aligned}$$

which is also exact. It has the added advantage that at critical points of g, i.e. at a point where $dg = \beta = 0$ one has

$$Y_k = y_k, \quad X_k = x_k$$

i.e. a fixed point, provided that the differentials dX_k, dy_k are linearly independent there. This is certainly the case if the given mapping is close to the identity.

Before giving the proof in § 3 and § 4 we describe a simple application of our theorem to a time dependent nonlinear perturbation of a system of oscillators. The theorem can also be used to show the preservation of homoclinic orbits of Hamiltonian systems, even if these homoclinic orbits are degenerate. For systems of two degrees of freedom such results were known (see, for example, [4]) but for higher degrees of freedom they seem to be new. Application of this nature giving the existence of homoclinic orbits can be found in the work by Easton and McGehee [1]. In fact, this investigation was prompted by discussion with R. McGehee on this topic.

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§ 2. An application

We consider a system of n harmonic oscillators given by the Hamiltonian

$$H_0 = \tfrac{1}{2} \sum_{k=1}^{n} \alpha_k (x_k^2 + y_k^2)$$

in $\mathbf{R}^{2k}$, where the frequencies α_k are positive numbers, so that the energy surface $H = c$ for $c > 0$ is compact. In the special case where all α_k are integer multiples of a positive number a all solutions are periodic of period $2\pi/a$; we refer to this case as the periodic case.

We perturb this system by a time dependent perturbation Hamiltonian $P = P(x, y, t)$ where we assume that $P \in C^2$ and $\int_{-\infty}^{\infty} \|P\|_{C^2} dt$ is small. Here $\| \;\|_{C^2}$ refers to the space variables x, y only. Moreover, we assume that $P(x, y, t) = 0$ for $|t| \geqslant T$ where T is a fixed positive number.

The perturbed system is governed by the Hamiltonian $H = H_0 + P$. It describes a slightly coupled time dependent system and, of course, the energy is not any more conserved for the system.

We study the connection between the solutions for $t < -T$ and for $t > T$ and will show that on any energy surface there exists at least one orbit which for $t > T$ agrees up to a phase shift with the continuation of the unperturbed orbit coinciding for $t < -T$. In particular, for this orbit the energy is strictly preserved from $t < -T$ to $t > T$.

To describe the mapping in question more precisely we introduce cocoordinates

$z_k = x_k$, $z_{k+n} = y_k$ $(k = 1, 2, ..., n)$ in $\mathbf{R}^{2n}$ and set $z = (z_j)$. Denote by $z = \varphi^{t, t_0}(\zeta)$ the solution of the t-dependent system

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}$$

which for $t = t_0$ takes on the value $z = \zeta$. Then

$$\varphi^{t_1, t_2} \circ \varphi^{t_2, t_3} = \varphi^{t_1, t_3},$$

if these mappings are defined. For the unperturbed system, given by H_0, we denote this mapping by $\varphi_0^{t_1, t_2} = \varphi_0^{t_1 - t_2}$; it depends on the difference $t_1 - t_2$ on account of the autonomous character.

We choose numbers a, b in $a < -T$, $b > T$ and define

$$\psi = (\varphi_0^{b, a})^{-1} \varphi^{b, a} = \varphi_0^{a-b} \circ \varphi^{b, a}$$

which is the mapping in question. Consider a solution

$$z = \varphi^{t, a}(\zeta)$$

so that

$$z = \varphi_0^{t-a}(\zeta) \quad \text{for } t < -T.$$

On the other hand, for $t > T$ this solution agrees with

$$z = \varphi^{t, b} \circ \varphi^{b, a}(\zeta) = \varphi_0^{t-b} \circ \varphi^{b, a}(\zeta) = \varphi_0^{t-a} \circ \varphi_0^{a-b} \circ \varphi^{b, a}(\zeta) = \varphi_0^{t-a}(\psi(\zeta)).$$

Thus the initial value ζ for $t = a$ is replaced by $\psi(\zeta)$.

It is good to notice that an orbit of the unperturbed system need not go into an orbit again, since the mapping depends on the phase. Otherwise ψ would have to commute with φ_0^t which need not be the case in general.

As is well known, the mapping $\varphi^{b,a}$ and φ_0^{b-a} and $\psi = \varphi_0^{a-b} \varphi^{b,a}$ are exact symplectic with respect to $\alpha = \sum_{k=1}^n y_k dx_k$. Moreover, ψ is C^1 close to the identity map since $\int_{-\infty}^{\infty} \|P\|_{C^2} dt$ is small. Thus our theorem applied to this mapping ψ and an energy surface $H_0 = c$, $c > 0$ as coisotropic manifold guarantees the existence of a point ζ^* on $H_0(\zeta^*) = c$ such that, $\psi(\zeta^*) = \varphi_0^s(\zeta^*)$ for some small s. In other words the solution

$$z = \varphi^{t, a}(\zeta^*)$$

which for $t < -T$ agrees with $\varphi_0^{t-a}(\zeta^*)$ is for $t > T$ equal to

$$z = \varphi_0^{t-a} \circ \psi(\zeta^*) = \varphi_0^{t-a+s}(\zeta^*),$$

i.e. agrees with the unperturbed solution up to a phase shift s. This is what we wanted to show.

We illustrate the case of a coisotropic manifold of codimension $r>1$. We note that the functions

$$G_\varrho = x_\varrho^2 + y_\varrho^2, \quad \varrho = 1, 2, \ldots, r-1$$

$$G_r = H_0 = \tfrac{1}{2} \sum_{k=1}^{n} \alpha_k (x_k^2 + y_k^2)$$

are integrals of the motion for the unperturbed problem. Moreover the Poisson brackets $\{G_\varrho, G_\sigma\} = 0$, which implies that the manifolds

$$G_\varrho = g_\varrho, \; c_\varrho > 0, \quad c_r > \sum_{\varrho=1}^{r-1} \alpha_\varrho c_\varrho, \quad \varrho = 1, 2, \ldots, r \tag{3}$$

are coisotropic, as will be shown in § 3. We included H_0 as one integral to ensure compactness of these manifolds. We use such a manifold as M in our theorem. It is of the topological type of $\mathbf{T}^{r-1} \times S^{2(n-r)+1}$ where $\mathbf{T}^{r-1}$ is an $(r-1)$-dimensional torus, and S^m an m dimensional sphere. Indeed, with polar coordinates

$$x_k + i y_k = r_k e^{i\theta_k}$$

one has $r_\varrho^2 = c_\varrho$ for $\varrho = 1, \ldots, r-1$ and $\sum_{k=r}^{n} \alpha_k r_k = c_r - \sum_{\varrho=0}^{r-1} \alpha_\varrho c_\varrho > 0$. The flow generated by G_ϱ is given by $\theta_k \to \theta_k + \delta_{k\varrho} \tau_\varrho$; $r_k \to r_k$, where $\delta_{k\varrho}$ is the Kronecker symbol. The leaves through a point (r^*, θ^*) are given by

$$r_k = r_k^*; \quad \theta_k = \theta_k^* + \sum_{\varrho=1}^{r-1} \delta_{k\varrho} \tau_\varrho + \alpha_k \tau_r, \quad k = 1, 2, \ldots, n$$

where $\tau_1, \tau_2, \ldots, \tau_r$ are r parameters on the leaf.

By the above theorem, on every manifold of the form (3) there exists a solution $\varphi_0^{t-a}(\zeta)$ for $t < -T$ which aside from phase shifts $\tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r$ of $\theta_1, \theta_2, \ldots, \theta_{r-1}, t$ returns to the continuation of the unperturbed orbit. In particular, for this orbit all integrals $G_1, G_2, \ldots, G_r$ have the same value for $t < -T$ and $t > T$. Of course, for other solutions the integrals $G_1, \ldots, G_r$ of the unperturbed system are generally not preserved.

In this example one could allow functions $P(x, y, t)$ which decay sufficiently rapidly as $|t| \to \infty$. Then we would have to describe the asymptotic behavior of the orbits for $t \to \pm\infty$ and study the scattering mapping relating these asymptotic data for $t = -\infty$ and $t = +\infty$. The above theorem yields orbits for which this mapping is the same as for the unperturbed flow aside from phase shifts.

Similarly the above theorem can be used to show that homoclinic orbits are conserved under small perturbation of the Hamiltonian vector field, even in degenerate situations. This can be found in Easton and McGehee [1], and we indicate a simple example only:

Consider the Hamiltonian

$$H_0 = \sum_{k=1}^{m} (x_k^2 + y_k^2) + x_0 y_0 + (x_0^2 + y_0^2)^2.$$

Here we have $\dim \Sigma = 2n = 2m + 2$.

The manifold N: $x_0 = y_0 = 0$, $H_0 = 1$ is a $2m-1$ dimensional sphere on which all orbits are periodic. Due to the term $x_0 y_0$ they are unstable and each periodic orbit S^1 possesses an unstable manifold $W^+(S^1)$ and a stable manifold $W^-(S^1)$ of dimension 2. In fact both these manifolds agree for this example and are given by

$$x_0 y_0 + (x_0^2 + y_0^2)^2 = 0$$

and $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ on the periodic orbit. However, if we subject H_0 to a perturbation, then these two manifolds need not agree any more.

In this situation the theorem of § 1 applies and shows if $P = P(x, y, x_0, y_0)$ is a smooth function of support outside of N and $\|P\|_{C^2}$ sufficiently small then there exists a periodic orbit S_*^1 on N such that

$$W^+(S_*^1) \cap W^-(S_*^1) \neq \emptyset.$$

An orbit on this intersection—a homoclinic orbit—approaches the same circle S_r for $t \to +\infty$ and $t \to -\infty$. In this respect this result is analogous to the earlier one of this section. For the details we refer to [1].

§ 3. Outline of proof for $\Sigma = \mathbf{R}^{2n}$

We first outline the proof in the special case when $\Sigma = \mathbf{R}^{2n}$. Using the coordinates $z_k = x_k$, $z_{k+n} = y_k$ for $k = 1, \ldots, n$ we introduce the exact symplectic structure by

$$\alpha = \tfrac{1}{2} \sum_{k=1}^{n} (y_k\, dx_k - x_k\, dy_k) = \tfrac{1}{2}\langle Jz, dz\rangle$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $\langle\, ,\rangle$ denotes the Euclidean inner product. Thus

$$\omega = d\alpha = \sum_{k=1}^{n} dy_k \wedge dx_k$$

and for two tangent vectors

$$V=\sum_{k=1}^{2n} V_k \frac{\partial}{\partial z_k}, \quad W=\sum_{k=1}^{2n} W_k \frac{\partial}{\partial z_k}$$

we can represent $\omega(V, W)$ in terms of the inner product by

$$\omega(V, W) = \tfrac{1}{2}\langle JV, W\rangle$$

where we identified V, W with the vectors with the components V_k, W_k.

To describe the idea of the proof we make the direct product $\Sigma\times\Sigma=\mathbf{R}^{4n}$ into an exact symplectic space (symplectic space) by introducing the forms

$$\tilde{\alpha} = \tfrac{1}{2}\langle JZ_1, dZ_1\rangle - \tfrac{1}{2}\langle JZ_2, dZ_2\rangle, \ (\tilde{\omega}=d\tilde{\alpha})$$

where $(Z_1, Z_2)\in\Sigma\times\Sigma$. The 1-form which is basic for the following is

$$\beta = \tfrac{1}{2}\langle J(Z_1-Z_2), (dZ_1+dZ_2)\rangle \tag{3.1}$$

which differs from $\tilde{\alpha}$ by the exact form

$$\tfrac{1}{2}d\langle JZ_1, Z_2\rangle,$$

and vanishes on the diagonal Δ: $Z_1=Z_2$. These two properties will be essential for the following:

(i) β vanishes on the diagonal $\Delta\subset\Sigma\times\Sigma$.

(ii) $\beta-\tilde{\alpha}$ is exact.

To describe the outline of the proof we consider the embedding $j\colon M\to\Sigma$ of the coisotropic manifold and the given exact symplectic mapping ψ, so that $\psi\circ j$ takes $M\to\Sigma$. Moreover, we will construct a mapping $\varphi\colon M\to M$ which preserves the leaves in M and is chosen so that $j\varphi(p)$ is the point on $j(L_p)$ closest to $\psi\circ j(p)$. We then look for points $\hat{p}$ for which $\psi\circ j(\hat{p})=j\circ\varphi(\hat{p})$, or $\psi(q)\in L_q$ for $q=j(\hat{p})$.

These points are found as critical points of a function which is constructed as follows: Let

$$\mu\colon M\to\Sigma\times\Sigma$$

be the mapping given by

$$\mu(p) = (j\varphi(p), \psi j(p)) \tag{3.2}$$

and consider the 1-form $\mu^*\beta$ on M. We will show that it is exact, i.e. equal dF with a function F on M, and the critical points of F turn out to be the desired points. In the special case when $M=\Sigma$, $j=\mathrm{id}$, $\varphi=\mathrm{id}$ this argument is precisely the one given at the end of § 1.

Before proceeding with the proof we characterize coisotropic submanifolds M in Σ locally and show that the distribution $(TM)^\omega$ is integrable. Here we identify $j(M)$ and M and describe $M\cap U$ in a sufficiently small neighborhood U of a point of M by the equations

$$G_\varrho(z)=0, \quad \varrho=1, \ldots, r,$$

where dG_ϱ are linearly independent in U.

PROPOSITION 1. *$M\cap U$ is coisotropic in Σ if and only if the Poisson brackets*

$$\{G_\varrho, G_\sigma\}=\langle J\nabla G_\varrho, \nabla G_\sigma\rangle$$

vanish on $M\cap U$.

To prove this we associate with a function G the Hamiltonian vector field $V_G=J^{-1}\nabla G$, whose Hamiltonian G is. For two such functions G, H we form the bilinear form

$$2\omega(V_G, V_H)=\langle JV_G, V_H\rangle=\langle JJ^{-1}\nabla G, J^{-1}\nabla H\rangle=\langle J\nabla G, \nabla H\rangle=\{G, H\} \tag{3.3}$$

which is the Poisson bracket of G and H. In particular, V_G, V_H are orthogonal with respect to ω if and only if $\{G, H\}=0$.

There is another way in which the Poisson bracket is related to G, H: The commutator $[V_G, V_H]$ is a Hamiltonian vector field with the Hamiltonian $-\{G, H\}$ i.e.

$$[V_G, V_H]=-V_{\{G,H\}}. \tag{3.4}$$

To prove the proposition we note that TM consists of those vectors ζ for which

$$\langle\nabla G_\varrho, \zeta\rangle=0$$

or equivalently, with $V_\varrho=J^{-1}\nabla G_\varrho$,

$$\omega(V_\varrho, \zeta)=0.$$

Hence $V_\varrho\in(TM)^\omega$ and since they are linearly independent they span $(TM)^\omega$. Therefore the condition $(TM)^\omega\subset TM$ is equivalent to

$$\omega(V_\varrho, V_\sigma)=0 \quad \text{on } M\cap U$$

which, by (3.3), proves the proposition.

Incidentally this proposition does not imply that the vector fields $V_\varrho=J^{-1}\nabla G_\varrho$ commute as one may expect from (3.4). Since the Poisson brackets $\{G_\varrho, G_\sigma\}$ vanish only on

$M \cap U$ one cannot conclude that the corresponding vector field vanishes. However, we have

PROPOSITION 2. *If $M \cap U$ is coisotropic then the distribution $(TM)^\omega$ is integrable.*

Proof. By Frobenius' theorem it suffices to show that $[V_\varrho, V_\sigma]$ belong to $(TM)^\omega$. We note that by Proposition 1 the function $H = \{G_\varrho, G_\sigma\}$ vanishes on $M \cap U$, and since the dG_ν are linearly independent we have

$$\nabla H = \sum_{\nu=1}^{r} \lambda_\nu \nabla G \quad \text{on } M \cap U.$$

Hence

$$V_H = \sum_{\nu=1}^{r} \lambda_\nu V_\nu$$

and by (3.4)

$$[V_\varrho, V_\sigma] = -V_H = -\sum_{\nu=1}^{r} \lambda_\nu V_\nu \in (TM)^\omega.$$

We note that Proposition 2 holds for general symplectic manifolds: Indeed, by a theorem of Darboux, locally one can introduce coordinates x_k, y_k such that the symplectic form has the standard form so that Proposition 2 becomes applicable in the general case.

By Proposition 1 one can construct examples of coisotropic manifolds by considering sets of the form $G_\varrho = c_\varrho$ for functions which are "in involution" in a domain. These are in two respects very special coisotropic manifolds. First, the corresponding vector fields V_ϱ commute by (3.4) which is a strong restriction. Secondly, these manifolds are given by global functions G_ϱ, which may not exist in general. It is one of the purposes of the proof of § 4 to be free from these restrictions.

More generally coisotropic manifolds are related to symplectic group actions also for noncommuting groups. In this connection we refer to [2], [3] and [5].

§ 4. Proof of Theorem

(a) In the exact symplectic manifold (Σ, α), which is also symplectic with respect to $\omega = d\alpha$ we introduce the standard notations: With any function H on Σ we associate a vector field V_H by

$$dH = \omega \lrcorner V_H \,(^1).$$

Then the Poisson bracket of two functions G, H is defined by

$$\{G, H\} = \omega(V_G, V_H).$$

(1) For notation, see [9]. The notation matches that of § 3 only up to irrelevant factors.

Moreover, one has

$$[V_G, V_H] = -V_{\{G,H\}}.$$

Let j: $M \to \Sigma$ be the embedding of a coisotropic manifold of codimension r, $0 \leqslant r \leqslant n$. Then $\alpha_M = j^* \alpha$ is a one-form on M and the corresponding two-form $d\alpha_M = j^* \omega = \omega_M$ has an r-dimensional nullspace $(TM)^\omega$ in TM which, by the remarks of the previous section, defines an integrable distribution. We denote by L_p the leaf through a point $p \in M$.

We need the following

PROPOSITION 3. *Let φ be a C^1-mapping of M into itself, C^1-close to the identity and $\varphi(p) \in L_p$ for all $p \in M$. Then the* 1-*form*

$$\varphi^* \alpha_M - \alpha_M = dg$$

is exact.

Proof. We interpolate φ by a family of such leaf-preserving mappings φ_s $(0 \leqslant s \leqslant 1)$ such that $\varphi_0 = \mathrm{id}$, $\varphi_1 = \varphi$, for example as follows: We introduce a metric in Σ and define the exponential map $\exp_q$: $T_q \Sigma \to \Sigma$.

Since φ is close to the identity the points $j(p)$ and $j\varphi(p)$ are close in Σ and can be represented by

$$j\varphi_s(p) = \lambda \exp_q(sA(q)), \quad q = j(p),$$

where λ is the projection of a neighborhood of $j(L_p)$ onto $j(L_p)$. It assigns to a point r near $j(L_p)$ the closest point $\lambda(r)$ to r on $j(L_p)$. This mapping λ is well defined and smooth in a sufficiently small neighborhood of $j(L_p)$.

Since $\varphi_s(p) \in L_p$ the vector field $W_s = ((d/ds)\varphi_s) \circ \varphi_s^{-1}$ is tangential to L_p and hence $W_s \in (TM)^\omega$ or

$$\omega_M \lrcorner W_s = 0.$$

We have to show that

$$\varphi^* \alpha_M - \alpha_M = \int_0^1 \frac{d}{ds}(\varphi_s^* \alpha)\, ds$$

is exact. Denoting the Lie-derivative along W by $\mathcal{L}_W$ we have

$$\frac{d}{ds}(\varphi_s^* \alpha_M) = \varphi_s^* \mathcal{L}_{W_s} \alpha_M$$

and, by a general identity, we have

$$\mathcal{L}_{W_s} \alpha_M = (d\alpha_M) \lrcorner W_s + d(\alpha_M \lrcorner W^s)$$

The first term is equal $\omega_M \lrcorner W_s$ and vanishes, the second is exact. Thus we have

$$\varphi^*\alpha_M - \alpha_M = d\left(\int_0^1 \varphi_s^*(\alpha_M \lrcorner W_s)\, ds\right)$$

proving the proposition.

(b) Next we use a theorem of A. Weinstein who constructed a 1-form β, analogous to that of the previous section, for general exact symplectic manifolds. For this purpose we define the product manifold $\Sigma \times \Sigma$ and the projections π_ν: $\Sigma \times \Sigma \to \Sigma$ in $\nu = 1, 2$ into the first and second component. Then

$$\tilde{\alpha} = \pi_1^* \alpha - \pi_2^* \alpha; \quad \tilde{\omega} = d\tilde{\alpha}$$

defines an exact symplectic structure on $\Sigma \times \Sigma$. The diagonal Δ of $\Sigma \times \Sigma$ is the manifold of points q satisfying $\pi_1(q) = \pi_2(q)$.

PROPOSITION 4. *If (Σ, α) is a simply connected, exact symplectic manifold then there exists a one-form β on a neighborhood $N(\Delta)$ of the diagonal in $\Sigma \times \Sigma$ such that*

(i) *$\beta = 0$ on the diagonal $\Delta \subset \Sigma \times \Sigma$.*

(ii) *$\beta - \tilde{\alpha} = df$ is exact.*

We outline the idea of the proof: A tubular neighborhood $N(\Delta)$ is differentiably equivalent to the cotangent bundle $T^*\Sigma$ of Σ and one constructs a diffeomorphism

$$k \colon N(\Delta) \to T^*\Sigma$$

which takes the diagonal Δ into the zero section of $T^*\Sigma$. If ν is the natural 1-form of $T^*\Sigma$ (which vanishes on the zero section and for which $d\nu$ is nondegenerate) then $k^*\nu$ is a 1-form on $N(\Delta)$ vanishing on the diagonal. Moreover, the diagonal Δ is a Lagrange manifold for the symplectic form $k^*(d\nu)$. By a deformation argument one shows that any two symplectic forms near a manifold which is a Lagrange manifold with respect to both forms are diffeomorphically equivalent. Applying this to $\tilde{\omega} = d\tilde{\alpha}$ and $k^*(d\nu)$ one sees that k can be so chosen that

$$d\tilde{\alpha} = k^*(d\nu)$$

hence

$$d(\tilde{\alpha} - k^*\nu) = 0.$$

Since Σ is simply connected $\tilde{\alpha} - k^*\nu$ is exact and we can take $\beta = k^*\nu$ to prove the proposition.

(c) Let ψ be the exact symplectic mapping of the theorem, so that

$$\psi^*\alpha - \alpha = dh \quad \text{in } U(jM). \tag{4.1}$$

Hence with $\alpha_M = j^*\alpha$

$$(\psi j)^*\alpha - \alpha_M = j^* dh = d(h\circ j).$$

With this mapping ψ we associate a leaf-preserving mapping $\varphi\colon M \to M$ which will be specified later. At this point we leave φ unspecified except to assume that φ is C^1-close to the identity and $\varphi(p) \in L_p$.

(d) With these mappings φ, ψ we define

$$\mu\colon M \to \Sigma \times \Sigma$$

by

$$\mu(p) = (j\circ\varphi(p),\ \psi\circ j(p)),$$

or

$$\pi_1\mu = j\circ\varphi, \quad \pi_2\mu = \psi\circ j.$$

Then $\mu^*\beta$ is a one-form on M. The proof of the theorem will follow from

PROPOSITION 5. *The form $\mu^*\beta$ is exact. Moreover, φ can be chosen in such a way that $\mu^*\beta$ vanishes only at points $\hat{p}$ for which $j\circ\varphi(\hat{p}) = \psi\circ j(\hat{p})$. Thus if $\mu^*\beta = dF$, the critical points, say the maximum and the minimum, of F give the desired two solutions. As a matter of fact the number of the solutions is at least equal to the category (in the sense of Liusternik-Schnirelman) of M.*

Proof. To show that $\mu^*\beta$ is exact we use Proposition 4 to write

$$\begin{aligned}\mu^*\beta &= \mu^*\pi_1^*\alpha - \mu^*\pi_2^*\alpha + d(f\circ\mu) = (j\varphi)^*\alpha - (\psi\circ j)^*\alpha + d(f\circ\mu) = \varphi^*\alpha_M - (\psi\circ j)^*\alpha + d(f\circ\mu)\\ &= (\varphi^*\alpha_M - \alpha_M) - ((\psi\circ j)^*\alpha - \alpha_M) + d(f\circ\mu).\end{aligned}$$

By Proposition 3 and (4.1) we have

$$\mu^*\beta = d(g - h\circ j + f\circ\mu) = dF.$$

To study the zeroes of the form $\mu^*\beta$ we represent $j\circ\varphi$ and $\psi\circ j$ by vector fields on $j(M)$, defined by

$$j\varphi(p) = \exp_q A(q); \quad \psi(q) = \exp_q B(q).$$

where $q = j(p)$. Because of the smallness condition $|A|_{C^1}$, $|B|_{C^1}$ are small and we will approximate $\mu^*\beta$ by the linear approximation in A, B. To formalize this we define the mapping $\mu_\varepsilon\colon M \to \Sigma\times\Sigma$ by $\pi_1\mu_\varepsilon = \exp_q(\varepsilon A(q))$, $\pi_2\mu_\varepsilon = \exp_q(\varepsilon B(q))$; $q = j(p)$ and approximate $\mu^*\beta$ by

$$\frac{d}{d\varepsilon}(\mu_\varepsilon^*\beta)\big|_{\varepsilon=0} = \mathcal{L}_W\beta,$$

where $W=(A, B)$ is a vector field in $T(\Sigma\times\Sigma)$ on $j(M)$. We need the following formula: For $v\in TM$ and $V=(dj)v$ one has

$$\mathfrak{L}_W\beta \lrcorner v=\omega(A-B, V). \tag{4.2}$$

To prove (4.2) we use the identity

$$\mathfrak{L}_W\beta=\mu_0^*\{(d\beta)\lrcorner W+d(\beta\lrcorner W)\}$$

and note that the second term vanishes, since β vanishes on the diagonal and the image of $\mu_0=(j, j)$ lies on the diagonal. Hence

$$\mathfrak{L}_W\beta=\mu_0^*(\tilde{\omega}\lrcorner W)$$

or

$$\mathfrak{L}_W\beta\lrcorner v=\tilde{\omega}(W, d\mu_0 v).$$

Since $\tilde{\omega}=\pi_1^*\omega-\pi_2^*\omega$ and $\pi_1\mu_0=\pi_2\mu_0=j$, $d\pi_1 W=A$, $d\pi_2 W=B$ we have

$$\mathfrak{L}_W\beta\lrcorner v=\omega(A, (dj)v)-\omega(B, (dj)v)=\omega(A-B, V),$$

where $V=(dj)v$, as we wanted to show.

Now we fix the mapping φ defined in terms of A by requiring that $(A-B)(q)\perp(dj(T_pM))^\omega$, for $q=j(p)$, i.e. that $A-B$ is orthogonal to the tangent space of the leaf. Since φ is assumed to be leaf-preserving this fixes A and hence φ uniquely. (One could have fixed by other choices, e.g. pick $j\circ\varphi(p)$ as the point on the leaf $j(L_p)$ closest to $\psi\circ j(p)$.)

Since $\sup_V\omega(X, V)=0$ where $V\in(dj)TM$ implies that $X\in(dj(TM))^\omega$ we conclude that for $X=A-B$

$$\sup_{|V|=1}\omega(X, V)\geqslant c|A-B|$$

where the norm is with respect to the chosen metric and c is a positive constant.

We recall that for a point p at which $(A-B)\circ j=0$ one has $\mu(p)\in\Delta$, hence $\mu^*\beta=0$. Therefore given $\eta>0$ we can choose $\psi\circ j$ so close to j that

$$|\mu^*\beta\lrcorner v-\omega(A-B, V)|\leqslant\eta|A-B|\,|V|.$$

Hence

$$\|\mu^*\beta\|=\sup_{|V|=1}\mu^*\beta\lrcorner v\geqslant\sup_{|V|=1}\omega(A-B, V)-\eta|A-B|$$
$$\geqslant(c-\eta)|A-B|.$$

Hence, for $0<\eta<c$ we conclude that $\mu^*\beta=0$ at a point $\hat{p}$ implies $A=B$ at $q=j(\hat{p})$ and hence $\psi(q)=j\circ\varphi(\hat{p})\in j(L_{\hat{p}})$ which concludes the proof of Proposition 5, and of the theorem.

We conclude with a remark which we owe to A. Weinstein: He observed that our result can be derived from Theorem 4.4 of his paper [10] about the intersection of Lagrange manifolds. We indicate his argument pointing out the connection to the above proof: Let

$$\mathcal{M} = \{p, q \in M, q \in L_p;\ d(p, q) < \delta\}$$

where d is an appropriate distance and δ a positive constant. We imposed the smallness restriction on the distance so that $\mathcal{M}$ is a manifold which is embedded in $\Sigma \times \Sigma$. Clearly dim $M = 2n$. The main observation is that $\mathcal{M}$ is a Lagrange manifold with respect to the symplectic structure $\tilde{\omega}$ in $\Sigma \times \Sigma$. For this purpose we construct for given $(p, q) \in \mathcal{M}$ a leaf preserving diffeomorphism φ taking p into $q = \varphi(p)$. Such a map can be constructed by modifying an arbitrary C^1-map near the identity taking p into q to one which preserves the leaves, as it was indicated in the proof of Proposition 3.

To show that $\mathcal{M}$ is a Lagrange manifold it suffices to show for fixed $(p, q) \in \mathcal{M}$ that

$$\tilde{\omega}(\zeta, \zeta') = 0 \quad \text{for } \zeta, \zeta' \in T_{p,q}\mathcal{M}. \tag{4.3}$$

Clearly, $\zeta = (\xi, \eta) \in T_{pq}\mathcal{M}$ if and only if

$$\eta - \Phi\xi \in (T_q M)^\omega \quad \text{with } \Phi = d\varphi$$

Now we note that $\omega(\eta, \eta') = \omega(\hat{\eta}, \hat{\eta}')$ if $\eta - \hat{\eta}, \eta' - \hat{\eta}' \in (T_q M)^\omega$ hence

$$\omega(\eta, \eta') = \omega(\Phi\xi, \Phi\xi').$$

By Proposition 3 this expression agrees with $\omega(\xi, \xi')$ hence $\omega(\xi, \xi') - \omega(\eta, \eta') = 0$. This proves the assertion (4.3).

Finally, the fact that $\mathcal{M}$ and Δ intersect "cleanly" in the terminology of [10] along $\mathcal{M} \cap \Delta = \{p, q \in M, p = q\}$ follows from Proposition 5.

Now the Lagrange manifold $\tilde{\Delta} = \{p, q \in M \times \Sigma,\ q = \psi(p)\}$ is C' close to Δ and by Theorem 4.4 of [10] $\tilde{\Delta} \cap \mathcal{M}$ is given by the zeros of a closed 1-form on $\Delta \cap \mathcal{M}$ which is diffeomorphic to M. If ψ is exact symplectic this form is exact, and the set $\tilde{\Delta} \cap \mathcal{M}$ given by the critical points of a function on M. Since $\tilde{\Delta} \cap \mathcal{M}$ consists of the points $(p, q) \in M \times \Sigma$ with $q = \psi(p) \in L_p$, the proof is finished.

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