SPIRALS AND THE UNIVERSAL TEICHMÜLLER SPACE

BY

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Dedicated to Professor L. V. Ahlfors on his seventieth birthday

1. Introduction

Suppose that D is a simply connected domain of hyperbolic type in the extended complex plane $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then the hyperbolic or noneuclidean metric ϱ_D in D is given by

$$\varrho_D(z) = (1 - |g(z)|^2)^{-1} |g'(z)|,$$

where g is any conformal mapping of D onto the unit disk $\{z: |z| < 1\}$. For each function φ defined in D we introduce the norm

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)| \varrho_D(z)^{-2}.$$

Next for each function f which is locally univalent and meromorphic in D we let S_f denote the Schwarzian derivative of f. At finite points of D which are not poles of f, S_f is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \tag{1}$$

and the definition is extended to ∞ and the poles of f by means of inversion.

Now let L denote the lower half plane, $L = \{z = x + iy: y < 0\}$, and let $B_2 = B_2(L, 1)$ denote the complex Banach space of functions φ analytic in L with the norm

$$\|\varphi\| = \|\varphi\|_L = \sup_{z \in L} 4y^2 |\varphi(z)| < \infty.$$

Next let S denote the family of functions $\varphi = S_g$ where g is conformal in L, and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ where g has a quasiconformal extension to $\bar{\mathbf{C}}$. Then

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$\|\varphi\| \leq 6$ for all $\varphi \in S$ by [11], and hence $T \subset S \subset B_2$. The set T is called the universal Teichmüller space. See [4], [5], [6], [7].

In a recent paper [8], the author established a result, which when combined with an extension theorem of Ahlfors [1], yields the following characterization of T.

THEOREM 1. *T is the interior of S.*

Theorem 1 is closely related to the following interesting open problem raised by Bers in [4], [5], [6], [7].

QUESTION. *Is S the closure of T?*

The purpose of this paper is to answer this question in the negative by establishing the following result.

THEOREM 2. *There exists a simply connected domain D of hyperbolic type and a positive constant δ with the following property. If f is conformal in D and if $\|S_f\|_D \leq \delta$, then $f(D)$ is not a Jordan domain.*

COROLLARY. *There exists a φ in S which does not lie in the closure of T.*

Proof of Corollary. Let D and δ be as in Theorem 2, and let g be any conformal mapping of L onto D. Then $\varphi = S_g \in S$. Choose $\psi \in S$ with $\|\psi - \varphi\| \leq \delta$. Then $\psi = S_h$, where h is conformal in L. Set $f = h \circ g^{-1}$. Then from the composition law

$$S_h(z) = S_f(g(z))\, g'(z)^2 + S_g(z)$$

it follows that

$$\|S_f\|_D = \|S_h - S_g\|_L = \|\psi - \varphi\| \leq \delta.$$

Hence $h(L) = f(D)$ is not a Jordan domain, h does not have even a homeomorphic extension to $\bar{L}$ and $\psi \notin T$. We conclude that φ is a point of S which does not lie in the closure of T.

The domain D in Theorem 2 can be described in a very explicit manner. Namely, $D = \bar{\mathbf{C}} - \gamma$, where γ is the arc

$$\gamma = \{z = \pm i e^{(-a+i)t}\colon t \in [0, \infty)\} \cup \{0\}$$

and $a \in (0, 1/8\pi)$. Hence it is not difficult to derive an analytic expression for the conformal mapping g of L onto D, and $\varphi = S_g$ turns out to be a rational function.

The idea behind the proof of Theorem 2 is quite simple. For $a \in (0, \infty)$ let

$$\alpha_1 = \{z = e^{(-a+i)t}\colon t \in (0, \infty)\}, \quad \alpha_2 = \{z\colon -z \in \alpha_1\}.$$

Then α_1 and α_2 are logarithmic spirals in D which converge onto the point 0 from opposite sides of ∂D. Next suppose that f is conformal in D and fixes the points $1, -1, \infty$. As $\|S_f\|_D$ approaches 0, f converges to the identity mapping in D. Hence for $\|S_f\|_D$ small, f maps α_1, α_2 onto a pair of disjoint open arcs α_1^*, α_2^* which spiral onto $f_1(0)$, $f_2(0)$, the points which $f(z)$ approaches as $z \to 0$ from the two sides of ∂D. This assertion follows from Lemmas 3, 5, 6 and 8.

Now the rate at which α_1 and α_2, and hence α_1^* and α_2^*, spiral depends on a. If a is sufficiently small, then α_1^*, α_2^* will spiral very slowly onto $f_1(0)$, $f_2(0)$. Since α_1^*, α_2^* are disjoint, the points $f_1(0)$, $f_2(0)$ will either coincide or be separated by a distance greater than a positive constant d. This is a consequence of Lemma 1.

Finally if we make $\|S_f\|_D$ still smaller, we can arrange by Lemma 9 that $f_1(0)$, $f_2(0)$ lie near 0 and hence within distance d of each other. Then $f_1(0)$ and $f_2(0)$ will coincide and $f(D)$ will not be a Jordan domain.

The complete proof for Theorem 2 is given in section 3. As indicated above, it depends on a number of results for a class of spirals. These are established in section 2.

2. Spirals

We derive here the results on spirals which will be needed in the proof of Theorem 2.

Definition. Suppose that α is an open arc in $\mathbf{C}$, that $z_1, z_2 \in \mathbf{C}$ and that $b \in (1, \infty)$. We say that α is a spiral from z_1 onto z_2 if α has the representation

$$z = z(t) = (z_1 - z_2)r(t)e^{it} + z_2, \quad t \in (0, \infty), \tag{2}$$

where $r(t)$ is positive and continuous with

$$\lim_{t \to 0} r(t) = 1, \quad \lim_{t \to \infty} r(t) = 0. \tag{3}$$

We say that α is a b-spiral if, in addition,

$$|z(t_1) - z_2| \leqslant b\,|z(t_2) - z_2| \tag{4}$$

for all $t_1, t_2 \in (0, \infty)$ with $|t_1 - t_2| \leqslant 2\pi$.

Example. Suppose that $a > 0$ and that α is the analytic open arc

$$z = e^{(-a+i)t}, \quad t \in (0, \infty).$$

Then α is an $e^{2\pi a}$-spiral from 1 onto 0 and

$$k(z)|z| = (a^2+1)^{-\frac{1}{2}}, \quad \frac{dk}{ds}(z)|z|^2 = a(a^2+1)^{-1} \tag{5}$$

for all $z \in \alpha$, where k denotes the curvature and s the arclength of α taken in the direction from 1 to 0.

PROPOSITION 1. *If α is a spiral from z_1 onto z_2 with the representation* (2), *then*

$$|z(t+2\pi)-z_2| < |z(t)-z_2| \tag{6}$$

for $t \in (0, \infty)$.

Proof. Let A denote the set of $t \in (0, \infty)$ for which (6) holds and let $B=(0, \infty)-A$. Since α is an open arc, B is the set of $t \in (0, \infty)$ for which the inequality in (6) is reversed. Hence A and B are both open. If $B \neq \emptyset$, then $B=(0, \infty)$ and

$$|z(2n\pi)-z_2| \geqslant |z(2\pi)-z_2| > 0$$

for all integers $n \geqslant 1$ contradicting (3). Thus $A=(0, \infty)$.

PROPOSITION 2. *If α is a b-spiral from z_1 onto z_2 and if f is a conformal similarity mapping, then $f(\alpha)$ is a b-spiral from $f(z_1)$ onto $f(z_2)$.*

Proof. This is an immediate consequence of the above definition.

The proof of Theorem 2 is based on a simple geometric fact. Namely that when $b \in (1, 2)$, the two points, onto which a pair of disjoint b-spirals converge, must either coincide or be separated by a distance greater than $\frac{1}{2}b^{-2}$ times the diameter of the smaller spiral. This observation is an immediate consequence of the following result.

LEMMA 1. *Suppose that α is a b-spiral from z_1 onto z_2, that β is a b-spiral from w_1 onto w_2 and that $\alpha \cap \beta = \emptyset$. If $b \in (1, 2)$, then either $z_2 = w_2$ or*

$$|z_2-w_2| > \frac{1}{b}\min(|z_1-z_2|, |w_1-w_2|).$$

Proof. Suppose otherwise. Then

$$0 < |z_2-w_2| \leqslant \frac{1}{b}\min(|z_1-z_2|, |w_1-w_2|). \tag{7}$$

If α has the representation (2), then $\arg(z(t_0)-z_2) = \arg(w_2-z_2)$ for some $t_0 \in (0, 2\pi]$, and we obtain

$$|z(t_0)-z_2| \geqslant \frac{1}{b}\lim_{t\to 0}|z(t)-z_2| = \frac{1}{b}|z_1-z_2| \geqslant |w_2-z_2|$$

from (3), (4) and (7). For each integer $m \geq 0$ let $t_1 = t_0 + 2m\pi$. Then $|z(t_1) - z_2|$ decreases to 0 as $m \to \infty$, and we can fix m so that

$$\begin{cases} \arg (z(t_1) - z_2) = \arg (w_2 - z_2) = \arg (z(t_1 + 2\pi) - z_2), \\ |z(t_1) - z_2| \geq |w_2 - z_2| > |z(t_1 + 2\pi) - z_2|. \end{cases} \tag{8}$$

Similarly if β has the representation $w = w(u)$, $u \in (0, \infty)$, then we can choose $u_1 \in (0, \infty)$ so that

$$\begin{cases} \arg (w(u_1) - w_2) = \arg (z_2 - w_2) = \arg (w(u_1 + 2\pi) - w_2), \\ |w(u_1) - w_2| \geq |z_2 - w_2| > |w(u_1 + 2\pi) - w_2|. \end{cases} \tag{9}$$

Now let λ denote the line through z_2 and w_2 directed from z_2 to w_2. Then (8) and (9) imply that $z(t_1)$, $z(t_1 + 2\pi)$, $w(u_1)$, $w(u_1 + 2\pi)$ lie on λ, that $w(u_1)$ precedes or coincides with z_2, that $z(t_1)$ coincides with or follows w_2, and that $z(t_1 + 2\pi)$ and $w(u_1 + 2\pi)$ lie between z_2 and w_2. We claim that

$$|z(t_1 + 2\pi) - z_2| \leq |w(u_1 + 2\pi) - z_2|. \tag{10}$$

To see this set

$$A = \{z = s(z(t) - z_2) + z_2 \colon s \in (0, 1),\ t \in (t_1 + \pi,\ t_1 + 3\pi)\},$$
$$B = \{z = s(z(t) - z_2) + z_2 \colon s \in (1, \infty),\ t \in (t_1 + \pi,\ t_1 + 3\pi)\},$$
$$\alpha_1 = \{z = z(t) \colon t \in (t_1 + \pi,\ t_1 + 3\pi)\} \subset \alpha,$$
$$\beta_1 = \{z = w(u) \colon u \in (u_1 + 2\pi,\ \infty)\} \subset \beta,$$
$$\lambda_1 = \{z = s(z(t_1 + \pi) - z_2) + z_2 \colon s \in [0, \infty)\} \subset \lambda.$$

Then A and B are open and disjoint, β_1 joins $w(u_1 + 2\pi)$ to $w_2 \in B$ in $\mathbf{C}$, and

$$\mathbf{C} = A \cup B \cup \alpha_1 \cup \lambda_1.$$

From Proposition 1 it follows that

$$\beta_1 \cap (\alpha_1 \cup \lambda_1) = \beta_1 \cap \lambda_1 = \emptyset$$

and hence that $\beta_1 \subset B$. Thus $w(u_1 + 2\pi) \notin A$ and we obtain (10).

Finally since α and β are b-spirals, Proposition 1 and (10) yield

$$\begin{aligned} |z(t_1) - z_2| &\leq b\,|z(t_1 + 2\pi) - z_2| \leq b\,|w(u_1 + 2\pi) - z_2| \\ &\leq b\,|w(u_1) - w(u_1 + 2\pi)| \\ &= b(|w(u_1) - w_2| - |w(u_1 + 2\pi) - w_2|) \\ &\leq (b - 1)\,|w(u_1) - w_2| < |w(u_1) - w_2|. \end{aligned}$$

Next we can reverse the roles of α and β in the above argument to obtain

$$|w(u_1)-w_2| < |z(t_1)-z_2|.$$

This contradiction shows that (7) cannot hold, completing the proof of Lemma 1.

We derive next in Lemmas 2 and 3 conditions, similar to (5), which guarantee that an analytic open arc is a spiral or a b-spiral, respectively. By Proposition 2, we may restrict our attention to the case where the arc has 1 and 0 as its endpoints.

LEMMA 2. *Suppose that c, $d \in (0, \infty)$, that α is an analytic open arc with* 1 *and* 0 *as endpoints, and that*

$$k(z)|z| \geqslant c, \quad \frac{dk}{ds}(z)|z|^2 \geqslant d$$

for $z \in \alpha$, where s is taken in the direction from 1 *to* 0. *Then α is a rectifiable spiral from* 1 *onto* 0.

Proof. For each $z \in \alpha$ let $\varrho(z)$ and $C(z)$ denote the radius and circle of curvature for α at z. Since k is positive and increasing in s, the part of α from z to 0 must lie inside $C(z)$ by a theorem due to A. Kneser. (See p. 48 in [9].) Hence

$$|z| \leqslant 2\varrho(z) = \frac{2}{k(z)}, \quad -\frac{d\varrho}{ds}(z) = \frac{\dfrac{dk}{ds}(z)}{k(z)^2} \geqslant \frac{d}{4}$$

for $z \in \alpha$. If β is any closed subarc of α from w_1 to w_2, then

$$l(\beta) = \int_\beta ds \leqslant \frac{4}{d}\int_\beta \left(-\frac{d\varrho}{ds}\right) ds < \frac{4}{d}\varrho(w_1) \leqslant \frac{4}{cd}|w_1|,$$

and hence α is rectifiable with length

$$l = l(\alpha) = \sup_{\beta \subset \alpha} l(\beta) \leqslant \frac{4}{cd}.$$

Let s denote the arclength of α from 1 to z, let $z = z(s)$, $s \in (0, l)$, denote the corresponding parametrization for α, and choose a continuous branch of $\log z(s)$ so that $\log z(s) \to 0$ as $s \to 0$. Then $t(s) = \operatorname{Im}(\log z(s))$ is continuously differentiable with

$$t'(s) = \operatorname{Im}\left(\frac{z'(s)}{z(s)}\right).$$

Suppose that $t'(s_0) = 0$ for some $s_0 \in (0, l)$. Then $z'(s_0) = az(s_0)$ where a is a real constant. This implies that the circle of curvature $C(z(s_0))$ is tangent to the ray from 0 through

$z(s_0)$ and hence that $C(z(s_0))$ cannot contain the point 0, thus contradicting the above mentioned theorem of Kneser. We conclude that

$$t'(s) = \operatorname{Im}\left(\frac{z'(s)}{z(s)}\right) \neq 0 \tag{11}$$

for $s \in (0, l)$ and hence that $t(s)$ is a strictly monotone function of s in $(0, l)$.

By (11) we can choose a continuous branch of $\log(z'(s)/z(s))$ such that

$$|\theta(s)| < \pi, \quad \theta(s) = \operatorname{Im}\left(\log \frac{z'(s)}{z(s)}\right) \tag{12}$$

in $(0, l)$. Then

$$\varphi(s) = t(s) + \theta(s) = \operatorname{Im}(\log z'(s)) \tag{13}$$

determines the angle of inclination for the tangent vector $z'(s)$ and

$$\varphi'(s) = k(z(s)) \geqslant c\,|z(s)|^{-1} \geqslant c(l-s)^{-1}$$

for $s \in (0, l)$. If $s_0 \in (0, l)$, then

$$\varphi(s) - \varphi(s_0) \geqslant \int_{s_0}^{s} c(l-s)^{-1}\,ds = c \log \frac{l - s_0}{l - s}$$

for $s \in (s_0, l)$, and $\varphi(s) \to \infty$ as $s \to l$. Thus $t(s) \to \infty$ as $s \to l$ by (12) and (13). Since $t(s) \to 0$ as $s \to 0$, we conclude that s is a strictly increasing function of t, $s = s(t)$, in $(0, \infty)$. Set $r(t) = |z(s(t))|$. Then

$$z = r(t)e^{it}, \quad t \in (0, \infty),$$

is a representation for α which shows that α is a spiral from 1 onto 0.

LEMMA 3. *Suppose that $c_1, c_2, d_1, d_2 \in (0, \infty)$ and that $4\pi d_2 < c_1^2$. Suppose also that α is an analytic open arc with 1 and 0 as endpoints and that*

$$c_1 \leqslant k(z)|z| \leqslant c_2, \quad d_1 \leqslant \frac{dk}{ds}(z)|z|^2 \leqslant d_2$$

for $z \in \alpha$, where s is taken in the direction from 1 to 0. Then α is a rectifiable b-spiral from 1 onto 0, where

$$b = \frac{c_1 c_2}{c_1^2 - 4\pi d_2} > 1.$$

Proof. Lemma 2 implies that α is a rectifiable spiral from 1 onto 0 with the representation

$$z = z(t) = r(t)e^{it}, \quad t \in (0, \infty).$$

It remains only to prove that $|z(t_1)| \leqslant b|z(t_2)|$ for all $t_1, t_2 \in (0, \infty)$ with $|t_1 - t_2| \leqslant 2\pi$. Let $\varrho(z)$ denote the radius of curvature for α at z. Then since

$$|z| \leqslant c_2 \varrho(z) \leqslant \frac{c_2}{c_1} |z|,$$

it suffices to show that

$$\varrho(z(t_1)) \leqslant \frac{c_1}{c_2} b \varrho(z(t_2)) \tag{14}$$

for all such t_1, t_2.

Fix t_1, $t_2 \in (0, \infty)$ with $|t_1 - t_2| \leqslant 2\pi$ and for $j=1$, 2 let $z_j = z(t_j)$, $s_j = s(t_j)$, $\theta_j = \theta(s_j)$ and $\varphi_j = \varphi(s_j)$ where $\theta(s)$ and $\varphi(s)$ are as in the proof of Lemma 3. Since

$$0 < -\frac{d\varrho}{ds}(z) = \frac{\dfrac{dk}{ds}(z)}{k(z)^2} \leqslant \frac{d_2}{c_1^2}$$

for $z \in \alpha$, $\varrho(z)$ is decreasing as a function of s. Suppose that $s_2 \leqslant s_1$. Then

$$\varrho(z_1) \leqslant \varrho(z_2) < \frac{c_1}{c_2} b \varrho(z_2)$$

and (14) holds. Suppose next that $s_1 < s_2$. Then

$$\varrho(z_1) - \varrho(z_2) = \int_{s_1}^{s_2} \left(-\frac{d\varrho}{ds}\right) ds \leqslant \frac{d_2}{c_1^2}(s_2 - s_1),$$

while

$$s_2 - s_1 = \int_{\varphi_1}^{\varphi_2} \left(\frac{ds}{d\varphi}\right) d\varphi = \int_{\varphi_1}^{\varphi_2} \varrho \, d\varphi \leqslant \varrho(z_1) |\varphi_2 - \varphi_1|.$$

Then (12) and (13) imply that

$$|\varphi_2 - \varphi_1| \leqslant |t_2 - t_1| + |\theta_2 - \theta_1| \leqslant 4\pi,$$

and we obtain

$$\varrho(z_1) - \varrho(z_2) \leqslant \frac{4\pi d_2}{c_1^2} \varrho(z_1),$$

from which (14) again follows. Hence the proof is complete.

We conclude this section with a result similar to Proposition 2. It implies that the image of a logarithmic spiral under a conformal mapping, which is nearly a similarity, is again a spiral. We require first the following result.

LEMMA 4. *Suppose that α is an analytic arc with the representation $z=z(t)$ where $z'(t)\neq 0$, and suppose that f maps a neighborhood of α conformally into* $\mathbf{C}$. *Then $\alpha^*=f(\alpha)$ is an analytic arc with the representation $w=f\circ z(t)$ and*

$$k^*(f(z))|f'(z)|-k(z)=\operatorname{Im}\left(\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right),$$

$$\frac{dk^*}{ds^*}(f(z))|f'(z)|^2-\frac{dk}{ds}(z)=\operatorname{Im}\left(S_f(z)\frac{z'(t)^2}{|z'(t)|^2}\right),$$

where k, k^* denote the curvatures and s, s^* the arclengths of α, α^* in the direction of increasing t.

Proof. If $w(t)=f\circ z(t)$, then $w'(t)=f'(z)z'(t)\neq 0$ and

$$\frac{w''(t)}{w'(t)}-\frac{z''(t)}{z'(t)}=\frac{f''(z)}{f'(z)}z'(t), \quad S_w(t)-S_z(t)=S_f(z)z'(t)^2, \tag{15}$$

where $z=z(t)$ and where S_w and S_z are defined exactly as in (1) with the differentiation now taken with respect to the real variable t. Then

$$k^*(w)|f'(z)|-k(z)=\operatorname{Im}\left(\frac{w''(t)}{w'(t)}-\frac{z''(t)}{z'(t)}\right)|z'(t)|^{-1} \tag{16}$$

by elementary differential geometry and

$$\frac{dk^*}{ds^*}(w)|f'(z)|^2-\frac{dk}{ds}(z)=\operatorname{Im}(S_w(t)-S_z(t))|z'(t)|^{-2} \tag{17}$$

by Exercise 3 on p. 21 of [3]. The desired conclusion now follows from (15), (16) and (17).

LEMMA 5. *Suppose that b, c_1, c_2, d_1, d_2 and α are as in Lemma 3 and that $b^*\in(b,\infty)$. Then there exists an $\varepsilon>0$, depending only on b^*, c_1, c_2, d_1, d_2, with the following property. If f maps a neighborhood of α conformally into* $\mathbf{C}$, *if $f(z)\to 1$ and 0 as $z\to 1$ and 0 on α, and if*

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leqslant\varepsilon, \quad \left|\frac{z^2f''(z)}{f(z)}\right|\leqslant\varepsilon, \quad \left|\frac{z^3f'''(z)}{f(z)}\right|\leqslant\varepsilon \tag{18}$$

for $z\in\alpha$, then $\alpha^*=f(\alpha)$ is a b^*-spiral from 1 onto 0.

Proof. By hypothesis we can choose $\eta\in(0,\min(c_1,d_1))$ so that

$$4\pi(d_2+\eta)<(c_1-\eta)^2, \quad \frac{(c_1-\eta)(c_2+\eta)}{(c_1-\eta)^2-4\pi(d_2+\eta)}\leqslant b^*.$$

Fix $\varepsilon \in (0, \frac{1}{2})$ so that $(4+2c_2)\varepsilon \leqslant \eta$ and $(20+6d_2)\varepsilon \leqslant \eta$, and suppose that f satisfies the hypotheses of Lemma 5. Then $\alpha^* = f(\alpha)$ is an analytic open arc with 1 and 0 as endpoints. If $w \in \alpha^*$, then $z = f^{-1}(w) \in \alpha$ and (18) implies that

$$\left| \left| \frac{f(z)}{f'(z)} \right| - |z| \right| \leqslant 2\varepsilon |z|, \quad \left| \frac{zf''(z)}{f'(z)} \right| \leqslant 2\varepsilon, \quad |z^2 S_f(z)| \leqslant 5\varepsilon.$$

Hence we obtain the inequalities

$$\begin{aligned} |k^*(w)|w| - k(z)|z|| &\leqslant \left| \frac{f(z)}{f'(z)} \right| |k^*(f(z))|f'(z)| - k(z)| + \left| \left| \frac{f(z)}{f'(z)} \right| - |z| \right| k(z) \\ &\leqslant 2|z| \left| \frac{f''(z)}{f'(z)} \right| + 2\varepsilon |z| k(z) \leqslant \eta \end{aligned}$$

and

$$\begin{aligned} \left| \frac{dk^*}{ds^*}(w)|w|^2 - \frac{dk}{ds}(z)|z|^2 \right| &\leqslant \left| \frac{f(z)}{f'(z)} \right|^2 \left| \frac{dk^*}{ds^*}(f(z))|f'(z)|^2 - \frac{dk}{ds}(z) \right| + \left| \left| \frac{f(z)}{f'(z)} \right|^2 - |z|^2 \right| \frac{dk}{ds}(z) \\ &\leqslant 4|z|^2 |S_f(z)| + 6\varepsilon |z|^2 \frac{dk}{ds}(z) \leqslant \eta \end{aligned}$$

from Lemma 4, where k^* and s^* denote the curvature and arclength of α^*. Thus

$$c_1 - \eta \leqslant k^*(w)|w| \leqslant c_2 + \eta, \quad d_1 - \eta \leqslant \frac{dk^*}{ds^*}(w)|w|^2 \leqslant d_2 + \eta$$

for $w \in \alpha^*$, and the desired conclusion follows from Lemma 3.

3. Proof of Theorem 2

For each $a \in (0, \infty)$ let

$$\begin{aligned} &\alpha_1 = \{z = e^{(-a+i)t} \colon t \in (0, \infty)\}, \quad \alpha_2 = \{z \colon -z \in \alpha_1\}, \\ &\beta = \{z = \pm i e^{(-a+i)t} \colon t \in (-\infty, \infty)\} \cup \{0, \infty\}, \\ &\gamma = \{z \colon z \in \beta, \ |z| \leqslant 1\}. \end{aligned}$$

Then β is a Jordan curve which separates α_1 and α_2. Let D_j denote the component of $\bar{\mathbf{C}} - \beta$ which contains α_j and set $D = \bar{\mathbf{C}} - \gamma$. Then D is a simply connected domain of hyperbolic type which contains $D_1 \cup D_2$ and hence $\alpha_1 \cup \alpha_2$.

Now suppose that $a \in (0, 1/8\pi)$ and that f is conformal in D. We shall show that there exists a $\delta = \delta(a) > 0$ such that $f(D)$ is not a Jordan domain whenever $\|S_f\|_D \leqslant \delta$; for this we may clearly assume that f is normalized so that it fixes the points $1, -1, \infty$. The argument then consists of three steps. First in Lemma 8 we show there exists a $\delta_2>0$ such that $f(\alpha_1)$ and $f(\alpha_2)$ are b^*-spirals with $b^*\in(1,2)$ whenever $\|S_f\|_D\leqslant\delta_2$. Next in Lemma 9 we show there exists a $\delta_3>0$ such that the points onto which $f(\alpha_1)$ and $f(\alpha_2)$ converge must lie in $\{z: |z|\leqslant\frac{1}{8}\}$ whenever $\|S_f\|_D\leqslant\delta_3$. Finally set $\delta=\min(\delta_2,\delta_3)$. Then Lemma 1 implies that $f(\alpha_1)$ and $f(\alpha_2)$ converge onto the same point and hence that $f(D)$ is not a Jordan domain whenever $\|S_f\|_D\leqslant\delta$.

We begin with an application of Ahlfors' extension theorem [1] to the domains D_1 and D_2.

LEMMA 6. *There exists a $\delta_1=\delta_1(a)>0$ with the following property. If f is conformal in D and if $\|S_f\|_D\leqslant\delta_1$, then for $j=1,2$ the mapping $f_j=f|D_j$ has a quasiconformal extension g_j to $\bar{\mathbf{C}}$ and*

$$K(g_j)\leqslant(1-c\|S_f\|_D)^{-1}, \tag{19}$$

where $c=c(a)$ and $K(g_j)$ denotes the maximal dilatation of g_j.

Proof. Let

$$h(re^{i\theta})=r^a e^{i(\theta-\log r)} \tag{20}$$

for $r\in(0,\infty)$, and set $h(0)=0$ and $h(\infty)=\infty$. Then it is easy to verify that h is a K-quasiconformal mapping of $\bar{\mathbf{C}}$, where $K=a+(2/a)$, and that h maps the imaginary axis onto β. Thus $\partial D_j=\beta$ is a K-quasiconformal circle. By the above mentioned theorem of Ahlfors, there exists a $\delta_1=\delta_1(a)$ such that each f_j conformal in D_j with $\|S_{f_j}\|_{D_j}\leqslant\delta_1$ has a quasiconformal extension g_j to $\bar{\mathbf{C}}$, where

$$\|\mu_{g_j}\|_\infty\leqslant c\|S_{f_j}\|_{D_j}(2-c\|S_{f_j}\|_{D_j})^{-1} \tag{21}$$

and $c=c(a)$. (For this last estimate see p. 22 in [10] or p. 132 in [2].)

Now suppose that f satisfies the hypotheses of Lemma 6 and let $f_j=f|D_j$. Then since $\varrho_D\leqslant\varrho_{D_j}$ in D_j,

$$\|S_{f_j}\|_{D_j}\leqslant\|S_f\|_D\leqslant\delta_1.$$

Thus f_j has a quasiconformal extension g_j to $\bar{\mathbf{C}}$ satisfying (21), and (19) follows directly.

Remark. If f is conformal in D with $\|S_f\|_D\leqslant\delta_1$, then Lemma 6 implies that $f_j=f|D_j$ has a homeomorphic extension to $D_j\cup\{0\}$ and hence that $f(z)$ has limits as $z\to0$ in D_1 and as $z\to0$ in D_2. We shall denote these limits by $f_1(0)$ and $f_2(0)$, respectively.

We require next the following consequence of a distortion theorem due to Teichmüller [13].

LEMMA 7. *For each $\eta > 0$ there exists a $K_1 = K_1(\eta) \in (1, \infty)$ with the following property. If g is a sense preserving quasiconformal mapping of $\bar{\mathbf{C}}$ with $K(g) \leq K_1$ and if g fixes three points z_1, z_2, ∞, then*

$$|g(z) - z| \leq \eta |z_1 - z_2| \tag{22}$$

for z with $|z - z_1| < |z_1 - z_2|$.

Proof. Let ϱ and σ denote respectively the hyperbolic metric and distance in $G = \bar{\mathbf{C}} - \{0, 1, \infty\}$ and set

$$b = \inf \{\varrho(z) \colon z \in G \cap B\}, \quad B = \{z \colon |z| \leq 2\}. \tag{23}$$

Then ϱ is positive and infinitely differentiable in G and $\varrho(z) \to \infty$ as $z \to 0$ or 1. (See, for example, p. 51 and p. 246 in [12].) Hence $b \in (0, \infty)$. Set

$$K_1 = \exp(2b \min(\eta, 1)) \in (1, \infty).$$

Now suppose that g is a sense preserving quasiconformal mapping of $\bar{\mathbf{C}}$ with $K(g) \leq K_1$, and suppose that g fixes the points 0, 1, ∞. Then by the above mentioned theorem of Teichmüller,

$$\sigma(g(z), z) \leq \tfrac{1}{2} \log K(g) \leq b \min(\eta, 1) \leq b$$

for $z \in G$. (See pp. 29–31 in [13].) If $|z| < 1$, then (23) implies that

$$\sigma(g(z), z) = \inf \int_\omega \varrho \, ds \geq \inf \int_{\omega \cap B} b \, ds \geq b \min(|g(z) - z|, 2 - |z|),$$

where the infima are taken over all rectifiable arcs ω joining z to $g(z)$ in G. Hence

$$|g(z) - z| = \min(|g(z) - z|, 2 - |z|) \leq \min(\eta, 1) \leq \eta$$

for $|z| < 1$ and we obtain (22) for the special case where $z_1 = 0$ and $z_2 = 1$. The general case then follows by applying what was proved above to the mapping

$$h(z) = \frac{g(z(z_2 - z_1) + z_1) - z_1}{z_2 - z_1}.$$

Remark. Lemma 7 also follows from a more elementary contra-positive normal family type argument. However this second method does not yield an explicit estimate for K_1 in terms of η.

LEMMA 8. *For each $a \in (0, 1/8\pi)$ there exists a $\delta_2 = \delta_2(a) \in (0, \delta_1]$ with the following property. If f is conformal in D with $\|S_f\|_D \leq \delta_2$ and if f fixes ∞, then for $j = 1, 2$, $\alpha_j^* = f(\alpha_j)$ is a b^*-spiral onto $f_j(0)$ where $b^* \in (1, 2)$.*

Proof. Set $c_1 = c_2 = (a^2+1)^{-\frac{1}{2}}$, $d_1 = d_2 = a(a^2+1)^{-1}$,

$$b = \frac{c_1 c_2}{c_1^2 - 4\pi d_2} = (1-4\pi a)^{-1} \in (1, 2),$$

and fix $b^* \in (b, 2)$. Next let ε be as in Lemma 5, set

$$\eta = \tfrac{1}{6}\varepsilon r^3, \quad r = \tfrac{1}{2} \operatorname{dist}(1, \partial D_1) < \tfrac{1}{2},$$

and choose $\delta_2 \in (0, \delta_1]$ so that $(1-c\delta_2)^{-1} \leqslant K_1$, where $c = c(a)$ and $K_1 = K_1(\eta)$ are as in Lemmas 6 and 7. Then δ_2 depends only on a.

Now suppose that f satisfies the hypotheses of Lemma 8. Then $\alpha_1^* = f(\alpha_1)$ and $\alpha_2^* = f(\alpha_2)$ are analytic open arcs with endpoints $f(1)$, $f_1(0)$ and $f(-1)$, $f_2(0)$ respectively. We shall show first that α_1^* is a b^*-spiral from $f(1)$ onto $f_1(0)$. By Proposition 2 we may assume without loss of generality that $f(1)=1$ and $f_1(0)=0$.

Let g_1 denote the quasiconformal extension of $f_1 = f\,|\,D_1$ to $\overline{\mathbf{C}}$ given by Lemma 6, fix $z_1 \in \alpha_1$ and set

$$h(z) = \frac{g_1(z_1 z)}{g_1(z_1)}.$$

Then h is a sense preserving quasiconformal mapping of $\overline{\mathbf{C}}$, $K(h) \leqslant K_1$ and h fixes the points $0, 1, \infty$. Hence

$$|h(z) - z| \leqslant \eta \tag{24}$$

for $|z-1| < 1$ by Lemma 7. Since $\varphi(z) = z_1 z$ maps D_1 onto D_1, $f(z_1 z) = g_1(z_1 z)$ for $z \in D_1$. Hence h is analytic in D_1 and

$$\left|\frac{z_1 f'(z_1)}{f(z_1)} - 1\right| = |h'(1) - 1| \leqslant \frac{1}{2\pi}\int_\omega \frac{|h(z)-z|}{|z-1|^2}\,|dz| \leqslant \frac{\eta}{r} < \varepsilon$$

by (24), where ω is the positively oriented circle $\{z\colon |z-1| = r\}$. Similarly we obtain

$$\left|\frac{z_1^2 f''(z_1)}{f(z_1)}\right| \leqslant \frac{2\eta}{r^2} < \varepsilon, \quad \left|\frac{z_1^3 f'''(z_1)}{f(z_1)}\right| \leqslant \frac{6\eta}{r^3} = \varepsilon.$$

Then (5) and Lemma 5 imply that α_1^* is a b^*-spiral from 1 onto 0.

Next let $g(z) = f(-z)$. Then g is conformal in D with $\|S_g\|_D \leqslant \delta_2$ and $g(\infty) = \infty$. Hence $\alpha_2^* = g(\alpha_1)$ is a b^*-spiral by what was shown above and the proof is complete.

LEMMA 9. *For each $\varepsilon > 0$ there exists a $\delta_3 = \delta_3(a, \varepsilon) \in (0, \delta_1]$ with the following property. If f is conformal in D with $\|S_f\|_D \leqslant \delta_3$ and if f fixes $1, -1, \infty$, then $|f_1(0)| \leqslant \varepsilon$ and $|f_2(0)| \leqslant \varepsilon$.*

Proof. Set $\eta=\min(\varepsilon/4, \frac{1}{2})$ and choose $\delta_3\in(0,\delta_1]$ so that $(1-c\delta_3)^{-2}\leqslant K_1$, where c and K_1 are as in Lemmas 6 and 7. Then δ_3 depends only on a and ε.

Now suppose that f satisfies the hypotheses of Lemma 9 and for $j=1, 2$ let g_j denote the quasiconformal extension of $f_j=f|D_j$ to $\bar{\mathbf{C}}$ given by Lemma 6. Then $g=g_2\circ g_1^{-1}$ is a sense preserving quasiconformal mapping of $\bar{\mathbf{C}}$ with $K(g)\leqslant K_1$. If $z_0\in\beta-\gamma$, then for $j=1, 2$, $z_0\in D_j$ and

$$g_j(z_0)=\lim g_j(z)=\lim f_j(z)=f(z_0),$$

where the limits are taken as $z\to z_0$ in D_j. Thus g fixes points in $\beta-\gamma$ and hence, by continuity, the points i, $-i$, ∞. Thus

$$|g_2(1)-1|=|g(1)-1|\leqslant 2\eta,\quad 0<|g_2(1)+1|\leqslant 3$$

by Lemma 7. Set

$$h(z)=\frac{2}{g_2(1)+1}g_2(z)-\frac{g_2(1)-1}{g_2(1)+1}.$$

Again h is a sense preserving quasiconformal mapping of $\bar{\mathbf{C}}$, $K(h)\leqslant K_1$ and h fixes 1, -1, ∞. Thus $|h(0)|\leqslant 2\eta$ by Lemma 7 and

$$|f_2(0)|=|g_2(0)|\leqslant\tfrac{1}{2}|g_2(1)+1|\,|h(0)|+\tfrac{1}{2}|g_2(1)-1|\leqslant\varepsilon.$$

Finally applying what was proved above to the mapping $-f(-z)$ yields the inequality $|f_1(0)|\leqslant\varepsilon$.

Proof of Theorem 2. Suppose that $a\in(0, 1/8\pi)$ and set

$$\delta=\min(\delta_2(a),\delta_3(a,\tfrac{1}{5}))\leqslant\delta_1,$$

where δ_2 and δ_3 are as in Lemmas 8 and 9. Next suppose that f is conformal in D with $\|S_f\|_D\leqslant\delta$. We shall show that $f(D)$ is not a Jordan domain. By following f by a Möbius transformation, we may assume without loss of generality that f fixes the points 1, -1, ∞.

Now Lemma 8 implies that $\alpha_1^*=f(\alpha_1)$ and $\alpha_2^*=f(\alpha_2)$ are disjoint b^*-spirals from 1 onto $f_1(0)$ and from -1 onto $f_2(0)$, respectively, where $b^*\in(1, 2)$. Next Lemma 9 implies that $|f_1(0)|\leqslant\frac{1}{5}$ and $|f_2(0)|\leqslant\frac{1}{5}$. Thus

$$|f_1(0)-f_2(0)|\leqslant\tfrac{2}{5}<\frac{1}{b^*}\min(|1-f_1(0)|, |-1-f_2(0)|),$$

and we conclude from Lemma 1 that $f_1(0)=f_2(0)$.

Next let $B=\{z\colon |z|<1\}$ and for $z\in B$ set $g(z)=h\left(\frac{i}{2}\left(z+\frac{1}{z}\right)\right)$, where h is the quasiconformal mapping of $\bar{\mathbf{C}}$ defined in (20) in the proof of Lemma 6. Then $f\circ g$ is a quasiconformal mapping of B onto $f(D)$ and $f\circ g(z)\to f_1(0)$, $f_2(0)$ as $z\to i$, $-i$ respectively in B. Hence $f(D)$ cannot be a Jordan domain, since otherwise $f\circ g$ would have a homeomorphic extension to $\bar{B}$ and $f_1(0)\neq f_2(0)$.

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