A CUTTING AND PASTING OF NONCOMPACT POLYGONS WITH APPLICATIONS TO FUCHSIAN GROUPS

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1. Introduction

In several classical textbooks on Algebraic Topology and Discontinuous Groups [cf. 7, 12] one finds the technique of cutting and pasting finite polygons to a canonical form. With Theorem 1 available we can use this classical technique to prove some important classical theorems in the theory of fuchsian groups as well as prove some interesting new ones. Thereby, we not only gain new insight into the theory of fuchsian groups, but also unify part of the classical theory.

Most of our applications will be to infinitely generated fuchsian groups. We do, however, prove some missed theorems about finitely generated groups along the way.

We should like to mention here that the techniques in this paper, especially Theorem 1, have many more applications to infinitely generated fuchsian groups whose detail is currently being worked out.

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2. Definitions

A *fuchsian group* Γ will be a group acting discontinuously on the unit disk Δ. A *set of generators* for Γ will be denoted by $\{A_1, A_2, \dots\}$. In this case we write $\Gamma = \langle A_1, A_2 \dots\rangle$. A *fundamental domain* for Γ is a domain $D \subseteq \bar{\Delta}$, the closure of Δ, such that (1) ∂D is accessible (accessible means that one can draw an arc from a point in D to any point in ∂D, where ∂D is the boundary of D); (2) $\partial D \cap \Delta$ can be written as the union of a countable (possibly infinite) number of Jordan arcs, which are identified in pairs by elements of Γ;

(3) $V(D) \cap D = \emptyset$ for all $V \in \Gamma$ and $V \neq I$, where $I(z) = z$ for all z; (4) for each point $z \in \Delta$ there exists a $V \in \Gamma$ such that $V(z) \in \bar{D}$. The Jordan arcs in (2) will be denoted by the set $\{\gamma_i, \gamma_i'\}_i$, where i runs through a subset of the positive integers and the arc labeled γ_i' is related to the arc γ_i by $\gamma_i' = A(\gamma_i)$ for some $A \in \Gamma$. The arcs γ_i, γ_i' for $i = 1, 2, \ldots$ are called sides of D.

P is a *fundamental polygon* if (1) P is a fundamental domain; (2) each γ_i or γ_i' is a hyperbolic line or line segment; (3) P is hyperbolically convex. A group Γ is said to be of *classical Schottky type* if there is a fundamental polygon for Γ whose only vertex cycles (see [7], for the definition) in Δ are cycles of exactly one elliptic fixed point. Such a polygon will be called a *Schottky polygon*. We say that a generating set $\{A_1, A_2, \ldots\}$ for Γ is a *classical Schottky generating set* if there is a Schottky polygon P for Γ such that the set $\{A_1^{\pm 1}, \ldots\}$ is precisely the set of elements of Γ which identify pairs of sides of P. A *Schottky pair of sides* for Γ is a pair of sides identified by some $A \in \Gamma$ which begin and end in $\partial\Delta$, when A is not elliptic, or, if A is elliptic, then one endpoint is the elliptic fixed point of A and the other endpoint lies in $\partial\Delta$.

If $A \in \Gamma$ and is elliptic, then $A^n = I$ for some positive integer n. The smallest positive integer for which $A^n = I$ is called the *order of* A. A is of minimal rotation if $|\operatorname{tr}(A)| = 2\cos(\pi/n)$ for some $n \geq 2$. Note n in this case is also the order of A. A *free hyperbolic transformation* $H \in \Gamma$ is a hyperbolic transformation in which one of the two open intervals of $\partial\Delta$ determined by the two fixed points of H lies in the set of ordinary points of Γ.

Let $S = \{A_1, A_2, \ldots\}$ be a set of transformations in $SL(2, \mathbf{R})$. We denote by n_j the order of A_j, where $n_j = +\infty$ if A is not of finite order. Calling the elements of S letters, we define the *length of a word W in the letters from S* by first writing W in *reduced form*:

$$W = A_{i_1}^{\alpha_1} \cdots A_{i_k}^{\alpha_k}, \tag{*}$$

where $0 < |\alpha_j| \leq \frac{1}{2} n_{i_j}$ $(j = 1, 2, \ldots, k)$ and $A_{i_r} \neq A_{i_{r+1}}^{\pm 1}$ for $r = 1, 2, \ldots, k-1$. Then $L(W)$, the length, is defined by $L(W) = \sum_{j=1}^{k} |\alpha_j|$. We note that if $A_{i_j} = I$, then $n_{i_j} = 1$, and hence $\alpha_j = 0$. So $L(W)$ is well defined. In the case $\langle A_1 \ldots \rangle = *_i \langle A_i \rangle$, where $*_i \langle A_i \rangle$ denotes the free product of the cyclic groups $\langle A_i \rangle$, $L(W)$ is the length of any element of $\langle A_1, A_2, \ldots \rangle$ with respect to the generators $\{A_1, A_2, \ldots\}$. Finally set $\sigma_A(W) = \sum_{A_{i_j} = A} \alpha_j$ for each $A \in \{A_1, \ldots, A_n, \ldots\}$ and $W = I$, if $L(W) = 0$.

3. The basic theorem

THEOREM 1. *Let* $\{C_i, C_i'\}_i$ *be a collection of circles perpendicular to* $\partial\Delta$ *which are exterior to each other except for possible external tangencies and the possibility that*

$C_i \cap C_i' \cap \Delta \neq \emptyset$. Let Γ be a fuchsian group for which to each C_i there exists an $A_i \in \Gamma$ such that $A_i(C_i) = C_i'$ with the outside of C_i being mapped by A_i onto the inside of C_i'. If an A_i is elliptic, then we assume it is of minimal rotation and that $C_i \cap C_i' \cap \Delta$ is the fixed point of A_i in Δ, when $n_i \neq 2$, and that $C_i = C_i'$, when $n_i = 2$. Then the polygon P formed by the intersection of Δ with the region exterior to all the circles from $\{C_i, C_i'\}_i$ is a fundamental polygon for Γ if and only if

(1) *the group $\langle A_1, A_2, ...\rangle = \Gamma$, and*

(2) *Γ has no intervals of discontinuity on $\partial\Delta$, or $\bar{P} \cap \partial\Delta$ contains a fundamental set for the ordinary points, Ω_Γ, of Γ in $\partial\Delta$.*

Remarks. (1) In the proof it will be clear that when there are ordinary points on $\partial\Delta$, then a fundamental set of Γ in $\partial\Delta$ in $\bar{P} \cap \partial\Delta$ will be expressable as a disjoint union of intervals of positive length.

(2) A similar theorem can be proved for the case where we allow accidental or more general elliptic vertex cycles in Δ. The proof, however, is not as easy as the one given here for a weaker, but for our purposes sufficient, Theorem 1.

(3) Theorem 1 is in some sense a generalized Klein's combination theorem [2]. It is known that Klein's combination theorem is not true if one allows the circles which are exterior to each other to have tangencies. Also, Klein's theorem does not make any assertion in the case there are infinitely many circles. Theorem 1 gives precisely the condition for the images of the polygon P exterior to the $\{C_i, C_i'\}_i$ in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ to be a covering of the set of ordinary points of $\Gamma = \langle A_1, A_2, ...\rangle$. Although Theorem 1 deals only with the fuchsian group case, Lemma 1 and Lemma 2 below are still valid in the general case and hence so is the proof of Theorem 1. This means that P is a fundamental domain of Γ if and only if $\hat{\mathbf{C}} \setminus \bigcup_{V \in \Gamma} V(P)$ does not contain a nontrivial disc. The problem with Theorem 1 is essentially the same problem that one has with Maskit's version of the Poincaré Theorem [11], and Theorem 1 is in a way a special case of the Poincaré Theorem. In both theorems there is no way to check the conditions which make P a fundamental polygon, except in the most trivial cases. However, Theorem 1 has the advantage of saying how far one can carry the ideas of cutting and pasting.

(4) At the suggestion of the referee we now give an example to show that condition (1) and (2) of Theorem 1 are independent. We choose a trivial example to keep these remarks short. Let $A = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ and $\Gamma = \langle A \rangle$. Let $C = \{|z-1| = 1\}$ and $C' = A(C)$. Then the polygon P in $H^+ = \{z: \text{Im}(z) > 0\}$ determined by C and C' satisfies (1) but not (2). If we set $C_1 = \{|z| = 1\}$ and $C_1' = A^2(C_1)$, then the polygon determined by C_1 and C_1' satisfies

(2) but not (1). So in neither case do we have a fundamental polygon of Γ. This brings us to an interesting question. Suppose P satisfies (2) in a nontrivial way, i.e. $\Omega_\Gamma \cap \partial\Delta \neq \emptyset$ and except for at most countably many points $x \in \Omega_\Gamma \cap \partial\Delta$, the $V \in \Gamma$ for which $V(x) \in \bar{P}$ is unique. Does P then also satisfy (1)? If Γ is finitely generated the answer is yes. There are examples in the infinitely generated case to contradict this conjecture.

Before proving Theorem 1 we must first prove some lemmas and introduce some notation. For each $C \in \{C_i, C_i'\}$ we let $D(C)$ be the closed bounded disk determined by C. We now define a collection of arcs $\{\beta_i, \beta_i'\}_i$ by $\beta_i = C_i$ and $\beta_i' = C_i'$, when A_i is not elliptic, and, when A_i is elliptic, we set $\beta_i = C_i \cup C_i' \setminus [D(C_i)^\circ \cup D(C_i')^\circ]$ and $\beta_i' = \emptyset$, where S° is the interior of S, $T \setminus S = \{x \in T: x \notin S\}$, and $\emptyset$ is the empty set. Now let $D(\beta)$ be the closed bounded region determined by β, where $\beta \in \{\beta_i, \beta_i'\}_i, \beta \neq \emptyset$.

LEMMA 1. *Let $\{C_i, C_i'\}_i$ and $\{A_i\}_i$ be as in the hypothesis of Theorem 1. Set $G = \langle A_1, \ldots\rangle$. Then the following are true.*

(1) *For any two words V, W in the letters from $\{A_i\}_i$ we have that $V(\bar{P}) \cap W(P) \neq \emptyset$ if and only if $L(W^{-1}V) = 0$; hence $G = *_i\langle A_i\rangle$ and is a fuchsian group.*

(2) *For $V, W \in G$, $W(\beta_i) \cap V(\beta_j) \cap \Delta \neq \emptyset$ if and only if $i = j$ and either $W = V$, or, A_i is elliptic and $W = VA_i^q$ for some integer q.*

(2′) *For $V, W \in G$, $W(\beta_i') \cap V(\beta_j') \cap \Delta \neq \emptyset$ if and only if $i = j$, $W = V$, and W is not elliptic.*

(3) *For $V, W \in G$, $W(\beta_i) \cap V(\beta_j') \cap \Delta \neq \emptyset$ if and only if $i = j$ and $W = VA_i$, where A_i is not elliptic.*

(4) *For $\beta \in \{\beta_i, \beta_i'\}_i$, $\beta \neq \emptyset$, $V \in G$, and $z \in D(V(\beta))^\circ \cap \bar{\Delta}$ there exists a $W \in G$ such that $W(\bar{P}) \subseteq D(V(\beta))$ and either $z \in W(\bar{P})$ or $z \in D(W(\alpha)) \subset D(V(\beta))$ for some $\alpha \in \{\beta_i, \beta_i'\}_i$, $\alpha \neq \emptyset$.*

Proof. (1) is just the discontinuous part of the Klein combination theorem, which is proven in [7, 2].

To prove (2) it suffices to show $W(\beta_i) \cap \beta_k \cap \Delta \neq \emptyset$ implies $W = I$, if A_j is not elliptic, and $W = A_j^q$, if A is elliptic. Considering W (see (*)) as a reduced word of positive length, we have from [7, 2] that $W(\bar{P})$ is contained in $D(C_{i_1})$ or $D(C_{i_1}')$, as $\alpha_1 < 0$ or $\alpha_1 > 0$, respectively. Hence $\beta_j = \beta_{i_1}$ or $j = i_1$. If A is not elliptic, then $\alpha_1 < 0$. Moreover, since $A_j^{-1}(C_j') = C_j$, we have $W(\beta_i) \subseteq D(C_j)^\circ$, except when $L(W) = 0$. Hence, $W = I$ and $i = j$. If A_j is elliptic, then we observe that $A_j^\alpha(\bar{\Delta} \setminus (D(C_j) \cup D(C_j'))) \subseteq D(C_j)^\circ$ for $0 < \alpha \leqslant n_j/2$, and $A_j^\alpha(\bar{\Delta} \setminus (D(C_j) \cup D(C_j'))) \subseteq D(C_j')^\circ$ for $0 > \alpha \geqslant -n_j/2$. Hence $W(\beta_i) \cap \beta_j \cap \Delta \neq \emptyset$ if and only if $W = A_j^\alpha$ and $\beta_i = \beta_j$.

The proof of (2′) is similar.

The proof of (3) is very similar to the proof of (2). Since either $W(P)\subseteq D(C_{i_1})^\circ$ or $W(P)\subseteq D(C'_{i_1})^\circ$ and $\beta'_j=\emptyset$, if A_j is elliptic, it follows that the only possibility for $W(\beta_i)\cap\beta'_j\cap\Delta\neq\emptyset$ is that A_j is not elliptic and $W(\beta_i)=\beta'_j$. Hence, $W=A_j$ and $i=j$, as claimed in (3).

We now prove (4). Let $z\in D(V(\beta))^\circ\cap\bar{\Delta}$. Now $V(\beta)\cap\bar{\Delta}$ is either a side or the union of two sides of the polygon $V(P)$. If $V(\beta)$ is a side, say $\beta=C_i$, then either $V(P)\subseteq D(V(\beta))^\circ$ or $VA_i^{-1}(P)\subseteq D(V(\beta))^\circ$. Let W be the transformation such that $W(P)\subseteq D(V(\beta))^\circ$. Here, we note that for either choice of W, $V(\beta)$ is a side of $W(P)$. Since $z\notin V(\beta)$, we have either $z\notin W(\bar{P})$ or z belongs to one of the connected components of the compliment of $W(\bar{P})$ with respect to $\bar{\Delta}$ which lie in $D(V(\beta))$. Hence, if $z\notin W(\bar{P})$, then $z\in D(W(\alpha))\subset D(V(\beta))$ for some $\alpha\in\{\beta_i,\beta'_i\}$. A similar argument holds for $\beta=C'_i$.

In the case $\beta=\beta_i$ is the union of two sides of P, we have A_i is elliptic. Now if $V(P)\cap D(V(\beta))=\emptyset$, then $D(V(\beta))=V(D(\beta))$. This last statement is easily seen from the fact that $V(D(\beta))$ is either $D(V(\beta))$ or $\hat{\mathbb{C}}\setminus D(V(\beta))$ and $V(0)\in\hat{\mathbb{C}}\setminus D(V(\beta))$. Writing $D(\beta)=\bigcup_{k=1}^{n_i-1}D(A_i^k(\beta))$, we see that, since $V(D(A_i^k(\beta)))$ is bounded for $k=1,2,\dots n_i-1$, $V(D(\beta))=\bigcup_{k=1}^{n_i-1}V(D(A_i^k(\beta)))=\bigcup_{k=1}^{n_i-1}D(VA_i^k(\beta))$. Hence $z\in D(VA_i^k(\beta))$ for some k. Moreover, $VA_i^k(P)\subseteq D(VA_i^k(\beta))$. Thus we have that either $z\in VA_i^k(P)$, or there exists an arc $\alpha\in\{\beta_i,\beta'_i\}$, $\alpha\neq\emptyset$, such that $z\in D(VA_i^k(\alpha))\subset D(VA_i^k(\beta))\subseteq D(V(\beta))$. Q.E.D.

COROLLARY. *$D(W(\alpha))$ in (4) can be chosen to be hyperbolically convex.*

Proof. The only case that need be considered is $\alpha=\beta_i$ and A_i is elliptic. The proof of (4) for this case can be used here to choose $D(W(\alpha))$ convex. Q.E.D.

Let

$$n(\gamma,z)=\frac{1}{2\pi}\left|\int_\gamma\frac{d\xi}{\xi-z}\right|$$

and $n(z)=\Sigma_{V,\beta}\,n(V(\beta),z)$, where V runs through $\langle A_1,A_2,\dots\rangle$ and β runs through the set $\{\beta_i\}_i$ and $z\notin\bigcup_{V,\beta}V(\beta)$.

LEMMA 2. *$n(z)$ is finite if and only if z is in an image of $\bar{P}$ under the group $\langle A_1,\dots\rangle$.*

Proof. Suppose $n(z)$ is finite, then z is contained in only finitely many sets $D(V(\beta))$, where $V\in\langle A_1,A_2,\dots\rangle$ and $\beta\in\{\beta_i,\beta'_i\}_i$, $\beta\neq\emptyset$. If $n(z)=0$, then it is clear $z\in P$. If $n(z)>0$, let $V_0(\beta_i)$ be such that $z\in D(V_0(\beta_i))$ and has minimal Euclidean area. We choose a $W\in\langle A_1,\dots\rangle$ so that either $z\in W(\bar{P})\subseteq D(V_0(\beta_i))$ or $z\in D(W(\alpha))\subset D(V_0(\beta_i))$. If $z\notin W(P)$, then $D(W(\alpha))$ has smaller area than $D(V_0(\beta_i))$.

Now let $z\in W(P)$ for some $W\in\langle A_1,\dots\rangle$. We proceed by induction on $L(W)$. If $L(W)=0$, then $W=I$ and it is clear that $n(z)=0$. Suppose the assertion is true for all words of length less than n. Let $L(W)=n+1$. We write W in reduced form. Let $W=VA_i^\varepsilon$, where $\varepsilon=\pm 1$ and $L(V)=n$, be an abbreviation of W. Since $V(P)$ and $W(P)$ share a common side γ, we see that $V(P)$ and $W(P)$ both lie in $D(V(\beta))$ unless $\gamma\subseteq V(\beta)$. From (2) and (3) of Lemma 1 we see that there can be at most only a finite number of $V(\beta)$ which contain γ. Hence $n(z)$ and $n(VA_j^{-\varepsilon}V^{-1}(z))$ are both finite or infinite together. Nothing $VA_j^{-\varepsilon}V^{-1}(z)\in V(P)$, we see from the induction hypothesis $n(z)<\infty$.

Proof of Theorem 1. Suppose $z\in\bar{\Delta}$ is an ordinary point of Γ and P is a fundamental domain of Γ. If

$$z\notin \bigcup_{V\in\langle A_1,\ldots\rangle} V(\bar{P}),$$

then with the aid of Lemma 1 part (4) we can inductively choose an infinite sequence $D(V(\beta)_j)$ ($V\in\langle A_1,\ldots\rangle$) such that (1) $D(V(\beta)_j)$ is convex, (2) $z\in D(V(\beta)_j)^\circ$, (3) there exists a $W_j\in\langle A_1,\ldots\rangle$ such that $W_j(\bar{P})\subseteq D(V(\beta)_j)$ and $z\in D(V(\beta)_{j+1})=D(W_j(\alpha))\subset D(V(\beta)_j)$. Set

$$D_z=\bigcap_{j=1}^{\infty} D(V(\beta)_j).$$

Now there are two necessary conditions stated in Theorem 1. Assuming P is a fundamental domain, we must show $\langle A_1,\ldots\rangle=\Gamma$ and $\bar{P}\cap\partial\Delta$ contains a fundamental set for Γ in $\partial\Delta$. We show first $\langle A_1,\ldots\rangle=\Gamma$. Here we choose the point $z=V(0)$, where $V\in\Gamma$. Now if $z\in W(\bar{P})$ for some $W\in\langle A_1,\ldots\rangle$, then because P is a fundamental domain for Γ, we have $W=V$. So we need only to show

$$z\in \bigcup_{W\in\langle A_1,\ldots\rangle} W(\bar{P}).$$

If z is not in the above union, then we choose a sequence $\{D(V(\beta)_j)\}_j$ as above and consider $D_z\cap\Delta\neq\emptyset$. Since $D_z\cap\Delta$ is neither $\emptyset$ nor Δ, $\partial D_z\cap\Delta\neq\emptyset$. Let $w\in\partial D_z\cap\Delta$ and $B\in\Gamma$ be such that $B(w)\in\bar{P}$. In each case possible, $B(w)\in P$, $B(w)\in\partial P$, or $B(w)$ is an elliptic fixed point, we can construct a connected open neighborhood of $B(w)$ by taking the interior of the union of a finite number of copies under $\langle A_1,\ldots\rangle$ of $\bar{P}$. Call this open set θ. Then $\theta_1=B^{-1}(\theta)$ intersects infinitely many $\partial D(V(\beta)_j)=V(\beta)_j$. In view of the facts that (1) every $V(\beta)_j\cap\bar{\Delta}$ is the image of a side or two adjacent sides of P under a transformation from $\langle A_1,\ldots\rangle\subseteq\Gamma$, and hence $V(\beta)_j$ cannot pass through any image of P, and (2) each $V(\beta)_j\cap\bar{\Delta}$ begins and ends in $\partial\Delta$, we see that $B(w)\notin P$ and that each $V(\beta)_j$ which passes through θ_1 must pass through one of the finitely many images of sides of P which intersect θ_1 (actually these sides lie in θ_1 except for the endpoints in $\partial\Delta$). From Lemma 1 parts (2) and (3) it follows that only finitely many $V(\beta)_j$ can pass through θ_1. Hence we have a contradiction and $V(0) \in W(P)$ for some $W \in \langle A_1, \ldots \rangle$.

In the proof of the second condition we choose any ordinary point $z \in \partial\Delta$ and assume that there is no $V \in \Gamma = \langle A_1, \ldots \rangle$ such that $V(z) \in \bar{P}$. Then $n(z) = +\infty$. Now as before we choose our nested sequence $D(V(\beta)_j)$ of compact sets and form D_z. Now let $I_j = D(V(\beta)_j) \cap \partial\Delta$. Then $z \in \bigcap_{j=1}^{\infty} I_j = D_z \cap \partial\Delta$. If $I = \{z\}$, then we consider the sequence $\{W_j(0)\}_j$, where W_j is as above. Noting that $W_j(0) \neq W_i(0)$ if $i \neq j$ and $|W_j(0)| \to 1$ as $j \to \infty$, we have $W_j(0) \to z$, the only point in I. Thus there exists a $w \neq z$ in I. Consider the circle C perpendicular to $\partial\Delta$ through z and w. Since both w and z are both in $D(V(\beta)_j)$ for all j and $D(V(\beta)_j)$ is convex, it follows that the inside of C is in each $D(V(\beta)_j)$. Therefore, the inside of C is in D_z. For these points q we have that $n(q) = +\infty$. Thus by Lemma 1 P is not a fundamental domain of Γ. This contradiction proves the necessity of (2) in Theorem 1.

To prove the sufficiency of (1) and (2) we need only show that $\Delta = \bigcup_{V \in \Gamma} V(\bar{P})$. The condition $V(P) \cap P = \emptyset$ for all $V \in \Gamma \setminus \{I\}$ was proved in Lemma 1 part (1). Let $z \in \Delta$. If $z \notin V(\bar{P})$ for all $V \in \Gamma$, then $n(z)$ is defined and $n(z) = +\infty$. We next choose a nested sequence $D(V(\beta)_j)$, as before, and consider $I = D_z \cap \partial\Delta$. If $D_z \cap \partial\Delta$ contains at least two points, then by the argument in the previous paragraph D_z contains the inside of a circle C perpendicular to $\partial\Delta$. Now every point, including those in $\partial\Delta$, which lies inside C is not a limit of any sequence of the form $\{B_k(0)\}_{k=1}^{\infty}$, where $B_k \in \Gamma$. Here we note $0 \in \bar{P}$. Hence $\partial\Delta \cap D(C)^{\circ}$ is an interval of ordinary points of Γ in $\partial\Delta$ such that $n(w) = +\infty$ for each $w \in \partial\Delta \cap D(C)^{\circ}$. Thus by Lemma 1 each $w \in \partial\Delta \cap D(C)^{0}$ cannot be mapped by an element of $\langle A_1, \ldots \rangle$ to $\bar{P}$. Hence, either (1) or (2) does not hold. So, it remains to show that I contains at least two points.

Here, we consider two cases. The first is that infinitely many of the $V(\beta)$ are circles, say the subsequence $\{V(\beta)_{j_n}\}_n$. By the nesting property $D_z = \bigcap_{n=1}^{\infty} D(V(\beta)_{j_n})$. In this case, since $z \in D(V(\beta)_{j_n}) \cap \Delta$ for each $n = 1, 2, \ldots$, we have I contains an interval of $\partial\Delta$ of positive Euclidean length. If there are only finitely many circles in the sequence $\{V(\beta)_j\}_j$, then all but finitely many $V(\beta)_j$ contain elliptic fixed points of transformations from Γ. We may assume that each $V(\beta)$ contains an elliptic fixed point e_j in Δ. Then an easy calculation shows $|e_j| \to 1$ as $j \to \infty$. Let $\{p_j, q_j\} = V(\beta)_j \cap \partial\Delta$. Considering a subsequence if necessary, let $p_j \to p$, $q_j \to q$, $e_j \to e$, as $j \to \infty$. If $p \neq q$, then I has at least two points and hence P is a fundamental polygon of Γ. Otherwise, $p = q = e$. We now claim that if $p = q = e = x$, then $D(V(\beta)_j)$ does not contain z for j large enough. To see this we consider an open disk θ which contains x but not z and whose boundary is a circle perpendicular to $\partial\Delta$. For j large enough $p_j, q_j, e_j, \in \theta$. From the convexity of both θ and $D(V(\beta)_j)$, one easily verifies that $D(V(\beta)_j) \subseteq \theta$. Hence I contains at least two points and we have P is a fundamental polygon for Γ. Q.E.D.

4. Fundamental domains

In the papers [3, 6, 9] concerning Theorem 2 one imposes certain extra conditions on a fundamental domain D. The first is that D is locally finite, i.e. every compact subset of Δ intersects only finitely many Γ images of D. This property was proven by A. Beardon [1] to follow from the definition of a fundamental domain. The property of local finiteness has two important consequences. If we define $S(D)=\{A\in\Gamma$: there exists a side with the label γ_i for some i such that $A(\gamma_i)=\gamma_i'\}$, then $\langle S(D)\rangle=\Gamma$. The second consequence is that if $\{\beta_j\}_j$ is a sequence of sides and the corresponding sequence of $A_j\in\Gamma$, for which $A_j(\beta_j)$ is also a side of D, is a sequence of distinct transformations, then the β_j's can accumulate only in $\partial\Delta$. That is, for each $r\in(0,1)$ there is a j_0 such that for $j>j_0$ we have $\beta_j\subseteq\Delta\setminus\{0\leqslant|z|\leqslant r\}$. This is clear, since if $x=\lim_{j\to\infty}x_j$, $x_j\in\beta_j$ and $x\in\Delta$, then every compact neighborhood of x intersects infinitely many $A_j^{-1}(D)$, $j=1, 2, \ldots$.

The second assumption is that each transformation of Γ pairs at most a finite number of sides. Now, it is clear that one can take a particular side, say γ_1, and partition it into subarcs, say α_1, α_2, ..., and call each α_i a side. Then if $A(\gamma_1)=\gamma_1'$, we have that A pairs as many sides as there are α_j's, when we consider the α_j's as sides and not γ_1. We leave as a conjecture but claim that it can be shown that except for this artifically created situation, each $A\in S(F)$ can identify at most one pair of sides. We claim the techniques in Section 5 as applied in this paper almost show this assertion for the noncompact fundament a domains. However, *in the following we will assume*, when it is not automatic, *that each $A\in S(D)$ identifies exactly than one pair of sides*. In the proofs of Theorems 3 and 4 we can actually start with fundamental domains which have this property. That any side can be identified with at most one other side by at most one $A\in\Gamma$ is trivial.

5. Applications of Theorem 1

With Theorem 1 proved we can apply the techniques in Section 6 of cutting and pasting to prove the theorem listed below.

THEOREM 2. *Every fundamental domain of a finitely generated fuchsian group is finite sided. Thus, the covering Δ over Δ/Γ has at most only a finite number of branch points and Δ/Γ is of finite type.*

THEOREM 3. *Every finitely generated fuchsian group which is a free product of cyclic subgroups is of classical Schottky type.*

THEOREM 4. *Every infinitely generated fuchsian group Γ is of classical Schottky type. Hence, Γ is a free product of cyclic subgroups.*

THEOREM 5. *Every fundamental domain D of an infinitely generated group for which $\partial D \cap \Delta$ has finite hyperbolic area is of infinite hyperbolic area.*

THEOREM 6. *Let D be a fundamental domain of a finitely generated fuchsian group Γ. Let $\{\Delta_1, ..., A_n\}$ be a subset of $S(D)$ such that $\Gamma = *_{i=1}^{n}\langle A_i \rangle$. Then the set $\{A_1, ..., A_n\}$ is a set of classical Schottky generators.*

THEOREM 7. *Let Γ be the free product of cyclic subgroups. Let θ be the set of all possible orders of elements of Γ. Then Γ has a torsion free subgroup of finite index if and only if θ is a finite set. In particular, every infinitely generated group satisfies the Fenchel conjecture if and only if θ is finite.*

Remark. Theorems 2 and 5 are classical. For Theorem 2 see [3, 6, 9]. For Theorem 5 see [15]. Theorem 4 was the original theorem to be proved. The motivation came from the result of Macbeath and Hoare [8]: every infinitely generated noneuclidean crystalographic group is algebraically the free product of cyclic subgroups. In the case of a fuchsian group Greenberg [4] has also shown this to be true. Theorem 4, however, shows that for fuchsian groups, Γ is also "geometrically" a free product. The reader should also note that Theorem 7 is an extension of a theorem of Fox [5] but not of the Selberg extension [14].

6. Cutting and pasting

Let D be a fundamental domain of Γ. Recall $S(D)$ is precisely those transformations in Γ which maps a side of D labeled γ_i to its corresponding image γ_i'. A cross cut of D is a Jordan arc $\alpha \subseteq \bar{D}$ such that $\alpha \cap \partial D$ is precisely the two endpoints of α. We now define a cutting pasting of D and the effect it has on $S(D)$. First we choose a side $\gamma = \gamma_{i_0}$ of D. Let $A(\gamma) = \gamma' = \gamma_{i_0}'$, where $A \in S(D)$. We next draw a cross cut α of D such that $D \setminus \alpha = D_1 \cup D_2$, where D_1, D_2 are disjoint subregions of D and $\gamma \subseteq \bar{D}_1$, $\gamma' \subseteq \bar{D}_2$. Then *a cutting and pasting of D along α by A* is the new fundamental domain $D' = [D_2 \cup A(\bar{D}_1)]^{\circ}$. It is classical [7] that this new region is in fact a fundamental domain. The effect of this cutting and pasting can be described as follows. For $X \in S(D)$, $X \neq A$, let $X(\beta) = \beta'$, where $\beta' = \gamma_i$ for some $i \neq i_0$. We remark here that it is easily seen that $i = i_0$ if and only if $X = A$. Then $A^{\varepsilon}XA^{\delta} \in S(D')$, where ε, $\delta = 0$, ± 1 accordingly to the following rules:

$R(1)$ if $\beta, \beta' \subseteq \bar{D}_2$, then $\varepsilon=\delta=0$;

$R(2)$ if $\beta \subseteq \bar{D}_1$ and $\beta' \subseteq \bar{D}_2$, then $\varepsilon=0$, $\delta=-1$;

$R(3)$ if $\beta' \subseteq \bar{D}_1$ and $\beta \subseteq \bar{D}_2$, then $\varepsilon=1$, $\delta=0$;

$R(4)$ if $\beta, \beta' \subseteq \bar{D}_1$, then $\varepsilon=1$, $\delta=-1$;

$R(5)$ if $\beta_1=\beta \cap \bar{D}_1$ and $\beta_2=\beta \cap \bar{D}_2$ are arcs, then we split β' into two corresponding arcs β_1' and β_2', respectively, and apply the rules (1)–(4) to X twice: once considered as the transformation which identifies β_1 to β_1' and the other as the transformation which identifies β_2 to β_2'.

Finally, it is clear that $A \in S(D')$. In this case we say $\varepsilon=\delta=0$. A cut and paste by A^{-1} can be similarly defined.

This brings us to the group theoretical aspects of cutting and pasting. Let $\{A_1, ...\}$ generate Γ. Then it is clear that the set $\{A_i^{\varepsilon_1} A_1 A_i^{\delta_1} ..., A_i, ... A_i^{\varepsilon_n} A_n A_i^{\delta_n} ...\}$ also generates Γ.

Any finite combination of replacements of this form is called a *Nielsen transformation*. In our situation we shall need only consider the cases where ε_j, δ_j assume only the values $0, 1, -1$. From the above considerations we see that if $S(D)$ generates Γ, then so does $S(D')$ for any cutting and pasting D' of D.

In our applications situation $R(5)$ *above will never arise. We will eliminate accidental cycles in* Δ *and* $R(5)$ *creates them.* We next describe the three types of cutting and pasting that will be used in this paper.

Type 1. Let D be a fundamental domain and $A \in S(D)$. Suppose $\gamma=\gamma_{t_0}$, $A(\gamma)=\gamma'$. If we can, we draw a cross cut α which begins and ends in $\partial\Delta$ and which separates γ and γ'. If A is elliptic with fixed point p, $\{p\}=\gamma \cap \gamma'$, then p is an endpoint of α and the other endpoint lies in $\partial\Delta$. We remind the reader that γ and γ' are to be in different connected components of $\bar{D} \setminus \alpha$. Then we cut and paste D along α by A. (See diagram 1.) In the resulting domain A identifies a Schottky pair of sides.

Type 2. If we cannot draw α, then by the remarks in Section 4 there exist a finite sequence of sides which to avoid a double subscript we simply denote by $\beta_0=\gamma, \beta_1, ..., \beta_{n+1}=\gamma'$ such that $\beta_j \cap \beta_{j+1} \cap \Delta \neq \emptyset$, for $j=0, 1, ..., n$. The Type 2 situation is when β_j for some j is identified with β_j', and β_j' is not a β_k for any $k=0, 1, ..., n+1$. Then we draw a cross cut α_1 from a point in $\partial\Delta$ to the endpoint of β_j' chosen so that one can cut and paste along α_1 by A_j^{-1}, where $A_j(\beta_j)=\beta_j'$. This cutting and pasting yields a fundamental domain D_1 in which γ and γ' are sides identified by A, but they are now of Type 1. Now we apply the cutting and pasting described in Type 1, first to A and then to A_j. (See diagram 2.)

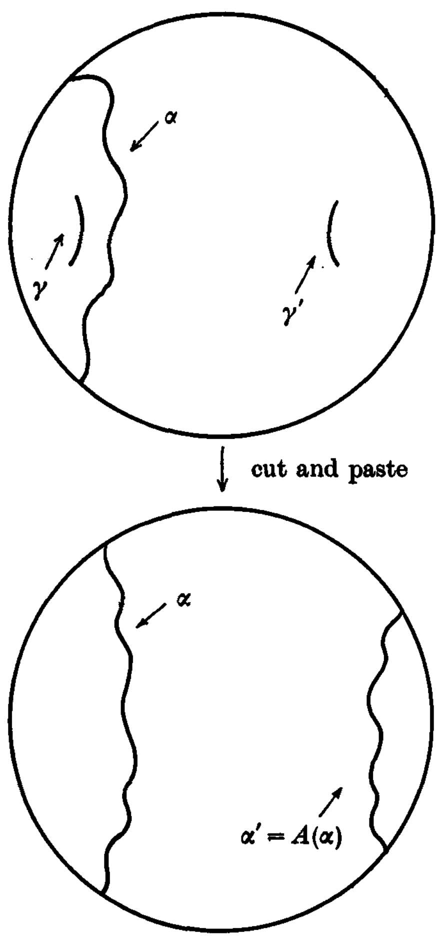

Diagram 1

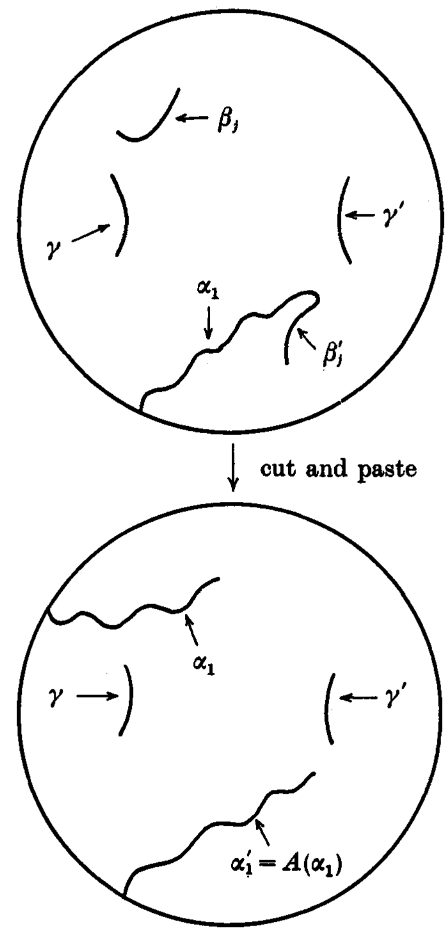

Diagram 2

Type 3. If γ, γ' are connected by a finite sequence as in the Type 2 situation, but is not of Type 2, then each β_j' is a β_k in the sequence $\beta_0=\gamma, \beta_1, \ldots, \beta_{n+1}=\gamma'$. We let $\{p_0\}=\beta_0 \cap \beta_1$ and $q \in \partial\Delta \cap \bar{D}$. Draw α from q to p_0 and cut and paste long α by A. In the resulting domain, which we call D again, A identifies α and $\alpha'=A(\alpha)$. Note that $\beta_1, \ldots, \beta_n$ remain fixed. We prove a lemma below which will be the Type 3 cut and paste sequence. Let $A_j \in \Gamma$, $A_j(\beta_j)=\beta_j'$. Recall $\beta_j'=\beta_k$ for some $k=1, 2, \ldots, n$ and that α, α' each have one vertex in $\partial\Delta$.

The following lemma describes a cutting and pasting which can be applied to any finite sided noncompact polygon in which the sides are paired. This lemma actually proves Theorem 3. It has application to the infinitely generated case when geometrically speaking the sides of the fundamental domain are arranged so that Γ is the free product of the group generated by a set of elements which pair sides whose union is the border of a polygon which contains all the "other" sides, and the elements which pair the "other" sides.

LEMMA 3. *Let D, $\alpha=\beta_0$, $\alpha'=\beta_0'$, $A=A_0$, $\{\beta_j\}_{j=1}^n$ be as above, except that here we allow the β_j $(j=0, \ldots, n)$ to have one or both endpoints in $\partial\Delta$, whereas above we assumed that both of the endpoints of β_j, $j=1, \ldots, n$ were in Δ. Then there exists a cutting and pasting of D to D' such that* (1) $S(D)\setminus S(D')=\{A_{j_1}, \ldots, A_{j_p}\}=T$; (2) *each $A_j \in S(D')$ identifies a Schottky pair of sides of D';* (3) *either $S(D')\subseteq S(D)$ and $T\subseteq\langle\{A_0, A_1, \ldots, A_n\}\cap S(D')\rangle$ or $S(D')\setminus S(D)=\{B_1, \ldots, B_r\}$ and $T\subseteq\langle\{A_0, A_1, \ldots, A_n, B_1, \ldots, B_r\}\cap S(D')\rangle$, where each B_i, $i=1, 2, \ldots, r$, identifies a Schottky pair of sides.*

Proof. Let $\beta_i\cap\beta_{i+1}=\{p_i\}$, if $\beta_i\cap\beta_{i+1}\neq\emptyset$, $i=0, 1, 2, \ldots, n-1$, $\alpha'\cap\beta_n=\{p\}_n$, if $\alpha'\cap\beta_n\neq\emptyset$. The proof is by induction on the number of p_j's in Δ which are part of a vertex cycle of more than one point. If no such p_j's exist, then we already have the conclusion of our lemma. Assume the lemma is true for $n-1$ or fewer such p_j's in Δ. We may without loss of generality assume p_0 is a point in Δ which is part of a vertex cycle of more than one point. Then $p_n=A_0(p_0)$ is also such a point. We draw a cross cut θ from $q\in\partial\Delta\cap\beta_0$ to the endpoint of β_1 which is not p_0. We cut and paste along θ by A_1. In the resulting fundamental domain there are fewer p_j's in Δ which are part of a vertex cycle of more than one point. Now apply the induction hypothesis. Q.E.D.

Notation. We set $U(A)$ to be either the set $\{A_0, \ldots, A_n\}\cap S(D')$, $\{A_0, \ldots, A_n, B_1, \ldots, B_r\}\cap S(D')$, whichever occurs in (3) of Lemma 3, $\{A, A_j\}$ of the Type 2 cutting and pasting, or $\{A\}$, as dictated by the type of cutting and pasting.

Remarks. It should be noted here that all the sides of D which are not an α, α' or β_j, β_j' $(j=1, \ldots, n)$ remain on D' unaltered, i.e. we have cut and pasted around these sides. Also, when applying Lemma 3 to A, we have $A\notin S(D')$.

We remark here that for any simply connected domain D and any cross cut α in D we have $D\setminus\alpha=D_1\cup D_2$, where D_1, D_2 are disjoint simply connected subregions of D [cf. 13]. That every fundamental domain is simply connected follows from the arguments in [1].

LEMMA 4. *Let D be a fundamental domain. Let $I_1, I_2, \ldots, I_n$ be a finite set of disjoint intervals contained in $\partial\Delta\cap\partial D$ and $\{\gamma_1, \gamma_1', \ldots, \gamma_p, \gamma_p'\}=S$ be a finite subset of sides of D, where each γ_i, γ_i' $i=1, \ldots, p$ is a Schottky pair of sides. Let $\gamma, \gamma'\in S$ be a pair of identified sides of D. Let $A\in\Gamma$ be such that $A(\gamma)=\gamma'$. Then each α in each step of each type of cutting and pasting* *above can be chosen so that* $\gamma_1, \gamma'_1, ..., \gamma_p, \gamma'_p$ *are sides of* D', *the resulting fundamental domain after cutting and pasting along* α *by* A, *and* $I_1, ..., I_n$ *are intervals contained in* $\partial\Delta \cap \partial D'$.

Proof. We proceed by induction on $2p+n$. If $2p+n=0$, then we have nothing to prove. Suppose the lemma is true for $2p+n-1$ sides and intervals. Then there exists an α so that $D \setminus \alpha = D_1 \cup D_2$, where D_1, D_2 are disjoint subregions, $\gamma \subseteq \bar{D}_1$, $\gamma' \subseteq \bar{D}_2$, and $I_2, ..., I_n$, $\gamma_1, \gamma'_1, ..., \gamma_p, \gamma'_p \subseteq \bar{D}_2$. If $I_1 \subseteq \bar{D}_2$, then we can cut and paste along α. So we assume $I_1 \cap D_1 \neq \emptyset$. We remark that if $n=0$, we assume $\gamma_1 \subseteq \bar{D}_1$ and the proof is similar. Let $z_0 \in \alpha \cap D$. Let β_1 be a cross cut in D_1 which begins at z_0 and ends at one endpoint of the interval $I_1 \cap D_1$. Then $D_1 \setminus \beta_1 = R_1 \cup R_2$, where R_1 and R_2 are disjoint subregions of D_1. Let $\gamma \subseteq \bar{R}_1$. If $I_1 \cap D_1 \subseteq R_2$ then we cut and paste along $\partial R_1 \cap D = \beta_1 \alpha_{z_0}$, where α_{z_0} is that part of α which begins at the endpoint of α in $\bar{R}_1$ ends at z_0, and the multiplication is the usual product of two arcs [see 12]. If, however, $I_1 \cap D_1 \subseteq R_1$, then we draw β_2, a cross cut of R_1, from z_0 to the other endpoint of $I_1 \cap D_1$. Then $R_1 \setminus \beta_2 = \theta_1 \cup \theta_2$, where θ_1 and θ_2 are disjoint subregions of R_1. Now, since the interval $I_1 \cap D_1$ and the arcs β_1, β_2 bound a subregion of R_1, we see that if, say, $\gamma \subseteq \bar{\theta}_1$ then $I_1 \cap D_1 \cap \theta_1 = \emptyset$. Hence, we cut and paste along $\partial\theta_1 \cap \bar{D} = \beta_2 \alpha_{z_0}$. Q.E.D.

7. Proofs of Theorems 2–7

In this section we begin by introducing an algorithm of cutting and pasting which can be directly applied to prove Theorems 2, 3, 4.

Let D_0 be a fundamental domain. Enumerate the pairs of sides of D_0, say, $\{\gamma_i, \gamma'_i\}_i$ and $S(D_0)$ by $S(D_0) = \{A_i\}_i$, where $A_i(\gamma_i) = \gamma'_i$. Also enumerate the disjoint intervals of $\partial\Delta \cap \partial D$ which are of more than one point, say $I_1, I_2, \dots$. Consider A_1 and γ_1, γ'_1, where $A_1(\gamma_1) = \gamma'_1$. We cut and paste D_0 as determined by γ_1, γ'_1 in such a way that $I_1 \subseteq D_1 \cap \partial\Delta$, where D_1 is the resulting fundamental domain. Let $S_1 = U(A_1)$. Now we must enumerate $S(D_1)$, but the enumeration must be the *derived enumeration from* $S(D_0)$. To be precise, we first note that each pair of identified sides β, β' of D_1 are of the form $\beta = W(\gamma_i)$, $\beta' = V(\gamma'_i)$ for some pair of sides γ_i, γ'_i of D_0, where W, $V \in \langle S_1 \rangle$. Then the sides β_i, β'_i will be the *unique pair* of sides of D_1 which are of this form. We remind the reader that rule $R(5)$ of Section 6 is never used, and thus the uniqueness of β_i, β'_i. We enumerate $S(D_1) = \{X_i\}_i$, where $X_i(\beta_i) = \beta'_i$. Note $X_i = VA_iW^{-1}$. It may happen that for a particular i no such pair β_i, β'_i exists. Then in this case the W and V satisfy the equation $W(\beta_i) = V(\beta'_i)$ and $VA_iW^{-1} = I$. In this case the $A_i \in \langle S_1 \rangle$ and, of course, the integer i disappears from the indexing set of $S(D_1)$. Let Q_1 be the sides of D_1 corresponding to generators in S_1 and $T_1 = \{I_1, I'_2\}$, where $I'_2 = W(I_2) \subseteq \partial D_1$ for some $W \in \langle S_1 \rangle$. We remark here that it is possible that the points equivalent to I_2 in $\partial D_1 \cap \partial\Delta$ may no longer be expressable as one interval. This difficulty can be avoided by simply never choosing any endpoint of any α of Section 6 to be an interior point of any I_j. That this choice is possible is easy, in fact a part of the proof of Lemma 4 can be applied. Now let i_1 be the smallest index among the generators of $S(D_1)\setminus S_1$. Now cut and paste D_1 as dictated by $\beta_{i_1}, \beta'_{i_1}$ so that the sides of Q_1 and intervals of T_1 are sides and intervals of $\bar{D}_2$, where D_2 is the resulting fundamental domain, and the intervals $I_3, I_4, \ldots$ have corresponding intervals in $\bar{D}_2$. Now given D_n we set $S_n = S_{n-1} \cup U(A_{i_{n-1}})$. Also, Q_n is the set of sides corresponding to the generators of S_n; $T_n = \{I_1, I'_2, I_3^{(2)}, \ldots, I_{n+1}^{(n)}\}$, where $I_{j+1}^{(j)} = W_j(I_j)$, $W_j \in \langle S_j \rangle$, $j = 1, \ldots, n$; and i_n is the smallest index of the generators of $S(D_n)\setminus S_n$, where $S(D_n)$ has the enumeration derived from $S(D_{n-1})$. Now cut and paste as dictated by the sides $\alpha_{i_n}, \alpha'_{i_n}$ of D_n with the above restrictions to obtain D_{n+1}.

Let $S = \bigcup_{n=1}^{\infty} S_n$ and $Q = \bigcup_{n=1}^{\infty} Q_n$. Since for any finite set $\{X_1, X_i, \ldots, X_n\}$ from S there exists a D_p such that each X_i $(i = 1, \ldots, n)$ identifies a pair of Schottky sides, we have $\langle S \rangle = \ast_{X \in S} \langle X \rangle$. Also, it is obvious that each $A_i \in \langle S_i \rangle$. Hence $\langle S \rangle = \langle A_1, A_2, \ldots \rangle$.

Now we consider Q. Each pair of arcs $\gamma, \gamma' \in Q$ correspond to an $X \in S$ in the sense that $X(\gamma) = \gamma'$. Also, either γ and γ' begin and end in $\partial\Delta$, or $\gamma \cap \gamma' \cap \Delta = \{p\}$ is the fixed point of X in Δ and one endpoint of both γ, γ' is in $\partial\Delta$. In the first case, if $\gamma \neq \gamma'$, we *define* the *outside of* γ to be that component of $\Delta\setminus\gamma$ which contains $\Delta \cap \gamma'$. Similarly, the outside of γ' is defined. In the second case we rotate γ by an angle of π around the elliptic fixed point p. Let $\pi(\gamma)$ be the image of γ under this rotation and we see that $\pi(\gamma) \cup \gamma$ can be expressed as an arc which begins and ends in $\partial\Delta$. Now, since X is a rotation about p, we see that $\gamma' \cap \pi(\gamma) = \{p\}$. We *define* the *outside* of γ to be that component of $\Delta\setminus[\gamma \cup \pi(\gamma)]$, which contains $\Delta \cap \gamma' \setminus \{p\}$. Similarly, the outside of γ' is defined. If $\gamma = \gamma'$, then we *define* the outside of γ to that component of $\Delta\setminus\gamma$ which contains the other arcs from Q. The *inside* of γ will be the complementary component of the outside.

Finally, a few remarks about $T = \bigcup_{n=1}^{\infty} T_n$ are in order. In particular, each $I \in T$ is outside each $\gamma, \gamma' \in Q$. Moreover, each point $x \in \bigcup_n I_n$ is equivalent under $\langle S \rangle$ to a point in $\bigcup_{I \in T} I$.

Proof of Theorem 2. Let D_0 be any fundamental domain of Γ with infinitely many sides. Then the above algorithm yields infinitely many generators in S. Thus Γ is the free product of infinitely many cyclic subgroups and hence not finitely generated.

Proof of Theorem 3. We choose a Dirichlet polygon P for Γ. We note P is not compact in Δ, but it is finite sided from Theorem 2. We apply the above algorithm and after finitely many steps we exhaust the set of sides and intervals of positive length, if any. So now we have D_n, $S_n = S(D_n)$, and Q_n. We now replace each pair $\gamma, \gamma' \in Q_n$ by the geodesics C_γ, $C_{\gamma'}$ which begin and end at the endpoints of γ, γ', respectively. *For the inside and outside of C_γ, $C_{\gamma'}$, $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$, below we use the definition given in this section*, which may disagree with the normal definition. Now, if $X(\gamma) = \gamma'$, X maps the outside of C_γ onto the inside of $C_{\gamma'}$. This is most easily seen by considering the action of X on $\partial\Delta$. Thus, if we replace C_γ, $C_{\gamma'}$ by the full circles $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$ corresponding to C_γ, $C_{\gamma'}$ then we see that for all pairs $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$, where $\gamma, \gamma' \in Q$, the outside of $\hat{C}_\gamma$ is mapped by X onto the inside of $\hat{C}_{\gamma}$. From the remarks in the development of the algorithm at the beginning of section we see that the region P in Δ outside all the circles $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$ satisfies all the hypothesis of Theorem 1, except possibly $0 \notin P$. After a conjugation of Γ by $V \in SL(2, \mathbf{R})$ so that $0 \in V(P)$, we see that $V(P)$ is a fundamental domain of $V\Gamma V^{-1}$. Thus P is a fundamental domain of Γ.

Proof of Theorem 4. We proceed as in the proof of Theorem 3. Starting with a Dirichlet fundamental polygon, we apply the above algorithm. Only in this case the algorithm does not stop and we obtain infinite sets $S = \bigcup_n S_n$, $Q = \bigcup_n Q_n$, $T = \bigcup_n T_n$, then replace each $\gamma, \gamma' \in Q$ by $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$ as defined above. Then with the same observations and arguments as above, we see that P, the exterior in Δ of all the $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$, where $\gamma, \gamma' \in Q$, is a fundamental domain for Γ.

Proof of Theorem 5. It is clear that the P in Theorem 4 has infinite area. Let D be any fundamental domain such that $\partial D \cap \Delta$ has finite hyperbolic area $m(\partial D \cap \Delta)$. Then

$$\begin{aligned}
\infty = m(P) &= m\Big(\bigcup_{V\in\Gamma} P \cap V(D) \cup \bigcup_{V\in\Gamma} P \cap V(\partial D \cap \Delta)\Big) \\
&\leqslant \sum_{V\in\Gamma} m(P \cap V(D)) + \sum_{V\in\Gamma} m(P \cap V(\partial D \cap \Delta)) \\
&\leqslant \sum_{V\in\Gamma} m(V^{-1}(P) \cap D) + \sum_{V\in\Gamma} m(V^{-1}(P) \cap \partial D \cap \Delta) \\
&= m(D) + m(\partial D \cap \Delta).
\end{aligned}$$

Hence $m(D) = +\infty$.

Proof of Theorem 6. Let D be a fundamental domain, $\{A_1, \ldots, A_n\}$ be transformations in Γ which identify certain sides of D and such that $\Gamma = *_{i=1}^{n} \langle A_i \rangle$. We note immediately that there are no cycles of fixed points of elliptic transformation from Γ which are of more than one point. If there are no accidental cycles on ∂D, then we replace each pair of sides γ, γ' by $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$. If the finite sided polygon P, formed from the exterior to all $\hat{C}_\gamma$, $\hat{C}_{\gamma'}$, where γ, γ' are sides of D, is not a fundamental polygon, then from the Poincaré Theorem [11] we have that there is a vertex v on ∂P which is the fixed point of some hyperbolic element of Γ. Let $v \in \hat{C}_\beta \cap \hat{C}_\alpha \cap \partial\Delta$ be the fixed point of $H \in \Gamma$, and let p, the other fixed point of H, belong to the inside of $\hat{C}_\alpha$. It is easy to see that p is inside either $\hat{C}_\alpha$ or $\hat{C}_\beta$. Here inside and outside are defined as above in this section. Let x belong to the open interval I determined by p and v which lies inside $\hat{C}_\alpha$. Now draw the circle C, perpendicular to $\partial\Delta$ through x and the endpoint of α which is not v. Let $A(\hat{C}_\alpha)=\hat{C}_{\alpha'}$. Then $A(C)$ intersects the circle adjacent to $\hat{C}_{\alpha'}$. This is easily seen from the fact that $A(I)$ lies in this circle. From $A(x)$ repeat the above construction and continue. Since the polygon is finite sided we eventually, after a finite number of steps, map x back into I. In fact, following v through this construction we see that x is mapped back into I precisely when v is mapped back to v. Thus $H(x)$ or $H^{-1}(x)$ is the image of x referred to above. Since there is no normalization with regard to H, we may assume $H(x)$ is the image of x in I referred to above. Then noting that $H(\hat{C}_\alpha)$ is inside $\hat{C}_\beta$ we conclude that v is the attractive fixed point of H (i.e. $\lim_{n\to\infty}H^n(0)=v$). Thus the above construction yields a new finite sided hyperbolic convex polygon, whereby the fixed point v is replaced by an interval of discontinuity and the A_i's pair Schottky sides. We do this for each hyperbolic fixed point in $\bar{P}$. After all hyperbolic fixed points have been eliminated, the resulting polygon will satisfy the hypothesis of the Poincaré Theorem.

If D has accidental cycles in Δ, then the set $S(D)\setminus\{A_1, ..., A_n\}\neq\emptyset$. Write $S(D)\setminus\{A_1, ..., A_n\}=\{B_1, ..., B_r\}$. We see that in a vertex cycle relation a letter X appears twice, once as X and then as X^{-1}, precisely when the endpoints of the sides identified by X are both in the same cycle. We assume for the moment that each B_i appears twice in any possible vertex relation in which it is found. Since D is not compact, in every vertex cycle relation at least one generator must appear only once. Let the word W in the letter from $S(D)$ be a relation derived from an accidental cycle in Δ. Since all the B_j's that appear in this accidental cycle relation (and there may not be a B_j in this relation) appear twice, an A_i must appear once. We now call this letter A. Then $\sigma_A(W)=\pm 1$. Writing the B_i's as words in the letters $A_1, ..., A_n$, we see that W is a non-trivial relation between the letters $A_1, ..., A_n$. This contradicts the assumption that $\Gamma=*_{i=1}^n\langle A_i\rangle$.

So we now know that at least one $B_i=B$ identifies sides whose endpoints are in two different vertex cycles and at least one of these vertex cycles is in Δ.

The next lemma will essentially finish this proof.

LEMMA 5. *Let D and B be as above. Then there exists a D' such that $S(D)=S(D')\cup\{B\}$.*

Proof. We proceed by induction on the number k of vertices in the *shortest* nonempty accidental cycle *in* Δ to which the endpoints of the sides γ, γ' identified by B are members. We note that accidental cycles in Δ of fewer than three points are artificial in the sense of the example in Section 4 and by a relabeling of the arcs we may assume that such a cycle does not exist. So we start with $k=3$. Let p, q be the endpoints of γ and say q is the point in an accidental cycle of three vertices. Now draw α from p to the other endpoint of the side β adjacent to γ at q. Let $X(\beta)=\beta'$. Cut and paste along α by X. In the resulting fundamental domain D' we see that $S(D')=[S(D)\setminus\{B\}]\cup\{BX^{-1}\}$. However, considering the accidental vertex cycle to which $X(q)$ belongs, we see that there are fewer than three points in this cycle. Hence $BX^{-1}\in S(D)$ and the lemma is true for $k=3$.

To go from k points in a vertex cycle to $k-1$ points the same construction is used. Q.E.D.

To conclude the proof of Theorem 6 we just note that with Lemma 5 available we can use induction on the number of B_j's which identify a pair of sides whose endpoints are in two different vertex cycles, one of which is in Δ.

Proof of Theorem 7. It is clear that if the set of orders of elements in Γ is infinite no subgroup of finite index can be torsion free. So assume that $n_1, \ldots, n_k$ are the only finite orders possible. Let G be the abelian group freely generated by $\{B_1, \ldots, B_k\}$, where, n_j is the order of B_j. Write $\Gamma = *_i\langle A_i\rangle$. Then define a homomorphism $\varphi: \Gamma\to G$ by $\varphi: A_i\to B_r$ if and only if order $A_i=$order $B_r=n_r$ and otherwise $A_i\to 1$, the identity element of G. We take the kernel of φ as the subgroup of finite index which is torsion free. Q.E.D.

Remark. The author would like to thank Prof. L. Greenberg for this proof of Theorem 7, which is shorter than the one he originally had.

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