SOME RIGIDITY THEOREMS FOR MINIMAL SUBMANIFOLDS OF THE SPHERE

BY

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Introduction

It is well known that the regularity of minimal submanifolds can be reduced to the study of minimal cones and hence to compact minimal submanifolds of the sphere. A phenomenon related to regularity was discovered by Bernstein [3] in 1915. The Bernstein Theorem says that an entire solution to the minimal surface equation in two variables is a plane. An answer to the extrinsic rigidity question of whether a minimal submanifold which lies in some neighborhood of a standard sphere must actually be a standard sphere is of interest in relation to the above topics as well as in its own right.

Efforts to generalize Bernstein's Theorem were made by many authors. The work of Simons [16] completed the proof in codimension one up to dimension 7. Bombieri, de Giorgi and Guisti [4] gave a counterexample in dimension 8. For two-dimensional graphs, Osserman [14] proved a version of Bernstein's theorem assuming the normal vectors omit a neighborhood of the sphere. Simons [16] proved that a minimal cone whose normal planes lie in a sufficiently small neighborhood is a plane and hence Bernstein's theorem is true for a graph whose normals satisfy the same condition. Reilly [15] enlarged the neighborhood. His estimate says that if a cone has the property that the normals satisfy $\langle N, A\rangle > \sqrt{(2k-2)/(3k-2)}$ for some fixed k-plane A, then the cone is a plane. For two-dimensional minimal graphs Barbosa [2] improved the neighborhoods to an open hemisphere. More specifically, Barbosa showed that a compact minimally immersed sphere in S^{k+2} such that its normal satisfy $\langle N, A\rangle > 0$ for some fixed A is totally geodesic. The theorem was also proved by S. T. Yau [17] for S^2 in S^4 and by Kenmatsu [10] under the stronger assumption of a bound on $\langle N, A\rangle$. Lawson and Osserman [12] constructed a series of examples of

(1) Supported in part by NSF grant MGS 77-18723 A01.

minimal graphs which are cones regular away from the origin, thus showing that Barbosa's Theorem fails to hold in general dimensions and codimensions and regularity does not hold for Lipschitz solutions to the minimal surface system. This paper deals with the study of minimal cones whose normals satisfy conditions of this type. We improve the estimates of Reilly and we generalize Barbosa's result.

In the two-dimensional case, the technique is to use a local computation to show $\log\langle N, A\rangle$ is a superharmonic function whenever $\langle N, A\rangle > 0$. In the general case for M^n a compact minimal immersion in S^{n+k} we show if $\langle N, A\rangle > \cos^p(\pi/2\sqrt{2p})$ where $p = \min(k, n+1)$, then M is a totally geodesic sphere. The first example of Lawson and Osserman occurs in dimension 3, codimension 3. In that example, $\langle N, A\rangle = 1/9$. Reilly's number is $2/\sqrt{7} \approx 0.74$ and this new estimate is $\cos^3(\pi/2\sqrt{6}) \approx 0.51$. This estimate improves previous ones in all dimensions and codimensions. The technique is to use facts about harmonic maps and information about the Grassmannian. One fact is that the Gauss map of a submanifold of $\mathbf{R}^n$ with parallel mean curvature is harmonic. The other is that the composition of a convex function with a harmonic map is subharmonic. The idea is then to determine the neighborhood of a point in the Grassmannian on which the distance function is convex, and compare this function with the Gauss map to give a subharmonic function on M.

I am indebted to my advisor H. Blaine Lawson for his unending support, encouragement and advice. I thank S. Hildebrandt for pointing out an error in an earlier version.

1. Preliminaries

We first establish basic notations.

Let $\bar{M}^m$ be a Riemannian manifold. We define a submanifold M^n of $\bar{M}^m$ to be an immersion $f: M^n \to \bar{M}^m$ of a differentiable manifold M. For any point p in $\bar{M}$, $T_p\bar{M}$ denotes the tangent space to $\bar{M}$ at p and $\mathfrak{X}_p\bar{M}$ the set of locally defined vector fields on $\bar{M}$ near p. The metric in the tangent space $T_p\bar{M}$ is denoted by $\langle\cdot,\cdot\rangle$. With reference to a submanifold M of $\bar{M}$, we may write $T_p\bar{M} = T_pM + N_pM$ where N_p is the normal space to M at p. With this splitting, $X = X^T + X^N$ where $X \in T_p\bar{M}$ and superscripts T and N denote the projection onto T_pM and N_pM respectively. The connection $\bar{\nabla}$ on $\bar{M}$ can be expressed as

$$\bar{\nabla}_X Y = (\bar{\nabla}_X Y)^T + (\bar{\nabla}_X Y)^N$$

for X and Y in $\mathfrak{X}_p\bar{M}$. The induced connection ∇ on M is given by

$$\nabla_X Y = (\bar{\nabla}_X Y)^T$$

for X and Y in $\mathfrak{X}_pM$. The curvature tensor $\bar{R}$ or $\bar{\nabla}$ on $\bar{M}$ is

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$$

and the sectional curvature $\bar{K}(X, Y)$ for orthonormal X and Y is

$$\bar{K}(X, Y) = \langle \bar{R}(X, Y)X, Y\rangle.$$

Note that $\bar{R}=0$ for $\bar{M}=\mathbf{R}^m$.

The second fundamental form B on M is a symmetric bilinear form on T_pM with values in $N_p(M)$ defined by

$$B_p(X, Y) = (\bar{\nabla}_X Y)^N.$$

The trace of the second fundamental form, tr B_p, pointwise defines a normal vector H_p. H is a smooth normal vector field on M called the mean curvature vector field. H_p is given locally as follows. Let $e_1, ..., e_n, v_1, ..., v_k$ be an orthonormal frame on $\bar{M}$ such that $e_1, ..., e_n$ is a frame on M. For $p\in M$,

$$H_p = \operatorname{tr} B_p = \sum_{i=1}^{n} (\bar{\nabla}_{e_i} e_i)^N.$$

If H is a parallel section of the normal bundle, i.e. if $(\bar{\nabla}_{X_p}H)^N=0$ for all $X_p\in T_pM$, then we say that M has parallel mean curvature. M is a minimal submanifold if $H\equiv 0$.

For notational convenience, we define $h^\alpha_{ij}=\langle\bar{\nabla}_{e_i}v_\alpha, e_j\rangle = -<v_\alpha, \bar{\nabla}_{e_i}e_j\rangle$, where e_i and v_α are as above. With this definition of h^α_{ij} we have

$$B_p(e_i, e_j) = -\sum_\alpha h^\alpha_{ij}(p)\, v_\alpha(p).$$

The h^α_{ij} are the components of B with respect to the chosen bases, with a sign change. Note that $h^\alpha_{ij}=h^\alpha_{ji}$. The minimality condition on M is $\sum_i h^\alpha_{ii}=0$ for $\alpha=1, ..., k$. We adopt the notation that roman indices i, j, l run from 1 to n and Greek indices α, β, γ run from 1 to k.

We will primarily be considering minimal submanifolds of the sphere. S^{k+n} will always denote the standard unit sphere in $\mathbf{R}^{k+n+1}$. For any n-dimensional submanifold M^n of S^{k+n}, the cone CM on M is the $(n+1)$-dimensional submanifold of $\mathbf{R}^{k+n+1}$ given by $CM=\{rx\,|\,x\in M, 0<r\leqslant 1\}$. M and CM have many of the same geometric properties. Define an orthonormal frame on $\mathbf{R}^{k+n+1}$ by $e_1, ..., e_{n+1}, v_1, ..., v_k$ where $e_1, ..., e_n$ is a frame for M, e_{n+1} is the position vector for M and $v_1, ..., v_k$ gives the normal space to CM in $\mathbf{R}^{k+n+1}$. Note that $v_1, ..., v_k$ also gives the normal space to M in S^{k+n}. Compare the second fundamental form B on M as a submanifold of the sphere and $\tilde{B}$ on CM as a submanifold of Euclidean space:

$$\tilde{B}(X, e_{n+1}) = 0 \qquad \text{for } X\in T_pCM$$
$$\tilde{B}_p(X, Y) = B_p(X, Y) \quad \text{for } X, Y\in T_pM.$$

Therefore we will not distinguish between the second form on M and on CM. Note that M is minimal in S^{n+k} if and only if CM is minimal in $\mathbf{R}^{k+n+1}$.

The Laplacian of a function φ on M is defined by

$$\Delta_M \varphi = \sum_i e_i e_i \varphi - \nabla_{e_i} e_i \varphi.$$

It is clear from this definition that if φ is extended to be constant on radial lines on CM, we have $\Delta_M = \Delta_{CM}$.

For simple k-vectors $N = v_1 \wedge \ldots \wedge v_k$, $A = u_1 \wedge \ldots \wedge u_k$, the inner product is defined as usual by $\langle N, A \rangle = \det(\langle v_i, u_j \rangle)$.

We investigate $\langle N, A\rangle$, where N is the normal k-plane field on M and A is a fixed simple k-plane. If $\langle N, A\rangle > 0$, $\log\langle N, A\rangle$ is well-defined. To compute $\Delta \log\langle N, A\rangle$ on M or on CM we use the standard fact that

$$\Delta \log \langle N, A\rangle = \frac{\langle N, A\rangle \Delta\langle N, A\rangle - |\nabla\langle N, A\rangle|^2}{\langle N, A\rangle^2}.$$

Since A is fixed, $\Delta\langle N, A\rangle = \langle \Delta N, A\rangle$ where ΔN means Laplacian with respect to the connection on k-vectors induced from the Euclidean connection $\overline{\nabla}$. Therefore to compute $\Delta \log\langle N, A\rangle$, we must determine $\langle \Delta N, A\rangle$ and $|\nabla\langle N, A\rangle|^2$.

LEMMA 1.1. *For a minimal immersion M in $\mathbf{R}^{k+n}$ with normal N and a fixed k-plane A*

$$\Delta\langle N, A\rangle = -|B|^2\langle N, A\rangle + \sum_{\substack{i,j,l\\ \alpha\neq\beta}} \langle h^\alpha_{ij} h^\beta_{il} v_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_l \wedge \ldots \wedge v_k, A\rangle$$

$$|\nabla\langle N, A\rangle|^2 = \sum_i \Big(\Big\langle \sum_{j,\alpha} h^\alpha_{ij} v_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge v_k, A\Big\rangle\Big)^2,$$

where e_j and e_l are in the α and β position respectively and $|B|^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2$.

Proof. Choose an orthonormal basis as above with the additional assumption that $(\overline{\nabla}_{e_i} e_j)^T(p) = 0$ and $(\overline{\nabla}_{e_i} v_\alpha)^N(p) = 0$. Since $\overline{\nabla}_{e_i}(v_1 \wedge \ldots \wedge v_k) = \sum_\alpha v_1 \wedge \ldots \wedge \overline{\nabla}_{e_i} v_\alpha \wedge \ldots \wedge v_k$ where $\overline{\nabla}_{e_i} v_\alpha$ occurs in the α position in the wedge product,

$$\begin{aligned}
\Delta(v_1 \wedge \ldots \wedge v_k) &= \sum_i \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} (v_1 \wedge \ldots \wedge v_k)\\
&= \sum_{i,\alpha\neq\beta} v_1 \wedge \ldots \wedge \overline{\nabla}_{e_i} v_\alpha \wedge \ldots \wedge \overline{\nabla}_{e_i} v_\beta \wedge \ldots \wedge v_k\\
&\quad + \sum_{i,\alpha} v_1 \wedge \ldots \wedge \overline{\nabla}_{e_i}\overline{\nabla}_{e_i} v_\alpha \wedge \ldots \wedge v_k\\
&= \sum_{i,\alpha\neq\beta} v_1 \wedge \ldots \wedge (\overline{\nabla}_{e_i} v_\alpha)^T \wedge \ldots \wedge (\overline{\nabla}_{e_i} v_\beta)^T \wedge \ldots \wedge v_k && (1.2)\\
&\quad + \sum_{i,\alpha} v_1 \wedge \ldots \wedge (\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} v_\alpha)^T \wedge \ldots \wedge v_k && (1.3)\\
&\quad + \sum_{i,\alpha} v_1 \wedge \ldots \wedge (\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} v_\alpha)^N \wedge \ldots \wedge v_k && (1.4)
\end{aligned}$$

where the terms with v_α appear in the α position.

The following equalities hold for the given choice of frame.

$$h_{ij}^{\alpha}=\langle\overline{\nabla}_{e_i}v_\alpha, e_j\rangle=h_{ji}^{\alpha},\quad \overline{\nabla}_{e_i}v_\alpha=\sum_j h_{ij}^{\alpha}e_j$$

$$0=e_i\langle v_\alpha, v_\alpha\rangle=2\langle\overline{\nabla}_{e_i}v_\alpha, v_\alpha\rangle$$

for each α so

$$0=e_i\langle\overline{\nabla}_{e_i}v_\alpha, v_\alpha\rangle=\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}v_\alpha, v_\alpha\rangle+\langle\overline{\nabla}_{e_i}v_\alpha, \overline{\nabla}_{e_i}v_\alpha\rangle$$
$$=\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}v_\alpha, v_\alpha\rangle+\sum(h_{ij}^{\alpha})^2$$

$$[e_i, e_j](p)=(\overline{\nabla}_{e_i}e_j)^T(p)-(\overline{\nabla}_{e_j}e_i)^T(p)=0.$$

Also in $\mathbf{R}^n$

$$0=\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}-\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}-\overline{\nabla}_{[e_i,e_j]},$$

so at p

$$\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}=\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}$$

and

$$\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}v_\alpha, e_j\rangle=e_i\langle\overline{\nabla}_{e_i}v_\alpha, e_j\rangle-\langle\overline{\nabla}_{e_i}v_\alpha, \overline{\nabla}_{e_i}e_j\rangle$$
$$=e_i\langle\overline{\nabla}_{e_j}v_\alpha, e_i\rangle=\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}v_\alpha, e_i\rangle.$$

With these equalities the above terms may be rewritten

$$(1.2)=\sum_{\substack{i,j,l\\ \alpha\neq\beta}}h_{ij}^{\alpha}h_{il}^{\beta}v_1\wedge\ldots\wedge e_j\wedge\ldots\wedge e_l\wedge\ldots\wedge v_k,$$

where e_j and e_l appear in the α and β position respectively.

$$(1.3)=\sum_{i,j,\alpha}\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}v_\alpha, e_j\rangle v_1\wedge\ldots\wedge e_j\wedge\ldots\wedge v_k$$
$$=\sum_{i,j,\alpha}\langle\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}v_\alpha, e_i\rangle v_1\wedge\ldots\wedge e_j\wedge\ldots\wedge v_k$$
$$=\sum_{j,\alpha}e_j\Big(\sum_i h_{ii}^{\alpha}\Big)v_\alpha\wedge\ldots\wedge e_j\wedge\ldots\wedge v_k=0$$

$$(1.4)=\sum_{i,\alpha}\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}v_\alpha, v_\alpha\rangle v_1\wedge\ldots\wedge v_k=-\sum_{i,j,\alpha}(h_{ij}^{\alpha})^2v_1\wedge\ldots\wedge v_k.$$

So

$$\Delta N=-|B|^2N+\sum_{\substack{i,j,l\\ \alpha\neq\beta}}h_{ji}^{\alpha}h_{li}^{\alpha}v_1\wedge\ldots\wedge e_j\wedge\ldots\wedge e_l\wedge\ldots\wedge v_k.$$

e_j and e_l are in the α and β position respectively. Also

$$|\nabla\langle N, A\rangle|^2=\sum_i\Big(\sum_\alpha\langle v_1\wedge\ldots\wedge\overline{\nabla}_{e_i}v_\alpha\wedge\ldots\wedge v_k, A\rangle\Big)^2$$
$$=\sum_i\Big(\sum_{j,\alpha}\langle\overline{\nabla}_{e_i}v_\alpha, e_j\rangle\langle v_1\wedge\ldots\wedge e_j\wedge\ldots\wedge v_k, A\rangle\Big)^2$$
$$=\sum_i\Big(\sum_{j,\alpha}h_{ij}^{\alpha}\langle v_1\wedge\ldots\wedge e_j\wedge\ldots\wedge v_k, A\rangle\Big)^2,$$

where e_j appears in the α position. This gives the desired formulae. □

Remark. The above proof actually gives the same formulae for a minimal immersion M^n in $\bar{M}^{k+n}$ where A is a parallel k-plane field and $\sum_i \langle \bar{R}(e_i, e_j)e_i, v_\alpha\rangle = 0$ on $\bar{M}$ for each j and α.

2. Two-dimensional case

The fact that a compact minimal hypersurface of the sphere which has the property that the normal vectors all lie within some fixed hemisphere is a totally geodesic sphere was first proved by de Giorgi [8]. We prove a statement for complete parabolic minimal immersions in the sphere which has as a consequence the result for surfaces analogous to the above.

We now consider a compact surface M^2 minimally immersed in S^{k+2}, the standard unit sphere in $\mathbf{R}^{k+3}$. We will compute $\Delta \log\langle N, A\rangle$ in the same setting as in the previous section. Choose an orthonormal frame $e_1, e_2, e_3, v_1, \ldots, v_k$ for $\mathbf{R}^{k+3}$ such that $e_1 \wedge e_2$ locally is the tangent space to M^2, e_3 is the unit normal to S^{k+2} and $v_1 \wedge \ldots \wedge v_k = N$ is the normal k-plane to M in S^{k+2}, all considered as simple vectors in $\mathbf{R}^{k+3}$. Let A be a fixed k-plane in $\mathbf{R}^{k+3}$ represented as a simple k-vector by $u_4 \wedge \ldots \wedge u_{k+3}$. We must consider $\langle N, A\rangle$. For simplicity, we consider instead the equivalent expression $\langle *N, *A\rangle$, where $*: \Lambda^k\mathbf{R}^{k+3} \to \Lambda^3\mathbf{R}^{k+3}$ is the Hodge star operator, $*N = e_1 \wedge e_2 \wedge e_3$ and $*A = A^\perp = u_1 \wedge u_2 \wedge u_3$. Furthermore, by the remarks in the previous section, $\Delta_{CM} \log\langle N, A\rangle = \Delta_M \log\langle N, A\rangle$. Hence to determine $\Delta_M\langle N, A\rangle$ we will compute $\Delta_{CM}\langle *N, *A\rangle$. The following Plücker identities hold for $G_{3,k} \subset \Lambda^3\mathbf{R}^{k+3}$.

Lemma 2.1. *Let e_i, v_α and $*A$ be as above where $i = 1, 2, 3$ and $\alpha, \beta = 1, \ldots, k$. Then*

$$\begin{aligned}&\langle e_1 \wedge e_2 \wedge e_3, *A\rangle\langle v_\alpha \wedge v_\beta \wedge e_3, *A\rangle + \langle v_\alpha \wedge e_1 \wedge e_3, *A\rangle\langle e_2 \wedge v_\beta \wedge e_3, *A\rangle \\ &\quad + \langle e_1 \wedge v_\beta \wedge e_3, *A\rangle\langle e_2 \wedge v_\alpha \wedge e_3, *A\rangle = 0.\end{aligned}$$

Proof. Let $*A = u_1 \wedge u_2 \wedge u_3$. Then $\tilde{A} = e_3 \llcorner *A = (e_3 \cdot u_1) u_2 \wedge u_3 - (e_3 \cdot u_2) u_1 \wedge u_3 + (e_3 \cdot u_3) u_1 \wedge u_2$ is also a simple vector. Here $\llcorner$ denotes contraction. In fact, say $e_3 \cdot u_1 \neq 0$, then

$$e_3 \llcorner *A = ((e_3 \cdot u_1) u_2 - (e_3 \cdot u_2) u_1) \wedge \left(-\frac{(e_3 \cdot u_3)}{(e_3 \cdot u_1)} u_1 + u_3\right).$$

Furthermore, by the definition of contraction, $\langle e_1 \wedge e_2 \wedge e_3, *A\rangle = \langle e_1 \wedge e_2, e_3 \llcorner *A\rangle$. Hence it suffices to show that for simple 2-vectors

$$\langle e_1 \wedge e_2, \tilde{A}\rangle\langle v_\alpha \wedge v_\beta, \tilde{A}\rangle + \langle v_\alpha \wedge e_1, \tilde{A}\rangle\langle e_2 \wedge v_\beta, \tilde{A}\rangle + \langle e_1 \wedge v_\beta, \tilde{A}\rangle\langle e_2 \wedge v_\alpha, \tilde{A}\rangle = 0.$$

Let $\tilde{A} = w_1 \wedge w_2$; then evaluating the inner products gives

$$\begin{aligned}&((e_1\cdot w_1)(e_2\cdot w_2)-(e_1\cdot w_2)(e_2\cdot w_1))((v_\alpha\cdot w_1)(v_\beta\cdot w_2)-(v_\alpha\cdot w_2)(v_\beta\cdot w_1))\\ &\quad+((v_\alpha\cdot w_1)(e_1\cdot w_2)-(v_\alpha\cdot w_2)(e_1\cdot w_1))((e_2\cdot w_1)(v_\beta\cdot w_2)-(e_2\cdot w_2)(v_\beta\cdot w_1))\\ &\quad+((e_1\cdot w_1)(v_\beta\cdot w_2)-(e_1\cdot w_2)(v_\beta\cdot w_1))((e_2\cdot w_1)(v_\alpha\cdot w_2)-(e_2\cdot w_2)(v_\alpha\cdot w_1))=0.\end{aligned}$$

This equation gives the desired relation. □

PROPOSITION 2.2. *Let M be a minimal surface in S^{k+2}. If $\langle N, A\rangle > 0$ at a point, then locally*

$$\Delta \log\langle N, A\rangle \leqslant -|B|^2.$$

In particular, $\log\langle N, A\rangle$ is superharmonic wherever $\langle N, A\rangle > 0$.

Proof. Let $\psi = \langle N, A\rangle = \langle *N, *A\rangle$.

Let $e_1, \dots, v_k$ be the orthonormal frame defined above. Then e_3 gives the radial direction on CM and $h^\alpha_{ij} = \langle \overline{\nabla}_{e_i} v_\alpha, e_j\rangle = 0$ if $i=3$ or $j=3$. Here $\overline{\nabla}$ denotes the connection on $\mathbf{R}^{k+3}$ and h^α_{ij} are components of the second fundamental form on CM. Recall when i and j are 1 or 2, the components h^α_{ij} for M are the same as for CM. Since

$$\Delta \log \psi = \frac{\psi\Delta\psi - |\overline{\nabla}\psi|^2}{\psi^2}$$

we must determine $\psi\Delta\psi - |\overline{\nabla}\psi|^2$. From section 1 we have that

$$\psi\Delta\psi = -|B|^2\psi^2 + 2\sum_{\substack{i=1,2\\ \alpha,\beta=1,k}} h^\alpha_{i1} h^\beta_{i2}\langle v_\alpha \wedge v_\beta \wedge e_3, *A\rangle\psi$$

$$|\overline{\nabla}\psi|^2 = \sum_{i=1,2}\Big(\sum_{\alpha=1,\dots,k} h^\alpha_{i1}\langle v_\alpha\wedge e_2\wedge e_3, *A\rangle + \sum_{\alpha=1,\dots,k} h^\alpha_{i2}\langle e_1\wedge v_\alpha\wedge e_3, *A\rangle\Big)^2.$$

By Lemma 2.1, we can rewrite $\psi\Delta\psi$ to get

$$\begin{aligned}\psi\Delta\psi - |\overline{\nabla}\psi|^2 = {}& -|B|^2\psi^2 + 2\sum_{i,\alpha,\beta} h^\alpha_{i1}h^\beta_{i2}\langle e_1\wedge v_\beta\wedge e_3, *A\rangle\langle v_\alpha\wedge e_2\wedge e_3, *A\rangle\\ & -2\sum_{i,\alpha,\beta} h^\alpha_{i1}h^\beta_{i2}\langle e_1\wedge v_\alpha\wedge e_3, *A\rangle\langle v_\beta\wedge e_2\wedge e_3, *A\rangle\\ & -\sum_i\Big(\sum_\alpha h^\alpha_{i1}\langle v_\alpha\wedge e_2\wedge e_3, *A\rangle\Big)^2 - \sum_i\Big(\sum_\alpha h^\alpha_{i2}\langle e_1\wedge v_\alpha\wedge e_3, *A\rangle\Big)^2\\ & -2\sum_{i,\alpha,\beta} h^\alpha_{i1}h^\beta_{i2}\langle v_\alpha\wedge e_2\wedge e_3, *A\rangle\langle e_1\wedge v_\beta\wedge e_3, *A\rangle.\end{aligned}$$

The second and last terms cancel and from section 1 we know that three dimensional minimal cones satisfy $\sum_i h^\alpha_{i1}h^\beta_{i2} = -\sum_i h^\alpha_{i2}h^\beta_{i1}$ so the third term may be rewritten as

$$2\sum_{i,\alpha,\beta} h^\alpha_{i1}h^\beta_{i2}\langle v_\alpha\wedge e_2\wedge e_3, *A\rangle\langle e_1\wedge v_\beta\wedge e_3, *A\rangle.$$

Therefore,

$$\psi\Delta\psi - |\nabla\psi|^2 = -|B|^2\psi^2 - \sum_i \Big(\sum_\alpha h_{i1}^\alpha \langle v_\alpha \wedge e_2 \wedge e_3, *A\rangle - \sum_\alpha h_{i2}^\alpha \langle e_1 \wedge v_\alpha \wedge e_3, *A\rangle\Big)^2.$$

So

$$\Delta \log \psi = -|B|^2 - \frac{1}{\psi^2}\sum_i \Big(\sum_\alpha h_{i1}^\alpha \langle v_\alpha \wedge e_2 \wedge e_3, *A\rangle - \sum_\alpha h_{i2}^\alpha \langle e_1 \wedge v_\alpha \wedge e_3, *A\rangle\Big)^2.$$

In particular $\Delta \log \psi \leqslant -|B|^2$. □

A surface M is *parabolic* if every bounded subharmonic function is constant. We now have the following theorem.

THEOREM 2.3. *If M^2 is a complete, parabolic, oriented immersed minimal surface in S^{k+2} such that, for some $\varepsilon > 0$, $\langle N, A\rangle > \varepsilon$ on M, then M is a totally geodesic two-sphere.*

Proof. We have shown that $\varphi = \log\langle N, A\rangle$ is a superharmonic function when $\langle N, A\rangle > 0$. Since M is parabolic and $\langle N, A\rangle > 0$, $\log\langle N, A\rangle$ is constant, so $\Delta \log\langle N, A\rangle = 0$. The above formula then gives $0 = \Delta \log\langle N, A\rangle \leqslant -|B|^2$. Thus $|B|^2 = 0$ and M is totally geodesic. □

In the case that M is compact it was pointed out by J. Milnor that the condition $\langle N, A\rangle > 0$ implies that M is homeomorphic to S^2, hence in the compact case the theorem gives another proof of Barbosa's theorem.

In the unorientable case, the following corollary holds.

COROLLARY 2.4. *If M^2 is a non-orientable complete parabolic immersed minimal surface in S^{k+2} then for any fixed k-plane A, there is some normal k-plane N such that $\langle N, A\rangle = 0$.*

3. Properties of the Grassmannian

The proof in the general case relies on information about the Grassmannian of k-planes in $(k+n+1)$-space in conjunction with some facts about harmonic maps. In this section we summarize a few properties of the Grassmannian which we will need.

Given a k-plane P we can choose an oriented orthonormal basis $e_1, ..., e_k$ for P and an orthonormal basis $v_1, ..., v_m$ for the orthogonal complement of P in $\mathbf{R}^{k+m}$. The $(k+m)\times(k+m)$ matrix whose columns are $e_1, ..., e_k, v_1, ..., v_m$ is then an orthogonal matrix O of determinant $+1$, that is, an element of the Lie group $SO(k+m)$. The plane P is invariant under orthogonal change of the basis $\{e_1, ..., e_k\}$ and of the normal basis $\{v_1, ..., v_m\}$; it is determined up to right multiplication of O by an element of $\mathrm{SO}(k)\times\mathrm{SO}(m)$. Thus $G_{k,m}$ is identified with the quotient space

$$G_{k,m}=\mathrm{SO}(k+m)/\mathrm{SO}(k)\times\mathrm{SO}(m).$$

It is in fact a symmetric space. There is an involutive automorphism σ on $\mathrm{SO}(k+m)$ given by $\sigma(O)=SOS^{-1}$ where

$$S=\begin{pmatrix}-I_k & 0\\ 0 & I_m\end{pmatrix}$$

and I_k is the $k\times k$ identity matrix. The subgroup $\mathrm{SO}(k)\times\mathrm{SO}(m)$ coincides with the identity component of the subgroup of all fixed elements of σ, so $G_{k,m}$ is symmetric.

The tangent space $\mathfrak{g}$ at the identity to $\mathrm{SO}(k+m)$ is the space of skew symmetric matrices which we denote $\square(k+m)$. The Riemannian metric on $\mathrm{SO}(k+m)$ can be described by

$$\langle A, B\rangle=\tfrac{1}{2}\operatorname{tr}A^tB=-\tfrac{1}{2}\operatorname{tr}AB.$$

The tangent space $\mathfrak{h}$ to $\mathrm{SO}(k)\times\mathrm{SO}(m)$ is the subspace $\square(k)\times\square(m)$ given by matrices

$$\begin{pmatrix}U & 0\\ 0 & V\end{pmatrix}$$

where $U\in\square(k)$, $V\in\square(m)$. The orthogonal complement to $\square(k)\times\square(m)$ which we denote by $\mathfrak{m}$ consists of matrices of the form

$$\begin{pmatrix}0 & -{}^tX\\ X & 0\end{pmatrix}$$

where X is a $(m\times k)$-matrix. $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ is the canonical decomposition of the Lie algebra $\mathfrak{g}$.

Let $\pi\colon \mathrm{SO}(k+m)\to G_{k,m}$ be the quotient map. The differential of π, $d\pi$, at the identity identifies $\mathfrak{m}$ with the tangent space to $G_{k,m}$. This map defines the metric on $G_{k,m}$ at the identity coset and invariance under left multiplication by $\mathrm{SO}(k+m)$ defines the metric on all of $G_{k,m}$. The geodesic through the identity coset with initial tangent $\bar{X}\in\mathfrak{m}$ is the image under π of the one parameter subgroup $e^{t\bar{X}}$ for $t\in\mathbf{R}$.

LEMMA 3.1. *The sectional curvatures K of $G_{k,m}$ satisfy $0\leqslant K\leqslant 2$. If $m=1$ or $k=1$ then $K=1$.*

Proof. By a standard formula (see e.g. [5, p. 76]) the curvature $K(AB)$ is given by $\|[A,B]\|^2$ for $A, B\in\mathfrak{m}$. First apply the Maximal Torus Theorem so that $A\in\mathfrak{m}$ has its $(m\times k)$-submatrix X in diagonal form with entries $\lambda_1,\dots,\lambda_p$, where $p=\min(k,m)$. $B\in\mathfrak{m}$ has submatrix Y with entries y_{ij}. In this notation it suffices to maximize

$$K = \frac{1}{2} \sum_{i,j=1}^{p} (\lambda_j y_{ji} - \lambda_i y_{ij})^2 + (\lambda_j y_{ij} - \lambda_i y_{ji})^2,$$

where $\sum_{i,j} y_{ij}^2 = 1$ and $\sum_i \lambda_i^2 = 1$.

This maximization is an exercise in Langrange multipliers which we omit here. If $p > 1$ one concludes that a maximum occurs when there are two equal non zero λ_i's and furthermore that this maximum is 2. Hence $0 \leq K \leq 2$. If $p = 1$ then $G_{k,m}$ is just the standard sphere and it is well known that $K = 1$. □

Definitions. Let f be a function defined on Riemannian manifold M. Define $Hf(X, Y)$, the *Hessian* of f, to be the quadratic tensor which assigns to vectors X and Y in $T_p(M)$ the value

$$Hf(X, Y) = XYf - (\nabla_X Y) \cdot f$$

where X and Y are extended as smooth vector fields to a neighborhood of p. It is easy to see that $Hf(X, Y)$ is independent of the extension of X and Y and that

$$Hf(X, Y) = Hf(Y, X).$$

A function f is (*strictly*) *convex* if $Hf(\cdot, \cdot)$ is a (strictly) positive definite quadratic form.

Let ϱ_p: $M \to \mathbf{R}$ be the distance function $\varrho_p(q) =$ the distance from p to q. Let γ: $[0, \infty) \to M$ be a geodesic with $\gamma(0) = p$, then $q = \gamma(t_0)$ is a *cut point* of p along γ if the set of t such that $\varrho_p(\gamma(t)) = t$ is $[0, t_0]$.

LEMMA 3.2 (Whitehead). *Let $B_r(p)$, the ball centered at $p \in M$ of radius r, be contained in some compact set $c \subset M$. Let $\tau = \inf_{p \in c}\{\varrho_p(q)$: q is a cut point of $p\}$. Let $\varkappa$ denote the supremum of all sectional curvatures at points of c. If $r < \frac{1}{2} \min\{\pi/\sqrt{\varkappa}, \tau\}$ then ϱ_p^2 is a strictly convex function. In fact, there is an $\varepsilon > 0$ depending on r such that*

$$H\varrho_p^2(X, X) \geq \varepsilon \cdot \langle X, X \rangle$$

for all X and Y in $T_p M$ and for all $q \in B_r(p)$.

Proof. See Cheeger and Ebin [5]. □

We are interested in specializing this result to the case of the Grassmann manifold $G_{k,m}$. We first note that $G_{k,m}$ is simply connected and $\varkappa = 2$.

LEMMA 3.3. *For $G_{k,m}$, $\tau \geq \pi$ where τ is as in Lemma 3.2.*

Proof. A result of Crittenden [7] shows that on a simply connected symmetric space, the first cut point along a geodesic is a conjugate point. By standard comparison theorem arguments, the first conjugate point cannot occur before the distance π is achieved. Therefore $\tau \geqslant \pi$. □

The rest of this section is devoted to computing the maximum of the inner product between a fixed k-plane A and any other k-plane N on the boundary of the ball $B_{\pi/2\sqrt{2}}(A)$ in $G_{k,m}$. This computation gives a comparison between the intrinsic distance on $G_{k,m}$ and the "sphere" distance, i.e. the distance on the sphere in $\Lambda^k \mathbf{R}^{k+m}$. The boundary $B_{\pi/2\sqrt{2}}(A)$ is given by moving along any geodesic emanating from A for time $t = \pi/2\sqrt{2}$. The following remarks show that we need consider only certain geodesics to maximize $\langle \cdot, \cdot \rangle$.

The inner product $\langle N, A\rangle$ is the usual one defined by $\langle N, A\rangle = \det(\langle u_i, v_j\rangle)$ where $N = v_1 \wedge \ldots \wedge v_k$, $A = u_1 \wedge \ldots \wedge u_k$ for $u_i, v_j \in \mathbf{R}^{k+m}$. Computation will be simplified by noting that the inner product is invariant under isometry. More precisely, if A and N are in $G_{k,m}$ and P is in $\mathrm{SO}(k+m)$ then P acts as an isometry of $G_{k,m}$ by left multiplication and the definition of inner product gives that $\langle N, A\rangle = \langle PN, PA\rangle$.

For simplicity, fix A to be the identity coset. The tangent space at A is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} : X \text{ is an } (m \times k)\text{-matrix} \right\}.$$

By standard facts about $G_{k,m}$, one can choose a maximal flat totally geodesic submanifold $\mathfrak{J}$ of $G_{k,m}$ containing A having tangent space at A of the form

$$\mathfrak{m}' = \left\{ \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} : X \text{ is an } (m \times k)\text{-matrix with entries } x_{ij} = 0 \quad \text{for } i \neq j \right\}.$$

By the Maximal Torus Theorem any geodesic γ can be translated into a geodesic γ' contained in $\mathfrak{J}$. By the invariance of $\langle \cdot, \cdot \rangle$ under isometry, we have $\langle \gamma(\pi/2\sqrt{2}), A\rangle = \langle \gamma'(\pi/2\sqrt{2}), A\rangle$. Hence in order to maximize $\langle N, A\rangle$ on the boundary of $B_{\pi/2\sqrt{2}}(A)$, it suffices to maximize $\langle N, A\rangle$ over the set

$$L = \{\mathfrak{J} \cap B_{\pi/2\sqrt{2}}(A)\}.$$

On this set the inner product can be explicitly computed.

We compute the geodesics $\pi(e^{t\bar{X}})$ for $\bar{X}$ in $\mathfrak{m}'$ where $\bar{X}$ has unit length. In general, $p = \min(m, k)$, but for notational convenience, assume $p = k$. Let $\lambda_1, \ldots, \lambda_p$ be the diagonal entries of $\bar{X}$. Then $\bar{X}$ is of the form

$$\left[\begin{array}{c|c}
\begin{matrix}(k\times k)\\ \\ 0\end{matrix} & \begin{matrix}-\lambda_1 & & 0 & \\ \vdots & \ddots & \vdots & 0\\ 0 & \dots & -\lambda_p & \end{matrix}\\
\hline
\begin{matrix}\lambda_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \lambda_p\\ & 0 & \end{matrix} & \begin{matrix}\\ 0\\ \\ (m\times m)\end{matrix}
\end{array}\right]$$

so

$$e^{(\pi/2\sqrt{2})\bar{X}}=\left[\begin{array}{c|c}
\begin{matrix}\cos\frac{\pi\lambda_1}{2\sqrt{2}} & & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \cos\frac{\pi\lambda_p}{2\sqrt{2}}\end{matrix} & \begin{matrix}-\sin\frac{\pi\lambda_1}{2\sqrt{2}} & & 0 & \\ & \ddots & \vdots & 0\\ 0 & \dots & -\sin\frac{\pi\lambda_p}{2\sqrt{2}} & \end{matrix}\\
\hline
\begin{matrix}\sin\frac{\pi\lambda_1}{2\sqrt{2}} & & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \sin\frac{\pi\lambda_p}{2\sqrt{2}}\\ & 0 & \end{matrix} & \begin{matrix}\cos\frac{\pi\lambda_1}{2\sqrt{2}} & & 0 & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & \cos\frac{\pi\lambda_p}{2\sqrt{2}} & \\ & & & 0\end{matrix}
\end{array}\right].$$

The image under π, $\pi(e^{(\pi/2\sqrt{2})\bar{X}})$, is the k-plane having as orthonormal basis the first k columns of $e^{(\pi/2\sqrt{2})\bar{X}}$.

Let A be the plane spanned by the first k standard coordinate vectors in $\mathbf{R}^{k+m}$. The inner product $\langle\pi(e^{(\pi/2\sqrt{2})\bar{X}}), A\rangle$ is

$$\langle\pi(e^{(\pi/2\sqrt{2})\bar{X}}), A\rangle=\cos\frac{\pi\lambda_1}{2\sqrt{2}}\dots\cos\frac{\pi\lambda_p}{2\sqrt{2}}$$

on the boundary of the ball of radius $\pi/\sqrt{2}$.

We want to determine $\nu=$maximum of $\langle N, A\rangle$ for $N\in L$. Then for N satisfying

$$\langle N, A\rangle>\nu$$

we have that $\varrho_A^2(N)$ is strictly convex.

LEMMA 3.4. *Let A be the fixed point in $G_{k,m}$ given above. Let $N\in L$; then*

$$\max_{N\in L}\langle N, A\rangle=\cos^p\left(\frac{\pi}{2\sqrt{2p}}\right).$$

Proof. We must maximize $\langle N, A\rangle$ so we must maximize

$$f(\lambda_1, \ldots, \lambda_p) = \cos\left(\frac{\pi\lambda_1}{2\sqrt{2}}\right) \ldots \cos\left(\frac{\pi\lambda_p}{2\sqrt{2}}\right) \quad \text{for } \sum_{i=1}^{p} \lambda_i^2 = 1.$$

Let

$$F(\mu, \lambda) = \prod_{i=1}^{p} \cos\left(\frac{\pi\lambda_i}{2\sqrt{2}}\right) + \mu\left(1 - \sum_{i=1}^{p} \lambda_i^2\right) \quad \text{where } \mu \neq 0$$

is another variable. Then

$$F_{\lambda_i} = -\frac{\pi}{2\sqrt{2}} \prod_{\substack{j=1\\ j\neq i}}^{p} \cos\frac{\pi\lambda_j}{2\sqrt{2}} \sin\frac{\pi\lambda_i}{2\sqrt{2}} - 2\mu\lambda_i.$$

At a critical point $(\lambda_1, \ldots, \lambda_p)$ of f

$$-\frac{\pi}{2\sqrt{2}} \prod_{\substack{j=1\\ j\neq i}}^{p} \cos\frac{\pi\lambda_j}{2\sqrt{2}} \sin\frac{\pi\lambda_i}{2\sqrt{2}} - 2\mu\lambda_i = 0,$$

so for $\lambda_i \neq 0$

$$-\frac{\pi}{4\sqrt{2}\mu} \prod_{j=1}^{p} \cos\frac{\pi\lambda_j}{2\sqrt{2}} = \lambda_i \cot\frac{\pi\lambda_i}{2\sqrt{2}}.$$

In particular,

$$-\frac{\pi}{4\sqrt{2}\mu} f(\lambda) = \lambda_i \cot\frac{\pi\lambda_i}{2\sqrt{2}}$$

for all i with $0 < \lambda_i < 1$. For all critical points of f with nonzero λ_i it follows that

$$\lambda_i \cot\frac{\pi\lambda_i}{2\sqrt{2}} = \lambda_j \cot\frac{\pi\lambda_j}{2\sqrt{2}}.$$

Note $g(\lambda) = \lambda \cot(\pi\lambda/2\sqrt{2})$ is a decreasing function for $0 < \lambda < 1$ since

$$g'(\lambda) = \frac{\cos\frac{\pi\lambda}{2\sqrt{2}} \sin\frac{\pi\lambda}{2\sqrt{2}} - \frac{\pi\lambda}{2\sqrt{2}}}{\sin^2\frac{\pi\lambda}{2\sqrt{2}}} = \frac{1}{2}\,\frac{\sin\frac{\pi\lambda}{\sqrt{2}} - \frac{\pi\lambda}{\sqrt{2}}}{\sin^2\frac{\pi\lambda}{2\sqrt{2}}} < 0.$$

Since $g(\lambda)$ is a decreasing function,

$$g(\lambda_i) = g(\lambda_j) \quad \text{only when } \lambda_i = \lambda_j.$$

Therefore the only critical point of $f(\lambda)$ with all $\lambda_i \neq 0$ occurs when $\lambda_1 = \ldots = \lambda_p = (1/\sqrt{p})$.

If $\lambda_i = 0$ for some i then the same argument as above shows that the remaining λ_i's are equal. But $\cos^l(\pi/2\sqrt{2l})$ is an increasing function in l, so the maximum is reached when all $\lambda_i \neq 0$, hence it is $\cos^p(\pi/2\sqrt{2p})$. □

The results in this section may be summarized by the following statement.

THEOREM 3.5. *The function ϱ_A^2 is a smooth strictly convex function on the set $\{N \in G_{k,m}: \langle N, A\rangle > \cos^p(\pi/2\sqrt{2p})\}$. In fact $H\varrho_A^2(\cdot,\cdot) \geq \varepsilon\langle\cdot,\cdot\rangle$ on the set $\{N \in G_{k,m}: \langle N, A\rangle > \lambda\}$ for any $\lambda > \cos^p(\pi/2\sqrt{2p})$, where ε is a positive number depending on λ.*

4. Proof in general dimensions and codimensions

The above result for minimal surfaces in S^{k+2} is false for higher dimensions. In this section we derive an estimate which improves previous estimates in all dimensions and codimensions. We prove the following.

THEOREM 4.1. *Let M^n be a compact manifold of dimension n minimally immersed in S^{k+n}. Let $p = \min(n+1, k)$. If there exists a constant k-plane A such that $\langle N, A\rangle > \cos^p(\pi/2\sqrt{2p})$ for all normals N, then M is a totally geodesic subsphere.*

We begin with some remarks about harmonic maps. Let M and $\bar{M}$ be smooth oriented Riemannian manifolds of dimensions m and $\bar{m}$ respectively. Let ∇ and D be connections on M and $\bar{M}$, and let $e_1, \ldots, e_m$ be a local orthonormal tangent frame on M at p. Let $f: M \to \bar{M}$ be a smooth mapping.

Definition. $f: M \to \bar{M}$ is *harmonic* if $\sum_{i=1}^m (D_{f_* e_i}(f_* e_i) - f_*(\nabla_{e_i} e_i)) = 0$ where f_* is the differential of f.

We need some known facts about harmonic functions.

LEMMA 4.2. *If $M^m \hookrightarrow \mathbf{R}^{m+k}$ is an immersion and $\eta: M \to G_{m,k}$ is the Gauss map then f is harmonic if and only if M has parallel mean curvature.*

Proof. See Chern and Goldberg [6]. □

An immediate corollary is the following.

COROLLARY 4.3. *Let $M^n \to S^{n+k}$ be a minimal immersion. Let $\eta: M^n \to G_{k,n+1}$ be the Gauss map on M given by mapping a point p on M to the k-plane normal to M in S^{n+k} at p. Then η is a harmonic map.*

We also need the following lemma which is well known and easily checked.

LEMMA 4.4. *Let U and $\bar{U}$ be open sets in M and $\bar{M}$ respectively. If $h\colon U \to \bar{U}$ is a harmonic map and $f\colon \bar{U} \to \mathbf{R}$ is a convex function, then $f \circ h\colon U \to \mathbf{R}$ is a subharmonic function.*

Now let M be a minimally immersed submanifold of S^{n+k} and $\bar{M} = G_{k,n+1}$. Let $h = \eta$ be the Gauss map on M, considered as a submanifold of S^{n+k}, into $G_{k,n+1}$. Let $f = \varrho_A^2\colon G_{k,n+1} \to \mathbf{R}$ be the square of the distance from A to X in $G_{k,n+1}$.

THEOREM 4.1. *If M^n is a compact minimal submanifold of S^{n+k} with normal k-plane field N and if A is a fixed k-plane in $\mathbf{R}^{n+k+1}$ such that*

$$\langle N, A\rangle > \cos^p(\pi/2\sqrt{2p})$$

where $p = \min(n+1, k)$, then M is a totally geodesic subsphere. If $p=1$ the result holds for $\langle N, A\rangle > 0$.

Proof. Combining Theorem 3.5, Corollary 4.3 and Lemma 4.4, we have that $\varrho_A^2 \circ \eta$ is a subharmonic function on a compact manifold M and hence constant. But $\varrho_A^2(N)$ is strictly convex when $\langle N, A\rangle > \cos^p(\pi/2\sqrt{2p})$ so by Theorem 3.5, $H\varrho_A^2(\cdot,\cdot) \geqslant \varepsilon\langle\cdot,\cdot\rangle > 0$. By (4), $0 = \Delta\varrho_A^2 \circ \eta = \sum_{i=1}^n H\varrho_A^2(\eta_* e_i, \eta_* e_i)$ so that $\eta_* \equiv 0$. Hence η must be constant, so N is fixed and the theorem follows. The same proof works for $p=1$. $\square$

Remark. The same proofs also show that there is no compact submanifold M^n of $\mathbf{R}^{n+k+1}$ with parallel mean curvature all of whose normals satisfy $\langle N, A\rangle > \cos^p(\pi/2\sqrt{2p})$ for any fixed $(k+1)$-plane A.

5. Applications

In this final section we give some applications of the previous results.

Let Ω be a convex open set in $\mathbf{R}^n$. Let $F\colon \Omega \to \mathbf{R}^{n+k}$ be a non-parametric immersion, i.e. let $F(x)$ be of the form $F(x) = (x, f(x))$ for some $f\colon \Omega \to \mathbf{R}^k$. F is a minimal immersion if f satisfies the minimal surface system, a system of second order non-linear elliptic partial differential equations given by

$$\sum_{i=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ij}) = 0 \quad \text{for } j = 1, \ldots, n$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j}\right) = 0$$

where $g_{ij} = \delta_{ij} + \langle \partial f/\partial x^i, \partial f/\partial x^j\rangle$, $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. This system is also defined in the weak sense for a Lipschitz function. A Lipschitz function f on Ω is a weak solution to the minimal surface system if

$$\left.\begin{aligned} \int_\Omega \sum_{i=1}^{n} (\sqrt{g}\, g^{ij}) \frac{\partial \varphi}{\partial x^i} = 0 \\ \int_\Omega \sum_{\substack{i,j=1,\dots,n \\ \alpha=1,\dots,k}} \left(\sqrt{g}\, g^{ij} \frac{\partial f_\alpha}{\partial x^i}\right) \frac{\partial \varphi_\alpha}{\partial x^j} = 0 \end{aligned}\right\} \quad \text{for } j=1, \dots, n$$

where φ is any C^∞ function with compact support on Ω.

A result of Morrey [13] shows that any C^1 function which satisfies the minimal surface system is real analytic. In the case $n=3$, Barbosa stated without proof that a Lipschitz solution is real analytic. We present the proof here. We use Theorem 2.3 and a regularity result of Allard [1]. We first establish some definitions and notation.

Let $f\colon \Omega \to \mathbf{R}^k$ be a Lipschitz function on an open bounded convex set Ω in $\mathbf{R}^n$. Let $\Gamma = \{(x, f(x)) \in \mathbf{R}^{n+1}\}$ be the graph of f.

We now suppose $0 \in \Gamma$ and define the tangent cone of Γ at 0. For any $r>0$, let $f_r(x) = (1/r)f(rx)$. Now f_r is directly seen to be a weak solution of the minimal surface system and moreover

$$|f_r(x) - f_r(y)| = \frac{1}{r}\, f(rx) - f(ry)| \leqslant \frac{1}{r} M|rx - ry| = M|x-y|. \tag{5.1}$$

This shows $\{f_r\}_r$ is an equicontinuous family on any compact subset of $\mathbf{R}^n$, so the Arzela-Ascoli Theorem implies there is a convergent subsequence $\{f_{r_i}\}$ with $\lim_{r_i \to 0} f_{r_i}(x) = h(x)$. From (5.1) we see that h is a Lipschitz function with constant M. Let $C(\Gamma, 0)$ be the graph of h.

LEMMA 5.1. *The function h is a weak solution of the minimal surface system and $C(\Gamma, 0)$ is a cone, i.e. if $x \in C(\Gamma, 0)$ and $t \in \mathbf{R}$ then $tx \in C(\Gamma, 0)$.*

Proof. See, for example, Lawson [11]. □

We call $C(\Gamma, 0)$ the *tangent cone* to Γ at 0. By translation, for any $x \in \Gamma$ one can define $C(\Gamma, x)$, the tangent cone of Γ at x.

The regularity result we need is given in its full form in Allard [1]. The relevant portion of our purposes is

LEMMA 5.2 (Allard regularity [1]). *If the tangent cone at $x \in \Gamma$ is a linear space, then Γ is a $C^{1,\alpha}$ submanifold near x.*

Morrey's theorem on the regularity of C^1 minimal immersions then gives real analyticity of f at x.

THEOREM 5.3. *Let $f: \Omega \to \mathbf{R}^k$ be a Lipschitz solution to the minimal surface system, where Ω is a domain in $\mathbf{R}^3$. Then f is real analytic.*

Proof. Let Γ be the graph of f and $C(\Gamma, x)$ the tangent cone at a point x.

If $C(\Gamma, x)$ is regular except at x, then it is the cone on a minimal two dimensional sphere S^2 in S^{k+2} centered at x. Let N_x be the k-dimensional normal space to S^2 in S^{k+2}. Let A be the k-dimensional space orthogonal to the Euclidean 3-space containing S^2. Then since $C(\Gamma, x)$ is given by a graph, $\langle N_x, A\rangle > 0$, so Theorem 2.3 implies S^2 is a totally geodesic subsphere in S^{k+2}. Hence $C(\Gamma, x)$ must be a plane. Using Allard regularity (together with Morrey's regularity) we conclude that f is real analytic.

Claim. $C(\Gamma, x)$ is regular for $y \neq x$, where $y \in C(\Gamma, x)$.

It follows from Federer [9], p. 456 that

$$C(C(\Gamma, x), y) = \text{line} \times 2\text{-dimensional cone}.$$

It is straightforward to show that the two-dimensional cone is minimal and is the graph of a Lipschitz function. Since the intersection of the graph with the unit sphere is one-dimensional, it must be a great circle. Therefore $C(C(\Gamma, x), y)$ is a plane and hence $C(\Gamma, x)$ is regular away from x. □

As another application of Theorem 2.3, we generalize Bernstein's Theorem. Bernstein's original theorem states if $f: \mathbf{R}^2 \to \mathbf{R}$ is an entire solution to the minimal surface system, then f is linear. This theorem has been generalized to higher codimensions by Osserman and Chern. One implication of their result is that if $f: \mathbf{R}^2 \to \mathbf{R}^k$ is an entire solution to the minimal surface system having bounded gradient, then f is linear. We generalize this result to three dimensions.

THEOREM 5.4. *If $f: \mathbf{R}^3 \to \mathbf{R}^k$ is an entire solution to the minimal surface system, satisfying*

$$\|\nabla f\| \leqslant K \quad \text{on } \mathbf{R}^3$$

for some constant K, then f is linear.

Proof. The idea of the proof is to look at the tangent cone at ∞. Thus we define a sequence $f_r(x) = (1/r) f(rx)$. As above, f_r has uniformly bounded gradient. Again, there is a subsequence $r_j \to \infty$ such that

$$\lim_{r_j \to \infty} f_r(x) = h(x).$$

$h(x)$ is a solution to the minimal surface system and the graph is a cone. By the same arguments as above, the cone is a plane. Allard's regularity estimate then implies the original graph is a plane, hence f is linear. □

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Received January 27, 1979