THE STRUCTURE OF THE PRYM MAP

BY

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Introduction

The study of Prym varieties has always served to link moduli-questions on curves and on abelian varieties. The "Prym differentials", as they appeared to the ancients, provided the main ingredient in the work of Schottky and Jung on the equations of the locus $\mathcal{J}_g$ of Jacobians in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties ([Sk], [SJ]). They kept precisely the same role through the extension by Farkas and Rauch ([FR] and many other works) of Schottky's computations to higher genus, and evolved through works of Fay, Mumford, Tjurin and others to a wide body of geometric knowledge with applications to both curves and abelian varieties, culminating in Beauville's refinement of the results of Andreotti and Mayer [AM] in genera 4, 5, and proof of the irreducibility of $\mathcal{A}_4$, $\mathcal{A}_5$ (in all characteristics).

In recent years, especially through works of Beauville ([B1], [B2]) and Masiewicki [Ma] and the directing hand of Clemens from behind the scenes, there evolved an increasing recognition that in order to extract the most information about moduli spaces, the correct way to use Prym varieties is to view them globally, as setting a correspondence between $\mathcal{M}_g$, $\mathcal{A}_{g-1}$, or as a map

$$\mathcal{P}: \mathcal{R}_g \to \mathcal{A}_{g-1}$$

from a certain finite cover $\mathcal{R}_g$ of $\mathcal{M}_g$, to $\mathcal{A}_{g-1}$. The present paper attempts a systematic and unified study, for all $g \geqslant 6$, of the structure of this map along its various exceptional loci, including those that are mapped to Jacobians $\mathcal{J}_{g-1} \subset \mathcal{A}_{g-1}$. Using the results of Wirtinger, Mumford, Tjurin, Recillas, Fay, Masiewicki and Beauville, one can describe all of the latter loci, including those components which lie in the boundary of an appropriate partial compactification of $\mathcal{R}_g$. Parts II, III and IV analyze $\mathcal{P}$ near these components.

In genus 6, the Prym map is equidimensional ($\dim \mathcal{M}_6 = \dim \mathcal{A}_5 = 15$). Wirtinger proved in [W] that it is generically finite; a motivating question for our work and its unpublished predecessor [S] was to compute its degree. Combining the local computations along the various loci over Jacobians, we are able to show in Part I that this degree is 27 (although the fiber over $\mathcal{J}_5$ itself is infinite with components of relative dimensions 0, 1 and 2).

Wake an algebraic geometer in the dead of night, whispering: "27". Chances are, he will respond: "lines on a cubic surface". In Part V, we amass substantial evidence towards the conjecture that the *general* fiber of $\mathcal{P}$ carries the structure of the intersection—configuration of lines on a cubic surface. (A good understanding of this general fiber is expected to help understand the Schottky problem and related moduli questions in genus 5.) Recently the conjecture has been proven by one of us, and will appear elsewhere.

In Part V we also study the other loci, in genus 6, where $\mathcal{P}$ is not of maximal rank. (It is true, though not proven here, that these are *all* such loci.) It is well-known [CG] that the intermediate Jacobian of a cubic threefold is, in many ways, the Prym of a plane quintic curve; we show it is not the Prym of anything else. A similar result holds for the intermediate Jacobians of a more general family, of "quartic double solids" studied recently by Clemens [C]. Part of the argument here was completed, upon our commission, by Clemens. While we are at it, we settle a question posed in [CG] by explicitly constructing a cubic threefold whose intermediate Jacobian is the Prym of a given plane quintic.

In a couple of places we make use of a result, obtained jointly with M. Green three years ago, describing the tangent space to the subvariety in of $\mathcal{M}_g$ curves admitting a g^1_d. This has since been subsumed in the general Brill–Noether theory of [GH2], yet it seemed simple and pretty enough to be included here, as an appendix.

As to the generality of our results: Parts III and IV deal with loci which exist in all genera. Our results there are also valid for all g, though in Part IV, § 4, we are able to make the necessary constructions much more explicit in genus 6; in particular, we obtain in this case a new compactification $\mathcal{M}'$ of $\mathcal{M}_6$, based on the stability of plane (rather than pluricanonical) models of the curves. Infinitesimally along the relevant components, $\mathcal{M}'$ "looks like" the corresponding versal deformation spaces. In it the locus of curves with "elliptic-tails" is blown down to cuspidal curves; the precise structure of this map is given in IV.4.3. It should be interesting to construct analogous objects in all genera.

Part II is anomalous: it deals with a locus present only for $g=6$. Moreover, the results we really need for the degree computation were obtained by Tjurin and Beauville. We included it for completeness, but also to establish some ideas that reccur in later sections, thus unifying the treatment of the various components. We also prove there an irreducibility result which might be new.

To be on the safe side, we only claim our results over the field $\mathbf{C}$ of complex numbers. We did make some efforts, though, to use only "algebraic" constructions and arguments, so we naively hope that there is no real obstruction to rewriting everything over an algebraically closed field of any characteristic ($\neq 2$ or 3, perhaps). We tried to keep the level of this work within reach of any mature reader. Thus when some high-powered machinery is needed, such as formal deformation theory in Part IV, § 2, we review it with plenty of examples covering our actual application of the theory.

Both of us wish to acknowledge our gratitude to our teacher, Herb Clemens, who introduced us to this fascinating subject, prodded us along, and gave a helping hand whenever we were stuck. We are also indebted to A. Beauville who caught some early inaccuracies and encouraged the project, to D. Gieseker for his help with understanding the relevant deformation theory and the subtle points of the factorization in Part IV, and to K. Chakiris, M. Green, P. Griffiths, J. Harris, D. Morrison and S. Ramanan for many conversations, ideas, good advice and much patience.

Notations

In Part X, "Y.3.5" refers to § 3 of Part Y, while 3.5 refers to § 3 of Part X.

$\mathcal{M}_g$ — moduli space of (smooth, complete) curves of genus g.

$\mathcal{R}_g$ — moduli space of unramified double covers of curves in $\mathcal{M}_g$.

$\bar{\mathcal{M}}_g$ — the Deligne–Mumford compactification of $\mathcal{M}_g$, allowing stable curves of arithmetic genus g.

$\bar{\bar{\mathcal{R}}}_g$ — the compactification of $\mathcal{R}_g$ constructed in I.1.2.

$\bar{\mathcal{R}}_g$ — moduli of "allowable" double covers in $\bar{\bar{\mathcal{R}}}_g$.

$\mathcal{A}_g$ — moduli space of principally polarized abelian varieties of dimension g.

$\mathcal{J}_g$ — the subvariety of $\mathcal{A}_g$ parametrizing Jacobians, birational image of $\mathcal{M}_g$ under the Jacobi map.
C — curve of genus g.
$\tilde{C}$ — double cover of C.
$\mathcal{P}$ — the Prym map.
η — a semi-period on C.
X — object whose Jacobian is $\mathcal{P}(\tilde{C})$; a curve in Parts I–IV, a threefold in Part V.
Φ — canonical map of X.
Ψ' — Prym-canonical map of C.
φ — Abel map of X.
ψ — Abel–Prym map of $\tilde{C}$.
$\mathcal{M}_T$, $\mathcal{R}_T$ — subvarieties of $\bar{\mathcal{M}}_g$, $\bar{\mathcal{R}}_g$ where C is trigonal.
$\mathcal{M}_S$, $\mathcal{R}_S$ — subvarieties of $\bar{\mathcal{M}}_g$, $\bar{\mathcal{R}}_g$ where C has an ordinary double point, $\tilde{C}$ a Wirtinger double cover.
$\mathcal{M}_E$, $\mathcal{R}_E$ — subvarieties of $\bar{\mathcal{M}}_g$, $\bar{\mathcal{R}}_g$ where C has an elliptic tail and η is supported on the tail.
$\mathcal{M}_{E,S}$, $\mathcal{R}_{E,S}$ — Intersection of previous two.
$\mathcal{M}_{\mathrm{har}}$, $\mathcal{R}_{\mathrm{har}}$ — subvarieties of $\mathcal{M}_E$, $\mathcal{R}_E$ where the elliptic tail E is harmonic (explained below).
$\mathcal{M}_{\mathrm{e.a.h.}}$, $\mathcal{R}_{\mathrm{e.a.h.}}$ — subvarieties of $\mathcal{M}_E$, $\mathcal{R}_E$ where the elliptic tail E is equianharmonic explained below).
$\mathcal{R}_Q$ — moduli space of plane quintic curves with even cover.
$\mathcal{R}_C$ — moduli space of plane quintic curves with odd cover.
$\mathbf{P}(V)$ — the projective space of one dimensional subspaces in V (no dualization).

Harmonic, equianharmonic—refer to elliptic curves with complex multiplication by $(-1)^{1/2}$, $1^{1/3}$. (Corresponding invariants: $j=1728$, 0. Equations: $y^2=x^3-x$, $y^2=x^3-1$.)

For $f: X\to Y$, the *branch* locus in Y is the image of the *ramification* locus in X.

Part I. The degree of the Prym map

After reviewing the definitions and some background material, due mostly to Beauville, we state our main result (Theorem 2) and outline its proof, or rather explain how the proof leads naturally to studying the local structure of the Prym map, in Parts II, III, IV.

§ 1. The partial compactification

Given a curve $C\in\mathcal{M}_g$ and a nowhere ramified double cover

$$\pi: \tilde{C}\to C$$

one constructs the "Prym variety" as follows. π induces a norm map

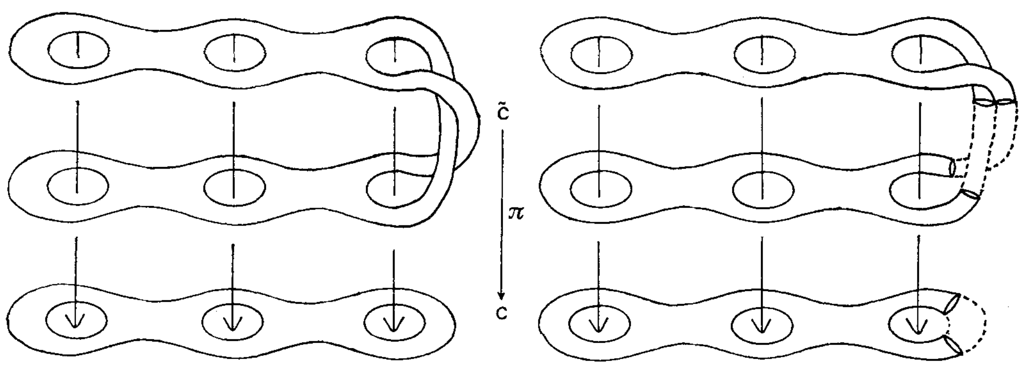

Fig. 1

$$Nm: J(\tilde{C}) \to J(C)$$

between the Jacobian varieties $J(C)$, $J(\tilde{C})$ of line bundles of degree 0 on C, $\tilde{C}$. The Prym variety $\mathcal{P}(\tilde{C}, C, \pi)$ is defined to be (the connected component of the origin in) ker (Nm). Mumford shows in [M2] that ker (Nm) has two components, and that the principal polarization of $J(\tilde{C})$ induces on $\mathcal{P}(\tilde{C}, C, \pi)$ twice a principal polarization, so that $\mathcal{P}(\tilde{C}, C, \pi)$ is a principally polarized abelian variety. By Hurwitz' formula, $\tilde{C}$ has genus $2g-1$; since Nm is surjective, we conclude that $\mathcal{P}(\tilde{C}, C, \pi)$ is $(g-1)$-dimensional.

Let $\mathcal{R}_g$ denote the moduli-space of curves C of genus g, together with an unramified double cover $\tilde{C}$. The above construction yields a morphism

$$\mathcal{P}: \mathcal{R}_g \to \mathcal{A}_{g-1}$$

which we call the Prym map.

The natural projection $p: \mathcal{R}_g \to \mathcal{M}_g$ is finite, of degree $2^{2g}-1$. Given a C, its double covers correspond to non-zero $(\mathbf{Z}/2\mathbf{Z})$ homology 1-classes. Topologically, one constructs a $\tilde{C}$ by pasting together two copies of C slit along a non-trivial loop, with sheets interchanged (see fig. 1, where $g=3$, $2g-1=5$). Algebraically, $\tilde{C}$ is determined by a "semi-period" on C, or a non-trivial line bundle η such that η^2 is trivial. η has a natural "two-valued section" which yields the double-cover $\tilde{C}$; vice versa, $\tilde{C}$ determines η as the only semi-period on C whose pullback to $\tilde{C}$ is trivial, ([M2]). We use various abreviations for $\mathcal{P}(\tilde{C}, C, \pi)$: $\mathcal{P}(\tilde{C})$, $\mathcal{P}(\pi)$ and also $\mathcal{P}(\eta)$ or $\mathcal{P}(C, \eta)$.

The map $\mathcal{P}: \mathcal{R}_g \to \mathcal{A}_{g-1}$ is dominant for $g \leqslant 6$ and generically finite for $g=6$. Wirtinger, in [W], showed this using the extension of $\mathcal{P}$ over a certain boundary component of $\mathcal{R}_g$ (cf. Part IV). The problem of extending $\mathcal{P}$ to possibly singular and ramified covers was attempted by Fay [F] and Mumford [M2]. Masiewicki in [Ma] had the correct notion of allowable double-cover, but applied it only in the special case of plane quintics (cf. Part II).

In [B1], § 6, Beauville obtained a proper map (although not necessarily finite) which on a dense open set factors through the previously described $\mathcal{P}$. For our purposes we need the following strengthening of Beauville's result. (The proof is technical and might better be skipped on first reading.)

THEOREM 1.1. (*"Prym is Proper".*) *The map $\mathcal{P}$ extends to a proper map*

$$\bar{\mathcal{P}}: \bar{\mathcal{R}}_g \to \mathcal{A}_{g-1}.$$

1.2. *Proof of Theorem* 1.1. First we construct a compactification $\bar{\bar{\mathcal{R}}}_g$ of $\mathcal{R}_g$. Let $\overline{\mathcal{M}}_g^{(n)}$ denote the moduli space of stable curves of genus g with a level-n structure. It is a complete, separated algebraic space ([P], Theorem 10.9 ff.) and for $n \geqslant 3$ it is actually a "fine" moduli space, i.e. a universal curve $\Gamma^{(n)} \to \overline{\mathcal{M}}_g^{(n)}$ exists and enjoys a universal property. For $n=1$, Mumford [M3] and Knudsen proved recently that $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g^{(1)}$ is in fact a projective variety, in particular a scheme. Since $\overline{\mathcal{M}}_g^{(n)} \to \overline{\mathcal{M}}_g$ (the forgetful map) is finite, we see by [K], Corollary II.6.16 that $\overline{\mathcal{M}}_g^{(n)}$ is also a scheme (separated and complete). The same holds for $\Gamma^{(n)}$ ($n \geqslant 3$) since there is a closed immersion $\Gamma^{(n)} \to \overline{\mathcal{M}}_{g+1}^{(n)}$. (A point of $\Gamma^{(n)}$ consists of a curve C with level-n structure and a marked point $p \in C$. Choose once and for all an elliptic curve E with level-n structure, and map

$$(C, p) \mapsto C \cup_p E \in \overline{\mathcal{M}}_{g+1}^{(n)},$$

compare IV.1.3.)

Consider, then, the morphism

$$q: \Gamma^{(n)} \to \overline{\mathcal{M}}_{2g-1}^{(n)} = T, \quad n \geqslant 3,$$

of complete, irreducible schemes. Following Beauville, we let $r: I \to T$ be the complete scheme, finite over T, representing the functor of T-involutions of $\Gamma^{(n)}$ ([DM], p. 84). The point-set underlying I parametrizes level-n curves $\tilde{C}$ of genus $2g-1$ together with an involution $i: \tilde{C} \to \tilde{C}$. Moreover, I has the following universal property ([DM], p. 84): "For any T- scheme $S \to T$, the set of S-involutions of $\Gamma^{(n)} \times_T S$ is naturally isomorphic to $\mathrm{Mor}_T(S, I)$ via pulling back the universal family

$$\Gamma^{(n)} \times_T I \to I$$

with its involution."

The symplectic group $\mathrm{Sp}(4g-2, \mathbf{Z}/n\mathbf{Z})$ acts on T with quotient $\overline{\mathcal{M}}_{2g-1}$. We claim the action lifts to I, or in other words, for each $\alpha \in \mathrm{Sp}(4g-2, \mathbf{Z}/n\mathbf{Z})$, there is a T-isomorphism $\alpha^* I \to I$. To produce such an isomorphism we only need, by the universal property, to exhibit an $\alpha^* I$-involution of $\Gamma^{(n)} \times_T \alpha^* I$. Let $\beta = \alpha^{-1}$. Since

$$\Gamma^{(n)} \times_T \alpha^* I \approx \alpha^*(\beta^* \Gamma^{(n)} \times_T I)$$

we need an I-involution of $\beta^*\Gamma^{(n)} \times_T I$. This is supplied by the "universal" involution of $\Gamma^{(n)} \times_T I$ using the following:

LEMMA 1.2.1. *The automorphism $\beta_*\colon T \to T$ lifts to $\Gamma^{(n)}$. Equivalently, there is a T-isomorphism $\beta^*\Gamma^{(n)} \xrightarrow{\approx} \Gamma^{(n)}$.*

Proof. By [P], Theorem 10.3, $\overline{\mathcal{M}}^{(n)}$ and $\Gamma^{(n)}$ are obtained as geometric quotients of the Hilbert scheme $H^{(n)}$ and the universal curve $\mathcal{C}^{(n)}$ over it by the action of PGL (N). Since

$$\mathcal{C}^{(n)} = \mathcal{C} \times_H H^{(n)}$$

is the pullback of the universal curve $\mathcal{C}$ over the level-1 Hilbert scheme H, we see that β lifts to $\mathcal{C}^{(n)}$ (acting as identity on $\mathcal{C}$, β_* on $H^{(n)}$). Its action commutes with PGL (N), hence descends to the quotient $\mathcal{C}^{(n)}/\mathrm{PGL}\,(N) \approx \Gamma^{(n)}$. Q.E.D.

By Deligne's theorem ([K], p. 183) quotients by finite group actions exist in the category of separated algebraic spaces. In our case, the quotient

$$\mathcal{R}' = I/\mathrm{Sp}\,(4g-2, \mathbf{Z}/n\mathbf{Z})$$

fits into a diagram

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{R}' \\ \downarrow & & \downarrow \\ T & \longrightarrow & \overline{\mathcal{M}}_{2g-1} \end{array}$$

where I is finite over $\overline{\mathcal{M}}_{2g-1}$. Hence $\mathcal{R}'$ is finite over $\overline{\mathcal{M}}_{2g-1}$ and by [K], II.6.16 $\mathcal{R}'$ is actually a scheme.

Restricting to smooth curves, we have a diagram

$$\begin{array}{ccc} \mathcal{R}_g^{(n)} & \hookrightarrow & I \\ \downarrow & & \downarrow \\ \mathcal{R}_g & \hookrightarrow & \mathcal{R}' \end{array}$$

which allows us to define $\bar{\bar{\mathcal{R}}}_g$ as the closure, in $\mathcal{R}'$, of $\mathcal{R}_g$. $\bar{\bar{\mathcal{R}}}_g$ is a complete, irreducible scheme containing $\mathcal{R}_g$ as dense open subset, and after finite base extension

$$\bar{S} = \bar{\bar{\mathcal{R}}}_g^{(n)} \to \bar{\bar{\mathcal{R}}}_g$$

(where $\bar{\bar{\mathcal{R}}}_g^{(n)}$ is the closure in I of $\mathcal{R}_g^{(n)}$) we have a family of stable curves

$$q\colon \tilde{\mathcal{C}} \to \bar{S}$$

and an $\bar{S}$-involution $i: \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ such that:

(a) For each $s \in \bar{S}$, the induced involution $i_s: \tilde{\mathcal{C}}_s \to \tilde{\mathcal{C}}_s$ is different from the identity on each component of $\tilde{\mathcal{C}}_s$.

(b) $\tilde{\mathcal{C}}_s$ has genus $2g-1$, and the quotient curve $\tilde{\mathcal{C}}_s/(i_s)$ has genus g.

(c) Any pair $(\tilde{C}, i)$ where $\tilde{C} \in \mathcal{M}_{2g-1}$, i a fixed-point free involution, is isomorphic to $(\tilde{\mathcal{C}}_s, i_s)$ for some $s \in \mathcal{R}_g^{(n)} \subset \bar{S}$.

Property (c) is clear; (a) and (b) are proved in [B1], 6.1.

1.3. Let $C \in \overline{\mathcal{M}}_g$, $\tilde{C} \in \overline{\mathcal{M}}_{2g-1}$ a (possibly branched) double cover, $i: \tilde{C} \to \tilde{C}$ the involution,

$$Nm: J(\tilde{C}) \to J(C)$$

the norm map on the generalized Jacobians, and $P = \ker (Nm)^0$.

Definition 1.3.1. $(\tilde{C}, i)$ is allowable if P is an abelian variety.

Definition 1.3.2. $(\tilde{C}, i)$ is allowable if the only fixed points of i are nodes where the two branches are *not* exchanged, and the number of nodes exchanged under i equals the number of irreducible components exchanged under i.

Definition 1.3.3. $(\tilde{C}, i)$ is allowable if the components of $\tilde{C}$ can be grouped as $\tilde{C} = A \cup A' \cup \tilde{B}$ where i interchanges A, A' and fixes $\tilde{B}$, each connected component of A is "tree-like" and either

(i) $\tilde{B} = \emptyset$, A connected, $\#(A \cap A') = 2$, or

(ii) $A \cap A' = \emptyset$, $\#(\tilde{B} \cap A_i) = 1$ for each connected component A_i of A, the fixed points of i in $\tilde{B}$ are precisely the nodes, and the two branches there are never exchanged (so that $B = \tilde{B}/(i)$ also has nodes at the corresponding points).

Definitions 1.3.1 and 1.3.2 are equivalent by [B1], Lemma 5.1. They imply Definition 1.3.3 by [B1], 5.2, and the converse is clear.

Let $\bar{\mathcal{R}}_g \subset \bar{\bar{\mathcal{R}}}_g$ be the open subset of allowable double covers, $S \subset \bar{S}$ the corresponding open subset in $\bar{\bar{\mathcal{R}}}_g^{(n)}$. Clearly $\mathcal{R}_g \subset \bar{\mathcal{R}}_g$ and $\mathcal{R}_g^{(n)} \subset S$. By [B1], 6.2 there is a Prym morphism

$$p: S \to \mathcal{A}_{g-1}.$$

(Beauville proves this whenever $\bar{S}$ satisfies conditions (a), (b), (c) in 1.2, and S is the open subset of Definition 1.3.1. We shall use this result for a different $\bar{S}$ in IV.4.4.4.) By [B1], Proposition 6.3, the map p is proper. Finally, $S \subset I$ is stable under Sp $(4g-2, \mathbf{Z}/n\mathbf{Z})$ with quotient $\bar{\mathcal{R}}_g$, and p commutes with this action, so we obtain an induced map

$$\bar{\mathcal{P}}: \bar{\mathcal{R}}_g \to \mathcal{A}_{g-1}.$$

$\bar{\mathcal{P}}$ is proper since p is, and $\bar{\mathcal{P}}|_{\mathcal{R}_g} = \mathcal{P}$. Q.E.D.

1.4. We briefly sketch another construction for $\bar{\bar{\mathcal{R}}}_g$. Recall that a level-$n$ structure on a stable $C \in \overline{\mathcal{M}}_g$ is a symplectic injection

$$H^1(C, \mathbf{Z}/n\mathbf{Z}) \hookrightarrow (\mathbf{Z}/n\mathbf{Z})^{2g}$$

where the right hand side has the standard symplectic structure. $G = \mathrm{Sp}\,(2g, \mathbf{Z}/n\mathbf{Z})$ acts by composition on the left. Let $G_0 \subset G$ be the stabilizer of

$$(1, 0, ..., 0) \in (\mathbf{Z}/2\mathbf{Z})^{2g} \quad (n=2).$$

The quotient $\overline{\mathcal{M}}_g^{(2)}/G_0$ is a complete algebraic space by Deligne's theorem and a scheme since it is finite over $\overline{\mathcal{M}}_g$. Its open subset covering $\mathcal{M}_g$ is naturally isomorphic to $\mathcal{R}_g$. Further, the examples worked out in IV.2.6 show that near all singular curves which we need, $\bar{\bar{\mathcal{R}}}_g$ is locally isomorphic to $\overline{\mathcal{M}}_g^{(2)}/G_0$.

§ 2. The main result

THEOREM 2.1. *The Prym map*

$$\mathcal{P}: \mathcal{R}_6 \to \mathcal{A}_5,$$

from unramified double covers of curves of genus 6 *to principally polarized abelian varieties of dimension* 5, *is generically* 27 *to* 1.

2.2. *Proof of Theorem 2.1.* Let $f: X \to Y$ be a generically-finite, proper map of degree d. This degree can be computed "at" any point $y \in Y$: $f^{-1}(y)$ breaks into finitely many connected components Y_i; to each of these we can associate the local degree d_i of the map f along Y_i, and we find

$$d = \sum d_i.$$

In our case we apply this to $\bar{\mathcal{P}}$, and compute the degree at a generic Jacobian

$$J(C) \in \mathcal{J}_5 \subset \mathcal{A}_5,$$

that is, for C a generic curve in $\mathcal{M}_5$.

2.3. In [M2], Mumford lists those double covers

$$\{\pi\colon \tilde{C} \to C\} \in \mathcal{R}_6$$

whose Prym could be a Jacobian. There are only 2 possibilities for obtaining a generic (not hyperelliptic etc.) Jacobian:

(1) C a smooth plane quintic curve, $\pi\colon \tilde{C} \to C$ an "even" double cover. (The distinction between "even" and "odd" covers of plane quintics is explained in Part II, § 1).

(2) C a trigonal curve, that is a 3-sheeted branched-cover of $\mathbf{P}^1$.

Note. The other family in his list, double covers of elliptic curves, is of dimension $2g-2=10$, thus too small to map onto the generic point of $\mathcal{M}_5$ which has dimension $3(g-1)-3=12$.

New types arise when the Prym map is partially-compactified. Beauville has extended Mumford's arguments to the boundary components of $\bar{\mathcal{R}}_g$, and obtained a substantially longer list ([B1], Theorems 4.10 and 5.4). However, using a dimension-count as above we eliminate all but the following two types (compare IV.1.4).

(3) C has an ordinary double point; let X be the normalization of C, then $C = X/(p=q)$ for 2 distinct points $p, q \in X$; $\tilde{C}$ is the union of 2 copies X_1, X_2 of X (with marked points p_1, q_1, respectively p_2, q_2):

$$\tilde{C} = X_1 \coprod X_2/(p_1 = q_2, p_2 = q_1)$$

and $\pi\colon \tilde{C} \to C$ is the natural projection. (This type arises from his Proposition 5.2 (i).)

(4) C is a reducible curve, consisting of a component X of genus 5 and an elliptic curve E, meeting in 1 point. $\tilde{C}$ has 3 components: X_1 and X_2 are copies of X (mapped by π to X), $\tilde{E}$ is another elliptic curve, doubly covering E. ([B1], Proposition 5.2 (ii).)

2.4. The bulk of this paper is devoted to computing the local degrees of $\mathcal{P}$ along the above four subvarieties of $\bar{\mathcal{R}}_6$. Combining II.4.4, III.3.4, and IV.5.1, we see that the contributions to the total degree are, respectively, 1, 10, 16, 0, adding up to 27. Q.E.D.

As discussed in Part IV, § 4, the vanishing contribution of the last family is possible as this is an irreducible component but not a connected component. In fact it intersects family #3, and $\mathcal{P}$ blows it down (to a subfamily of #3) before mapping to $\mathcal{A}_5$.

§ 3. Computation of local degrees

Let $f\colon X \to Y$ be a proper, dominant map between n-dimensional varieties. f is generically finite, say of degree d. Let $W \subset Y$ be an irreducible closed subvariety of codimension k in Y. $f^{-1}(W)$ breaks into finitely many connected components Z_i, of codimension l_i in X. The local degree d_i of f along Z_i is the degree of the map obtained from f by localizing X at Z_i. We have

$$d = \sum d_i.$$

Let Z be one of these components. Let $\tilde{X}$ be the blowup of X along Z, with exceptional divisor $\tilde{Z}$. Let $\tilde{Y}$ be the blowup of Y along W, with exceptional divisor $\tilde{W}$. The map f lifts to a rational map

$$\tilde{f}: \tilde{X} \to \tilde{Y}.$$

We would like this to induce a map on the exceptional divisors

$$f_*: \tilde{Z} \to \tilde{W}.$$

We restrict our attention to the smooth points of Z, W. First we notice that $\tilde{Z}$ is the (projectivized) normal bundle to Z in X, similarly for $\tilde{W}$, W, Y. Choose $z \in Z$, $w = f(z) \in W$. The differential

$$df: T_z X \to T_w Y$$

maps $T_z Z$ to $T_w W$, hence induces a linear map

$$f_{*,z}: N_{Z \setminus X, z} \to N_{W \setminus Y, w}.$$

The following observation is intuitively obvious.

LEMMA 3.1. *The rational map $\tilde{f}$ is regular at a generic point $\tilde{z} \in \tilde{Z}$ if and only if the differential $f_{*,z}$ is not identically zero at a generic $z \in Z$. $\tilde{f}$ is regular for all $\tilde{z}$ in the fiber of $\tilde{Z}$ over z if and only if $f_{*,z}$ is injective on the normal space to Z at z. In this case the restriction of $\tilde{f}$ to $\tilde{Z}|_z$ is the projectivization of the linear map $f_{*,z}$.*

The lemma follows immediately from the universal property of blowups (see, e.g. [H], Chapter 2, Proposition 7.14). The universal property guarantees that $\tilde{f}$ is a morphism near $\tilde{Z}|_z$ if the ideal of W becomes invertible when pulled back to $\tilde{X}$. Injectivity of f_* on normal bundles is equivalent to surjectivity on conormals

$$I_W/I_W^2 \twoheadrightarrow I_z/I_z^2$$

so by the implicit function theorem or Nakayama's lemma,

$$f^{-1} I_W \cdot O_X = I_Z$$

and the pullback to $\tilde{X}$ becomes invertible as required.

LEMMA 3.2. *Assume $f_{*,z}$ is injective on the normal space $N_{Z\setminus X,z}$ at each $z\in Z$. Then the local degree of f along Z equals the degree of the induced equidimensional map on exceptional divisors,*

$$f_*\colon \tilde{Z}\to \tilde{W}.$$

Proof. Localize at Z, so that $Z=f^{-1}(W)$ and $\tilde{Z}=\tilde{f}^{-1}(\tilde{W})$. Now

$$\deg f|_Z = \deg \tilde{f}|_{\tilde{Z}}$$

since the regular maps f, $\tilde{f}$ agree on $X\setminus Z\approx \tilde{X}\setminus\tilde{Z}$. Since $\tilde{f}|_{\tilde{Z}}=f_*$ (followed by the embedding $\tilde{W}\hookrightarrow\tilde{Y}$), the desired equality

$$\deg \tilde{f}|_{\tilde{Z}} = \deg f_*$$

is equivalent to asserting that $\tilde{f}$ is unramified at (the generic point $\tilde{z}$ of) $\tilde{Z}$, i.e. that $d\tilde{f}_{\tilde{z}}$ is an isomorphism.

Let $L_{\tilde{z}}$ be the line in $N_{Z\setminus X,z}$ given by $\tilde{z}\in\tilde{Z}|_z$. We have the exact sequence

$$0\to T_{\tilde{z}}\tilde{Z}\to T_{\tilde{z}}\tilde{X}\to L_{\tilde{z}}\to 0$$

mapped by $d\tilde{f}_{\tilde{z}}$ to

$$0\to T_{\tilde{w}}\tilde{W}\to T_{\tilde{w}}\tilde{Y}\to L_{\tilde{w}}\to 0$$

where $\tilde{w}=\tilde{f}(\tilde{z})$. The induced map $L_{\tilde{z}}\to L_{\tilde{w}}$ is just the restriction to $L_{\tilde{z}}$ of $f_{*,z}$, injective by assumption. On the other hand, the map on $T_{\tilde{z}}\tilde{Z}$ is just

$$d(f_*)\colon T_{\tilde{z}}\tilde{Z}\to T_{\tilde{w}}\tilde{W}.$$

Now $\tilde{f}$ is proper and dominant since f is, hence $\tilde{f}$ is surjective, hence f_* is surjective since $\tilde{Z}=\tilde{f}^{-1}(\tilde{W})$. Therefore $d(f_*)$ must be an isomorphism at a generic $\tilde{z}\in\tilde{Z}$, and by the exact sequences so is $d\tilde{f}_{\tilde{z}}$. Q.E.D.

3.3. *Warning.* The lemma requires injectivity for *all* $z\in Z$, not only *generic* z. Otherwise $\tilde{f}$ is not regular on any neighborhood of $\tilde{Z}$, and could involve a blowup of some small-dimensional subvariety onto $\tilde{W}$, in which case $\tilde{Z}$ is only one of several components of the graph of $\tilde{f}$ over Z, hence possibly

$$\deg f_* < \deg \tilde{f}|_{\tilde{f}^{-1}(\text{neighborhood of } Z)} = \text{local degree of } f \text{ on } Z.$$

The connected component $\tilde{Z}$ in Part IV has two irreducible components and the conditions of Lemma 3.2 fail along the intersection, necessitating the blowdown in Part IV, § 4.

§ 4. The codifferential of Prym

Lemmas 3.1, 3.2 indicate that the first step towards understanding $\mathcal{P}$ is to compute its differential, or as is convenient in many variational problems, the dual map, the codifferential. Precisely this was done by Beauville in [B2], Proposition 7.5, for C, $\tilde{C}$ smooth.

Since $\mathcal{R}_g$ is an unramified cover of $\mathcal{M}_g$ near a smooth point $C \in \mathcal{M}_g$, we can identify $T^*\mathcal{R}_g$ at (C, η) with $T^*\mathcal{M}_g$ at C, hence with the space $H^0(C, \omega_C^2)$ of quadratic differentials on C, by standard identifications. Similarly $T^*\mathcal{A}_{g-1}$ at $A \in \mathcal{A}_{g-1}$ is identified with $S^2 T_0^* A$, the symmetric square of the cotangent space to A at the origin. When $A = \mathcal{P}(C, \eta)$ (η the line bundle of order 2 on C corresponding to the double cover $\tilde{C}$) there is the further identification

$$T_0^* A \approx H^0(C, \omega_C \otimes \eta)$$

which follows immediately from the definition of A as the skew-invariant subvariety of $J(\tilde{C})$, and the splitting

$$H^0(\tilde{C}, \omega_{\tilde{C}}) \approx H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta).$$

(Compare [M2].) Beauville's result is:

PROPOSITION 4.1. *Using these identifications, the codifferential*

$$\mathcal{P}^*: S^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2)$$

is just cup-product

$$S^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2 \otimes \eta^2)$$

followed by the identification

$$H^0(C, \omega_C^2 \otimes \eta^2) \xrightarrow{\approx} H^0(C, \omega_C^2)$$

deduced from the isomorphism

$$\eta^2 \approx \mathcal{O}_C.$$

4.2. *Pause.* As stated, this result holds for (automorphism-free) $C \in \mathcal{M}_g$, or $\tilde{C} \in \mathcal{R}_g$ over it. In Part IV we shall extend it to singular curves. Here is a pictorial interpretation which could motivate the generalization.

An object $X \in \mathcal{M}_g$ is represented uniquely up to projective automorphisms by the canonical image $\Phi(X)$ of X in $\mathbf{P}^{g-1} = \mathbf{P}(H^0(X, \omega_X)^*)$, unless X is "special" (=hyperelliptic). The analogous concept for $(C, \eta) \in \mathcal{R}_g$ is the Prym-canonical image $\Psi(C)$, namely the image of C in $\mathbf{P}^{g-2}$ given by the linear system $\omega_C \otimes \eta$.

Using this Prym-canonical map Ψ, the codifferential $\mathcal{P}^*$ as given in Proposition 4.1 is just the restriction

$$H^0(\mathbf{P}^{g-2}, \mathcal{O}(2)) \to H^0(C, \omega_C^2)$$

sending a quadric to the quadratic differential it cuts on $\Psi(C)$. In particular, ker $(\mathcal{P}^*)$ can be identified with the system of quadrics in $\mathbf{P}^{g-2}$ containing $\Psi(C)$.

To obtain an analogue for singular (C, η), one could study the limiting position of $\Psi(C_t)$ as $t \to 0$, where $\{C_t\}$ is a family of curves, smooth over a punctured neighborhood of 0, where $C_0 = C$. For Wirtinger double covers $C = X/(p \sim q)$ etc. (as in 2.3(3)) the limiting object is the canonical image $\Phi(X)$ together with a chord $\overline{\Phi(p), \Phi(q)}$ (the map Ψ is ill-defined at the double point, and it blows it up to this line) and for elliptic-tail curves $C = X \cup_p E$ (2.3(4)) it is $\Phi(X)$ together with its tangent line at $\Phi(p)$. This suggests what ker $(\mathcal{P}^*)$ should be in these cases, though the precise statements (IV.3.4, proved in 3.1–3.3, and Propositions IV.4.4.5, IV.4.3.1(iv) (explained and proved in IV.4.3.3)) are rather delicate.

4.3. We stress another analogy between canonical and Prym-canonical maps. The canonical map

$$\Phi: X \to \mathbf{P}^{g-1}$$

is the "derivative" of the Abel-map

$$\varphi: X \to J(X).$$

In other words, $H^0(X, \omega_X)^*$ can be identified with the tangent space to $J(X)$ at any point of $J(X)$, and the canonical image of $p \in X$ is the point of $\mathbf{P}^{g-1}$ given by the one-dimensional subspace of $T_{\varphi(p)}J(X)$ which is the tangent line, at $\varphi(p)$, to $\varphi(X)$. In the same sense, the Prym-canonical map

$$\Psi: C \to \mathbf{P}^{g-2}$$

is the derivative of the "Abel–Prym" map

$$\psi: \tilde{C} \to \mathcal{P}(C, \tilde{C}, \pi).$$

In other words, $H^0(C, \omega_C \otimes \eta)^*$ can be identified with the tangent space to $\mathcal{P}(\tilde{C})$ at any of its points, and the Prym-canonical image of $p \in C$ is the projectivized tangent direction to $\psi(\tilde{C})$ at either point of $\pi^{-1}(p)$.

Both of these facts follow trivially from the definitions, by differentiating the Abel (respectively Abel–Prym) map.

4.4. Pushing this analogy further, we arrive at the following result, which should have become a basic tool in Prym-theory.

Masiewicki's Universal Property. Let (P, Ξ) be a g-dimensional principally polarized abelian variety, and $\tilde{C}$ a symmetric curve in P representing the homology class $(2/(g-1)!)[\Xi]^{g-1}$. Let $C=\tilde{C}/i$ be the quotient of $\tilde{C}$ by the involution of P given by multiplication by -1. Then:

(1) $\pi: \tilde{C} \to C$ is allowable.

(2) $\mathcal{P}(C, \tilde{C}) \approx (P, \Xi)$.

(3) $\tilde{C} \hookrightarrow P$ is the Abel-Prym map.

This is a perfect analogue of Matsusaka's theorem, characterizing Jacobians as g-dimensional principally polarized abelian varieties (J, Θ) in which the homology class $1/(g-1)![\Theta]^{g-1}$ is represented by an (effective) curve.

Part II. Plane Quintics

For the component of plane quintic curves there is actually enough information in the literature to conclude that the local degree of the Prym map is one; we explain this below. With an eye to future applications, we also prove irreducibility of the family of even covers of plane quintics (Proposition 3.3) and describe the codifferential of $\mathcal{P}$ explicitly along it.

§ 1. Mumford's results

Let $\mathcal{R}'_Q$ be the subvariety of $\mathcal{R}_6$ parametrizing double covers of smooth plane quintics. It is a basic fact in the subject that R'_Q splits into two components (at least): we can designate each double cover as either "odd" or "even".

Indeed, even and odd theta-characteristics on arbitrary Jacobians have been distinguished since the birth of theta-function theory. In modern terms [M1] a theta-characteristic is a point of order 2 on the Jacobian $J^{g-1}(C)$ of line-bundles of degree $g-1$ on C, i.e. a line-bundle μ such that $\mu \otimes \mu = \omega_C$, the canonical class of C. (A semi-period is a point of order 2 on $J^0(C)$, the Jacobian of line bundles of degree 0 on C.) A theta-characteristic μ is even or odd according to the parity of $h^0(C, \mu)$.

Let C be a smooth plane quintic (or, with obvious modifications, a smooth plane curve of any odd degree). On C there is a natural theta-characteristic, given by the hyperplane class L. (By adjunction, ω_C is cut by conics.) Thus we can identify $\mathcal{R}'_Q$ with the family of theta-characteristics on plane quintics, and assign to η the parity of $L \otimes \eta$. Let

$\mathcal{R}_Q$ denote the even part of $\mathcal{R}'_Q$, and R_C the odd. In this section we are interested in $\mathcal{R}_Q$, leaving $\mathcal{R}_C$ to Part V (on cubic threefolds). The following is a special case of Mumford's theorem. ([M2], Section 7.)

PROPOSITION 1.1. *For* $(C,\eta)\in\mathcal{R}'_Q$, $\mathcal{P}(C,\eta)$ *is in the Andreotti–Mayer locus* N_1 *if and only if* η *is even, or* $(C,\eta)\in\mathcal{R}_Q$. *(A principally polarized abelian variety* (A,Θ) *is in* N_k *if the singular locus* Θ_{sing} *of* Θ *has dimension* $\geqslant k$.*)*

Moreover, the proof of the proposition shows that in this case Θ_{sing} can be identified with the double cover $\tilde{C}$ of C corresponding to η. The involution on Θ_{sing} giving the map to C is the restriction to Θ_{sing} of the involution on $\mathcal{P}(C,\eta)$ given by multiplication by -1 (when the origin is determined by the "Riemann constants"). Hence:

COROLLARY 1.2. $\mathcal{P}|_{\mathcal{R}_Q}$ *is bijective.*

§ 2. Tjurin's results

Tjurin [T1, T2] proved that the Jacobian of a generic $C\in\mathcal{M}_5$ is the Prym of a double cover in $\mathcal{R}_Q$. Masiewicki [Ma] extended the result to all $C\in\mathcal{M}_5$, allowing the double cover to be in $\bar{\mathcal{R}}_Q$, the closure of $\mathcal{R}_Q$ in $\bar{\mathcal{R}}_6$. The proof uses the maps

$$\alpha\colon \mathcal{M}_5\to\bar{\mathcal{R}}'_Q$$
$$\beta\colon \bar{\mathcal{R}}'_Q\to\mathcal{A}_5.$$

Here β is just the restriction to $\bar{\mathcal{R}}'_Q$ of $\mathcal{P}$. For $X\in\mathcal{M}_5$ (non-hyperelliptic and not trigonal), $\alpha(X)$ is the curve $\tilde{C}=\Theta_{\mathrm{sing}}$ in the Jacobian $J(X)$ of X, mapping two-to-one to

$$C=\Theta_{\mathrm{sing}}/(\pm 1)$$

as in Proposition 1.1. To realize C as a plane quintic, we note that points of Θ_{sing} correspond to g^1_4's on X (i.e. linear systems of degree 4 and projective dimension 1); and that when X is mapped canonically to $\mathbf{P}^4$, as the complete intersection of three quadrics, any g^1_4 on X is cut out by a 1-parameter family of planes sweeping out a quadric (of rank 3 or 4) in $\mathbf{P}^4$, containing X. Let Π be the abstract plane $\mathbf{P}^2$ parametrizing the quadrics containing X; the discriminant locus in Π, parametrizing singular quadrics, is a quintic curve (given by the vanishing of a 5×5 linear determinant). Now our $\tilde{C}$ maps to this quintic curve, by sending a g^1_4 to the point of Π parametrizing the singular quadric whose planes cut the g^1_4. The map is 2–1 (since a (generic) quadric in the family has rank 4 [AM] and this contains two plane-systems) and can be identified with the double cover

$$\pi\colon \tilde{C}\to\tilde{C}/(\pm 1)=C.$$

Remark 2.1. $\tilde{C}$ is smooth and π is unramified, for X generic. In fact both fail precisely when X posesses a "vanishing theta-null"; Masiewicki's work asserts that the resulting singular, ramified cover is still "allowable".

Tjurin's result states:

2.2. $\beta\circ\alpha=\mathrm{id}_{\mathcal{M}_5}$ (more precisely, this is the Jacobi map $\mathcal{M}_5\to\mathcal{J}_5\subset\mathcal{A}_5$).

We can reformulate Proposition 1.1 and Corollary 1.2:

2.3. $\beta^{-1}(N_1)=\mathcal{R}_Q$ and $\alpha\circ\beta|_{\bar{\mathcal{R}}_Q}=\mathrm{id}_{\bar{\mathcal{R}}_Q}$.

Let

$$\mathcal{R}_Q^0=\beta^{-1}(\mathcal{J}_5)$$
$$\beta_0=\beta|_{\mathcal{R}_Q^0}.$$

Combining 2.2, 2.3, we find:

2.4. $\mathcal{R}_Q^0=\beta^{-1}(\mathcal{J}_5)=\alpha(\mathcal{M}_5)$ is irreducible, and the maps α, β_0 are birational inverses of each other.

§ 3. Irreducibility

We prove that $\bar{\mathcal{R}}_Q=\mathcal{R}_Q^0$, i.e. $\mathcal{R}_Q$ is irreducible.

LEMMA 3.1. *Given $(C,\eta)\in\mathcal{R}_Q$ and a divisor*

$$D\in|\omega_C\otimes\eta|$$

there is a unique quartic curve Q intersecting C tangentially at points of D.

Proof. The restriction map:

$$H^0(\mathbf{P}^2, \mathcal{O}(4))\to H^0(C, \omega_C\otimes\omega_C)$$

is injective (since C is not contained in a quartic curve) between 15-dimensional vector spaces, hence an isomorphism. Thus there is a unique quartic cutting C in $2D$. Q.E.D.

LEMMA 3.2. *$h^0(C,L\otimes\eta)=0$, and the 10 points of D do not lie on a cubic. (L is the hyperplane class, cut on C by lines.)*

Proof. $h^0(C,L^3(-D))=h^0(C,L\otimes\eta)$ is even. On the other hand, $h^0(C,L^3(-D))=h^0(\mathbf{P}^2,L^3\otimes I_D)$ via restriction, and this number is $\leqslant 1$ since the intersection of two cubics has degree $9<\deg(D)$. Hence it is zero. (The two cubics cannot have a common component, for then nine of the points of D must lie on a conic, i.e. for some $p_1,p_2\in C$,

$$\mathcal{O}_C(D)\approx\mathcal{O}_C(2)(p_1-p_2)$$

but then

$$O_C(p_1-p_2) \approx \eta,$$

i.e.

$$O_C(2p_1) \approx O_C(2p_2).$$

Since a smooth plane quintic is not hyperelliptic, we must have $p_1=p_2$, i.e. $\eta \approx O_C$.) Q.E.D.

Note. This argument shows that for $(C, \eta) \in \mathcal{R}_C$, odd,

$$h^0(C, L \otimes \eta) = 1$$

and the points of any $D \in |\omega_C \otimes \eta|$ lie on a (unique) cubic.

PROPOSITION 3.3. *The space $\mathcal{R}_Q$ of even covers of plane quintics is irreducible.*

Proof. Since $h^0(C, \omega_C \otimes \eta)=5$ for *all* $(C, \eta) \in \mathcal{R}_Q$, the irreducibility is equivalent by Lemmata 3.1, 3.2 to that of the space of pairs (C, Q) consisting of a smooth plane quintic and a smooth plane quartic, intersecting tangentially in 10 points D, not on a cubic. By arguments analogous to Lemmata 3.1, 3.2 the divisor class [D] of D on Q satisfies

$$[D]^2 = L^5, \quad h^0(Q, [D] \otimes L^{-1}) = 4, \quad h^0(Q, [D] \otimes L^{-2}) = 0$$

(On Q, $L=\omega_Q$) and for any $D' \in |[D] \otimes L^{-1}|$, there is a unique cubic cutting Q (tangentially) in $2D'$. The irreducibility of $\mathcal{R}_Q$ thus reduces to that of the space of pairs (Q, C') of a quartic and an everywhere tangent cubic such that their 6 points of tangency do not lie on a conic. Repeating this argument once more, we end with the space of pairs (C', C'') of a cubic and a conic, meeting tangentially in 3 non-colinear points. This latter is indeed irreducible: having chosen the smooth conic C'', we are completely free to choose the 3 points on it, and for each choice we find a $\mathbf{P}^3$ of possibilities for C'. Q.E.D.

§ 4. Non-ramification

In view of Proposition I.4.1, the assertion that the local degree of $\mathcal{P}$ on $\mathcal{R}_Q$ equals 1 ("non-ramification") is equivalent to

PROPOSITION 4.1. *The Prym-canonical image $\Psi(C)$, for generic $(C, \eta) \in \mathcal{R}_Q$, is contained in no quadrics.*

In [B2], Beauville proved the following:

COROLLARY 4.2. *For $X \in \mathcal{M}_5$ (non-hyperelliptic), the Prym-canonical image of $\alpha(X)$ (=the even cover of a plane quintic, constructed from X in § 2) is contained in no quadrics.*

In fact, by the irreducibility of Proposition 3.3 the corollary implies the proposition. We give an independent proof of Proposition 4.1, similar to that of Proposition 3.3.

Proof. We want to show that the map

$$S^2H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^2)$$

is an isomorphism (of 15-dimensional vector spaces). Let $D \in |\omega_C \otimes \eta|$, Q the corresponding quartic as in Lemma 3.1. By genericity Q can be assumed smooth. (In particular, since a smooth plane quartic is embedded canonically in the plane, Q is non-hyperelliptic.) In the commutative diagram

$$\begin{array}{ccc} S^2H^0(\mathbf{P}^2, \mathcal{O}(4) \otimes I_D) & \longrightarrow & S^2H^0(C, \omega_C \otimes \eta) \\ \downarrow & & \downarrow \\ H^0(\mathbf{P}^2, \mathcal{O}(8) \otimes I_D^2) & \longrightarrow & H^0(C, \omega_C^2) \end{array}$$

(horizontal maps given by restriction) the top map is an isomorphism since quartics cut the complete linear system $|\omega_C^2|$ on C. The bottom is injective by Lemma 3.2, since an octic passing doubly through D and containing C must contain residually a cubic through D. We are left with proving that the vertical map on the left is an isomorphism. It fits into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^2, \mathcal{O}(4) \otimes I_D) & \xrightarrow{\otimes Q} & S^2H^0(\mathbf{P}^2, \mathcal{O}(4) \otimes I_D) & \longrightarrow & S^2H^0(Q, \mathcal{O}(4) \otimes I_D) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathbf{P}^2, \mathcal{O}(4) \otimes I_D) & \xrightarrow{\otimes Q} & H^0(\mathbf{P}^2, \mathcal{O}(8) \otimes I_D^2) & \longrightarrow & H^0(Q, \mathcal{O}(8) \otimes I_{2D}) & \longrightarrow & 0 \end{array}$$

(where the horizontal maps on the right are restriction to Q). So we are reduced to checking that the vertical map on the right is an isomorphism. This map is cup-product

$$S^2H^0(Q, \omega_Q \otimes \mu) \to H^0(Q, \omega_Q^3)$$

where μ is the theta-characteristic on Q such that $D \in |\mu \otimes \omega_Q^2|$. By Lemma 3.2 again, μ is not effective on Q. (Hence μ is an even theta-characteristic.) We conclude by:

LEMMA 4.3. *For a non-hyperelliptic $Q \in \mathfrak{M}_3$ and line bundle μ of degree 2 such that $h^0(Q, \mu) = 0$, the image $\chi(Q) \subset \mathbf{P}^3$ of Q under $\mu \otimes \omega_Q$ is contained in no quadric.*

Proof. (1) If $\chi(Q)$ is singular, with branches p_1, p_2 (distinct or coincident) through a singular point p, we compose χ with projection from p to obtain a map of Q to $\mathbf{P}^2$ of degree $\leqslant 4$; such a map must be the canonical map, so that

$$\mu \otimes \omega_Q(-p_1 - p_2) \approx \omega_Q$$

so μ is effective, contradiction.

(2) If $\chi(Q)$ is contained in a smooth quadric A, it has type (d, e) on it with $d+e=6$, but $d>2$, $e>2$ since Q is not hyperelliptic. Hence $d=e=3$, and the arithmetic genus of $\chi(Q)$ is

$$(d-1)(e-1)=4$$

so $\chi(Q)$ must be singular.

(3) If the quadric A is singular, it must be an ordinary cone with point-vertex p (since $\chi(Q)$ is not contained in any plane.) Upon projection from p, Q becomes a cover of a conic, hence of even degree, so the multiplicity at p is even, hence 0. Let L be a generator of the cone A (that is, a line through p in A) meeting $\chi(Q)$ in q_1, q_2, q_3. By projecting from q_3, we represent Q as a plane quintic with a unique singularity, a simple tacnode with branches q_1, q_2 (assuming $\chi(Q)$ smooth): this is since no line through q_3, other than L, meets A, hence $\chi(Q)$, elsewhere in more than one point or tangentially. The genus formula gives $g(Q)$ as 4, contradiction. Q.E.D.

COROLLARY 4.4. *Along the component of plane quintics, $\mathcal{P}$ has local degree* 1.

§ 5. The conormal map

We describe explicitly the codifferential of the Prym map. Let $J(X)=\mathcal{P}(C,\eta)$, $X\in\mathcal{M}_5$, $(C,\eta)\in\mathcal{R}_6$. Then the codifferential is

$$\mathcal{P}^*: T^*_{J(X)}\mathcal{A}_5\to T^*_{(C,\eta)}\mathcal{R}_6=T^*_C\mathcal{M}_6.$$

By standard identifications

$$\begin{aligned} T^*_{J(X)}\mathcal{A}_5 &= S^2H^0(X,\omega_X)\\ T^*_{(C,\eta)}\mathcal{R}_6 &= H^0(C,\omega_C^2)\end{aligned} \tag{5.1}$$

and $\mathcal{P}^*$ is the restriction map Ψ^{-1}, where $\Psi: C\to\mathbf{P}^4$ is the Prym-canonical map. Since the restriction of $\mathcal{P}$ (denoted above by β) maps $\mathcal{R}_Q$ to $\mathcal{J}_5$, there is an induced map

$$\beta^*: N^*_{J(X)}(\mathcal{M}_5\setminus\mathcal{A}_5)\xrightarrow{\sim} N^*_{(C,\eta)}(\mathcal{R}_Q\setminus\mathcal{R}_6) \tag{5.2}$$

on conormal bundles. Both of these are rank-3 bundles and β^*, being the restriction of $\mathcal{P}^*$, is injective, hence an isomorphism. According to [G], the conormal spaces are:

$$\begin{aligned} N^*_{J(X)}(\mathcal{M}_5\setminus\mathcal{A}_5) &= \operatorname{Ker}(S^2H^0(X,\omega_X)\to H^0(X,\omega_X^2))\\ N^*_{(C,\eta)}(\mathcal{R}_Q\setminus\mathcal{R}_6) &= \{\text{quadratic differentials cut on } C\subset\mathbf{P}^2 \text{ by}\\ &\qquad \text{those plane quartics which are polars of } C\\ &\qquad \text{with respect to points of } \mathbf{P}^2.\}\end{aligned} \tag{5.3}$$

(The second identity also follows from the result of our appendix, since for C near $\mathcal{M}_Q$, C is in $\mathcal{M}_Q$ if and only if it posesses a 1-parameter family of g_4^1's (obtained by projection from points of the plane quintic). Hence $T_C\mathcal{M}_Q$ is cut by the obstructions to deforming these g_4^1's on C; but the ramification divisor of each g_4^1 is cut on C by the corresponding polar quartic.)

We obtain the following description: the two maps

$$\begin{aligned} H^0(\mathbf{P}^4, \mathcal{O}(2)) &\xrightarrow{\approx} H^0(C, \omega_C^2) \\ H^0(\mathbf{P}^2, \mathcal{O}(4)) &\xrightarrow{\approx} H^0(C, \omega_C^2) \end{aligned} \tag{5.4}$$

are isomorphisms, setting an identification of quadrics in $\mathbf{P}^4$ with quartics in $\mathbf{P}^2$. To the $\mathbf{P}^2$ of quartics C_p polar to C with respect to some $p \in \mathbf{P}^2$ corresponds a net Π of quadrics in $\mathbf{P}^4$, with an isomorphism

$$\gamma: \Pi \to \mathbf{P}^2.$$

Since $(\mathcal{D}^*)^{-1}(N^*_{(C,\eta)}(\mathcal{R}_Q \setminus \mathcal{R}_6)) = N^*_{J(X)}(\mathcal{M}_5 \setminus \mathcal{A}_5)$ by (5.2), and using (5.3) we recover X as the base locus of the linear system of quadrics Π; γ is (the projectivization of) β^*. A priori we have two maps

$$\begin{aligned} \gamma&: \Pi \to \mathbf{P}^2 \\ \delta&: \Pi \to \mathbf{P}^2 \end{aligned}$$

where $\gamma(q) = p$ if the quadric A_q corresponding to $q \in \Pi$ satisfies

$$\Psi^{-1}(A_q) = C \cap C_p$$

while δ gives the identification of Π with $\mathbf{P}^2$ used to define C in $\mathbf{P}^2$.

LEMMA 5.5. $\gamma = \delta$; *in particular*, $\gamma(q) = p \in C$ *if and only if* A_q *is singular (at the point* $\Psi(p) \in \mathbf{P}^4$).

Proof. If A_q is singular, it is singular at $\Psi(\delta(q))$. Then

$$2\delta(q) \leqslant \Psi^{-1}(A_q) = C \cap C_{\gamma(q)}$$

hence $\gamma(q) = \delta(q)$ and is on C. Thus

$$\{q \mid A_q \text{ singular}\} \subset \gamma^{-1}(C)$$

since both sides are quintics we have equality. Since γ, δ agree on $\gamma^{-1}(C)$ and are linear, they coincide. Q.E.D.

Part III. Trigonal curves

§ 1. Surjectivity and irreducibility

We work in arbitrary genus following the work of Recillas [R] and extending it "to the boundary". Let X be a curve of genus $g-1$ which is a four-sheeted branched covering of $\mathbf{P}^1$ ("tetragonal curve")

$$f\colon X \to \mathbf{P}^1.$$

For now, assume f has only simple ramification points. Let

$$\tilde{C} = S^2_{\mathbf{P}^1} X \tag{1.1}$$

be the relative second symmetric product of X over $\mathbf{P}^1$, with induced map

$$f^{(2)}\colon \tilde{C} \to \mathbf{P}^1$$

of degree 6. If for $p \in \mathbf{P}^1$, $f^{-1}(p) = \{a, b, c, d\}$ then

$$(f^{(2)})^{-1}(p) = \{\{ab\}, \{ac\}, \{ad\}, \{bc\}, \{bd\}, \{cd\}\}. \tag{1.2}$$

On $\tilde{C}$ there is a natural involution σ, sending a pair of points over p to the complementary pair in $f^{-1}(p)$. Let

$$C = \tilde{C}/\sigma$$

with induced map

$$f_*\colon C \to \mathbf{P}^1$$

of degree 3. In the above notation,

$$(f_*)^{-1}(p) = \{\{ab\} \sim \{cd\}, \{ac\} \sim \{bd\}, \{ad\} \sim \{bc\}\}.$$

The following facts are obvious:

1.3. (1) C is trigonal.

(2) The natural map $\pi\colon \tilde{C} \to C$ is an unramified double cover ($\Leftrightarrow \sigma$ is fixed point free).

(3) The branch points of f_* are precisely those of f; hence by the Riemann–Hurwitz formula, C has genus g.

1.4. We analyze the possible degenerations of Recillas' construction.

(0) For a generic four-sheeted cover f, with $2g+4$ distinct simple ramification points, 1.3 applies.

(i) A double ramification point of f. This means 3 sheets of X "come together" over $p \in \mathbf{P}^1$, with the local monodromy around p permuting them cyclically. Locally over p,

$\tilde{C}$ has 2 disjoint irreducible components, each consisting of 3 sheets joined in a double ramification point. σ interchanges these components, hence C has a double ramification point over p, and 1.3 still holds.

(ii) Two distinct simple ramification points mapping to the same branch point $p \in \mathbf{P}^1$. The local monodromy around p acts on the fiber $\{a, b, c, d\}$ near p by, say, the permutation

$$(a, b)(c, d).$$

Locally over p, $\tilde{C}$ has 4 distinct irreducible components: the pair $\{ab\}$ (respectively $\{cd\}$) forms a 1-sheeted local component, while $\{ac\}$ and $\{bd\}$ (respectively $\{ad\}$ and $\{bc\}$) join in a 2-sheeted component with simple ramification. Moreover, these last 2 components intersect in the ramification point. σ interchanges the 1-sheeted components and acts on each of the 2-sheeted components by interchanging their sheets. Hence C has 3 irreducible sheets near p: $\{ab\} \sim \{cd\}$ is covered by the 2 disjoint sheets $\{ab\}$, $\{cd\}$ of $\tilde{C}$, while the other 2, $\{ac\} \sim \{bd\}$ and $\{ad\} \sim \{bc\}$, intersect over p and the map $\pi: \tilde{C} \to C$ is a double cover of each, branched over p. By I.1.3, the double cover π is still allowable.

(iii) At worst, f could have a triple ramification point, with all 4 sheets coming together, with local monodromy

$$(a, b, c, d).$$

The induced monodromy on $\tilde{C} = S^2_{\mathbf{P}^1} X$ is

$$(\{ab\}, \{bc\}, \{cd\}, \{da\})(\{ac\}, \{bd\})$$

so $\tilde{C}$ has locally two intersecting components of degrees 4, 2. σ acts on each, so that C has two local components of degrees 2, 1 over $\mathbf{P}^1$; the two intersect over p, and $\pi: \tilde{C} \to C$ is a branched double cover of each. We thus combine the features of (i) and (ii): f_* is simply ramified, and π is simply singular-ramified, hence allowable. In both this and the previous case, C is trigonal and singular, of arithmetic genus g. (Its normalization has genus $g-1$.)

(iv) For computing the local degree of $\mathcal{D}$ we need consider only generic (tetragonal) curves X. Hence we study only one example where X itself degenerates. Namely, let X_0 be a trigonal curve, with trigonal divisor class T. Let $\{X_t\}$ be a smooth family of curves equipped with tetragonal divisor-classes D_t, and assume the limit divisor-class D_0 exists and equals $T + [p_0]$ for some $p_0 \in X_0$. In the limiting 4-sheeted cover

$$f: X \to \mathbf{P}^1$$

we have $X = X_0 \cup \mathbf{P}^1$, $f_0 = f|_{X_0}$ the trigonal map, $f_1 = f|_{\mathbf{P}^1}$ the identity, and $X_0 \cap \mathbf{P}^1 = \{p_0\}$. The resulting C is just X_0; $\tilde{C} = X_0' \cup X_0''$ with 2 pairs of points identified:

$$p_1' \sim p_2'', \; p_1'' \sim p_2'$$

where p_1', p_1'' are the points of X_0', X_0'' corresponding to p_1 (and similarly for p_2), and

$$\{p_0, p_1, p_2\} = f_0^{-1}(p) = f_0^{-1}(f_0(p_0)).$$

We see that even in this far-out degeneration, the double cover $\pi: \tilde{C} \to C$ is allowable. (In fact is of "Wirtinger type" I.2.3(3).)

The foregoing allows us to construct a map

$$\tau: \mathcal{J}^1_{4,g-1} \to \bar{\mathcal{R}}_g$$

from the space of curves X of genus $g-1$ with a marked g^1_4, to $\bar{\mathcal{R}}_g$. We claim $\mathcal{P}\circ\tau =$ Jacobi map $X \to J(X)$.

PROPOSITION 1.5. *For* $(\pi: \tilde{C} \to C) = \tau(X)$ *as above,*

(1) $\mathcal{P}(\tau(X)) \approx J(X)$ *(as principally polarized abelian varieties).*

(2) *The map* $\psi: \tilde{C} \to J(X)$ *given by*

$$\psi(\{ab\}) = \varphi(a) + \varphi(b)$$

(where $\varphi: X \to J(X)$ *is the Abel map) is the Abel–Prym map.*

Proof. We want to use the universal property I.4.4 of Prym varieties. $J(X)$ has the correct dimension $g-1$, and the image $\psi(\tilde{C})$ is symmetric since if $f^{-1}(p) = \{a, b, c, d\}$, then

$$\psi(\{ab\}) + \psi(\{cd\}) = \varphi(a) + \varphi(b) + \varphi(c) + \varphi(d)$$

depends only on $p \in \mathbf{P}^1$, hence is constant in $J(X)$. Hence we are reduced to showing that the homology class $\in H_2(J(X), \mathbf{Z})$ of $\psi(\tilde{C})$ is $(2/(g-2)!)\,\Theta^{(g-2)}$ where $\Theta = \Theta(X)$ is the principal polarization of $J(X)$. The class of $\varphi(X)$ is $(1/(g-2)!)\,\Theta^{(g-2)}$, suggesting we use a degeneration argument: Let X degenerate to a trigonal curve as in 1.4(iv). (This can always be achieved.) Since $H_2(J(X), \mathbf{Z})$ is discrete, the class of $\psi(\tilde{C})$ does not change in the degeneration. In the limit, $\tilde{C} = X_0' \cup X_0''$, and clearly ψ maps one component isomorphically to $\varphi(X_0)$, the other to its mirror-image. Q.E.D.

1.6. Let $\mathcal{R}_{T,g}$ be the subvariety of $\mathcal{R}_g$ parametrizing double-covers of trigonal curves, $\bar{\mathcal{R}}_{T,g}$ its closure in $\bar{\mathcal{R}}_g$. Before computing the degree of $\mathcal{P}$ on $\bar{\mathcal{R}}_T$ (we omit the genus for brevity) we sketch a construction showing that the image of τ is all of $\bar{\mathcal{R}}_T$, so we do not have to worry about $J(X)$ arising as a Prym in ways other than Proposition 1.5.

Start with a double cover

$$\pi: \tilde{C} \to C$$

of a trigonal curve C; for simplicity we may assume C, $\tilde{C}$ smooth, π unramified. Thus $\tilde{C}$ is a $\mathbf{Z}/2$-bundle over C, and the 3–1 map

$$f: C \to \mathbf{P}^1$$

yields a $(\mathbf{Z}/2)^3$-bundle $f_*\tilde{C}$ over $\mathbf{P}^1$. The involution

$$\sigma: \tilde{C} \to \tilde{C}$$

induces an involution σ_* on $f_*\tilde{C}$; let X be the quotient

$$X = f_*\tilde{C}/\sigma_*$$

since $f_*\tilde{C}$ is an 8-sheeted cover of $\mathbf{P}^1$, X comes with a 4–1 map to $\mathbf{P}^1$ and is tetragonal. One verifies directly that $\tau(X)$ can be identified with

$$\pi: \tilde{C} \to C.$$

COROLLARY 1.7. *$\bar{R}_T$ is irreducible.*

Proof. The irreducibility of $\mathcal{J}_4^1$ is well-known [EC].

§ 2. Projective lemmata

We need some lemmata on the projective geometry of a Prym-canonical trigonal curve. Assume $g \geqslant 6$. We fix a smooth, unramified double cover $\pi: \tilde{C} \to C$ of a trigonal curve of genus g and the quadrigonal curve X such that $\tau(X) = (\pi: \tilde{C} \to C)$. Also assume X neither hyperelliptic, trigonal nor (when $g = 7$) a smooth plane quintic. Let $\Phi: X \to \mathbf{P}^{g-2}$, $\Psi: C \to \mathbf{P}^{g-2}$ be the canonical, Prym-canonical maps.

LEMMA 2.1. (1) *In $\mathbf{P}^{g-2}$, the images of the 4 points $a, b, c, d \in X$ of each divisor $D \in g_4^1$ are coplanar.*

(2) *In each of these planes the 3 points of intersection of opposite lines $\overline{ab} \times \overline{cd}$, $\overline{ac} \times \overline{bd}$, $\overline{ad} \times \overline{bc}$ are on $\Psi(C)$, and as D varies in g_4^1 they trace $\Psi(C)$ once.*

Proof. (1) follows from Riemann–Roch for X.

(2) The diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{d\psi} & T_0\mathcal{P}(\tilde{C}) \setminus \{0\} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\text{projection}} \\ C & \xrightarrow{\Psi} & \mathbf{P}^{g-2} \end{array}$$

commutes by I.4.3, where ψ is the Abel–Jacobi map. Let

$$p = (\{ab\} \sim \{cd\}) \in C$$

$$\pi^{-1}(p) = \{\tilde{p}, \sigma\tilde{p}\} \quad \tilde{p} = \{ab\} \quad \sigma\tilde{p} = \{cd\}.$$

By Proposition 1.5(2),

$$\Psi'(p) = \mathrm{proj}\ (d\psi(\tilde{p})) = \mathrm{proj}\ (d\varphi(a) + d\varphi(b))$$

so the point $\Psi'(p)$ is on the line in $\mathbf{P}^{g-2}$ joining $\Phi(a)$, $\Phi(b)$. (Using I.4.3 again, this time for the canonical map.) The same argument applied to $\sigma\tilde{p}$ shows that $\Psi'(p)$ is on the line $\overline{\Phi(c), \Phi(d)}$. Q.E.D.

LEMMA 2.2. *A point $a \in X$ is contained in a unique $\mathbf{P}^2$ of our family.*

Proof. If a is contained in the plane spanned by a divisor $D \in g^1_4$ other than the unique D containing a, then $\Phi(X)$ has 5 coplanar points, or a g^2_5 by Riemann–Roch. Hence X is a plane quintic, and a smooth one since we are assuming X non-trigonal. Q.E.D.

LEMMA 2.3. *The intersection in $\mathbf{P}^{g-2}$ of $\Phi(X)$, $\Psi'(C)$ consists of $2g+4$ points. On C, respectively X, these are the ramification points of the trigonal, respectively tetragonal, map.*

Proof. Consider a plane Π spanned by the 4 points a, b, c, d of a divisor $D \in g^1_4$ on X. By Riemann–Roch, no 3 of the points can be colinear if the 4 are pairwise distinct, and an analogous statement holds if 2, 3, or two pairs of the points coincide (namely, if $D = 2a + b + c$, then b, c are not on the tangent line to X at a; etc.). By the analysis in 1.4, only simple or double ramification points occur for C smooth.

If the 4 points are distinct, they differ from the 3 points of intersection of opposing lines joining them. By Lemma 2.2, they also differ from the points of $\Psi'(C)$ in any plane other than Π.

If in the plane Π_0 the points a, b coincide while c, d remain separate, the line $\overline{ab}$ is replaced by $T_a X$, which by non-colinearity does not pass through c or d. The points

$$\overline{ac} \times \overline{bd}, \quad \overline{ad} \times \overline{bc}$$

both tend to the point $a = b$, while the third point

$$\overline{ab} \times \overline{cd}$$

remains apart. Thus we are at a simple ramification point of the trigonal map on C, and the points in $\mathbf{P}^{g-2}$ do coincide. By continuity, the same coincidence takes place if the ramification in Π_0 is higher. Q.E.D.

We are left with the sticky question of the multiplicity of intersection of $\Psi(C)$, $\Phi(X)$ at the ramification points.

LEMMA 2.4. (1) *Φ is an embedding (unless X is hyperelliptic).*

(2) *Ψ is an embedding unless C has a g_4^1 two of whose divisors have 2 coinciding pairs of points each, i.e. there are 4 points p, q, r, s on C such that $2p+2q\sim 2r+2s$, $\{p, q\}\neq\{r, s\}$.*

Proof. (1) is standard. For (2), Ψ fails to be an embedding if and only if there are 2 points $p, q\in C$ (distinct or not) such that

$$h^0(C, \omega_C\otimes\eta)-h^0(C, \omega_C\otimes\eta(-p-q))<2$$

or equivalently

$$h^0(C, \eta(p+q))>0$$

so there should exist r, s such that

$$\eta\approx O(p+q-r-s)$$

and the lemma follows since $\eta^2\approx O$. Q.E.D.

Remark 2.4.1. We shall see that when $g=6$ and X is generic, there are infinitely many corresponding C and they *all* satisfy the condition of Lemma 2.4. For the rest of § 2 we assume Ψ is an embedding so $\Psi(C)$ is smooth. We compute the intersection-multiplicities separately in cases (0), (i), (ii), of 1.4. These "codimension one" phenomena are the only ones we shall need.

LEMMA 2.5. *Assume the $2g+4$ ramification points (of C, X) are simple (=distinct). Then at each of these, the tangent lines to C, X lie in the limiting plane Π_0 and are distinct. (Hence C, X intersect transversely.)*

Proof. By assumption $\Psi(C)$, $\Phi(X)$ are smooth, and they each intersect Π_0 at one point less than they should, hence the first statement.

Let Π_t be the family of planes spanned by the $D_t\in g_4^1$ near Π_0, and consider the degenerating family of complete quadrilaterals. Using a generic projection (from a fixed linear subspace disjoint from Π_0 and of complementary dimension) we identify all the Π_t, for t near 0, with a fixed quotient plane. This converts the question to one about moving points in the plane: Let a, b, c, d be four points in the plane, with affine coordinates

$$a(t)=(Z, Z)\quad b(t)=(-Z, -Z)\quad C(t)=(0, 1)\quad d(t)=(1, 0) \tag{2.5.1}$$

where $Z^2=t$; describe the limit quadrilateral as $t\to 0$. (Since we are only asking for the limiting lines, we need consider, in a, b, c, d, only the leading terms in Z. A moment's reflection shows that coordinates can be chosen to give (2.5.1).)

Solving the linear equations, we find

$$\begin{aligned}\overline{ab}\times\overline{cd} &= (\tfrac{1}{2}, \tfrac{1}{2})\\ \overline{ac}\times\overline{bd} &= (2Z^2+Z, 2Z^2-Z)\\ \overline{ad}\times\overline{bc} &= (2Z^2-Z, 2Z^2+Z)\end{aligned}$$

as $t\to 0$ (hence $Z\to 0$) the points a, b tend to $(0, 0)$ with tangent direction $(1, 1)$. Of the three cross points, the first remains fixed while the other two tend to $(0, 0)$ with tangent direction $(1, -1)$, transversal to $(1, 1)$ as required. (For what it is worth, this shows that in the limit the four coplanar lines $\overline{ac}$, $\overline{ad}$, T_aC, T_aX are harmonic, i.e. have cross-ratio -1.)

Q.E.D.

LEMMA 2.6. *If the quadrigonal map on X has a double ramification point, so does the trigonal map on C, and their images in $\mathbf{P}^{g-2}$ have contact of order precisely 2 there.*

Proof. The first statement is 1.4(i). Clearly $\Psi(C)$, $\Phi(X)$ meet tangentially (contact $\geqslant 2$) at the images of the ramification points, and the subtle point is to check that contact is not higher. Since projection can only increase the order of contact, we can follow the procedure of Lemma 2.5. This time 3 of the 4 points come together, and we can choose affine coordinates in the plane so that, ignoring higher terms,

$$a(t)=(Z, Z^2)\quad b(t)=(\omega Z, \omega^2Z^2)\quad c(t)=(\omega^2 Z, \omega Z^2)\quad d(t)=(0, 1), \tag{2.6.1}$$

where $Z^3=t$, $\omega^3=1$, $\omega\neq 1$.

Now the cross-points are:

$$\begin{aligned}\overline{ab}\times\overline{cd} &= \left(\omega^2 Z\cdot\frac{1+\omega Z^2}{1-2\omega Z^2}, -\omega Z^2\frac{2-\omega Z^2}{1-2\omega Z^2}\right)\\ \overline{ac}\times\overline{bd} &= \left(\omega Z\cdot\frac{1+\omega^2 Z^2}{1-2\omega^2 Z^2}, -\omega^2 Z^2\cdot\frac{2-\omega^2 Z^2}{1-2\omega^2 Z^2}\right)\\ \overline{ad}\times\overline{bc} &= \left(Z\cdot\frac{1+Z^2}{1-2Z^2}, -Z^2\cdot\frac{2-Z^2}{1-2Z^2}\right).\end{aligned}$$

We see that up to second order, the points a, b, c tend to $(0, 0)$ along the parabola $y=x^2$, while the cross-points tend to $(0, 0)$ along $y=-2X^2$. These two parabolas have intersection-number 2 at the origin (linear terms agree, quadratics differ) hence the same applies to $\Psi(C)$, $\Phi(X)$.

Q.E.D.

The case 1.4(ii) of two distinct ramification points over one branch point, requires special consideration. The linear system of Prym-differentials on the singular curve C has the double-point of C as base-point; hence the Prym-canonical map Ψ is not defined at the double point. A more careful analysis shows that this point "blows-up" to a line l; Ψ lifts to a smooth map of the normalization NC of C into $\mathbf{P}^{g-2}$, and l is the chord of $\Psi(NC)$ joining the 2 points of NC over the double point. The same geometric picture arises if we fix an X with a one-parameter family of g_4^1's, one of which is special in the sense of 1.4(ii). The corresponding one-parameter family of trigonal curves C is described in Lemma 2.1(2). The limit curve corresponding to the special g_4^1 consists of $\Psi(NC)$ (of degree $2g-4$) plus twice the line l in the special plane. All that we shall actually need is contained in the

LEMMA 2.7. *In case* 1.4(ii), $\Psi(NC)$ *is of degree* $2g-4$. *It is the closure in* $\mathbf{P}^{g-2}$ *of the points obtained via Lemma* 2.1(2) *outside the special plane. In the special plane,* $\Psi(NC)$ *does not meet* $\Phi(X)$.

Proof. All is clear except for the last statement. We can take

$$a(t)=(Z,0)\quad b(t)=(-Z,0)\quad c(t)=(Z,1)\quad d(t)=(-Z,1)\quad (Z^2=t)$$

up to higher terms; the cross points are

$$\begin{aligned}\overline{ab}\times\overline{cd}&=(\infty,0)\\ \overline{ac}\times\overline{bd}&=(0,\infty)\\ \overline{ad}\times\overline{bc}&=(0,\tfrac{1}{2})\end{aligned}$$

and these are different than $a(0)=b(0)=(0,0)$, $c(0)=d(0)=(0,1)$. Q.E.D.

§ 3. The degree

We now specialize to the case $g=6$, and compute the local degree of the Prym map on the trigonal locus. We work over a generic curve $X\in\mathcal{M}_5$. In particular we assume X is neither hyperelliptic nor trigonal, and does not have a vanishing even theta-null. By Remark II.2.1, this guarantees that $\alpha(X)=\Theta_{\text{sing}}$ is a smooth curve of genus 11 with a natural involution, such that the quotient is a smooth plane quintic curve. By the discussion in Part II, § 2, X has a 1-parameter family of g_4^1's, parametrized by Θ_{sing}. These g_4^1's correspond to plane-families on singular quadrics containing $\Phi(X)$: such a quadric is a point-cone over a smooth (no vanishing thetanulls!) quadric surface in $\mathbf{P}^3$, thus contains two families of planes, each cutting a g_4^1 on X. Thus the restricted Prym map

$$\mathcal{P}:\bar{\mathcal{R}}_T\to\mathcal{J}_5\subset\mathcal{A}_5$$

has 1-dimensional fibers.

LEMMA 3.1. *$\omega_C \otimes L^{-2}$ is the only base-point-free g^1_4 on a trigonal curve (C, L) of genus* 6.

Proof. Let M be a base-point-free g^1_4 on C. Map C into $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ using the two maps $C \to \mathbf{P}^1$ given by sections of L, M. The resulting curve C is of type (3, 4) on a smooth quadric in $\mathbf{P}^3$. Since the degrees 3, 4 are relatively prime, the map does not factor through a quotient of C. The image of C is smooth since its arithmetic genus

$$(4-1)\cdot(3-1)=6$$

equals its geometric genus. By adjunction, ω_C is cut on C by curves of type (1, 2) on the quadric, i.e.

$$\omega_C \approx M \otimes L^2. \qquad \text{Q.E.D.}$$

3.2. We can now prove Remark 2.4.1. In fact for

$$C \in \mathcal{P}^{-1}(X) \subset \bar{\mathcal{R}}_T$$

with trigonal bundle L, there is not even one pair $p, q \in C$ such that the unique g^1_4 on C (Lemma 3.1) has a divisor $2p+2q$. Otherwise

$$O_C(2p+2q) \approx \omega_C \otimes L^{-2}$$

so

$$L' = L(p+q)$$

satisfies

$$(L')^2 \approx \omega_C$$

and

$$h^0(C, L') = 2$$

($h^0(C, L') \geqslant h^0(C, L) = 2$, while if $h^0(C, L') \geqslant 3$ then C is a plane quintic, hence possesses an infinity of tetragonal bundles, contradicting Lemma 3.1.) This means that L' is a vanishing (even) thetanull on C. The Schottky relations ([M2, § 5, Cor. 2], [F], [FR], [Sk]) then imply that X must also have a vanishing (even) thetanull, contradicting our genericity assumption on X.

A simple dimension count shows that only an 11-dimensional family of curves $X \in \mathcal{M}_5$ can be represented as a 4-sheeted branched cover of $\mathbf{P}^1$ with a total ramification point. Hence for the generic X case 1.4(iii) will not arise.

Along the lines of Part I, § 3 we compute the local degree by blowing up $\mathcal{A}_5$ along $\mathcal{J}_5$ and $\bar{\mathcal{R}}_6$ along $\bar{\mathcal{R}}_T$. We have the diagram

$$\begin{array}{ccc} \tilde{\mathcal{P}}: \tilde{\mathcal{R}}_6 & \longrightarrow & \tilde{\mathcal{A}}_5 \\ \downarrow & & \downarrow \\ \mathcal{P}: \bar{\mathcal{R}}_6 & \longrightarrow & \mathcal{A}_5 \end{array}$$

and its restriction to the exceptional loci:

$$\begin{array}{ccc} \tilde{\mathcal{D}}_e = \tilde{\mathcal{R}}_T & \longrightarrow & \tilde{\mathcal{M}}_5 \\ \downarrow & & \downarrow \\ \mathcal{D}_e = \bar{\mathcal{R}}_T & \longrightarrow & \mathcal{M}_5 \end{array}$$

where $\tilde{\mathcal{M}}_5$ is a $\mathbf{P}^2$-bundle over $\mathcal{M}_5$ (projectivized normal bundle of $\mathcal{M}_5$ in $\mathcal{A}_5$) and $\tilde{\mathcal{R}}_T$ a $\mathbf{P}^1$-bundle over $\bar{\mathcal{R}}_T$. In view of Lemma I.3.1, the main ingredient is the computation of the codifferential

$$\mathcal{D}^*\colon T^*\mathcal{A}_5 \to T^*\bar{\mathcal{R}}_6$$

which by Proposition I.4.1 is cup product

$$S^2H^0(C, \omega_C\otimes\eta) \to H^0(C, \omega_C^2).$$

Our purpose is to show dim (ker ($\mathcal{D}^*$)) is as small as possible.

PROPOSITION 3.3. Ker ($\mathcal{D}^*$) *is one dimensional, corresponding to the unique quadric in* $\mathbf{P}^4$ *containing* ($\Phi(X)$ *and) the family of planes cutting the given* g_4^1 *on* X.

Proof. Ker ($\mathcal{D}^*$) consists of those quadrics in $\mathbf{P}^4$ which contain $\Psi(C)$. Clearly the quadric Q, one of whose (two) families of planes cuts g_4^1 on X, contains $\Psi(C)$ since $\Psi(C)$ is contained in the union of these planes by Lemma 2.1(2). It is also clear that any quadric in ker ($\mathcal{D}^*$) contains $\Phi(X)$.

Assume $\Psi(C)\subset Q\cap Q'$, and let Q'' be a third quadric such that

$$\Phi(X) = Q\cap Q'\cap Q''.$$

In case 1.4(0), $\Psi(C)$ has degree

$$2g-2 = 10$$

hence

$$\Psi(C)\cap\Phi(X) = \Psi(C)\cap Q''$$

has degree 20. This contradicts Lemmata 2.3, 2.5 since the trigonal map has only

$$2g+4 = 16 < 20$$

ramification points.

In case 1.4(i), we obtain the same contradiction: two of the ramification points come together, and at the resulting point the contact is only of order 2, thus $\#(\Psi(C)\cap\Phi(X))$ is still $16<20$.

By 3.2, the only remaining case is 1.4(ii). In this case we are dealing with singular curves; luckily we are able to avoid the delicate problems arising in Part IV by the following reasoning. If Q' is in ker $(\mathcal{P}^*)$, it must clearly contain $\Psi(NC)$. The degree of this curve is 8, so

$$\#\Psi(NC) \cap \Phi(X) = \#\Psi(NC) \cap Q'' = 16$$

but by Lemma 2.7, two of the 16 ramification points do not contribute to the intersection while the others contribute exactly one intersection each, hence,

$$\#\Psi(NC) \cap \Phi(X) = 14 < 16$$

completing our proof. (The subtlety in Part IV arises since the condition "Q' contains $\Psi(NC)$" does *not* suffice to determine Q' in the situation there, and it is necessary to consider higher-order information, centered at the double point.) Q.E.D.

THEOREM 3.4. *The local degree of $\mathcal{P}$ on the trigonal locus is* 10.

Proof. By Lemma I.3.1, $\tilde{\mathcal{P}}$ is a well-defined regular map, and $\tilde{\mathcal{P}}_e$ is the projectivization of its differential $\mathcal{P}_*$. Indeed Proposition 3.3 shows that ker $(\mathcal{P}_*)$ is one-dimensional, hence contained in the tangent space $T(\bar{\mathcal{R}}_T)$, so $\mathcal{P}_*$ is injective on the normal bundle. By Lemma I.3.2 the required local degree equals that of

$$\tilde{\mathcal{P}}_e\colon \tilde{\mathcal{R}}_T \to \tilde{\mathcal{M}}_5.$$

For generic $X \in \mathcal{M}_5$, let $\mathbf{P}^2$ denote the fiber of $\tilde{\mathcal{M}}_5$ over X, and $\mathcal{R}$ its inverse image in $\tilde{\mathcal{R}}_T$. We proceed to describe the map

$$\tilde{\mathcal{P}}_e\colon \mathcal{R} \to \mathbf{P}^2.$$

We use the identifications made in Part II, § 5. Our $\mathbf{P}^2$, being the projectivized normal (rather than conormal) space, is dual to the plane Π containing the plane quintic $F = \alpha(X)/(\pm 1)$. Thus a point of $\mathbf{P}^2$ is a line in Π, or a pencil of quadrics containing $\Phi(X)$, and vice versa: a line in $\mathbf{P}^2$ corresponds to a quadric Q containing $\Phi(X)$. By Lemma II.5.4, the quadric Q is singular if and only if the point $p \in \Pi$ dual to l is in the plane quintic F. $\mathcal{R}$ is a $\mathbf{P}^1$-bundle over $\alpha(X) = \Theta_{\text{sing}}(X)$, the curve of genus 11 which is the double cover of the quintic curve F. Let $C \in \alpha(X)$, that is, C is a trigonal curve (in $\bar{\mathcal{M}}_6$) together with a double-cover, whose Prym is $J(X)$. The restricted map

$$\tilde{\mathcal{P}}_e\colon \mathbf{P}^1 \to \mathbf{P}^2$$

(on the fiber $\mathbf{P}^1$ of $\mathcal{R}$ over C) is injective, and its image is the line in $\mathbf{P}^2$ corresponding to the singular quadric Q in ker $(\mathcal{P}^*|_C)$. (Since $\tilde{\mathcal{P}}_e$ is the projectivization of $\mathcal{P}_*$, its image is the dual of ker ($\mathcal{D}^*$.) Now,

$$\begin{aligned}\text{degree } \tilde{\mathcal{D}}_e &= \#\tilde{\mathcal{D}}_e^{-1}(p) \quad (p \text{ generic point } \in \mathbf{P}^2)\\ &= \#\{C\in\alpha(X)\,|\,\tilde{\mathcal{D}}_e(\mathbf{P}^1_C)\ni p\}\\ &= \#\{C\in\alpha(X)\,|\,l(p)\ni\pi(C)\}\end{aligned}$$

where $l(p)$ is the line in Π dual to p, and

$$\pi\colon \alpha(X)\to F$$

is the double cover $\alpha(X)\to\alpha(X)/(\pm 1)$. Therefore,

$$\begin{aligned}\text{degree } \tilde{\mathcal{D}}_e &= \text{degree } (\alpha(X)\to\Pi)\\ &= (\text{degree } \pi)\cdot(\text{degree } F)\\ &= 2\cdot 5 = 10. \qquad \text{Q.E.D.}\end{aligned}$$

§ 4. Refinements

4.1. In the proof of Theorem 3.4 we did not need a description, in geometric terms, of the normal bundle to $\mathcal{R}_T$ in $\mathcal{R}_6$ (and the fiber $\mathbf{P}^1$ of $\mathcal{R}$ over C, etc.). Such a description is readily available, though, and helps to complete our picture of the Prym map near the trigonals.

By the result of the appendix, the conormal space at C to $\mathcal{R}_T$ in $\mathcal{R}_6$, as a subspace of the cotangent space to $\mathcal{R}_6$, is the vector space of quadratic differentials on C which vanish on the ramification divisor R of the trigonal map on C,

$$N^*_{\mathcal{R}_T\setminus\mathcal{R}_6,C}\approx H^0(C,\omega_C^2(-R)).$$

Let $L\in\mathrm{Pic}^3(C)$ be the trigonal line bundle. By Hurwitz' formula $\mathcal{O}(R)\approx\omega_C\otimes L^2$. Hence

$$N^*_{\mathcal{R}_T\setminus\mathcal{R}_6,C}\approx H^0(C,\omega_C\otimes L^{-2})$$

and its dual projectivization, the fiber of $\mathcal{R}$ over C, is naturally isomorphic with the $\mathbf{P}^1$ on which C is mapped by sections of $\omega_C\otimes L^{-2}$.

We now obtain another proof that the map of C to $\mathbf{P}^4$ via Lemma 2.1 is Prym-canonical.

COROLLARY 4.2. *Quadrics in $\mathbf{P}^4$ cut on C quadratic differentials. The quadrics containing $\Phi(X)$ cut the unique $g^1_4=\omega_C\otimes L^{-2}$ on C, plus a base-locus equal to the ramification R.*

Proof. The second statement implies the first. The base locus is clearly $\Psi'(C) \cap \Phi(X) = R$, of degree 16. Since a unique quadric contains $\Psi'(C)$, the residual intersection is of projective dimension 1, and degree $20-16=4$. We conclude by Lemma 3.1. Q.E.D.

Combining these identifications, we find:

COROLLARY 4.3. *The dual of the restricted map*

$$(\mathbf{P}^1)^* = \mathcal{R}|_C \to \mathbf{P}^2$$

is the projection

$$\Pi \to \mathbf{P}^1$$

(from the point in Π *corresponding to the singular quadric* $Q=\pi(C)$ *where* π: $\alpha(X) = \Theta_{\text{sing}} \to \Theta_{\text{sing}}/\pm 1 \hookrightarrow \Pi$*) and can be identified with the restriction map*

$$\Pi \approx \{\textit{quadrics through } \Phi(X)\} \to \{\textit{divisors of } g_4^1 \approx \omega_C \otimes L^{-2}\} \approx \mathbf{P}^1.$$

Part IV. The boundary components

In studying the Prym map near singular curves, the finer points of moduli theory come into play: at certain points, the "universal family" of curves does not exist over any punctured neighborhood; at others, the "versal deformation" is not universal; the map $\mathcal{R}_g \to \mathcal{M}_g$ can be ramified; instead of the canonical bundle, we need to distinguish the dualizing sheaf ω from the Kähler differentials Ω. And when one finally overcomes the confusion and computes $\mathcal{P}^*$ correctly, its kernel turns out, as already mentioned in I.3.3, to be too large for Part I, § 3, to apply!

In § 1 we see that there are essentially two boundary components of interest to us, $\mathcal{R}_S$ and $\mathcal{R}_E$. In § 3 we compute the codifferential along $\mathcal{R}_S$, a computation which leads in § 5 to the desired local degree of 16. Most of the difficulties mentioned above arise along $\mathcal{R}_E$, and especially the intersection $\mathcal{R}_{E,S} = \mathcal{R}_E \cap \mathcal{R}_S$. To avoid these we construct in § 4 a new compactification $\mathcal{M}'$ of $\mathcal{M}_g$ (actually this is done only for $g=6$, and "infinitesimally" for all g) and show that $\mathcal{P}$ factors through the new space $\mathcal{R}'$, in which $\mathcal{R}_E$ has been "blown down". The computation of § 5 then yields the degree on the whole boundary.

In § 2 we gathered some definitions, facts, and specific examples which might help a reader previously unfamiliar with deformations and moduli of singular curves.

§ 1. The loci

We start with a "generic" curve

$$X \in \mathcal{M}_{g-1}$$

of genus $g-1$, and describe, following Beauville's list [B1], all the ways $J(X)$ can arise as a Prym of a double-cover in the boundary of $\mathcal{R}_g$, that is a singular double cover, missing from Mumford's list [M2]. The "generic" choice will be used twice: to ignore families in $\bar{\mathcal{R}}_g$ of dimension $<3(g-1)-3$, and later on to insure that X corresponds to a smooth point of $\mathcal{M}_{g-1}$; in particular we assume that X has no non-trivial automorphisms.

1.1. Singulars. Let $p, q \in X$ be two distinct points, and

$$C = X/(p \sim q)$$

the singular curve of genus g obtained from X by identifying p, q. C has an ordinary double point $p=q$, and the natural map

$$N: X \to C$$

is its normalization. We let

$$\tilde{C} = X_1 \coprod X_2/(p_1 \sim q_2, p_2 \sim q_1)$$

where X_1, X_2 are copies of X. $\tilde{C}$ has two ordinary double points, and like C is a stable curve. The natural map

$$\pi: \tilde{C} \to C,$$

obtained by identifying each X_i with X, is an allowable double cover in the sense of Beauville. (I.1.3.) Hence

$$(\pi: \tilde{C} \to C) \in \bar{\mathcal{R}}_g$$

and $\bar{\mathcal{P}}(\pi) \in \mathcal{A}_{g-1}$ is well-defined. It follows from the definitions (and is proven in [B1], Theorem 5.4, case (i), and much earlier in Wirtinger's work [W]) that

$$\bar{\mathcal{P}}(\pi) \approx J(X).$$

Letting X, p, q vary and allowing all limits, we find a closed subvariety

$$\mathcal{R}_S \subset \bar{\mathcal{R}}_g$$

parametrizing singular covers. It has dimension

$$3(g-1)-3+2 = 3g-4$$

and is therefore a hypersurface in $\bar{\mathcal{R}}_g$, in fact a boundary-component sitting over one of the several boundary-components $\mathcal{M}_S$ of $\mathcal{M}_g$ in the standard ("Deligne–Mumford" or "stable") compactification $\bar{\mathcal{M}}_g$.

The restricted Prym map

$$\mathcal{R}_S \to \mathcal{J}_{g-1}$$

is proper and surjective. For an automorphism-free X, its fiber over $J(X)$ is the symmetric product S^2X. Outside of the hyperelliptic locus (excluded by the generic choice of X) we can identify $\mathcal{J}_{g-1}$ with $\mathcal{M}_{g-1}$; the resulting map over $(\mathcal{M}_{g-1})_{\mathrm{smooth}}$ can be identified with the (relative) second symmetric product of the universal curve (which exists over $(\mathcal{M})_{\mathrm{smooth}}$).

1.2. Singular elliptic tails. In taking $\mathcal{R}_S$ to be a closed subvariety of $\bar{\mathcal{R}}_g$, we need in particular to allow p, q to coincide; in the fiber S^2X this corresponds to the diagonal. Somewhat contrary to intuition, the limiting curve C is not cuspidal, as stable curves can have ordinary double points at worst. Instead we get the reducible curve

$$C = X \coprod \mathbf{P}^1/(p \sim 1, 0 \sim \infty)$$

where $0, 1, \infty \in \mathbf{P}^1$. Equivalently,

$$C = X \bigcup\nolimits_p E$$

where E is the rational-elliptic curve $\mathbf{P}^1/(0 \sim \infty)$. In this case we have

$$\tilde{C} = (X_1 \cup E_1 \cup E_2 \cup X_2)/(p_1 \sim 1_1, 0_1 \sim \infty_2, \infty_1 \sim 0_2, 1_2 \sim p_2)$$

and each X_i (respectively E_i) is mapped isomorphically to X (respectively E) by $\pi: \tilde{C} \to C$. These are the correct limits since they are stable (each $\mathbf{P}^1$ has $\geqslant 3$ points fixed) and fit into one-parameter families where all nearby curves are as in 1.1, with p and q tending to a common limit (a deformation of C smoothing the singularity p but preserving the other double point). We shall analyze this situation carefully in § 4.

1.3. Elliptic tails. Let $p \in X$ and let E be an elliptic curve. Then

$$C = X \bigcup\nolimits_p E$$

is a stable curve, as is

$$\tilde{C} = X_1 \bigcup\nolimits_{p_1} \tilde{E} \bigcup\nolimits_{p_2} X_2$$

where X_i are copies of X, $\tilde{E}$ an unramified, irreducible double cover of E, hence an elliptic curve itself. (There are three such, corresponding to the three nonzero elements of $H_1(E, \mathbf{Z}/2\mathbf{Z})$.) Since E has a one-parameter family of automorphisms (translations in the group structure) it does not matter which point e of E we identify to $p \in X$; but once e is chosen, the identification in $\tilde{C}$ is: $p_i \sim e_i$, where e_1, e_2 are the points of $\tilde{E}$ over e. Now there is a natural map

$$\pi: \tilde{C} \to C$$

which is an allowable double cover since it satisfies the conditions of [B1], Lemma 5.1. By [B1], Theorem 5.4(ii), we have

$$\mathcal{P}(\pi) \approx J(X) \times Q$$

where Q is the Prym of

$$\tilde{E} \to E.$$

Clearly $Q = (0)$ so that $\mathcal{P}(\pi) \approx J(X)$.

We let $\mathcal{M}_E$, $\mathcal{R}_E$ denote the hypersurfaces in $\overline{\mathcal{M}}_g$, $\bar{\mathcal{R}}_g$ respectively parametrizing these curves C with an "elliptic tail", and their degenerations. For fixed X the only degeneration is the one already discussed in 1.2. The two hypersurfaces $\mathcal{M}_E$, $\mathcal{M}_S$, respectively $\mathcal{R}_E$, $\mathcal{R}_S$, intersect in the subvariety $\mathcal{M}_{E,S}$, respectively $\mathcal{R}_{E,S}$, of their common degenerations, curves with an elliptic tail which is itself singular. In fact it follows from Schlessinger's local deformation theory (cf. 2.5) that the intersection is transverse.

LEMMA 1.3.1. *The fiber of the restricted Prym map*

$$\mathcal{R}_E \to \mathcal{M}_{g-1},$$

over a generic $X \in \mathcal{M}_{g-1}$, *is isomorphic to* $X \times \mathbf{P}^1$.

Proof. We are free to vary p and $\tilde{E}$, independently. Clearly the possible p are parametrized by X, E by $\mathbf{P}^1$ (the "j-invariant") and $\tilde{E}$ by a three-sheeted cover of $\mathbf{P}^1$. This cover has a simple ramification point over the harmonic E ($j = 1728$, complex multiplication by $\sqrt{-1}$) and over the singular E ($j = \infty$, monodromy around E interchanges 0, ∞ while fixing -1) and has a double ramification point (all three sheets permuted, cyclically) over the equianharmonic E ($j = 0$, complex multiplication by $\omega = \sqrt[3]{1}$). By Hurwitz' formula this three-sheeted cover of $\mathbf{P}^1$ is itself rational. Q.E.D.

Remark 1.3.2. In all three cases 1.1, 1.2, 1.3, there is a line bundle η on C such that $\eta \neq \mathcal{O}$, $\eta^2 \approx \mathcal{O}$, and $\pi^*\eta = \mathcal{O}_{\tilde{C}}$. For example, in 1.1, (respectively 1.2), η is the line bundle which is trivial on X (respectively $X \cup_p \mathbf{P}^1$) and whose fibers over p, q (respectively 0, ∞) are identified after multiplication by -1. (Compare 4.4.2.)

1.4. The complete fiber.

LEMMA 1.4. *The only irreducible components of* $\bar{\mathcal{R}}_g \setminus \mathcal{R}_g$, *for* $g = 6$, *whose image under* $\mathcal{P}$ *contains* $\mathcal{J}_{g-1}$ *are* $\mathcal{R}_S$, $\mathcal{R}_E$ *and (for* $g = 6$ *only)* $\bar{\mathcal{R}}_T - \mathcal{R}_T$.

Proof. We use the notation and results of [B1], (5.2) and (5.4). If for generic

$$\pi\colon \tilde{C} \to C$$

in a family mapping onto $\mathcal{J}_{g-1}$,

$$\tilde{C} = A \cup A' \cup B$$

with B fixed by the involution on $\tilde{C}$, A and A' interchanged, then as in I.1.3 we have either

(i) $B=\emptyset$, $\#(A \cap A')=2$, $\mathcal{P}(\pi) \approx J(A)$, so $J(A)$ is a generic Jacobian, hence A a generic curve, and the family is $\mathcal{R}_S$, or

(ii) $B \neq \emptyset$, $A \cap A' = \emptyset$, B satisfies condition (*), $\mathcal{P}(\pi) \approx J(A) \times Q$, Q the Prym of B. Since the generic Jacobian is simple, we have either

(ii) (1) $Q=(0)$, hence $g(B)=1+\dim(Q)=1$, B is elliptic and our family is $\mathcal{R}_E$, or

(ii) (2) $J(A)=(0)$, hence $\tilde{C}=B$ satisfies condition (*) and $\mathcal{P}(\tilde{C})$ is a generic Jacobian. By [B1], Theorem 4.10, the curves C, $\tilde{C}$ must belong to one of the families (a)–(j) described there. The families (a), (b), (c), (f), (h), (i), respectively, have codimensions $g-4$, $g-2$, $g-1$, $g-2$, 4, $g-1$ in $\bar{\mathcal{R}}_g$; families (d), (g) exist only for $g=5$; families (e), (j) exist only for $g=6$ and have codimensions 3, 6 in $\bar{\mathcal{R}}_6$. Such a family can surject onto $\mathcal{J}_{g-1}$ only if its codimension is $\leqslant 3$. This happens only for (a) and (e), both for $g=6$. The first corresponds to the 13-dimensional family $\bar{\mathcal{R}}_T$ of trigonals, which does contain a 12-dimensional subfamily $\bar{\mathcal{R}}_T \setminus \mathcal{R}_T$ mapped onto $\mathcal{J}_5$ (as we saw in Part III, § 3). The second is 12-dimensional and its generic element is a smooth plane quintic, thus the singular members form a subvariety of dimension $\leqslant 11 < \dim(\mathcal{M}_5)$. Case (a) has codimension $\leqslant 3$ (actually $=3$) also when $g=7$, but its generic element is smooth, hence the singular curves of type (a) have codimension $\geqslant g-3=4$ when $g=7$. Q.E.D.

§ 2. Moduli at the boundary

In order to compute the differential of $\mathcal{P}$ at boundary points, it is necessary to understand some generalities on the structure of the moduli space near singular curves. Almost everything we need is in § 1 of [DM]. We go through it here not for completeness' sake (our discussion is fragmented and incomplete) but with the hope of giving just enough working tools that a non-expert but believing reader might be able to follow the rest of the paper.

2.1. Deformations. The most convenient object to work with is the *versal formal* deformation $\mathcal{C}$ of a curve C. It is a family of curves parametrized by the complete germ M of the origin in a smooth $(3g-3)$-dimensional variety, or

$$M = \operatorname{Spec} \mathbf{C}[[t_1, \ldots, t_{3g-3}]].$$

a "deformation" of C is such a family, where the central fiber is isomorphic to C and the base space M has a unique point (but (preferably) lots of tangent directions). The "formal" part refers to the base being defined by formal power series or being a "complete" germ, and the "versal" indicates our family is as close to being "universal" as can be. Roughly, the germ of any infinitesimal family of curves (with central fiber C) comes via pullback of $\mathcal{C}$, by a map of its base to M; and this map is uniquely determined to first order. (Cf. 2.2(2) for a non-uni-versal deformation.) In [SC] Schlessinger introduced the notion of versal deformation, and proved its existence under very general conditions, in particular for all "curves".

Concentrating attention at a point $p \in C$, one can isolate the interesting part of $\mathcal{C}$ via the concept of local deformation, or deformation of the "curve" which is the localization of C at p. Essentially, what happens is that M splits,

$$M \approx M_p \times M'$$

where M_p is base to the versal local deformation of C_p, and the action in M' happens away from p.

2.2. Examples. (0) p a smooth point of C, then $M_p = (0)$, $M' = M \sim \mathbf{A}^{3g-3}$, affine $(3g-3)$-space. ($\sim$ reads = "germ of".)

(1) p an ordinary double point of C. Then $M_p \sim \mathbf{A}^1 = \operatorname{Spec} \mathbf{C}[t]$ and the local deformation looks like

$$\mathcal{C}_p \sim \operatorname{Spec} \mathbf{C}[u, v, t]/(uv-t).$$

(2) p a cusp of C. Then $M_p \sim A^2 = \operatorname{Spec} \mathbf{C}[a, b]$, and

$$\mathcal{C}_p \sim \operatorname{Spec} \mathbf{C}[a, b, x, y]/(x^3+ax+b-y^2).$$

The local deformation here is 2-dimensional, and contains a 1-dimensional subfamily, parametrized by points of the cuspidal curve $4a^3+27b^2=0$, of curves with ordinary double points.

We note that the versal deformation here is not universal. We can give different maps of the affine line to $\mathbf{A}^2$ inducing the same family of curves over the line. For example,

$$t \mapsto (ct^2, dt^3)$$

induces the family of curves with equations

$$y^2 = x^3+ct^2x+dt^3.$$

These curves, for all $t \neq 0$, are all isomorphic to each other (multiplying y by s^3, x by s^2, has the effect of dividing t by s^2). Still the family itself is non-trivial, as the trivializing transformations "blow-up" as $t \to 0$; we encounter a "jump phenomenon" where the central fiber, cuspidal, differs from the (isomorphic) curves near it. However, this family depends only on the ratio c^3/d^2, so there is a 1-parameter family of these maps inducing isomorphic jump-deformations. The versal condition still holds since all these maps vanish to first order in t.

(3) One can also form the local deformation M_{local} for the simultaneous localization of C at several singular points. The result is nothing but the product of the local deformations. This applies, for example, to the curves

$$C_0 = X \bigcup_p E \in \mathcal{R}_{E,S}$$

of 1.2. Here $M_{\text{local}} \sim \mathbf{A}^2 = \operatorname{Spec} \mathbf{C}[s, t]$. Varying one parameter will smooth E, and the other will smooth p, yielding an irreducible neighboring C.

2.3. Canonical sheaves.

2.3.1. The versal property clearly determines M uniquely, up to non-natural isomorphism which is "natural to first order" (compare 2.2(2)). In particular, the tangent and cotangent spaces to M at the origin are natural objects, computable in terms of C. The tangent space is computed in [Sc] to be $\operatorname{Ext}^1_{O_C}(\Omega_C, O_C)$ so by duality the cotangent space is $H^0(C, \Omega_C \otimes \omega_C)$, involving the two "canonical" sheaves Ω_C, ω_C. We discuss these next.

2.3.2. The locally free sheaf ω_C is the dualizing sheaf of C, or the object that makes duality and Riemann–Roch work for a possibly singular C. Ω_C is the sheaf of Kähler differentials, generalizing the cotangent bundle of smooth curves. At a smooth point of C, sections of either of these sheaves are regular differentials on C. At any point, both sheaves can be computed given a projective embedding of C: sections of Ω are restrictions of differential 1-forms on the ambiant space, modulo differentials of the defining ideal; sections of ω come from meromorphic differentials "with logarithmic singularities" (at worst) along C via Poincaré residue. Thus for a plane curve C defined locally by $f(x, y) = 0$ at the origin p, we have the explicit formulae:

$$\Omega_{C,p} \approx O_{C,p}(dx, dy)/(f_x\,dx + f_y\,dy)$$

$$\omega_{C,p} \approx O_{C,p}\left(\frac{dx}{f_y}, \frac{dy}{f_x}\right) \Big/ \left(\frac{dx}{f_y} + \frac{dy}{f_x}\right).$$

We note that ω_C is locally-free of rank 1, since its two generators are set equal. On the other hand, Ω_C is invertible only for smooth C; in fact it has torsion sections, supported at the singular point p.

2.3.3. There is a natural map

$$j\colon \Omega \to \omega,$$

which is the identity at smooth points. One way to describe j is by comparing with the normalization

$$N: NC \to C.$$

The pullback induces a map from sections of Ω_C onto sections of Ω_{NC}; pushforward maps sections of ω_{NC} to sections of ω_C; and the sheaves Ω_{NC}, ω_{NC} are naturally isomorphic.

Clearly j induces an isomorphism on stalks at smooth points. Thus its kernel and cokernel are skyscraper sheaves supported on Sing (C), in fact of equal ranks. ker (j) consists precisely of the torsion-sections in Ω_C. Im (j) consists of differentials regular on NC.

For example, at an ordinary double point with local equation $f(x, y) = xy = 0$,

$$\omega_p \approx O_p\left(\frac{dx}{x}, \frac{dy}{y}\right) \Big/ \left(\frac{dx}{x} + \frac{dy}{y}\right)$$

$$\Omega_p \approx O_p(dx, dy)/(y\,dx + x\,dy)$$

and ker (j), coker (j) are 1-dimensional, spanned by $y\,dx = -x\,dy$, $dx/x = -(dy/y)$ respectively. (We label the generators so that formally, $j(dx) = dx = x(dx/x)$, etc.)

2.3.4. We return to the description of T^*M as $H^0(C, \Omega_C \otimes \omega_C)$, proven in [Sc]. The splitting

$$M \approx M_{\text{local}} \times M'$$

is natural at least in the sense that TM' is a well-defined subspace of TM, corresponding to those directions along which the deformations are trivial when localized at the singular points. Thus T^*M_{local} is a well-defined subspace of $T^*M \approx H^0(C, \Omega_C \otimes \omega_C)$. It is related to T^*M by the "local to global" spectral sequence, which in this case degenerates allowing the computation of T^*M_{local}; it turns out to be precisely the torsion subspace. (More precisely, the space of sections supported on the finite set where we have localized.)

Since ω_C is invertible, there is an isomorphism of the torsion spaces for $\Omega_C, \omega_C \otimes \Omega_C$, deter mined by the choice of a (meromorphic) section of ω_C which is regular and non-zero at the singular points of localization. Thus T^*M_{local} becomes identified with the torsion sections of Ω_C, or with ker (j).

We saw above that for an ordinary double point this space is one-dimensional in accord with 2.2(1). Similarly for a cusp, whose equation is $f(x, y) = y^2 - x^3 = 0$:

$$\Omega_p \approx O_p(dx, dy)/(2y\,dy - 3x^2\,dx)$$

$$\omega_p \approx O_p\left(\frac{dx}{2y}, \frac{dy}{3x^2}\right) \Big/ \left(\frac{dx}{2y} - \frac{dy}{3x^2}\right)$$

and ker (j) is 2-dimensional, spanned by

$$3y\,dx-2x\,dy, \quad 3xy\,dx-2x^2dy.$$

This agrees with 2.2(2).

2.4. Deformations vs. Moduli. Finally, the relation between versal deformations and local moduli of a stable curve. There is a natural map

$$\mu\colon M\to\overline{\mathcal{M}}_g$$

sending the origin of M to the point of $\overline{\mathcal{M}}_g$ corresponding to C. The degree of μ, at the origin, is the order of the group G of automorphisms of C (which is finite for C stable). Indeed, by the versal property, each non-trivial automorphism of C induces a non-trivial automorphism of M commuting with μ, and vice versa.

While M is always smooth, its quotient by μ may be singular. In fact, one sees that when G is cyclic this happens precisely when the locus in M parametrizing curves on which G persists fails to be a smooth (possibly empty) hypersurface. Thus $\mathcal{M}_g$ is singular along the locus of hyperelliptic curves precisely for $g\geqslant 4$, since the hyperellipties have codimension $g-2$. Another example: $\overline{\mathcal{M}}_g$ is not singular at a generic point of the hypersurface $\mathcal{R}_E$, even though these curves have an automorphism (identity on X, -1 on E).

For the case of the cuspidal curves we shall work out the relation between M, $\mathcal{M}$ in full detail in § 4.

2.5. Moduli at a singular elliptic tail. As an application of these ideas, we prove the remark made in 1.3 that the hypersurfaces $\mathcal{M}_E$, $\mathcal{M}_S$ intersect transversely in $\mathcal{M}_{E,S}$ over a generic $X\in\mathcal{M}_{g-1}$.

For $C\in\mathcal{M}_{E,S}$, the map

$$\mu\colon M\to\overline{\mathcal{M}}_g$$

is two-to-one due to the automorphism of E. It is the sum of two maps,

$$\mu_{\text{local}}\colon M_{\text{local}}\to\overline{\mathcal{M}}_g$$

and

$$\mu'\colon M'\to\overline{\mathcal{M}}_g$$

where μ' is injective, μ_{local} is two-to-one, and as seen in 2.2(3), $M_{\text{local}}\sim\mathbf{A}^2$. (Both maps go into the germ of $\overline{\mathcal{M}}_g$ at C which is a large vector space. The "sum" is taken in this vector space.) Clearly the germs of $\mathcal{M}_E$, $\mathcal{M}_S$ are products, with the image of μ', of the images of the corresponding hypersurfaces (=curves) M_E, M_S in M_{local}. By 2.2(3), the curves M_E, $M_S \subset M_{\text{local}}$ are transversal. The involution of M_{local} commuting with μ_{local} induces an involution (with the origin as fixed point) on M_S; but on M_E it acts as the identity since (2.4) the automorphism of C persists there. Hence, in formal coordinates (x, y) on M_{local} such that

$$M_S = \{y = 0\}, \quad M_E = \{x = 0\}$$

the map μ_{local} is

$$(x, y) \mapsto (x^2, y).$$

Therefore, (x^2, y) are formal coordinates on the quotient, and

$$\mathcal{M}_S = \{y = 0\}, \quad \mathcal{M}_E = \{x^2 = 0\}$$

are transversal, being given by the vanishing of two independent coordinates.

2.6. Double covers. Given any type of "level structure", we can construct a corresponding (branched) cover of M. In particular, there is the family of allowable double-covers, parametrized by a space R mapping onto M with degree $\leqslant 2^{2g}-1$. Further, the map μ lifts to a commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\pi}} & M \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \mu} \\ \mathcal{R} & \xrightarrow{\pi} & \mathcal{M} \end{array}$$

The basic examples are:

(0) For a smooth curve $C \in \mathcal{M}_g$, R consists of $2^{2g}-1$ disjoint copies of M. Note that this is true even in the presence of automorphisms of C; in that case, π, μ are ramified, $\tilde{\pi}$ is not, and ϱ could be ramified though it tends not to be, or at least to be "less" ramified than μ. This indicates a general phenomenon: replacing double-covers by sufficiently high level-structures, the corresponding ϱ becomes unramified, so π factors through M allowing the construction of a universal curve over a high-level moduli space.

(i) For $C \in \mathcal{M}_S - \mathcal{M}_{E,S}$, R is a $(2^{2g-1}+1)$-sheeted cover of M consisting of one component R_S of degree 1 (mapping isomorphically to M) and 2^{2g-2} components of degree 2, each a double cover of M branched along $M_{\text{sing}} = (0) \times M' \subset M_{\text{local}} \times M' = M$. Here R_S parametrizes deformations of a Wirtinger double-cover $(C, \tilde{C}) \in \mathcal{R}_S$, and the other components R_{odd} parametrize deformations of double covers whose corresponding $(\mathbf{Z}/2\mathbf{Z})$-homology classes meet the vanishing cycle of C in one point, hence are acted upon non-trivially by the monodromy. (The remaining covers are not allowable.)

(ii) For $C \in \mathcal{M}_E \setminus \mathcal{M}_{E,S}$ generic, R consists of $2^{2g-2}+2$ copies of M, and $\mathcal{R}$ is locally étale of degree $2^{2g-2}+2$ over $\mathcal{M}$. (Since the vanishing cycle is homologous to 0 there is no monodromy, hence no branching in $\pi, \bar{\pi}$.) $2^{2g-2}-1$ of the corresponding curves look like $E_1 \cup \tilde{X} \cup E_2$, and the remaining 3 look like $X_1 \cup \tilde{E} \cup X_2$. The corresponding 3 sheeted covers of M, $\mathcal{M}$ will be denoted R_E, $\mathcal{R}_E$ respectively. The maps μ, ϱ are 2 to 1, branched along $\mathcal{M}_E, \pi^{-1}(\mathcal{M}_E)$ respectively.

(iii) Let $C = X \cup E$ where X is still generic but E is one of the special elliptic curves.

For the equianharmonic E, $j=0$, Aut $(E) \approx \mathbf{Z}/6\mathbf{Z}$ acts on $\mathcal{R}_E$ by permuting the 3 non-zero elements of $H(E, \mathbf{Z}/2\mathbf{Z})$ cyclically, hence $\mathcal{R}_E$ is totally ramified over $\mathcal{M}_{E,\text{e.a.h.}}$ ($\mathcal{R} \setminus \mathcal{R}_E$ is still étale as in (ii)). Deg (μ) is 6, and in fact μ factors through $\mathcal{R}_E$, hence $R = R_E \times_{\mathcal{M}} M$ is still étale of degree $2^{2g-2}+2$ over M.

For the harmonic E, $j=1728$, Aut $(E) \approx \mathbf{Z}/4\mathbf{Z}$ acts on $\mathcal{R}_E$ by permuting 2 of the 3 semiperiods and fixing the third. Hence $\mathcal{R}_E$ has two components: an étale component (of degree 1 over $\mathcal{M}$) and a double cover ramified over $\mathcal{M}_{E,\text{har}}$. $\mathcal{R} \setminus \mathcal{R}_E$ is as before. Deg $(\mu)=$ 4 and again μ factors through $\mathcal{R}_E$, so R is étale.

Finally, consider $C \in \mathcal{M}_{E,S}$. Since being allowable is an open property, R has only those components that are allowed by both (i) and (ii); this amounts to 3 sheets only; namely $\mathcal{R} = \mathcal{R}_E$. Aut $(E) \approx \mathbf{Z}/2\mathbf{Z}$ as in (ii), but there is an extra monodromy action on $H(C, \mathbf{Z}/2\mathbf{Z})$ coming from the degenerating Jacobian, as in (i). Hence $\mathcal{R}_E$ has 2 components: an étale component $\mathcal{R}_S$, of degree 1 over $\mathcal{M}$, and a double cover $\mathcal{R}_{\text{odd}}$ branched along $\mathcal{M}_S$ On the other hand, $M \to \mathcal{M}$ has degree 2 and is branched along $\mathcal{M}_E$ as in (ii); hence R has degree 3 over M, and consists of an étale component of degree 1 and a double cover branched along M_S. The "elliptic tail" is in the étale component.

§ 3. The codifferential

We attempt to extend Beauville's result, Proposition I.4.1, to the boundary. Let $X \in \mathcal{M}_{g-1}$ be a generic curve, and let $(C_0, \tilde{C}_0) \in \bar{\mathcal{R}}_g$ be either $X/(p \sim q)$ or $X \cup_p E$, with the double cover chosen as in § 1. Let $\eta \in \text{Pic}^0(C_0)$ be the line bundle corresponding to $\tilde{C}_0$, $\eta^2 \approx O_{C_0}$. (Compare Remark 1.3.2.)

3.1. The spaces

$$S^2H^0(C_0, \omega \otimes \eta), \quad H^0(C_0, \Omega \otimes \omega), \quad H^0(C_0, \omega \otimes \omega)$$

are of dimensions

$$\binom{g}{2}, 3g-3, 3g-3,$$

as they would be for a smooth C. Further, by general nonsense they vary smoothly over families of curves C, in particular they each form a vector-bundle over the base M of the versal deformation of C. (Each of these sheaves $\mathcal{F}_{C_0}$ is the restriction to C_0 of a similar sheaf $\mathcal{F}$ defined over $\mathcal{C}$ (or over the total space of any family deforming C_0.) The above vector bundles are associated to the locally-free-sheaves

$$\mathcal{R}^0 f_*(\mathcal{F})$$

on M, where $f\colon \mathcal{C}\to M$ is the deformation.)

By the examples in 2.6 we see that whether (C_0, η) is in $\mathcal{R}_S$, $\mathcal{R}_E$ on $\mathcal{R}_{E,S}$, the base R of its versal deformation is isomorphic, via $\tilde{\pi}$, to that of C_0 alone, M. In particular we can identify

$$T^*_{(C_0,\eta)}R \approx T^*_{C_0}M \approx H^0(C_0, \Omega\otimes\omega).$$

The Prym codifferential at (C_0, η) becomes a map

$$\mathcal{P}^*\colon S^2H^0(C_0, \omega\otimes\eta)\to H^0(C_0, \Omega\otimes\omega).$$

This clearly extends to a sheaf map of the locally-free-sheaves over M,

$$S^2\mathcal{R}^0 f_*(\omega_{C/M}\otimes\eta)\to \mathcal{R}^0 f_*(\Omega_{C/M}\otimes\omega_{C/M})$$

and can thus be computed, at C_0, if we know its values at all nearby C.

3.2. The map $j\colon \Omega\to\omega$ described in 2.3.3 induces

$$j_*\colon H^0(C, \Omega\otimes\omega)\to H^0(C, \omega\otimes\omega).$$

We compose this with $\mathcal{P}^*$, obtaining a map

$$j_*\circ\mathcal{P}^*\colon S^2H^0(C, \omega\otimes\eta)\to H^0(C, \omega\otimes\omega)$$

which varies continuously with (C, η), and equals cup-product for C generic (recall $\eta^2\approx \mathcal{O}$), hence for all C including our C_0.

LEMMA 3.2.1. $\mathbf{P}$ (ker $(j_*\circ\mathcal{P}^*)$) *is* (*naturally identifiable with*) *the family of quadrics in the canonical space of* X *containing the canonical curve* $\Phi(X)$.

Proof. The isomorphism $P(C,\eta)\approx J(X)$ induces on cotangent spaces an isomorphism

$$H^0(C, \omega\otimes\eta)\xrightarrow{\approx} H^0(X, \omega_X).$$

Hence $\mathbf{P}(S^2H^0(C, \omega\otimes\eta))$ is identified with the family of quadrics Q in the canonical space.

If $j_*(\mathcal{P}^*(Q))=0$ as a section of $\omega\otimes\omega$ on C, it vanishes in particular at each point of X, hence $\Phi(X)\subset Q$.

Vice versa, since $\omega\otimes\omega$ is locally free on C, in particular torsion-free, it is enough to show that if $\Phi(X)\subset Q$ then $j_*\circ\mathcal{D}^*(Q)$ vanishes at each smooth point of C. For C irreducible (case of $\mathcal{R}_S$) this is equivalent to $\Phi(X)\subset Q$. In the elliptic-tail case clearly $j_*\circ\mathcal{D}^*(Q)$ vanishes on $X\subset C$. On E,

$$\omega\otimes\eta|_E\approx\omega_E(p)\otimes\eta\approx\eta(p)$$

This has a unique non-zero section, which is non-zero at p; but we saw already that $j_*\circ\mathcal{D}^*(Q)$ is zero on X, hence at p, hence everywhere on E.

COROLLARY 3.2.2. $\mathbf{P}$ (ker $(\mathcal{D}^*))\subset\{$*quadrics through* $\Phi(X)\}$.

3.3. Since j induces isomorphisms on stalks away from the double point, we see that for $Q\in\ker(j_*\circ\mathcal{D}^*)$:

$$Q\in\ker\mathcal{D}^*\Leftrightarrow\mathcal{D}^*(Q)|_{\text{double point}}=0.$$

This reduces the search for ker $(\mathcal{D}^*)$ to a *local* computation in the stalks at the double point p of the various sheaves involved. We write down the stalks of all sheaves in sight, labeling the generators so that the natural maps preserve them,

$$dx\mapsto dx,\ dx\otimes dy\mapsto dx\,dy,\quad \text{etc.}$$

$$\omega\colon O_p\left\langle\frac{dx}{x},\frac{dy}{y}\right\rangle\Big/\left(\frac{dx}{x}+\frac{dy}{y}\right)$$

$$\Omega\colon O_p\langle dx,dy\rangle/(y\,dx+x\,dy)$$

$$\eta\colon O_p$$

$$\Omega\otimes\Omega\colon O_p\langle dx^2,dx\,dy,dy\,dx,dy^2\rangle/(y\,dx^2+x\,dx\,dy,\ y\,dx\,dy + x\,dy^2,\ x(dx\,dy-dy\,dx),\ y(dx\,dy-dy\,dx))$$

$$\Omega\otimes\omega\colon O_p\left\langle\frac{dx^2}{x},\frac{dy^2}{y}\right\rangle\Big/\left(y\frac{dx^2}{x}-x\frac{dy^2}{y}\right)$$

$$\omega\otimes\omega\colon O_p\left\langle\frac{dx^2}{x^2},\frac{dx\,dy}{xy},\frac{dy^2}{y^2}\right\rangle\Big/\left(\frac{dx^2}{x^2}+\frac{dx\,dy}{xy},\ \frac{dx\,dy}{xy}+\frac{dy^2}{y^2}\right)\quad\text{etc.}$$

In view of the first sentence in 3.3 and the commutative diagram

$$\begin{array}{ccc} H^0(\Omega\otimes\eta)\otimes H^0(\Omega\otimes\eta) & \longrightarrow & H^0(\Omega\otimes\Omega)\\ \downarrow & & \downarrow\\ \mathcal{D}^*\colon H^0(\omega\otimes\eta)\otimes H^0(\omega\otimes\eta) & \longrightarrow & H^0(\Omega\otimes\omega)\end{array}$$

(where vertical maps are induced by j) our problem lifts to computing the map

$$(\Omega\otimes\eta)_p\otimes(\Omega\otimes\eta)_p\to(\Omega\otimes\Omega)_p\to(\Omega\otimes\omega)_p.$$

Since η is trivial near p, this is the same as

$$\Omega_p\otimes\Omega_p\to(\Omega\otimes\Omega)_p\to(\Omega\otimes\omega)_p$$

where the first map is induced by cup product, the second by j. As an O_p-module, the object on the left has generators

$$dx^2,\ dx\,dy,\ dy\,dx,\ dy^2.$$

By our convention, their images in $(\Omega\otimes\omega)_p$ still bear the same names, except now

$$dx\,dy=dy\,dx=-y\frac{dx^2}{x}=-x\frac{dy^2}{y}$$

is the (unique) torsion-element of $\Omega\otimes\omega$. We conclude: $\mathcal{D}^*(Q)$ vanishes at p if and only if the coefficients in Q of dx^2, $dx\,dy$, dy^2 all vanish.

3.4. The codifferential for $(C,\eta)\in\mathcal{R}_S$. For $C=X/(p\sim q)$ we can take x, y as coordinates on X at p, q; dx, dy are the corresponding differentials. The coefficients of dx^2, $dx\,dy$, dy^2 furnish three linear functions α, β, γ on $S^2H^0(C,\omega\otimes\eta)$. By Corollary 3.2.2 and 3.3, Q is in $\ker(\mathcal{D}^*)$ if and only if:

(1) $Q\supset\Phi(X)$

(2) $\alpha(Q)=\beta(Q)=\gamma(Q)=0$.

Now, condition (1) *implies* $\alpha(Q)=\gamma(Q)=0$, as these two give the values of Q at p, q. Assuming this holds, $\beta(Q)=0$ iff Q is identically zero on the line $\overline{pq}$. We have proven:

PROPOSITION 3.4.1. *For $(C,\eta)\in\mathcal{R}_S\setminus\mathcal{R}_{E,S}$ over a generic $X\in\mathcal{M}_{g-1}$, $C=X/(p\sim q)$, $\mathbf{P}(\ker(\mathcal{D}^*))$ can be naturally identified with the space of quadrics containing the canonical curve $\Phi(X)$ and its chord $\overline{\Phi(p),\Phi(q)}$.*

3.5. The codifferential for elliptic-tails. If we perform the same computation for $(C,\eta)\in\mathcal{R}_E$, taking x, y as coordinates on X, E near p, we see that β, γ vanish identically; in fact, so does the coefficient of dy, as a linear function on $H^0(C,\omega\otimes\eta)$. (Proof:

$$\omega_C\otimes\eta\,|_X=\omega_X(p)\otimes O=\omega_X(p)$$
$$\omega_C\otimes\eta\,|_E=\omega_E(p)\otimes\eta_E=\eta_E(p)$$

and for a section of $\omega_C \otimes \eta$ on C, the residues at p should be opposite. But

$$h^0(X, \omega_X(p)) = h^0(X, \omega_X) = g-1$$

so all sections of $\omega_C \otimes \eta$ vanish at p, and when restricted to E yield sections of η_E, necessarily zero.) Since if $Q \supset \Phi(X)$ then $\alpha(Q) = 0$ as the value of Q at p, we conclude that for $(C, \eta) \in \mathcal{R}_E$,

$$\ker(\mathcal{P}^*) = \ker(j_* \circ \mathcal{P}^*).$$

This is larger, by one dimension, than the kernel for $(C, \eta) \in \mathcal{R}_S$, and would play havoc with our argument in § 5. Luckily, we have computed here the codifferential of the wrong map! The point is that (cf. 2.6(ii))

$$\mu\colon M \to \mathcal{M}, \quad \varrho\colon R \to \mathcal{R}$$

are 2-to-1, and we were only able to compute the codifferential of the real $\mathcal{P}$,

$$\mathcal{P}\colon \mathcal{R} \to \mathcal{A}_{g-1}$$

after it is composed with ϱ. Instead of pushing the local arguments further to factor $\mathcal{P}^*\colon T^*\mathcal{A}_{g-1} \to T^*R$ through $T^*\mathcal{R}$, we describe in the next section a more global approach, showing that for computing deg $(\mathcal{P})$, the locus $\mathcal{R}_E$ is an "exceptional divisor" which can be ignored.

§ 4. Factoring out the elliptic tails

Our goal in this section is to show that the Prym map involves blowing down the elliptic-tail locus $\mathcal{R}_E \subset \bar{\mathcal{R}}_g$ to a "cuspidal locus". More precisely, for $X \in \mathcal{M}_{g-1}$, $p \in X$, let $F_{X,p}$ (respectively $\tilde{F}_{X,p}$) be the family in $\mathcal{M}_E$ (respectively $\mathcal{R}_E$) of curves consisting of an elliptic tail attached to X at p. (By Lemma 1.3.1 $\tilde{F}_{X,p}$ is a rational curve.) We want to show that the Prym map, from a (formal) neighborhood of $\tilde{F}_{X,p}$ in $\bar{\mathcal{R}}_g$ to a (formal) neighborhood of $J(X) \in \mathcal{J} \subset \mathcal{A}_{g-1}$, factors through the base-space R of the versal family of allowable double covers near the cuspidal curve $C = X/(2p)$. (See the discussion and examples in 2.6.) Thus the Prym map is composed of a map which blows down $\mathcal{R}_E$ (sending each rational curve $\tilde{F}_{X,p}$ to a point) and a regular map. In particular, deg $\mathcal{P}$ can be computed after $\mathcal{R}_E$ is blown down.

The natural steps to this end are:

(1) Construct a "natural" map from a formal neighborhood of $F_{X,p}$ in $\bar{\mathcal{M}}$ to the base M of the versal deformation $\mathcal{C}$ of C.
(2) Lift to double-covers.
(3) Analyze these maps.
(4) Construct a Prym map on R such that the composition is $\bar{\mathcal{P}}$.

We perform steps (2), (3), (4) in arbitrary genus. It seems that step (1) can also be done in general, using methods similar to those of [M3]; it depends on constructing a family of curves parametrized by a finite branched cover of a neighborhood of $\tilde{F}_{X,p}$ in $\bar{\mathcal{R}}_g$, and a line bundle over the total space whose sections embed the generic curve in the family, while restricting to the trivial bundle on the elliptic tails (hence mapping the curves over $\tilde{F}_{X,p}$ to cuspidal curves). However, there are some nasty technicalities involved in the existence and naturality of the above family. To avoid this, we present a fairly elementary argument which works only in the case we need, $g=6$. In this case we proceed to construct a compactification $\mathcal{M}'$ of the moduli space $\mathcal{M}_6$ which is based on families of plane curves, thus replacing the elliptic tails by cusps. As a result we can explicitly factor the Prym map on an honest (not just formal) neighborhood of $\tilde{F}_{X,p}$, through the space $\mathcal{R}'$ of allowable double covers of objects in $\mathcal{M}'$. $\mathcal{M}'$ is defined in 4.2.4 and described in 4.3.

4.1. Moduli of plane curves. In our construction of $\mathcal{M}'$, we use a bit of the general technique of geometric invariant theory [M4] together with the special symmetrics present for genus six. All that we need of the general theory is discussed in § 1 of [M3], up to Example 1.13 where Mumford constructs a compactified moduli space $\mathcal{M}'$, similar to the one we need, in genus three. We briefly recall his construction.

The natural action of SL (3) on $V=\mathbf{A}^3$ induces an action on $\mathbf{P}(S^4V)$, the linear system of plane quartic curves. We consider only the semi-stable curves in $\mathbf{P}(S^4V)$, and divide by the action of SL (3). The resulting projective variety $\mathcal{M}'$ is birationally equivalent to $\bar{\mathcal{M}}_3$, since a generic $C\in\mathcal{M}_3$ can be realized as a plane quartic in a unique way (the canonical) up to automorphisms of $\mathbf{P}^2$, and this plane quartic is automorphism-free and stable under SL (3). However, as is beautifully illustrated in [M3], the locus $\mathcal{M}_E$ in $\bar{\mathcal{M}}$ is blown down to the locus of cuspidal curves in $\mathcal{M}'$. (In Proposition 4.3.1 we shall find the precise structure of this blowdown.) In both $\bar{\mathcal{M}}$, $\mathcal{M}'$ there are other exceptional loci, which do not concern us here.

More generally, for fixed d and g we can start with the family of plane curves of degree d, whose geometric genus is $\leqslant g$. SL (3) acts on this, and the quotient of the semi-stable locus is a projective variety $\mathcal{M}_{g,d}$, which comes with a natural rational map

$$\varepsilon\colon \mathcal{M}_{g,d}\to\bar{\mathcal{M}}_g.$$

(Mumford notes that for $d\geqslant 4$, plane curves with only ordinary double points or cusps are actually stable.) For fixed g, $\mathcal{M}_{g,d}$ contains only singular curves if d is too small, and ε is surjective if d is sufficiently large. Mumford's example ($g=3$, $d=4$) is the *only* case where ε is birational.

For genus 6, it seems most profitable to take $d=6$, thus considering plane sextics with 4 double points, or a degeneration of these. In this case ε is generically finite (in fact, five to one; see below) and certainly for generic $X \in \mathcal{M}_5$ any map of degree 6 from $X \cup_p E$ into $\mathbf{P}^2$ which is not constant on X must be constant on E, so that $\varepsilon^{-1}(\mathcal{M}_E)$ consists of cuspidal curves as required. This suggests that $\mathcal{M}'$ should be a 5-sheeted quotient of $\mathcal{M}_{6,6}$; the problem is that there is no group action on $\mathcal{M}_{6,6}$ permuting the sheets. Fortunately such a group can be produced after $\mathcal{M}_{6,6}$ is replaced by an appropriate cover. We describe this first for a slightly different (but birationally equivalent) object, then return to $\mathcal{M}_{6,6}$.

4.2. Forming the quotient $\mathcal{M}'$. Fix four reference points e_1, e_2, e_3, e_4 in $\mathbf{P}^2$, in general position. Let $\mathbf{P}^{15}$ parametrize the linear system of plane sextic curves with double points at the four e_i.

In $\mathbf{PGL}(3)$ there are $4!=24$ elements which permute the e_i, forming a subgroup of $\mathbf{PGL}(3)$ isomorphic to S_4. Let G be the group of Cremona transformations of the plane generated by these linear transformations and the quadratic transformation based at any 3 of the e_i.

LEMMA 4.2.1. $\#G=120$.

Proof. Consider the linear system of cubic curves in $\mathbf{P}^2$ through e_1, e_2, e_3, e_4. It maps $\mathbf{P}^2$ (with the e_i blown up) to $\mathbf{P}^5$ as a quintic del Pezzo surface S. All generators of G preserve this linear system, hence G can be represented as (a subgroup of) the group of linear automorphisms of $\mathbf{P}^5$ preserving S. It is well-known [SR] that the quintic del Pezzo surface contains 10 lines (the 4 exceptional divisors and the 6 lines $\overline{e_i e_j}$) with a symmetric intersection-configuration, each meeting 3 others. Thus G becomes a subgroup of the group G' of permutations of the 10 lines preserving this configuration. Clearly G' is simply transitive on quadruplets of disjoint lines: any such quadruplet can be simultaneously blown down, recovering $\mathbf{P}^2$, and thus mapped to the four exceptional divisors; and once the labeling of $e_1, \ldots, e_4$ is decided, it determines the other six as pairs $e_i e_j$. Now, the first, second, third and fourth elements of a quadruplet can be chosen in 10, 6, 2, 1 ways; thus $\#G'=120$. Since G strictly contains S_4, we have $G=G'$. Q.E.D.

The group G of Cremona transformations of $\mathbf{P}^2$ acts on the system of cubics through the e_i, hence on sextics double at the e_i. We can thus form the quotient $\mathbf{P}^{15}/G$.

LEMMA 4.2.2. *$\mathbf{P}^{15}/G$ is birationally equivalent to $\overline{\mathcal{M}}_6$.*

Proof. There is a natural rational map $\mathbf{P}^{15} \to \overline{\mathcal{M}}_6$ sending a plane curve to its proper transform in S. This map clearly commutes with the action of G, hence factors through

$\mathbf{P}^{15}/G$. Both spaces have the same dimension (15) so we only need to check that a generic $C \in \overline{\mathcal{M}}_6$ is the image of precisely 120 curves of $\mathbf{P}^{15}$.

It follows from a formula of Brill and Noether (cf. [KL]) that a generic curve of genus 6 posesses precisely 5 g_4^1's. (For a generic curve in $\mathcal{M}_{6,S}$ we check this below.) By Riemann–Roch, a line bundle L on $C \in \mathcal{M}_6$ maps C as a plane sextic if and only if the complementary line bundle $\omega_C \otimes L^{-1}$ is a g_4^1 on C. A plane sextic of genus 6 necessarily has 4 double points (properly counted), and an easy dimension-count shows that the generic $C \in \mathcal{M}_6$ cannot be realized as a plane sextic whose 4 double points are not in general position (including a coincident pair, etc.) Further, there is a unique element of **PGL** (3) taking a given ordered quadruplet of points in general position in $\mathbf{P}^2$ to e_1, e_2, e_3, e_4, hence 4! such elements for an unordered quadruplet. We conclude that, indeed, a generic $C \in \mathcal{M}_6$ appears in $\mathbf{P}^{15}$ $5 \cdot 4! = 120$ times. Q.E.D.

Remarks 4.2.3. (i) Given one of these 120 plane representations of C, there are precisely 4 others which are projectively inequivalent, obtained from the original by quadratic transformations based on three of the four e_i (such triplets can be chosen in four ways). However, there is no natural way of re-embedding these four; this corresponds to the sad fact that S_4 is not a normal subgroup of S_5. The five-to-one map

$$f: \mathbf{P}^{15}/S_4 \to \mathbf{P}^{15}/G$$

is consequently not the quotient under a group action.

(ii) On the cheerful side, knowledge of the fiber of f at a point $x_0 \in \mathbf{P}^{15}/S_4$ is sufficient to construct the fiber of $\mathbf{P}^{15}$ over x_0: if

$$f^{-1}(f(x_0)) = \{x_0, x_1, x_2, x_3, x_4\}$$

then the fiber over x_0 can be naturally identified with the set of possible orderings of x_1, x_2, x_3, x_4. Indeed one checks easily that there is a unique element of $\mathbf{P}^{15}$ such that its projective equivalence class (as plane sextic) is x_0, and becomes x_i after quadratic transformation based at the 3 e_j other than e_i $(i = 1, 2, 3, 4)$.

(iii) $\mathbf{P}^{15}/G$ itself is not quite good enough to serve as an $\mathcal{M}'$; for example, Proposition 4.3.1(ii) below would hold only for *almost all* $C \in \mathbf{P}^{15}/G$ over a generic X.

4.2.4. We now return to $\mathcal{M}_{6,6}$. By the proof of Lemma 4.2.2 there is a Zariski-open subset $\mathcal{M}_{6,6}^0$ of $\mathcal{M}_{6,6}$ which is finite (of degree 5) over an open $\mathcal{M}_6^0$ in $\overline{\mathcal{M}}_0$. Let $\Pi^5 \mathcal{M}_{6,6}$ be the fifth Cartesian product, with open subvariety $\Pi^5 \mathcal{M}_{6,6}^0$. Let $\Pi^5{}_{\mathcal{M}_6^0} \mathcal{M}_{6,6}^0$ be the relative Cartesian product, i.e. the subvariety of $\Pi^5 \mathcal{M}_{6,6}^0$ parametrizing quintuplets mapping to single points in $\mathcal{M}_6^0$. Let $\tilde{\mathcal{M}}^0$ be the unique "off-diagonal" component of $\Pi^5{}_{\mathcal{M}_6^0} \mathcal{M}_{0,6}^0$, and

$\tilde{\mathcal{M}}$ its closure in $\Pi^5 \mathcal{M}_{6,6}$. The action of S_5 on $\Pi^5 \mathcal{M}_{6,6}$ clearly restricts to an action on $\tilde{\mathcal{M}}^0$, hence also on $\tilde{\mathcal{M}}$. By Remark 4.2.3(ii), this action commutes with the action of G on $\mathbf{P}^{15}$, hence the quotient

$$\mathcal{M}' = \tilde{\mathcal{M}}/G$$

is birationally equivalent to $\tilde{\mathcal{M}}_6$. By standard arguments, the quotient $\mathcal{M}'$ is a reduced, irreducible projective variety.

4.3. Structure of $\mathcal{M}'$.

PROPOSITION 4.3.1. (i) *The projective variety $\mathcal{M}'$ is birationally equivalent to $\overline{\mathcal{M}}_6$ via the natural map*

$$\varepsilon\colon \mathcal{M}' \to \overline{\mathcal{M}}_6.$$

(ii) *For generic $X \in \mathcal{M}_5$ and all $C \in \varepsilon^{-1}(\mathcal{M}_E \cup \mathcal{M}_S)$ over X, the germ of $\mathcal{M}'$ at C is naturally isomorphic to the base M of the versal deformation of C', where C' is the partial normalization of C given by its proper transform in the quintic del Pezzo surface S.*

(iii) *If C (as ablve) is nodal (i.e. $\varepsilon(C) \in \mathcal{M}_S \setminus \mathcal{M}_{E,S}$) then ε is a local isomorphism at C.*

(iv) *If C (as above) is cuspidal (i.e. $\varepsilon(C) \subset \mathcal{M}_E$) then on a neighborhood of $C \in \mathcal{M}'$, ε consists of three successive blowups (along the cuspidal locus $\mathcal{M}'_C \subset \mathcal{M}'$) followed by two disjoint blowdowns (onto the loci $\mathcal{M}_{\text{har}}$, $\mathcal{M}_{\text{e.a.h.}} \subset \overline{\mathcal{M}}_6$).*

4.3.2. Statement (i) was proved in 4.2. We prove (ii) and (iii) next. We fix a generic X, and restrict C to be a singular curve lying over X as in (ii). By definition there is a relative curve over $\mathbf{P}^{15}$, embedded as a hypersurface in $\mathbf{P}^{15} \times \mathbf{P}^2$. This induces, by definition of the versal deformation, a map from the germ of $\mathbf{P}^{15}$ at C to M. Since G acts on $\mathbf{P}^{15}$ freely near C, the map can be reinterpreted as

$$\zeta\colon (\mathbf{P}^{15}/G)|_C \to M.$$

Claim. ζ is an isomorphism.

Proof. The five-to-one map

$$\varepsilon\colon \mathcal{M}_{6,6} \to \overline{\mathcal{M}}_6$$

induces a five-sheeted cover of the base of any family of genus-6 curves, in particular M. On the one hand, this cover can clearly be identified with $(\mathbf{P}^{15}/S_4)|_C$; on the other, by Remark 4.2.3(ii) the five sheets over M are disjoint (whenever the four double points at C, in our case $e_1, \ldots, e_4$, are distinct) so that each one of them, given by $(\mathbf{P}^{15}/G)|_C$, is mapped isomorphically onto M by ζ. Q.E.D.

This proves statement (ii) for all C such that

$$\mathcal{M}'|_C \approx (\mathbf{P}^{15}/G)|_C$$

i.e. when the five g_4^1's on C' are distinct.

Claim. For generic $C \in \mathcal{M}_S$ over X, the 5 g_4^1's are distinct.

Proof. Recall from Part I that on X there is a 1-parameter family of g_4^1's parametrized by $\alpha(X) = \Theta_{\mathrm{sing}}(J(X))$, a double cover of the plane quintic parametrizing singular (=rank 4) quadrics containing the canonical image $\Phi(X) \subset \mathbf{P}^4$; choosing a g_4^1 on X is equivalent to choosing a singular quadric and a plane-system on it, cutting the g_4^1 on X. This g_4^1 lifts to $C = X/(p \sim q)$ if and only if it maps p, q to the same point in $\mathbf{P}^1$, that is, if and only if the line $\overline{pq}$ in $\mathbf{P}^4$ is contained in a plane of the system. Thus the 5 g_4^1's on C correspond to the 5 singular quadrics, generically distinct, in the pencil of quadrics in $\mathbf{P}^4$ containing $\Phi(X)$ and its chord $\overline{pq}$. Q.E.D.

Similarly we see that for $C = X/(2p)$ cuspidal, the 5 g_4^1's correspond to the singular quadrics in the pencil of quadrics containing $\Phi(X)$ and its tangent line at $\Phi(p)$. For all but finitely many $p \in X$, these are distinct.

Claim. For all C over X, at least one of the five sheets of ε is disjoint from the others (=unramified). Hence $\mathcal{M}'|_C$ is mapped isomorphically to M.

Proof. The first statement follows from the previous claim by an easy dimension-count (keeping in mind that our claim is only for generic X). The map ζ': $\mathcal{M}'|_C \to \mathcal{M}$ is constructed like ζ above, replacing $\mathbf{P}^{15}|_C$ and its relative curve by $\tilde{\mathcal{M}}|_C$ and the relative curve pulled back from $\mathcal{M}_{6,6}|_C$ where it exists by the first statement. The inversion of ζ' is the same as for ζ. Q.E.D.

(If 2 of the 5 sheets of $\mathcal{M}_{6,6}$ come together, then the 120 sheets of $\tilde{\mathcal{M}}$ fit in 60 2-sheeted components, and there is a unique element of G interchanging the sheets in each pair, so the quotient $\mathcal{M}'$ is unramified; similarly in the other possible cases, as long as one sheet remains apart.)

This completes the proof of (ii); (iii) follows from (ii) and the isomorphism

$$\mu: M \to \overline{\mathcal{M}}_6|_C,$$

discussed in 2.4. Q.E.D.

4.3.3. Before proving (iv) we try to make it plausible by the following description. In a neighborhood of $F_{X,p} \subset \mathcal{M}_E \subset \overline{\mathcal{M}}$, the essential features are: The 2 hypersurfaces $\mathcal{M}_E$,

$\mathcal{M}_S$, meeting transversally (by 2.5) along $\mathcal{M}_{E,S}$; the codimension -2 loci $\mathcal{M}_{\mathrm{har}}$, $\mathcal{M}_{\mathrm{e.a.h.}}$ in $\mathcal{M}_E$, parametrizing curves whose elliptic tail is harmonic, respectively equianharmonic; and the fibration of $\mathcal{M}_E$ over $\mathcal{M}_{E,S}$ (or $\mathcal{M}_{\mathrm{har}}$, $\mathcal{M}_{\mathrm{e.a.h.}}$) by $\mathbf{P}^1$'s, the curves $F_{X,p}$. This whole configuration is taken by ε^{-1} to a neighborhood, in $\mathcal{M}'$, of the cuspidal C. Here the only feature is the hypersurface of singular curves, which has a cuspidal singularity (Example 2.2(2)) along $\mathcal{M}'_C$.

Now think of $\mathcal{M}'$ as being "essentially two-dimensional"—by taking a generic plane section, or by considering instead the local deformation M_{local} of the cusp. (Example 2.2(2).) We describe ε on this plane. (See Fig. 2.)

Start: Cuspidal curve in the plane.

First blowup: An exceptional divisor (self-intersection -1) and a smooth curve meeting it tangentially at a unique point.

Second blowup: Two exceptional divisors (self-intersections -2, -1) and a smooth curve, all passing pairwise transversally through a common point.

Third blowup: An exceptional divisor of self-intersection -1, met transversally at three distinct points by two exceptional divisors (self-intersections -3, -2) and the original curve.

Blow-downs: The first two exceptional divisors contract to a triple point (the equianharmonic tail) and a double point (the harmonic tail). The last exceptional divisor parametrizes, in $\overline{\mathcal{M}}$, elliptic tails. The original curve meets it transversally, and still parametrizes the singular curves.

Proof of (iv). What is needed is a method to determine, given a family of plane sextics degenerating to a cuspidal curve, which elliptic tail appears as the limit of the corresponding family of stable curves. The answer becomes evident when we replace again "plane sextics" by their complementary "four-sheeted branched-covers of $\mathbf{P}^1$":

In a branched cover, a double point (resp. cusp) appears as two (resp. three) ramification points coming together. After replacing the base by a finite cover if necessary, we can assume the 18 branch points of curves in our family to be single-valued functions

$$a_1(t), \ldots, a_{18}(t)$$

of the parameter t in the family, and say

$$a_1(0) = a_2(0) = a_3(0) = 0, \quad a_i(0) \neq 0, \quad i > 3.$$

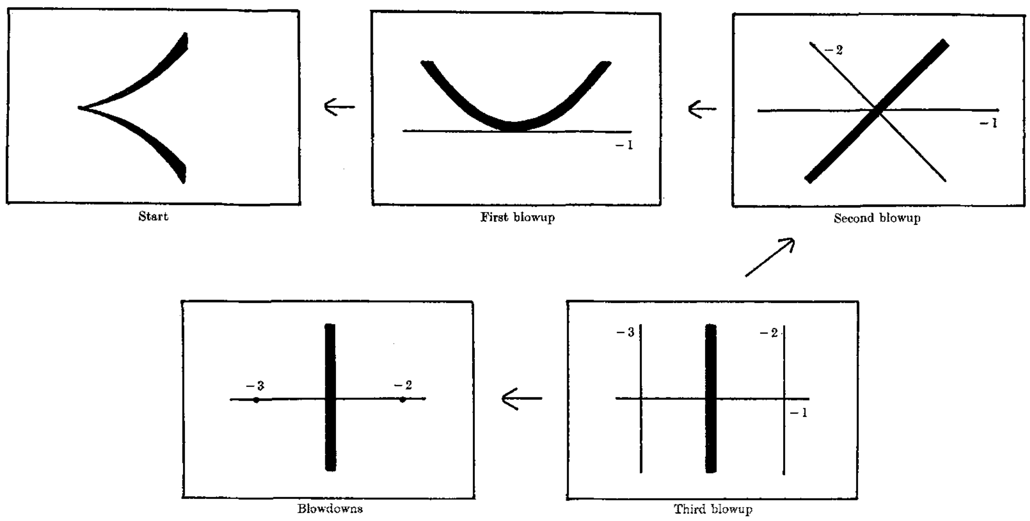

Fig. 2

We obtain an equivalent family, for $t \neq 0$, by composing the covering map with a linear map on $\mathbf{P}^1$ taking $a_1(t)$, $a_2(t)$, $a_3(t)$ to 0, 1, ∞. In the limit all the $a_i(0)$, $i>3$, will go to the same point $\gamma \in \mathbf{P}^1$. The required elliptic tail is clearly the double cover of $\mathbf{P}^1$ branched at 0, 1, ∞, γ.

It now becomes straightforward to verify the proposition. For example, to check that the first exceptional divisor goes entirely to the equianharmonic point: In the (a, b)-plane of M_{local} (Example 2.2(2)) we approach the origin along a straight line of generic slope $b/a=c$; say

$$a = t \quad b = ct$$

then the three coinciding branch points are given, modulo higher terms, by the three solutions of

$$0 = x^3 + ax + b = x^3 + tx + ct$$

and as $t \to 0$ the ratio of any two solutions tends to a cubic root of 1; hence the limit curve is equianharmonic. Similarly we get harmonic curves from $b \sim t^2$, and finally by setting

$$a = t^2 \quad b = ct^3$$

we get triplets (x_1, x_2, x_3) whose ratio is independent of t and varies with c, showing that the third exceptional divisor maps birationally to the modulus-curve of elliptic curves.

Q.E.D.

Remark 4.3.4. Inspection of the proof of (iv) shows that the g_4^1 on the degenerating family (global data on C) is not fully used. In fact, all that is needed is its local piece near the cusp, a two-sheeted cover of a neighborhood of the origin branched at three (moving) points. This data is local, in fact can be read off the local deformation of the cusp. Consequently the proof is valid in any genus, provided $\mathcal{M}'$ (or locally, ε) can be constructed and satisfy (ii). (Same situation for (iii).)

4.4. The factor maps.

4.4.1. Whenever the space $\mathcal{M}'$ exists, in particular in genus 6, we can construct the analogous object $\mathcal{R}'$, a branched-cover of $\mathcal{M}'$ parametrizing allowable double covers of curves $C \in \mathcal{M}'$. We discuss $\mathcal{R}'$ first, leaving the construction to 4.4.2. (In arbitrary genus we settle for the germ R of $\mathcal{R}'$, described in 2.6.)

The only difference between "allowable" here and in Beauville's definition is that C, $\tilde{C}$ are "plane curves" instead of "stable curves". Thus the condition on the double-cover $\tilde{C}$ is that its (arithmetic) genus be $2g-1$, and that $P(C, \tilde{C}) = \ker^0 (Nm\, \pi)$ be an abelian variety. For a cuspidal C one can describe all the allowable $\tilde{C}$ as in the proof of Lemma 5.1 of [B1]. The only double cover $\tilde{C}$ we are interested in is the degeneration of the Writinger covers on nearby singular curves. Thus, for

$$C = X/(2p)$$

we have

$$\tilde{C} = X_1 \coprod X_2/(2p_1 \sim 2p_2),$$

that is, $\tilde{C}$ consists of two copies of X meeting tangentially over p. On each X_i the map $\pi: \tilde{C} \to C$ is the normalization of C.

One checks that the double cover $\pi: \tilde{C} \to C$ is allowable; indeed the (generalized) Jacobian of C is an extension of $J(X)$ by the 1-dimensional additive group $\mathbf{G}_a$; similarly, $J(\tilde{C})$ is a $\mathbf{G}_a$-extension of $J(X) \times J(X)$. The norm map is the identity on the fibers $\mathbf{G}_a$ of the normalization maps; hence ker (Nm) is an abelian variety, isomorphic, of course, to $J(X)$

Near $\tilde{C}$, $\mathcal{R}'$ is a 3-sheeted branched cover of $\mathcal{M}'$, branched along the hypersurface in $\mathcal{M}'$ of singular curves. The 3 covers of a nearby smooth curve which appear in $\mathcal{R}'$ correspond to the 3 non-zero $(\mathbf{Z}/2\mathbf{Z})$-homology 1-classes which are vanishing cycles for the cusp. (The vanishing subspace of $H_1(C, \mathbf{Z}/2\mathbf{Z})$ is a 2-dimensional $\mathbf{Z}/2\mathbf{Z}$ vector space, containing 3 non-zero elements.) For a nearby singular curve, the 3 covers correspond, in the notation of 2.6(i), to R_S and one of the ramified two-sheeted components R_{odd}. Over the whole neighborhood of C, $\mathcal{R}'$ can be identified with the 3-sheeted cover of $\mathcal{M}'$ given by the 3 coinciding ramification—points of the complementary g_4^1, as in 4.3.3. (Similarly for $R \to M$ in all genera, using Remark 4.3.4.) If a, b are local coordinates on M_{local}, as in Example 2.2(2) then a, x are local coordinates on R_{local}, where $x^3 + ax + b = 0$.

4.4.2. We pause to sketch a construction for $\mathcal{R}'$, based on the existence of compactified Picard schemes as constructed in [AK].

Given a family $\mathcal{C} \to \mathcal{S}$, one has the classical family $J(\mathcal{C})$ of "generalized Jacobians" over $\mathcal{S}$. The fiber corresponding to a curve C is an algebraic group parametrizing invertible sheaves on C. In particular, this is not complete unless C is smooth: One obtains extensions of Jacobians (of the normalization NC) by the groups $\mathbf{G}_m$, $\mathbf{G}_a$ corresponding to nodes, cusps in C, etc.

In order to obtain a compactification $\overline{J(\mathcal{C})}$ of this family, Altman and Kleiman consider rank-1 torsion-free sheaves on C instead of only locally-free sheaves. Under very general conditions they obtain a projective scheme $\overline{J(\mathcal{C})}$ parametrizing equivalence-classes of these. The singular locus of this compactification is precisely its boundary, parametrizing non-locally-free, torsion-free, rank-1 sheaves.

The easiest example is obtained from the standard family $\mathcal{C}$ of plane cubics,

$$y^2 = x^3 + ax + b.$$

Since these all pass through a fixed point (at ∞), the family of Jacobians (away from the discriminant locus) is isomorphic to the original family, if the fixed point is taken as origin. We see now that the family of generalized Jacobians embeds naturally in $\mathcal{C}$ as the complement of the locus of singular points, and the compactification is $\mathcal{C}$ itself. For a nodal cubic

$$y^2 = (x+2t)(x-t)^2$$

the non-invertible sheaf is the ideal-sheaf of the node, deforming along either branch. (The generalized Jacobian is $\mathbf{G}_m$ and its compactification a nodal $\mathbf{P}^1$.) For a cuspidal cubic

$$y^2 = x^3$$

the generalized Jacobian is $\mathbf{G}_a$, its compactification a cuspidal $\mathbf{P}^1$, and the funny sheaf is $(s, s^2)\mathcal{O}_C$, where s is a coordinate on the normalization near the origin (so $x = s^2$, $y = s^3$). These represent the main phenomena that occur in arbitrary genus.

To construct $\mathcal{R}'$, we start with a "universal curve" $\mathcal{C}' \to \mathcal{M}'$. (This exists in a neighborhood of our cuspidal curves by Proposition 4.3.1(ii), and globally after replacing $\mathcal{M}'$ by an appropriate finite branched cover $\tilde{\mathcal{M}}$.) Over $\mathcal{M}$ (=smooth curves) $\mathcal{R}$ is the kernel of the squaring map in the corresponding family of Jacobians, or the (non-zero) points of order 2. We now define $\mathcal{R}'$ to be the closure, in $\overline{J(\mathcal{C}')}$, of $\mathcal{R}$.

To describe this $\mathcal{R}'$ near nodal or singular curves, note that given an extension

$$0 \to E \to A \to J \to 0$$

of J by an elliptic curve E, the points of order 2 in A map onto those in J with fiber $(\mathbf{Z}/2\mathbf{Z})^2$; When E degenerates to a nodal $\mathbf{P}^1$, say with $0, \infty$ identified, half of these 2^{2g} points remain apart while the others join in 2^{2g-2} pairs corresponding to the double point in E; in addition there is a marked non-trivial point of order 2 mapping to $0 \in J$, namely the point $-1 \in E$. These latter $2^{2g-1}+1$ points correspond to the sheets of $\mathcal{R}'$, as in 2.6(i). We leave the details to the reader, as well as the verification that near cuspidal C, $\mathcal{R}'$ behaves as claimed in 4.4.1. (*Hint.* given an unordered pair $0, \infty \in \mathbf{P}^1$ and a point $1 \in \mathbf{P}^1$, one has the harmonic point -1. Fixing 1 and letting $0, \infty$ come together, we see that -1 also acquires the same limit as $0, \infty$. Thus the isolated sheet and one ramified pair of sheets, on the nodal curve, all come together on the cuspidal!)

COROLLARY 4.4.3. *The natural map*

$$p\colon \bar{\mathcal{R}} \to \mathcal{R}'$$

is birational. For generic X and all $q \in X$, arbitrarily small neighborhoods of $\tilde{F}_{X,q}$ in $\bar{\mathcal{R}}$ contain p^{-1} of some neighborhood of $C = X/(2q)$ in $\mathcal{R}'$. p can be factored in a sequence of blowups followed by blowdowns, so that the total transform of $\mathcal{R}_E \subset \bar{\mathcal{R}}$ under the blowups is eventually blown down into the cuspidal locus $\mathcal{R}_C \subset \mathcal{R}'$. On a neighborhood of $\mathcal{R}_S \setminus \mathcal{R}_{E,S}$, p is biregular.

In fact, from the previous description of $\mathcal{R}'$ and Proposition 4.3.1(iv), one can easily write down the precise sequence of blowups–blowdowns, as in 4.3.3.

In a neighborhood of $\tilde{F}_{X,p}$ in $\bar{\mathcal{R}}$, we have: The hypersurface $\mathcal{R}_E$ (mapping 3 to 1 to $\mathcal{M}_E$) (the infinitesimal version R_E is described in 2.6(ii)), the two hypersurfaces $\mathcal{R}_S$, $\mathcal{R}_{\text{odd}}$ (mapping onto $\mathcal{M}_S$ with degrees 1, 2) (cf. 2.6(i)), meeting $\mathcal{R}_E$ transversally in $\mathcal{R}_{E,S}$, $\mathcal{R}_{E,\text{odd}}$; three codimension-2 loci in $\mathcal{R}_E$, *one* for equianharmonic tails and *two* for the harmonic tails (since monodromy there interchanges two semi-periods and fixes the third), and the fibration by rational curves $\tilde{F}_{X,q}$.

In a neighborhood of $(C, \tilde{C})$ in $\mathcal{R}'$ we have two hypersurfaces $\mathcal{R}_{\text{odd}}$, $\mathcal{R}_S$ (over the cuspidal hypersurface $\mathcal{M}_S$ parametrizing singular curves), each smooth, meeting tangentially along the cuspidal locus $\mathcal{R}'_C$. ($\mathcal{M}_S$ is the branch locus for $\mathcal{R}' \to \mathcal{M}'$, $\mathcal{R}_{\text{odd}}$ the ramification.) In local coordinates (a, b on M_{local}; a, x on R_{local}, as above) the equations are:

$$\begin{aligned} &M_S\colon 4a^3+27b^2=0 && \text{(cf. Example 2.2(2))}\\ &R_S\colon 4a+3x^2=0 \\ &R_{\text{odd}}\colon a+3x^2=0. \end{aligned}$$

Finally, the steps of blowing up and down correspond naturally to those in 4.3.3, replacing loci in $\mathcal{M}$ by their analogues in $\mathcal{R}$.

4.4.4. It remains to construct the map

$$\mathcal{P}'\colon \mathcal{R}_g' \to \mathcal{A}_{g-1}.$$

We use the notation of 4.4.2. The family

$$\mathcal{C}' \to \mathcal{M}'$$

lifts to

$$\mathcal{C}' \to \mathcal{R}'$$

and this relative curve admits a double cover

$$\pi'\colon \tilde{\mathcal{C}}' \to \mathcal{C}'$$

such that over a point of $\mathcal{R}'$ we have the double cover it parametrizes. We form the families $J(\mathcal{C}')$, $J(\tilde{\mathcal{C}}')$ over $\mathcal{R}'$ of generalized Jacobians, with the norm map

$$Nm\colon J(\tilde{\mathcal{C}}') \to J(\mathcal{C}')$$

induced by π'. We define $P(\tilde{\mathcal{C}}')=\ker^0(Nm)$. A priori this is a family of algebraic groups, but by the restriction to allowable covers, all of these are abelian varieties, varying nicely over $\mathcal{R}'$, inducing the desired map $\mathcal{P}'$. By Beauville's theorem (cf. Theorem I.1.1 and I.1.3) $\mathcal{P}'$ is a proper map.

Clearly the composition

$$\mathcal{P}' \circ p$$

equals $\bar{\mathcal{P}}$ on a punctured neighborhood of $\mathcal{R}_E$ in $\bar{\mathcal{R}}$, hence equals $\bar{\mathcal{P}}$ everywhere by continuity. For $g=6$, since p is birational

$$\deg \bar{\mathcal{P}} = \deg p$$

so we only need compute the (local) degree of $\bar{\mathcal{P}}$ along $\mathcal{R}_S$, as claimed.

PROPOSITION 4.4.5. *For $(C, \tilde{C}) \in \mathcal{R}'_{\text{cusp}}$ over a generic $X \in \mathcal{M}_{g-1}$, $\ker(\mathcal{P}'^*)$ consists of quadrics in the canonical space of X containing $\Phi(X)$ and its tangent line $T_p\Phi(X)$ at the normalization of the cusp.*

Proof. We use the notation and method of § 3. For C near $C_0=X/(2p)$, the codifferential is

$$\mathcal{P}^*\colon S^2T^*(P(\tilde{C})) \to T^*_{\tilde{C}}\mathcal{R}'$$

as before, $T^*(P(\tilde{C}_0)) \approx T^*J(X) \approx H^0(X, \omega_x) \approx H^0(C_0, \omega_{C_0} \otimes \eta)$. However, there are two differences:

(1) η is no longer locally-free at p, but rather

$$\eta_p \approx (t, t^2)\, O_p \subset O_p$$

which is only torsion-free (cf. 4.4.2).

(2) $T^*_{\tilde{C}_0} \mathcal{R}' \not\approx T^*_{C_0} \mathcal{M}'$. Here $\mathcal{R}'$ is a 3-sheeted cover of $\mathcal{M}'$ (cf. 4.4.1, Example 2.2(2)) branched along the hypersurface

$$4a^3 + 27b^2 = 0$$

and a, x are the coordinates on R_{local} corresponding to a, b on M_{local}. ($x^3 + ax + b = 0$.) Letting $\pi\colon R \to M$ (or $\mathcal{R}' \to \mathcal{M}'$) denote the natural projection, we have

$$\pi^* da = da$$
$$\pi^* db = -x\, da - (3x^2 + a)\, dx.$$

This means that over $\mathcal{R}'$, $\pi^* T^* \mathcal{M}'$ is the subsheaf of $T^* \mathcal{R}'$ whose sections have a vanishing db-coefficient along $\mathcal{R}'_{\text{cusp}}$. Instead, we think of sections of $T^*\mathcal{R}'$ as certain meromorphic sections of $\pi^* T^* \mathcal{M}'$, a locally free sheaf whose fiber over $\tilde{C} \in \mathcal{R}'$ is $H^0(C, \Omega_C \otimes \omega_C)$.

We can now compute $\mathcal{D}^*$. As in § 3, ker ($\mathcal{D}^*$) is contained in the space of quadrics containing $\Phi(X)$. (This reflects the fact that "$\mathcal{D}$ looks the same near all (generic) X" so $\mathcal{D}_*$ surjects at least onto $T_* \mathcal{J}_{g-1} \subset T_* \mathcal{A}_{g-1}$.) Since for $C \in \mathcal{M}_S \setminus \mathcal{M}_{E,S}$ we know that $\mathcal{D}^*$ vanishes on quadrics containing $\Phi(X)$ and the relevant chord $\overline{\Phi(p), \Phi(q)}$, the same follows at a cuspidal C_0 for quadrics containing $\Phi(X)$ and its tangent line at p, by continuity. Hence, ker ($\mathcal{D}^*$) must be either the system of quadrics through $\Phi(X)$, or its codimension-1 subsystem of quadrics through $T_p\Phi(X)$. We claim it is the latter.

Fix a quadric Q, containing $\Phi(X)$ but not $T_p\Phi(X)$, hence not containing nearby chords either. Consider a family of singular curves C_t with common normalization X, degenerating to C_0. ($C_t = C/(p_1(t) \sim p_2(t))$ where in local coordinate s on X near p, $p_i(t)$ is the point with coordinate $s_i(t) = \pm t^{1/2}$.) This family maps to M_{local} by $a(t) = -3t^2$, $b(t) = 2t^3$ (clearly satisfying $4a^3 + 27b^2 = 0$) and lifts to R_{local} by $x(t) = -2t$. By Proposition 3.4.1, at C_t, $\mathcal{D}^*(Q)$ is a torsion-section of $H^0(C_t, \Omega \otimes \omega)$, hence in the deformation M_{C_t} it vanishes on the local deformation of the node and must be a multiple of $d(4a^3 + 27b^2) = 12a^2 da + 54b\, db = 108t^4 da + 108t^3 db$. As $t \to 0$, this should acquire a first-order pole. Hence, up to an invertible factor, $\mathcal{D}^*(Q)$ gives

$$da + t^{-1} db$$

in $H^0(\Omega\otimes\omega)$, and therefore by applying π^*

$$3da-9t\,dx$$

as a section of $T^*\mathcal{R}'$. In particular as $t\to 0$ we see that $\mathcal{P}^*(Q)$ is a non-zero multiple of da, as required. Q.E.D.

§ 5. The degree

As in Parts II, III, we restrict to the case $g=6$ and compute the local degree of $\mathcal{P}$ along $\mathcal{R}_S$, using the previous results on ker $(\mathcal{P}^*)$.

THEOREM 5.1. *The local degree of the Prym map*

$$\mathcal{P}\colon \bar{\mathcal{R}}_6\to\mathcal{A}_5$$

at the boundary equals 16.

Proof. By Lemma 1.4, the relevant components are $\mathcal{R}_S$, $\mathcal{R}_E$ and $\bar{\mathcal{R}}_T\setminus\mathcal{R}_T$. The latter is part of the trigonal component extending into $\mathcal{R}_6$ itself, and was treated in part III. By Corollary 4.4.3, $\mathcal{R}_E$ is blown down into $\mathcal{R}'_{\text{cusp}}$ in $\mathcal{R}'$, so we only need compute the degree along $\mathcal{R}'_S\subset\mathcal{R}'$. By Propositions 3.4.1 and 4.4.5, for $C=X/(p\sim q)$ where $p,q\in X$ are distinct or coincide, ker $(\mathcal{P}'^*)$ is 2-dimensional: $\Phi(X)$ is the complete intersection in $\mathbf{P}^4$ of 3 quadrics in general position, and the line $\overline{pq}$ (tangent at p, if $q=p$) imposes one linear condition on these quadrics.

By Lemma I.3.2, the degree of $\mathcal{P}'$ can be computed after blowing up $\mathcal{J}_5\subset\mathcal{A}_5$ and restricting to the exceptional fiber. By Lemma I.3.1, this is the projectivization of the differential $\mathcal{P}_*$. (Since $\mathcal{R}'_S$ is a hypersurface, it need not be blown up.)

Restricting to the fiber over a fixed, generic, $X\in\mathcal{M}_5$, the map becomes

$$f\colon S^2X\to\mathbf{P}^2.$$

Here the symmetric product S^2X parametrizes singular C over X, and $\mathbf{P}^2$ is the projective normal space to $\mathcal{M}_5$ in $\mathcal{A}_5$ at X. As in Part II, § 5, this is dual to the $\mathbf{P}^2$ parametrizing the quadrics through $\Phi(X)$. The map f takes $(p,q)\in S^2X$ to the pencil of quadrics through $\overline{\Phi(p),\Phi(q)}$, so that deg (f) equals the number of chords of $\Phi(X)$ contained in the intersection of two of the quadrics, in general position.

LEMMA 5.2. *The intersection of two (transversal) quadrics in* $\mathbf{P}^4$ *contains precisely* 16 *lines.*

LEMMA 5.3. *The canonical curve* $\Phi(X)$ *meets each of the* 16 *lines, twice.*

The combination of these two lemmas will clearly complete the proof of the theorem. Lemma 5.2 is classical: the number of lines on any del Pezzo surface is computed in [SR], in particular the intersection of 2 quadrics in $\mathbf{P}^4$ is shown there to be the blowup of $\mathbf{P}^2$ at 5 points in general position so that the 16 lines are given by the 5 exceptional divisors, 10 lines joining pairs of points, and the unique conic through all 5. (More directly, perhaps, one can use Schubert claculus to compute the number 4^n of linear spaces $\mathbf{P}^{n-1}$ in the intersection of 2 quadrics in $\mathbf{P}^{2n}$, cf. [Do].)

Recalling that $\Phi(X)$ is the complete intersection

$$Q_0 \cap Q_1 \cap Q_2$$

of three quadrics and letting l be a line in $Q_1 \cap Q_2$, Lemma 5.3 follows immediately:

$$\#(\Phi(X) \cap l)_{Q_1 \cap Q_2} = \#(Q_0 \cap l)_{\mathbf{P}^4} = 2. \qquad \text{Q.E.D.}$$

Remark 5.4. Precisely the same local picture holds in all g: $\bar{\mathcal{P}}$ factors through $\mathcal{R}'$ (or at least infinitesimally through R) and the factor map $\mathcal{P}'$ looks like

$$f = S^2X \to \mathbf{P}((\text{quadrics through } \Phi(X))^*).$$

But for $g \geqslant 7$, f is generically injective (the space on the right has dimension $((g-3)(g-4)/2) - 1$, which is $\geqslant 5$ for $g \geqslant 7$).

Part V. Applications

§ 1. Cubic threefolds

In much of Prym theory, a recurrent difficulty is the presence of sporadic loci, providing counterexamples to desired statements or preventing "generic" identities from holding everywhere. The point of our main result I.2.1 is to ensure, under favorable conditions, that no such sporadic components may pass undetected. In this section we prove a typical example.

THEOREM 1.1. *For a generic cubic threefold $X \subset \mathbf{P}^4$, if the intermediate Jacobian $J(X)$ is isomorphic (as principally polarized abelian varieties) to $P(C, \eta)$ for some $(C, \eta) \in \mathcal{R}_6$, then C is a plane quintic curve and η an odd semi-period, $(C, \eta) \in \mathcal{R}_C$.*

To prove this, we apply the technique developed in Parts II, III, IV to compute the degree of $\mathcal{P}$ along

$$\mathcal{R}_X = \mathcal{P}^{-1}(J(X)) \cap \mathcal{R}_C.$$

We see below that this degree is 27. Moreover, the codifferential is of maximal rank so $\mathcal{R}_X$ is isolated in $\mathcal{P}^{-1}(J(X))$, in other words it is a connected, as well as irreducible, component, hence there can be no collapsing components such as $\mathcal{R}_E$ in Part IV. These two facts together imply $\mathcal{R}_X = \mathcal{P}^{-1}(J(X))$. In the next section we shall see moreover that all of $\mathcal{R}_C$ is mapped by $\mathcal{P}$ to intermediate Jacobians of cubic threefolds.

1.2. We recall some standard facts on cubic threefolds. The original work [CG] or the expository [T1] are good references for the projective-geometric facts, while the identifications of cotangent spaces etc. are paraphrased from [G].

Let $X \subset \mathbf{P}^4$ be a smooth cubic hypersurface. X contains a two parameter family of lines, parametrized by the *Fano surface* $F(X)$. (*Proof.* A generic hyperplane section of X contains a finite number (27) of lines.) The intermediate Jacobian $J(X)$ is isomorphic, as principally polarized abelian varieties, to $A^0(F)$, the Albanese variety of zero-cycles on $F(X)$ of degree zero, modulo abelian equivalence. Fixing a line $l \in F$ we obtain a map

$$\varrho_l: F(X) \to J(X)$$

by

$$\varrho: l_1 \to [l_1] - [l].$$

The Θ-divisor in $J(X)$ is the image of $F \times F$ under the induced map

$$\varrho: F \times F \to J(X)$$
$$(l_1, l_2) \to [l_1] - [l_2].$$

Under ϱ the diagonal in $F \times F$ collapses to $0 \in J(X)$ which is a triple point of the image Θ, in fact its only singularity. The projectivized tangent space

$$\mathbf{P}(T_0(J(X)))$$

can be naturally identified with the ambient $\mathbf{P}^4$. (This identification leads to Torelli's theorem for cubic threefolds: X can be recovered as the tangent *cone* to Θ at its (only) triple point.)

1.3. The relation with Pryms is via Mumford's conic-bundle theory, cf. [B2] or the appendix to [CG]. Let $\tilde{X}_l$ be the blowup of X along l, and

$$f: \tilde{X}_l \to \mathbf{P}^2$$

the projection map.

(i) f makes $\tilde{X}_l$ into a conic-bundle: for generic $p \in \mathbf{P}^2$, $f^{-1}(p)$ is a conic in X (meeting l in two points). For p in a curve $C \subset \mathbf{P}^2$,

$$f^{-1}(p) = l_1(p) \cup l_2(p)$$

is the union of two lines coplanar with l. Thus there is an isomorphism

$$J(X) \approx \mathcal{P}(C, \tilde{C}, \eta)$$

where $\tilde{C}$ is the double cover of C given by the two components in $f^{-1}(p)$, $p \in C$, and η the corresponding semi-period.

(ii) C is a plane quintic, smooth for generic l. (*Proof.* A generic hyperplane section of X contains 5 line-pairs coplanar with l.) Similarly, $\tilde{C}$ is smooth and π: $\tilde{C} \to C$ unramified for generic l, and allowable in any case.

(iii) The map

$$\Psi: C \to \mathbf{P}^4,$$

sending $p \in C$ to the intersection

$$l_1(p) \cap l_2(p) \in X \subset \mathbf{P}^4,$$

is the Prym-canonical map of (C, η). Indeed the Abel–Prym map

$$\psi: \tilde{C} \to \mathcal{P}(C, \eta) \approx J(X) \approx A^0(F)$$

is just the restriction to $\tilde{C} \subset F(X)$ of ϱ_l. The Prym-canonical image of $l_1(p) \in \tilde{C}$ is the derivative of ψ at $l_1(p)$ (as in I.4.3) and corresponds to a point of $l_1(p) \subset \mathbf{P}^4$. We conclude as in Proposition III.1.5 that $\Psi(p)$ is indeed $l_1(p) \cap l_2(p)$, by repeating the argument for $l_2(p)$.

(iv) We now see that η is odd so that $(C, \eta) \in \mathcal{R}_C$. Indeed $\mathcal{O}_{\mathbf{P}^2}(1)|_C \otimes \eta$ has a unique effective divisor (compare note after Lemma II.3.2) given by the 5-point intersection $l \cap \Psi(C)$, since Ψ is given by

$$\omega_C \otimes \eta \approx \mathcal{O}_{\mathbf{P}^2}(2)|_C \otimes \eta$$

while after projection from l, C is mapped to $\mathbf{P}^2$ by

$$\mathcal{O}_{\mathbf{P}^2}(1)|_C.$$

1.4. It follows from the foregoing that, as abstract varieties,

$$F(X) \approx \mathcal{R}'_X \subset \mathcal{R}_X.$$

We let $\mathcal{R}'_C \subset \mathcal{R}_C$ denote the closure of the union of $\mathcal{R}'_X$ for all smooth cubic threefolds X. This is clearly an irreducible 12-dimensional subvariety of $\mathcal{R}_C$. (In § 2 we will see that actually $\mathcal{R}'_C = \mathcal{R}_C$.) Let $\mathcal{A}_C \subset \mathcal{A}_5$ denote the closure of the locus of intermediate Jacobians of cubic threefolds. To compute the local degree of $\mathcal{P}$ along $\mathcal{R}'_C$, and prove the theorem, we are led as in III.3.2 to the blowup diagrams:

$$\begin{tikzcd} \tilde{\mathcal{D}}\colon \tilde{\mathcal{R}}_6 \arrow[r] \arrow[d] & \tilde{\mathcal{A}}_5 \arrow[d] \\ \mathcal{D}\colon \mathcal{R}_6 \arrow[r] & \mathcal{A}_5 \end{tikzcd}$$

and on exceptional loci,

$$\begin{tikzcd} \tilde{\mathcal{D}}_e\colon \tilde{\mathcal{R}}'_C \arrow[r] \arrow[d, "\pi_1"] & \tilde{\mathcal{A}}_C \arrow[d, "\pi_2"] \\ \mathcal{D}_e\colon \mathcal{R}'_C \arrow[r] & \mathcal{A}_C \end{tikzcd}$$

where π_1 is a $\mathbf{P}^2$-bundle-map and π_2 a $\mathbf{P}^4$-bundle-map.

For a cubic threefold X, the fiber $\pi_2^{-1}(J(X))$ can be identified with the dual of the ambient $\mathbf{P}^4$ of X, as follows: The cotangent space

$$T^*_{J(X)}\mathcal{A}_5 = S^2T^*_0(J(X))$$

consists of all quadrics in $\mathbf{P}^4=\mathbf{P}(T^*_0(J(X)))$. Among these, Griffiths shows in [G] that the conormal space

$$N^*_{J(X)}(\mathcal{A}_C\setminus\mathcal{A}_5)$$

corresponds to those quadrics X_p polar to points p of $\mathbf{P}^4$ with respect to X. Hence naturally

$$\mathbf{P}(N^*)\approx\mathbf{P}^4$$

and

$$\pi_2^{-1}(X)\approx\mathbf{P}(N)\approx(\mathbf{P}^4)^*.$$

Since $\mathcal{R}'_C$ is an unramified cover of M_Q (the moduli space of plane quintics) we can identify the fiber of π_1 over $(C,\eta)\in\mathcal{R}'_C$, as in Part II, § 5, with the dual of the ambiant $\mathbf{P}^2$ of C. Fixing X, we can describe this $\mathbf{P}^2_l$ in terms of the line $l\in F(X)$: $\mathbf{P}^2_l$ is the space of planes through l in $\mathbf{P}^4$, and $(\mathbf{P}^2_l)^*$ is the subspace of $(\mathbf{P}^4)^*$ dual to l.

Let $\tilde{\mathcal{R}}_X$ denote $\pi_1^{-1}(\mathcal{R}'_X)$, and $\tilde{\mathcal{D}}_X$ the restricted map. One might guess the following:

LEMMA 1.5. *The map*

$$\tilde{\mathcal{D}}_X\colon \tilde{\mathcal{R}}_X=\bigcup_{l\in F(X)}(\mathbf{P}^2_l)^*\to(\mathbf{P}^4)^*$$

is the natural injection on each $(\mathbf{P}^2_l)^*$.

Proof. By 1.3(iii) the $\mathbf{P}^4$ ambiant for X is also ambiant for the Prym-canonical $\Psi(C)$, so that Proposition I.4.1 applies to this $\mathbf{P}^4$: the codifferential $\mathcal{D}^*$ is the restriction of quadrics from $\mathbf{P}^4$ to $\Psi(C)$. The lemma is equivalent to the following dual statement: For $p\in\mathbf{P}^4\setminus$

l, $\mathcal{D}^*(X_p)$ is the divisor on C cut by the quartic $C_{p'}$ polar with respect to C to the point $p' \in \mathbf{P}_l^2 = P^4/l$ corresponding to p. This is clear, recalling that for $q \in C$,

$$\Psi(q) \in X_p \Leftrightarrow \text{the line } \overline{p\Psi(q)} \text{ is tangent to } X \text{ at } \Psi(q) \Leftrightarrow p \in T_{\Psi(q)}X,$$

and that

$$q \in C_{p'} \Leftrightarrow \text{the line } \overline{p'q} \text{ is tangent to } C \text{ at } q \Leftrightarrow p' \in T_q C$$
$$\Leftrightarrow p \in \operatorname{span}(l, T_{\Psi(q)}\Psi(C)) = T_{\Psi(q)}X. \qquad \text{Q.E.D.}$$

1.6. The proof of the theorem can now be completed. By Lemma 1.5, $\tilde{\mathcal{D}}_e$ has the maximal possible rank. Its degree is 27 since a generic hyperplane section of X contains 27 lines, hence comes from 27 planes $(\mathbf{P}_l^2)^*$. This is the total degree of $\mathcal{D}$, so we have exhausted the fiber $\mathcal{D}^{-1}(\mathcal{A}_C)$. Q.E.D.

§ 2. Explicit construction

In appendix C to [CG], Clemens and Griffiths suggest that a cubic threefold should be recoverable from the data of a plane quintic C with an odd semi-period η. This is known for those (C, η) that come from cubic threefolds X, so the question is essentially one of irreducibility (almost-all $(C, \eta) \in \mathcal{R}_C$ arise from cubics) and degeneration. Our treatment, though, is purely synthetic.

2.1. Let

$$\Psi: C \to \mathbf{P}^4$$

be the Prym-canonical map of (C, η), and

$$D = p_1 + \dots + p_5$$

the effective divisor in $|\mathcal{O}_{\mathbf{P}^2}(1)|_C \otimes \eta$. The 5 points $\Psi(p_i)$ are colinear in $\mathbf{P}^4$, since projection from them yields the planar C. Let $l \subset \mathbf{P}^4$ be the corresponding line.

We build up a skeleton for X. X should contain $\Psi(C)$ and l, and for each $q \in \Psi(C)$ two lines through q meeting l; hence

$$T_q X = \operatorname{span}(l, T_q \Psi(C)).$$

Next we recover the pencil of quadrics X_p polar to X with respect to points $p \in l \subset X$. The system of cubics in $\mathbf{P}^2$ through D maps $\mathbf{P}^2$ birationally to a (quartic del Pezzo) surface $S \subset \mathbf{P}^4$. The image of C under this map spans $\mathbf{P}^4$ and the restricted linear system is $\omega_C \otimes \eta$, so this $\mathbf{P}^4$ can be identified with the Prym-canonical space. Thus we have

$$\Psi(C) \subset S \subset \mathbf{P}^4$$

and S is the transversal complete intersection of two quadrics in $\mathbf{P}^4$ (cf. [SR] and compare the discussion following Lemma IV.5.2). Note that for the desired X, S should be the base locus of the pencil of polar quadrics X_p, $p \in l$: Each X_p should contain $\Psi(C)$ and l since the tangent spaces to X at points of $\Psi(C)$, l contain l, and any quadric containing $\Psi(C)$ must contain S.

LEMMA 2.2. *In the system $\mathbf{P}^{34}$ of cubic threefolds in $\mathbf{P}^4$ there is a subsystem $\mathbf{P}^{10}$ of cubics through $\Psi(C)$. All of these contain l. There is a unique X for which all $X_p (p \in l)$ contain $\Psi(C)$.*

Proof. Consider the restriction map

$$H^0(\mathbf{P}^4, O(3)) \to H^0(S, O(3)|_S)$$

since S is $\mathbf{P}^2$ with D blownup,

$$H^0(S, O(3)|_S) \approx H^0(\mathbf{P}^2, O(9) \otimes I_D^3)$$

and is of dimension

$$\binom{9+2}{2} - 5 \cdot \binom{4}{2} = 25.$$

On the other hand,

$$\dim H^0(\mathbf{P}^4, O(3)) = 35$$

and the kernel of restriction is 10 dimensional, consisting of cubics of the form

$$H_1 Q_1 + H_2 Q_2$$

where H_i are linear and Q_1, Q_2 are two distinct quadrics through S. Hence restriction is surjective.

The further restriction

$$H^0(\mathbf{P}^2, O(9) \otimes I_D^3) \approx H^0(S, O(3)|_S) \to H^0(C, (\omega_C)^3 \otimes \eta)$$

has a 1-dimensional kernel, the only nonic curve with triple points at D being

$$C + 2 \cdot A$$

where A is the unique conic meeting C tangentially along D,

$$C \cdot A = 2D.$$

(Conics in $\mathbf{P}^2$ cut the complete $|\omega_C|$ on C.)

Altogether, there is an 11-dimensional subspace in $H^0(\mathbf{P}^4, \mathcal{O}(3))$ restricting to 0 on C, or a system $\mathbf{P}^{10}$ of cubics containing $\Psi(C)$. All of these contain l since they meet it in ≥ 5 points.

To prove the existence of X, it suffices to check that for fixed $p \in l$, the condition "X'_p contains $\Psi(C)$" imposes 5 linear conditions on a cubic $X' \in \mathbf{P}^{10}$ (containing $l, \Psi(C)$), so the same condition for all $p \in l$ imposes ≤ 10 conditions. (The X_p vary *linearly* in p.) Indeed, the quadric X'_p cuts on C a divisor in $H^0(C, (\omega_C)^2)$ which contains $D = l \cap \Psi(C)$, hence residually a divisor in $H^0(C, \omega_C \otimes \eta)$ which is 5-dimensional, so there is a subspace in $\mathbf{P}^{10}$ of codimension ≤ 5 of cubics X' such that $X'_p \supset \Psi(C)$, as required.

Finally, the uniqueness of X up to projective automorphisms follows from the following result and the Torelli theorem proven in [CG]. The absolute uniqueness follows, since all conditions on X are linear, while no non-trivial linear family of cubics can be of constant projective type (since, for example, it contains singular cubics). Q.E.D.

PROPOSITION 2.3. *For the cubic X of Lemma 2.2,*

$$J(X) \approx \mathcal{P}(C, \eta).$$

In fact, $\Psi(C)$ is the Prym-canonically embedded plane quintic derived from X, l.

Proof. For each $p \in C$, the plane $\Pi = \langle l, \Psi(p)\rangle$ is contained in $T_{\Psi(p)}X$; hence $\Pi \cap X$ consists of l and a conic singular at p. Q.E.D.

As the rich get richer, the existence of our construction proves another one.

PROPOSITION 2.4. *For $(C, \eta) \in \mathcal{R}_C$ (odd double cover of a plane quintic curve), the branch locus of the Gauss map*

$$C\colon \Theta \to (\mathbf{P}^4)^*$$

on the theta divisor Θ of $\mathcal{P}(C, \eta)$, is the projective dual variety of a cubic threefold $X \subset \mathbf{P}^4$, and $J(X) \approx \mathcal{P}(C, \eta)$.

Proof. Clemens and Griffiths prove this for those (C, η) which arise from an X. Q.E.D.

§ 3. The structure of the fiber

We saw in § 1 that for a cubic threefold X, $\mathcal{P}^{-1}(J(X)) \approx F_X$, the Fano surface of lines in X. After blowup, we found the fiber

$$(\tilde{\mathcal{P}}_e)^{-1}(A), \quad A \in \tilde{\mathcal{A}}_C$$

to have cardinality 27. In fact, this fiber comes equipped with an extra structure, that of the incidence-correspondence for lines on a cubic surface. Thus given one of the 27 objects in $(\tilde{\mathcal{P}}_e)^{-1}(A)$, the others break into 10 corresponding to the incident lines and 16 corresponding to skew lines to the original l. This suggests that the general fiber of $\mathcal{P}$ might also carry some extra structure.

The results over Jacobians fit with this possibility: the fiber, after blowing up, breaks into sets of 1, 10, 16 objects, as would the lines of a cubic surface after one of them is marked. The marked line can be blown down yielding a quartic del Pezzo surface, and indeed we saw in Part IV, § 5, that the 16 double-covers arising from the boundary components correspond naturally to lines on a quartic del Pezzo, hence carry the structure induced by the incidence correspondence on the 16 lines on a cubic surface not meeting a given one.

We formalize these indications in the following:

Conjecture. The Galois group of the Prym map

$$\mathcal{P}\colon \bar{\mathcal{R}}_6 \to \mathcal{A}_5,$$

as a subgroup of the symmetric group S_{27}, is isomorphic to the group of symmetries of the incidence-correspondence of lines on a smooth cubic surface.

This group is well-known, cf. [Di]. It has order 51840, and a subgroup of index 2 (the *even* permutations) which is simple.

In § 4 we discuss briefly a family of threefolds, the "double solids", which is larger than that of cubic threefolds, in the sense that any cubic threefold is a degeneration of "double solids". From work of Clemens we knew that the intermediate Jacobians of these threefolds form a hypersurface in $\mathcal{A}_5$ (containing both $\mathcal{A}_C$, $\mathcal{J}_5$). Further, Clemens could realize each of these intermediate Jacobians in 6 ways as a Prym variety, and by Beauville's criterion, Proposition I.4.1, all of these represented ramification points of the Prym map.

Comparing with our conjecture, it seemed that the intermediate Jacobians of Clemens' double solids formed the locus in $\mathcal{A}_5$ where the elusive cubic-surface became singular, thus the 27 lines degenerated to 6 double-lines (through the double point) and 15 simple lines. The 6 available Prym-constructions represented the double lines, and the other 15 were yet to be found, with a few hints as to where to look coming from a numerological comparison of the lines on a singular cubic with various objects attached to a double-solid.

To us, the conjecture became believable when we received Clemens' letter, providing a conic-bundle construction for the missing 15, fitting perfectly with the expected symmetries: each of the 15 lines is contained in the plane spanned by a unique pair of the 6, so we expect one of the 15 simple double-covers to correspond to a unique pair of the "ramified" 6. We sketch this in § 4.

The conjecture has recently been proven by one of us. Its proof will appear elsewhere.

§ 4. Clemens' double solids

4.1. Everything in this section is due to H. Clemens. We let X denote the double cover

$$\varphi: X \to \mathbf{P}^3$$

of $\mathbf{P}^3$, branched along a quartic surface $F \subset \mathbf{P}^3$ with 5 ordinary double points $p_1, \ldots, p_5$ in general position. These and more general double-covers of $\mathbf{P}^3$ are studied in [C]. It is shown there that, in this case, $J(X)$ is a 5-dimensional, principally polarized abelian variety,

$$J(X) \in \mathcal{A}_5.$$

The fastest way to see this is to exhibit $J(X)$ as a Prym variety, via a conic-bundle construction. Let

$$\varphi_i: X \to \mathbf{P}^2$$

be the composition of φ with projection (a rational map, blowing up p_i)

$$\pi_i': \mathbf{P}^3 \to \mathbf{P}^2$$

from $p_i \cdot \varphi_i$ is clearly a conic-bundle map: a point $l \in \mathbf{P}^2$ can be identified with a line in $\mathbf{P}^3$ through p_i, and $\varphi_i^{-1}(l)$ consists of a conic or pair of lines according as l meets F transversally or tangentially.

Thus we have

$$J(X) \approx \mathcal{P}(C_i, \tilde{C}_i, \eta)$$

where $C_i \subset \mathbf{P}^3$ is the branch locus of projection

$$\pi_i = \pi_i'|_F: F \to \mathbf{P}^2$$

from p_i, a two-to-one map. The 2 points of $\tilde{C}_i$ over $p \in C_i$ correspond, by Stein factorization, to the components of $\varphi^{-1}(\overline{pp_i})$.

So far we saw 5 ways of representing X as a conic-bundle. The sixth appears from symmetry considerations, analogous to those in IV.4.2. This time we are dealing with a group G of order 6! acting on $\mathbf{P}^3$ by Cremona transformations which preserve the system of quadrics through the p_i. It is generated by linear transformations and inversion in any four of the five p_i. Under this G, the quartic F is sent to various quartics, all with 5 ordinary double points. Of these there are $6=6!/5!$ projectively inequivalent types. Applying π_1 to five of these yields the 5 π_i on F. The sixth can be described as follows: Instead of π_i' we have the map

$$\pi_0'\colon \mathbf{P}^3 \to D$$

where D is the del Pezzo quintic surface parametrizing twisted-cubic curves in $\mathbf{P}^3$ through $p_1, ..., p_5$, and π_0' sends $p\in\mathbf{P}^3$ to the unique twisted cubic through $p_1, ..., p_5, p$. The resulting φ_0 is equivalent, via transformations in G, to the other φ_i.

Using the linear system of quadrics through $p_1, ..., p_5$, $\mathbf{P}^3$ is mapped to $\mathbf{P}^4$ (as a "Segre cubic primal") and the action of G is linearized. The image of each C_i $(i=0, 1, ..., 5)$ in $\mathbf{P}^4$ is Prym-canonical (cf. [C]), and there is a quadric in $\mathbf{P}^4$ cutting a divisor whose pullback in $\mathbf{P}^3$ is F. (The map

$$H^0(\mathbf{P}^4, \mathcal{O}(2)) \to H^0(\mathbf{P}^3, \mathcal{O}(4)\otimes(I_{p_1+\ldots+p_5})^2)$$

is injective, hence surjective since both sides are 15-dimensional.) Since $C_i\subset F$, we see that the canonical images $\Psi(C_i)$ are contained in quadrics, hence (C_i, η_i) are ramification points of $\mathcal{P}$, by Proposition I.4.1.

We shall need the following:

LEMMA 4.2. (i) *Let l_1, l_2 be lines on a smooth cubic surface $S\subset\mathbf{P}^3$, meeting at p. The projection from p,*

$$f\colon S \to \mathbf{P}^2$$

has degree 2, blows up p, blows down l_1, l_2 to points $p_1, p_2\in\mathbf{P}^2$, and its branch locus is a quartic F passing doubly through the p_i.

(ii) *Any plane quartic F with 2 double points arises this way.*

Proof. (i) The ramification curve of f is the space sextic $S\cap S_p$, where S_p is the polar quadric of S with respect to p. For $p\in S$, this has a double point at p, so the branch locus F is a quartic. For $p=l_1\cap l_2$, the ramification curve splits into l_1, l_2, and a residual quartic curve, cut on S_p by another quadric, hence meeting each l_i twice, so the projection in $\mathbf{P}^2$ passes through each p_i twice.

(ii) Let $\mathbf{P}'$ be $\mathbf{P}^2$ blown up at p_1, p_2. The proper transform of F in $\mathbf{P}'$ represents an even homology class (twice the proper transform of a conic through p_1, p_2) so the double cover $f: S' \to \mathbf{P}^2$ can be formed. In S' the inverse image of $l = \overline{p_1 p_2}$ splits into 2 curves l', l''; we have the intersection numbers:

$$l' \cdot l'' = 2$$

$$(l' + l'')^2 = 2l^2 = 2$$

hence

$$l' \cdot l' = l'' \cdot l'' = -1$$

so that l'' can be blown down to a point p yielding a smooth surface S. One sees that the linear system

$$|l_1 + l_2 + l'|$$

(where l_i is the inverse image of p_i) embeds s as a cubic in $\mathbf{P}^3$, and after projection from p the resulting map is given by $|l'|$, hence is the original double-cover f. Q.E.D.

4.3. As discussed in § 3, we expect the existence of 15 more conic-bundle structures on X, each corresponding to a pair of the previous six. Thus we start, for example, with a quartic F as in 4.1, with two of its double points marked, say p_1, p_2. (We discuss the other possibilities below.)

Let $l = \overline{p_1 p_2}$, l' and l'' the two lines over it in X, and K a variable plane through l in $\mathbf{P}^3$. By Lemma 4.2, $\varphi^{-1}(K)$ is a projected cubic surface, with the p_i blown up and l'' blown down to the point of projection p. The cubic is ruled by the pencil of conics in planes through l', and this ruling descends to $\varphi^{-1}(K)$. Letting K vary, we obtain a (birational) map of X to $\mathbf{P}^1 \times \mathbf{P}^1$ whose fibers are conics or line-pairs, as required. This structure is easily seen to differ from the previous six; for example, it is symmetric in p_1, p_2 and hence differs from the corresponding two structures.

4.4. The choice of one of the 6 projectively inequivalent quartics F corresponding to X is equivalent to marking $\tilde{C}_0$, one of the 6 ramification-points $\tilde{C}_i$ of Prym over $J(X)$. The 15 pairs $(\tilde{C}_i, \tilde{C}_j)$ then break into

$$10 = \binom{5}{2}$$

pairs where $i, j = 1, ..., 5$ and 5 where $j = 0$. In 4.3 we described the conic-bundle structures corresponding to the former 10; the latter can be deduced by applying transformations of G, and are as follows:

Instead of a 1-parameter family of planes K, one considers the pencil of quadric cones in $\mathbf{P}^3$, with vertex p_i and through the four other p_k, $k \neq i$. For each such cone K, $\varphi^{-1}(K)$ is again rational and can be ruled by conics.

4.5. Both of these cases can be described simultaneously when $\mathbf{P}^3$, F are replaced by the Segre Cubic Primal Y and the quadric Q cutting F on it. It is well-known [SR] that Y contains 15 planes (blowups of the 5 p_i, and images of the 10 planes $\overline{p_i p_j p_k}$) fitting in 6 ways into quintuplets, each quintuplet intersected by a two-parameter family of lines in Y, and the lines of each of these 6 families give a ruling of Y. The foregoing can be restated as follows:

Each of the six curves C_i is obtained as the locus in Y where lines of one of the rulings are tangent to Q.

The other 15 structures correspond to the 15 planes in Y: Given a plane Π, we consider the family of quadric surfaces K cut on Y, residually to Π, by spaces $\mathbf{P}^3$ containing Π. For each such K, $\varphi^{-1}(K)$ is the double cover of a quadric branched along its intersection with another quadric; projection from a generic point of K reduces to the situation of Lemma 4.2, so $\varphi^{-1}(K)$ is ruled by conics, yielding a conic-bundle structure on X. We leave to the reader the verification of details and the derivation of 4.3, 4.4 from the present, symmetric description.

§ 5. Geometric Schottky Problem

The Schottky problem asks to find equations in the coordinates on $\mathcal{A}_g$ ("thetanulls") or $\mathbf{H}_g$ (periods) characterizing those abelian varieties which are Jacobians. The geometric analogue is to find natural hypersurfaces in $\mathcal{A}_g$ which contain $\mathcal{J}_g$.

When $g=5$, the branch locus B of $\mathcal{D}$ is one such hypersurface. Indeed, by the results of Part IV, § 1, and Part V, § 1, $\mathcal{D}$ fails to be finite over $\mathcal{J}_5$ and over the locus of intermediate Jacobians of cubic threefolds, so both of these are in B. Further, as observed in Part V, § 4, B contains the intermediate Jacobians of Clemens' double solids. By a dimension count these form a hypersurface in $\mathcal{A}_5$, hence an irreducible component of B. Easy degeneration arguments show that these can specialize either to Jacobians or to intermediate Jacobians of cubic threefolds.

A more careful study reveals that B is irreducible (thus consists of nothing but intermediate Jacobians of double-solids and their degenerations); that $\mathcal{J}_5$ is in fact highly-singular in B; and finally, that $\mathcal{J}_5$ can be recovered from B, yielding a "geometric" solution of the Schottky problem in genus 5. The details will appear elsewhere.

APPENDIX: Families of Polygonal Curves

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Let C be a smooth, automorphism-free algebraic curve of genus g. C is called d-gonal if it possesses a base-point free linear system g_d^1, or equivalently if it can be represented as a d-sheeted branched cover

$$f: C \to \mathbf{P}^1.$$

For $d=3$, 4 etc. we have trigonal, tetragonal etc. (Hyperelliptic curves, $d=2$, always possess a non-trivial automorphism.)

A trivial count of degrees-of-freedom, a la Riemann, shows that modulo automorphisms (of $\mathbf{P}^1$), the family $\mathcal{J}_d^1$ of d-sheeted covers of $\mathbf{P}^1$ (of given genus g) depends on $2d+2g-5$ parameters. This proves: *The subvariety $\mathcal{M}_d^1 \subset \mathcal{M}_g$ of d-gonal curves has codimension $\geqslant g-2d+2$.* (The codimension is precisely $g-2d+2$ if a generic $C \in \mathcal{M}_d^1$ possesses only finitely many g_d^1's.)

In [Fa] Farkas proves that the precise codimension is

$$g-2d-1+h^0(C, \mathcal{O}(2D))$$

where D is any divisor in the given g_d^1 on C. Clearly $h^0(C, \mathcal{O}(2D)) \geqslant 3$, and one "expects" equality, at least for small d, but we do not assume this.

Our purpose is to exhibit the tangent space to $\mathcal{M}_d^1$, at a "nice" C as above. More precisely, we use the standard identification

$$T_C \mathcal{M}_g \approx H^1(C, \Theta) \approx (H^0(C, \omega_C^2))^*$$

(where Θ denotes the tangent sheaf of C) and exhibit the subspace of $H^0(C, \omega_C^2)$ which is the annihilator of $T_C \mathcal{M}_d^1$. The proof depends, naturally, on making the identifications explicit, and the simplest way to do this seems to be Kodaira's, reviewed below, which works only over $\mathbf{C}$. (The identification can be done homologically, cf. [GH2].)

THEOREM. *Let R be the ramification divisor in C of $f: C \to \mathbf{P}^1$. Then the subspace*

$$H^0(C, \omega_C^2(-R)) \subset H^0(C, \omega_C^2)$$

is the annihilator of $T_C \mathcal{M}_d^1$, under the standard identification.

Proof. Since $R \in |\omega_C \otimes \mathcal{O}(2D)|$ (where $D \in g_d^1$), we have

$$\begin{aligned} h^0(C, \omega_C^2(-R)) &= h^0(C, \omega_C \otimes \mathcal{O}(-2D)) = (2g-2-2d)-g+1+h^0(C, \mathcal{O}(2D)) \\ &= g-2d-1+h^0(C, \mathcal{O}(2D)) = \operatorname{codim}(\mathcal{M}_d^1). \end{aligned}$$

It suffices therefore to show that all quadratic differentials on C which vanish on R are annihilated by the tangent vector to any curve in $\mathcal{M}_d^1$ through C.

Consider a one-parameter family $\mathcal{C} \to \Delta$ deforming C, where Δ is "a small disc". Topologically $\mathcal{C}$ is trivial, and we think of $\mathcal{C}$ as a family of complex structures on a fixed curve C. Such a family can be given, for example, by a map F (subject to various topological restrictions):

$$F: C \times \Delta \to U$$

where U is some Riemann surface. (More generally, there could be a family of compatible maps F_i defined on an open covering of $C \times \Delta, \ldots$.) For each $t \in \Delta$, $f_t = F|_{C \times \{t\}}$ determines the complex structure on $C_t = C \times \{t\}$. We denote f_0 by z and think of it as a complex-analytic coordinate on $C_0 \approx C$. The quantity

$$\omega_t = \frac{\dfrac{\partial f_t}{\partial \bar{z}}}{\dfrac{\partial f_t}{\partial z}} \frac{\partial}{\partial z} \otimes d\bar{z}$$

is a Θ-valued $(0, 1)$-form on C, depending (for fixed t) only on the complex structures on C_0, C_t (that is, does not change when either f_0 or f_t are composed on the right with an analytic function) and measuring, in a way, the "distance" between these structures. Kodaira shows that the Θ-valued $(0, 1)$-form

$$\omega = \left.\frac{\partial}{\partial t}\right|_0 \omega_t$$

represents the class, in $H^1(C, \Theta)$, given by the desired tangent vector to $\mathcal{M}_g$ along $\mathcal{C}$.

Now assume the family $\mathcal{C}$ is in $\mathcal{M}_d^1$, so there is an algebraic map $F: \mathcal{C} \to \mathbf{P}^1$ which has degree d on each C_t. Via an identification $\mathcal{C} \approx C \times \Delta$, we obtain

$$F: C \times \Delta \to \mathbf{P}^1$$

which can be used as above to measure the variation of complex structure on C along $\mathcal{C}$. Let $q = q(z)dz^2$ be a quadratic differential on C, and $\omega = \omega(z)(\partial/\partial z)d\bar{z}$ a Θ-valued $(0, 1)$-form. The pairing

$$H^1(C, \Theta) \times H^0(C, \omega_C^2) \to H^1(C, \omega_C) \approx \mathbf{C}$$

is just integration,

$$\langle \omega, q \rangle = \int_C \omega(z)\, q(z)\, dz \wedge d\bar{z}.$$

In our case,

$$\omega(z) = \frac{\partial}{\partial t}\bigg|_{t=0}\left(\frac{\dfrac{\partial F}{\partial \bar z}}{\dfrac{\partial F}{\partial z}}\right) = \frac{\dfrac{\partial^2 F}{\partial t\,\partial \bar z}\bigg|_{t=0}}{\dfrac{\partial f}{\partial z}} \quad \left(\text{since } \frac{\partial F}{\partial \bar z}\bigg|_{t=0} = \frac{\partial f}{\partial \bar z} = 0\right)$$

so we set

$$\eta = \frac{\dfrac{\partial F}{\partial t}\bigg|_{t=0}}{\dfrac{\partial f}{\partial z}}\, q(z)\, dz,$$

a singular $(1, 0)$-form on C, which is regular if q vanishes where $\partial f/\partial z = 0$, i.e. on R. In this case

$$\begin{aligned} d\eta &= -\frac{\partial}{\partial \bar z}\left(\frac{\dfrac{\partial F}{\partial t}\bigg|_{t=0}}{\dfrac{\partial f}{\partial z}}\, q(z)\right) dz\, d\bar z \\ &= -\,\omega(z)\, q(z)\, dz\, d\bar z \end{aligned}$$

is exact, hence

$$\langle \omega, q \rangle = -\int_C d\eta = 0$$

as required. Q.E.D.

Remark. A similar argument shows that if q is perpendicular to all possible ω($\Leftrightarrow \partial F/\partial t$ arbitrary) then q must vanish on R, reproving Farkas' result.

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