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Continuous analogues of Fock space IV: essential states

by

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1. Introduction

In this paper, the last of our series [1], [4], [5], we present a new procedure for constructing examples of E_0-semigroups, and we show how these methods can be applied to settle a number of problems left open in [1], [4] and [5]. The central objects of study are semigroups $\alpha=\{\alpha_t: t\geqslant 0\}$ of normal *-endomorphisms of the algebra $\mathcal{B}(H)$ of all operators on a (separable) Hilbert space H, which are continuous in the sense that $\langle \alpha_t(A)\xi, \eta\rangle$ should be a continuous function of t for fixed A in $\mathcal{B}(H)$ and fixed ξ, η in H.

α is called an *E_0-semigroup* [11] if it is unital in the sense that $\alpha_t(1)=1$, for every $t\geqslant 0$. At the opposite extreme, α is called *singular* if the projections $P_t=\alpha_t(1)$ decrease to zero as $t\to\infty$. While it is E_0-semigroups that are of primary interest, much of our analysis will concern singular semigroups. In particular, we will show that the generator of a singular semigroup is injective, and that its inverse is an unbounded completely positive linear map which can be represented in very explicit terms. Perhaps surprisingly, the results of this analysis of singular semigroups can be applied directly to E_0-semigroups. This is accomplished by making appropriate use of the *spectral C^*-algebra* $C^*(E)$ associated with a product system E ([4], [6]).

Recall that a *product system* is a measurable family of Hilbert spaces $E=\{E_t: t>0\}$ over the open interval $(0, +\infty)$, on which there is defined a (measurable) associative

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multiplication $(x, y) \in E \times E \to xy \in E$ which acts like tensoring. This means that for every $s, t > 0$, the multiplication is a bilinear mapping of $E_s \times E_t$ into E_{s+t} which satisfies

$$\langle xy, x'y' \rangle = \langle x, x' \rangle \langle y, y' \rangle,$$

for $x, x' \in E_s$, $y, y' \in E_t$, and that E_{s+t} it spanned by $E_s E_t$.

Every continuous semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of *-endomorphisms of $\mathcal{B}(H)$ gives rise to a pair (E^α, ϕ) consisting a product system E^α and a canonical representation $\phi : E^\alpha \to \mathcal{B}(H)$ [1]. This means that, in addition to the product system E^α, we also have a canonical operator valued mapping ϕ such that $\phi(xy) = \phi(x)\phi(y)$, which is linear on fiber spaces, and which satisfies the "communication relations"

$$\phi(y)^* \phi(x) = \langle x, y \rangle 1, \quad x, y \in E_t,\ t > 0. \tag{1.1}$$

The construction of the pair (E^α, ϕ) from α is analogous to the procedure whereby, starting with a normal operator N, one constructs its spectrum $\sigma(N)$ together with a canonical representation ϕ of the commutative C^*-algebra $C(\sigma(N))$.

Conversely, if one is given an abstract product system E and a representation ϕ of E on a (separable) Hilbert space H, then there is a semigroup of *-endomorphisms of $\mathcal{B}(H)$ which is associated with E as above. In this way, one can approach difficult problems about semigroups of endomorphisms of $\mathcal{B}(H)$ by analyzing their product systems and various structures associated with them, and that is the basis of our approach.

For example, a representation $\phi : E \to \mathcal{B}(H)$ is called *essential* if $\phi(E_t) H$ spans H for every $t > 0$, and *singular* if the intersection of the subspaces $[\phi(E_t) H]$, $t \geq 0$, is trivial. It is quite easy to see that the semigroup associated to a given representation ϕ of E is an E_0-semigroup (resp., a singular semigroup) iff ϕ is essential (resp., singular) [1].

E_0-semigroups were introduced by Powers ([11], also see [12], [13], [14], [15]), and are the primary objects of interest. Thus one is led to ask if *every* product system is associated with E_0-semigroups or, what is the same, does every product system have essential representations? More generally, one would like to have a procedure whereby, starting with an arbitrary product system E one can construct not only one E_0-semigroup, but all possible ones whose canonical product systems are isomorphic to E. The difficulty with this program has been that a product system comes with only one "obvious" representation: the regular representation. There is also a regular antirepresentation. Unfortunately, both of these give rise to *singular* semigroups.

Our results below on constructing essential representations of product systems can be summarized as follows. Starting with an arbitrary product system E, we form the Hilbert space of all square integrable sections $L^2(E)$. An element of $L^2(E)$ is an (equivalence class) of measurable sections

$$\xi: t \in (0, \infty) \to \xi(t) \in E_t,$$

and the inner product is the natural one

$$\langle \xi, \eta \rangle = \int_0^\infty \langle \xi(t), \eta(t) \rangle \, dt.$$

$L^2(E)$ is a continuous analogue of Fock space [1]. Every integrable section $f \in L^1(E)$ gives rise to a left convolution operator $l(f)$, which acts on $L^2(E)$ by

$$l(f)\,\xi(x) = \int_0^x f(t)\,\xi(x-t)\,dt, \quad x > 0, \ \xi \in L^2(E).$$

The *spectral C^*-algebra* of E is defined to be the norm-closed linear span

$$C^*(E) = \operatorname{span}\{l(f)\,l(g)^*: f, g \in L^1(E)\}. \tag{1.2}$$

$C^*(E)$ is a separable nuclear antiliminal C^*-algebra without unit, which is in most (and perhaps all) cases simple ([4], or [6]). It is appropriate to think of the family of C^*-algebras $C^*(E)$ as continuous analogues of the Cuntz C^*-algebra O_∞ [7].

$C^*(E)$ has an important universal property, in that there is a bijective correspondence between the nondegenerate *-representations of $C^*(E)$ on separable Hilbert spaces and separable representations of E as defined above. It is not necessary to describe the precise nature of this correspondence here (see [6], section 2, for more detail). But the fact of its existence makes it meaningful to say that a representation $\pi: C^*(E) \to \mathcal{B}(H)$ is *essential* or *singular* if the corresponding representation $\phi: E \to \mathcal{B}(H)$ is essential or singular.

Accordingly, a bounded linear functional $\varrho \in C^*(E)^*$ is called *essential* or *singular* if the representation of $C^*(E)$ obtained from $|\varrho|$ by the GNS construction has the corresponding property. The set $\mathscr{E}$ (resp., $\mathscr{S}$) of all essential (resp., singular) elements of $C^*(E)^*$ is a norm-closed linear subspace of $C^*(E)$, each is an order ideal, and we have a direct sum decomposition ([5], section 2)

$$C^*(E)^* = \mathscr{E} \oplus \mathscr{S}. \tag{1.3}$$

In this way, the problem of constructing singular or essential representations of E is reduced to the problem of determining the structure of the summands $\mathcal{S}$ and $\mathcal{E}$. For example, to prove that there is an E_0-semigroup α for which E^α is isomorphic to E one has to show that $\mathcal{E}\neq\{0\}$. Our objective in [5] was to give an explicit description of the Banach space $\mathcal{S}$. We now describe a new method for constructing *positive* linear functionals on $C^*(E)$ which works as well for $\mathcal{E}$ as it does for $\mathcal{S}$. For every $t\geq 0$, let P_t denote the projection of $L^2(E)$ onto the subspace $L^2((0,t];E)$ consisting of all sections ξ satisfying $\xi(s)=0$ for almost every $s>t$. The family of projections $\{P_t: t\geq 0\}$ is strongly continuous, and increases from 0 to 1 as t increases from 0 to $+\infty$. We will say that an operator B has *bounded support* if there is a positive t such that $B=P_tBP_t$. Let

$$\mathcal{B}=\bigcup_{t>0} P_t\,\mathcal{B}(L^2(E))\,P_t$$

be the set of all such operators. $\mathcal{B}$ is a weakly dense *-subalgebra of $\mathcal{B}(L^2(E))$. A (perhaps unbounded) linear functional ω on $\mathcal{B}$ is called a *locally normal weight* if for every $t\geq 0$, the restriction of ω to $P_t\,\mathcal{B}(L^2(E))\,P_t$ is a positive normal linear functional.

We will make essential use of the semigroup $\beta=\{\beta_t: t\geq 0\}$ *-endomorphisms of $\mathcal{B}(L^2(E))$ associated with the *right regular antirepresentation* of E on $L^2(E)$ (cf. section 2 below). β is a singular semigroup of *-endomorphisms of $\mathcal{B}(L^2(E))$, and we have $\mathcal{B}\supseteq\beta_t(\mathcal{B})$ for every $t\geq 0$. We will say that a locally normal weight ω is *decreasing* if

$$\omega(\beta_t(B^*B))\leq\omega(B^*B), \quad \textit{for every} \quad B\in\mathcal{B},\ t\geq 0, \tag{1.4}$$

and *invariant* if equality holds in (1.4). Let $\mathcal{W}_\beta$ denote the set of all decreasing locally normal weights ω which satisfy the growth condition

$$\sup_{t>0}\frac{\omega(P_t)}{t}<+\infty.$$

$\mathcal{W}_\beta$ is a partially ordered cone of linear functionals on $\mathcal{B}$, and it can be described in rather concrete terms.

Our main results (summarized in Theorem 1.5 below) assert that there is an isomorphism of $\mathcal{W}_\beta$ onto the positive cone of $C^*(E)^*$ which carries the invariant weights in $\mathcal{W}_\beta$ onto the essential states $C^*(E)$. This isomorphism involves differentiation, and is defined as follows. Let δ be the generator of the semigroup β:

$$\delta(A)=\text{strong}\lim_{t\to 0}\frac{A-\beta_t(A)}{t}.$$

δ is a well-defined derivation on an appropriate domain (cf. section 2).

THEOREM 1.5. *Let $\mathscr{A}$ be the set of all operators A in the domain of δ such that $\delta(A)$ is a compact operator of bounded support. Then $\mathscr{A}$ is a self adjoint algebra whose norm closure is $C^*(E)$.*

Moreover, for every $\omega \in \mathscr{W}_\beta$, the linear functional $d\omega$ defined on $\mathscr{A}$ by

$$d\omega(A) = \omega(\delta(A)), \quad A \in \mathscr{A}$$

extends to a positive linear functional on $C^*(E)$ of norm

$$\|d\omega\| = \sup_{t>0} \frac{\omega(P_t)}{t}.$$

The map $d: \mathscr{W}_\beta \to C^*(E)^*$ induces an affine order isomorphism of $\mathscr{W}_\beta$ and the partially ordered cone of positive linear functionals on $C^*(E)$. This map carries the invariant elements of $\mathscr{W}_\beta$ onto the cone of essential positive linear functionals.

Theorem 1.5 implies that every product system has essential representations. In order to see this, one has only to show that there exist nonzero invariant locally normal weights on $\mathscr{B}$, and that is easily accomplished by elementary methods (see Theorem 5.9 and Corollary 5.17).

2. The generator and its inverse

Let $\alpha=\{\alpha_t: t\geq 0\}$ be a semigroup of *-endomorphisms of $\mathscr{B}(H)$. While the purpose of this paper is to construct examples of E_0-semigroups, the analysis in sections 2 through 4 will focus on *singular* semigroups and certain C^*-algebras associated with them. Applications of these results to E_0-semigroups will be given in section 5.

In this section we discuss the generator of a singular semigroup. We show that such a generator is an injective linear mapping, and that its inverse is an unbounded completely positive linear map having a very explicit integral representation. Let α be a singular semigroup of *-endomorphisms of $\mathscr{B}(H)$, which will be fixed throughout this section. We will write δ for the generator of α, and it is defined as follows. dom(δ) is the set of all operators $A \in \mathscr{B}$(H) for which the limit

$$\delta(A) = \lim_{t\to 0} t^{-1}(A-\alpha_t(A)) \tag{2.1}$$

exists in the strong operator topology. It is known that dom(δ) is a strongly dense *-subalgebra of $\mathscr{B}(H)$ and that δ: dom(δ)$\to\mathscr{B}(H)$ is an unbounded self-adjoint derivation

[11]. Notice that this definition of δ differs in sign from the definition of the generator of an E_0-semigroup given in [11].

For every $t \geq 0$, we will write P_t for the self-adjoint projection $1-\alpha_t(1)$. The family $\{P_t: t \geq 0\}$ increases from 0 to 1 as t increases from 0 to $+\infty$, and varies continuously in the strong operator topology. Some terminology will be convenient. We will say that a bounded operator B (resp., a vector ξ in H) is *supported in the interval* $[a, b]$ if $B=(P_b-P_a)B(P_b-P_a)$ (resp., $\xi=(P_b-P_a)\xi$). Let

$$\mathcal{B} = \bigcup_{t>0} P_t \mathcal{B}(H) P_t,$$

$$H_0 = \bigcup_{t>0} P_t H$$

be the set of all operators on H (resp., vectors in H) having bounded support. $\mathcal{B}$ is a weakly dense *-subalgebra of $\mathcal{B}(H)$, and H_0 is a dense linear subspace of H. The following result implies that every operator of bounded support is in the range of δ.

THEOREM 2.2. *For every $B \in \mathcal{B}$, there is a unique operator $\lambda(B) \in \mathcal{B}(H)$ satisfying*

$$\langle \lambda(B)\xi, \eta \rangle = \int_0^\infty \langle \alpha_s(B)\xi, \eta \rangle \, ds \tag{2.3}$$

for every ξ, $\eta \in H_0$. λ is a linear mapping of $\mathcal{B}$ into the domain of δ whose restriction to each local von Neumann algebra $\mathcal{B}_t = P_t \mathcal{B}(H) P_t$, $t>0$, is a normal completely positive map of norm t.

$\lambda(\mathcal{B})$ consists of all operators $A \in \mathrm{dom}(\delta)$ for which $\delta(A)$ $\mathcal{B}$, and we have $\delta(\lambda(B))=B$ for every $B \in \mathcal{B}$.

Remarks. Notice that if $\xi, \eta \in H_0$, then for any bounded operator B we have $\langle \alpha_s(B)\xi, \eta \rangle = 0$ for sufficiently large $s \geq 0$, so that there is no problem with the existence of the integral appearing in (2.3). In the course of proving Theorem 2.2 we will show that δ is injective; therefore λ is simply the inverse of δ restricted to a convenient domain.

Proof of Theorem 2.2. Fix $T>0$, and let B be any operator supported in an interval of the form $[a, b]$ with $b-a \leq T$. Then $\alpha_{nT}(B)$ is supported in $[na, nb]$ for every $n=0, 1, 2, \ldots$. Since the intervals $[ma, mb]$ and $[na, nb]$ are disjoint for $m \neq n$, the operators $\{\alpha_{nT}(B): n=0, 1, \ldots\}$ must have mutually orthogonal initial spaces and final spaces. Thus the series

(2.4)
$$\tilde{B} = \sum_{n=0}^{\infty} \alpha_{nT}(B)$$

converges strongly to an operator in $\mathcal{B}(H)$ satisfying $\|\tilde{B}\| = \|B\|$. Note too that for every $t \geq 0$, $\alpha_t(B)$ is supported in an interval $[a+t, b+t]$ having similar properties, and so we have

(2.5)
$$\alpha_t(\tilde{B}) = \sum_{n=0}^{\infty} \alpha_{nT}(\alpha_t(B)).$$

We claim now that

(2.6)
$$\int_0^t \langle \alpha_s(\tilde{B})\xi, \eta \rangle \, ds = \int_0^\infty \langle \alpha_s(B)\xi, \eta \rangle \, ds$$

for every $\xi, \eta \in H_0$. Indeed, the integral on the right can be decomposed as follows

$$\sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} \langle \alpha_s(\tilde{B})\xi, \eta \rangle \, ds = \sum_{n=0}^{\infty} \int_0^T \langle \alpha_{s+nT}(B)\xi, \eta \rangle \, ds = \int_0^T \langle \alpha_s(\tilde{B})\xi, \eta \rangle \, ds,$$

noting that all sums that appear are actually finite sums becaue $\xi, \eta \in H_0$.

Now that for T fixed, the map $B \in \mathcal{B}_T \to \tilde{B}$ is a normal *-homomorphism of the von Neumann algebra $\mathcal{B}_T$ into $\mathcal{B}(H)$, and so we can define a normal completely positive linear map λ_T of $\mathcal{B}_T$ into $\mathcal{B}(H)$ by

(2.7)
$$\lambda_T(B) = \int_0^T \alpha_s(\tilde{B}) \, ds.$$

Obviously, $\|\lambda_T\| \leq T$. Moreover, (2.6) implies that for every $\xi, \eta \in H_0$,

(2.8)
$$\langle \lambda_T(B)\xi, \eta \rangle = \int_0^T \langle \alpha_s(B)\xi, \eta \rangle \, ds, \quad B \in \mathcal{B}_T.$$

Since the right side of (2.8) does not depend on T, we conclude that the family $\{\lambda_T : T > 0\}$ is coherent, and defines a single linear map $\lambda: \mathcal{B} \to \mathcal{B}(H)$ with the property that for every $T > 0$, the restriction of λ to $\mathcal{B}_T$ is a normal completely positive linear map of norm at most T, and which satisfies the required equation (2.3).

In order to see that $\|\lambda|_{\mathscr{B}_T}\|=T$, we claim that $\|\lambda(P_T)\|=T$. Indeed, since

$$(P_T)^{\sim}=\sum_{n=0}^{\infty}\alpha_{nT}(P_T)=1,$$

we see from (2.7) that

$$\lambda(P_T)=\int_0^T \alpha_s(1)\,ds.$$

Since $\alpha_T(1)$ is a nonzero projection, we can find a unit vector $\xi\in H$ satisfying $\alpha_T(1)\xi=\xi$. Noting that $\alpha_s(1)\xi=\xi$ for every $0\leq s\leq T$, we see that

$$\lambda(P_T)\,\xi=\int_0^T \alpha_s(1)\,\xi\,ds=T\xi,$$

and thus $\|\lambda(P_T)\|=T$.

We claim now that

$$\alpha_t(\lambda(B))=\lambda(\alpha_t(B)) \tag{2.9}$$

for every $B\in\mathscr{B}$ and every $t\geq 0$. Indeed, if $B\in\mathscr{B}_T$ then (2.5) asserts that $\alpha_t(\tilde{B})=\alpha_t(B)^{\sim}$, hence

$$\alpha_t(\lambda(B))=\alpha_t\left(\int_0^T \alpha_s(\tilde{B})\,ds\right)=\int_0^T \alpha_{s+t}(\tilde{B})\,ds=\int_0^T \alpha_s(\alpha_t(B)^{\sim})\,ds=\lambda(\alpha_t(B)).$$

We show next that $\lambda(\mathscr{B})$ is contained in $\operatorname{dom}(\delta)$, and $\delta(\lambda(B))=B$ for every $B\in\mathscr{B}$. In order to prove this, we claim first that for every B in $\mathscr{B}$ and for every $t>0$, $\lambda(B)-\alpha_t(\lambda(B))$ is given by the weak integral

$$\lambda(B)-\alpha_t(\lambda(B))=\int_0^t \alpha_s(B)\,ds. \tag{2.10}$$

Indeed, using (2.9) we have for every $\xi,\eta\in H_0$,

$$\langle\alpha_t(\lambda(B))\,\xi,\eta\rangle=\langle\lambda(\alpha_t(B))\,\xi,\eta\rangle=\int_0^\infty \langle\alpha_{s+t}(B)\,\xi,\eta\rangle\,ds=\int_t^\infty \langle\alpha_s(B)\,\xi,\eta\rangle\,ds,$$

and hence

$$\langle(\lambda(B)-\alpha_t(\lambda(B)))\,\xi,\eta\rangle=\int_0^t \langle\alpha_s(B)\,\xi,\eta\rangle\,ds.$$

(2.10) follows because both sides of (2.10) are bounded operators and H_0 is dense in H. From (2.10) it is clear that $t^{-1}(\lambda(B)-\alpha_t(\lambda(B)))$ convrges strongly to B as $t\to 0+$. Hence $\lambda(B)$ belongs to dom(δ) and $\delta(\lambda(B))=B$.

In particular, this shows that $\lambda(\mathcal{B})$ is contained in

$$\{A\in \mathrm{dom}(\delta): \delta(A)\in \mathcal{B}\}.$$

To see that this inclusion is actually equality, we claim first that δ is injective in the sense that if $A\in\mathrm{dom}(\delta)$ and $\delta(A)=0$, then $A=0$. For that, choose any $A\in\mathrm{dom}(\delta)$. Then for every pair of vectors ξ, η in H, the function $f(t)=\langle \alpha_t(A)\xi,\eta\rangle$ is differentiable on $[0,+\infty)$ and $f'(t)=-\langle \alpha_t(\delta(A))\xi,\eta\rangle$. Hence

$$\langle (A-\alpha_t(A))\xi,\eta\rangle = f(0)-f(t) = -\int_0^t f'(s)\,ds = \int_0^t \langle \alpha_s(\delta(A))\xi,\eta\rangle\,ds.$$

It follows that $A-\alpha_t(A)$ is given by the weak integral

$$A-\alpha_t(A) = \int_0^t \alpha_s(\delta(A))\,ds. \tag{2.11}$$

(2.11) implies that if we also assume that $\delta(A)=0$, then $A=\alpha_t(A)$ for every $t\geq 0$. Since the projections $\alpha_t(1)$ tend strongly to zero as $t\to\infty$, $\alpha_t(A)=\alpha_t(1)\,\alpha_t(A)$ tends strongly to zero as $t\to\infty$, and we have the desired conclusion

$$A = \lim_{t\to\infty} \alpha_t(A) = 0.$$

Now choose any A in dom(δ) such that $\delta(A)\in\mathcal{B}$. In order to show that $A\in\lambda(\mathcal{B})$, it suffices to observe that $A=\lambda(\delta(A))$. But we know that $\lambda(\delta(A))$ belongs to dom(δ) and that $\delta(\lambda(\delta(A)))=\delta(A)$. Hence $A-\lambda(\delta(A))$ belongs to the kernel of δ, and we conclude from the preceding discussion that $A=\lambda(\delta(A))$. □

We collect the following observations for use in section 3.

PROPOSITION 2.12. (i) *Let A and B be operators in the domain of δ such that $\delta(A)$ and $\delta(B)$ are supported in $[0,a]$ and $[0,b]$ respectively. Then $\delta(AB)$ is supported in $[0,a+b]$.*

(ii) *For every $B\in\mathcal{B}$ and $t\geq 0$, we have $P_t\lambda(B)=P_t\lambda(P_tB)$.*

Proof. (i) Let $K=\delta(A)$, $L=\delta(B)$. Then $A=\lambda(K)$, $B=\lambda(L)$, and we have

$$\delta(AB) = \delta(A)\,B + A\,\delta(B) = K\lambda(L)+\lambda(K)\,L.$$

Because of the formula (2.3), $P_a\lambda(L)$ is given by the weak integral

$$P_a\lambda(L) = \int_0^a P_a\alpha_s(L)\,ds,$$

and hence

$$K\lambda(L) = KP_a\lambda(L) = \int_0^a KP_a\alpha_s(L)\,ds = \int_0^a K\alpha_s(L)\,ds.$$

Since L is supported in $[0, b]$, $\alpha_s(L)$ is supported in $[0, s+b]$, and therefore in $[0, a+b]$ for every $0 \leq s \leq a$. It follows that $K\lambda(L)$ is supported in $[0, a+b]$. A similar argument shows that $\lambda(K)L$ is supported in $[0, a+b]$.

(ii) Fix $B \in \mathscr{B}$, $t \geq 0$. We have to show that $P_t\lambda((1-P_t)B)=0$, i.e., $P_t\lambda(\alpha_t(1)B)=0$. But for every ξ, η in H_0 we can write

$$\begin{aligned}\langle P_t\lambda(\alpha_t(1)B)\xi, \eta\rangle &= \int_0^t \langle \alpha_s(\alpha_t(1)B)\xi, P_t\eta\rangle\,ds \\ &= \int_0^t \langle \alpha_{s+t}(1)\alpha_s(B)\xi, P_t\eta\rangle\,ds\end{aligned}$$

which vanishes because $P_t\alpha_{s+t}(1)=P_t(1-P_{s+t})=0$ for every $s \geq 0$. □

3. The algebra $\mathscr{A}$

As in section 2, $\alpha=\{\alpha_t : t \geq 0\}$ denotes a *singular* semigroup of *-endomorphisms of $\mathscr{B}(H)$ with generator β, fixed throughout this section. We now introduce a *-subalgebra of the domain of δ and we will construct a particular kind of approximate identity for it. We will see later, in section 5, that the C^*-algebra associated with any product system can be realized as the norm closure of one of these algebras.

Let $\mathscr{A}$ denote the set of operators $\lambda(\mathscr{K} \cap \mathscr{B})$. By Theorem 2.2, an operator A belongs to $\mathscr{A}$ iff $A \in \operatorname{dom}(\delta)$ and $\delta(A)$ is a compact operator of bounded support.

PROPOSITION 3.1. *$\mathscr{A}$ is a self adjoint subalgebra of $\mathscr{B}(H)$ and δ is a derivation of $\mathscr{A}$ onto the algebra of all compact operators having bounded support.*

Proof. If A and B belong to $\mathscr{A}$, then $\delta(AB)=\delta(A)B+A\delta(B)$ is clearly a compact operator, and Proposition 2.12(i) implies that $\delta(AB)$ has bounded support. Hence $\mathscr{A}$ is an algebra of operators. It is self adjoint because $\delta(A^*)=\delta(A)^*$ for every A in the domain of δ, and $\delta(\mathscr{A})=\mathscr{K}\cap\mathscr{B}$ follows from Theorem 2.2. □

The next results exhibits a family of approximate units for $\mathscr{A}$ with rather convenient properties.

THEOREM 3.3. *Let $t_1, t_2, \ldots$ be a sequence of positive real numbers which converges to zero. There is a sequence of finite rank projections $F_1, F_2, \ldots$ in $\mathscr{B}(H)$ such that F_n is supported in the interval $[0, t_n]$, and the sequence*

$$E_n = t_n^{-1}\lambda(F_n)$$

is a strong approximate unit for $\mathscr{A}$ in the sense that

$$\lim_{n\to\infty}(\|E_nA-A\|+\|\delta(E_nA-A)\|)=0, \quad A\in\mathscr{A}. \tag{3.4}$$

Remark. Notice that for any choice of $\{t_n\}$ *and* $\{F_n\}$ as above, Theorem 2.2 implies that each E_n is a positive operator in $\mathscr{A}$ of norm at most 1.

Proof of Theorem 3.3. Fix $\{t_n\}$. We show first that there is a sequence of finite rank projections F_n such that F_n is supported in $[0, t_n]$ and

$$E_n = t_n^{-1}\lambda(F_n)$$

tends strongly to the identity in $\mathscr{B}(H)$. To see this, let ϱ be a faithful normal state of $\mathscr{B}(H)$ and fix $n\geq 1$. Since $\lambda\colon \mathscr{B}\to\mathscr{B}(H)$ is a locally normal mapping, $\varrho\circ\lambda$ is a positive normal linear function on $P_{t_n}\mathscr{B}(H)P_{t_n}$. Thus we can find a finite rank projection $F_n\leq P_{t_n}$ such that

$$\varrho(\lambda(F_n))\geq\varrho(\lambda(P_{t_n}))-n^{-1}t_n.$$

Now for every $t>0$ we have

$$\varrho(\lambda(P_t))=\int_0^\infty \varrho(\alpha_s(P_t))\,ds=\int_0^\infty \varrho(\alpha_s(1)-\alpha_{s+t}(1))\,ds=\int_0^t \varrho(\alpha_s(1))\,ds,$$

since $s\to\varrho(\alpha_s(1))$ is a bounded nonnegative function which decreases to zero as $s\to\infty$. Putting $E_n=t_n^{-1}\lambda(F_n)$, we see that E_n is a positive operator of norm at most 1, and

$$\varrho(E_n)\geq\frac{1}{t_n}\int_0^{t_n}\varrho(\alpha_s(1))\,ds-\frac{1}{n}$$

for every $n \geq 1$. Since $\varrho(\alpha_s(1))$ is continuous in s and takes the value 1 at $s=0$, we see that

$$\lim_{n\to\infty} \varrho(E_n) = 1.$$

Because of the inequality

$$\varrho((1-E_n)^2) = 1+\varrho(E_n^2)-2\operatorname{Re}\varrho(E_n) \leq 2\operatorname{Re}\varrho(1-E_n),$$

we concluded that

$$\lim_{n\to\infty} \varrho((1-E_n)^2) = 0. \tag{3.5}$$

Since ϱ is a faithful normal state of $\mathcal{B}(H)$ and $\{1-E_n\}$ is a bounded sequence of self adjoint operators, (3.5) implies that E_n tends strongly to 1.

Now choose F_n as above and let

$$E_n = t_n^{-1}\lambda(F_n).$$

Then E_n tends strongly to 1 as $n\to\infty$, and it remains to prove (3.4). For that, fix $A \in \mathcal{A}$. If $\delta(A)$ is supported in $[0,T]$ then by Proposition 2.12(i), $\delta(E_n A)$ is supported in $[0,T+t_n]$, and so $\delta(E_n A)$ is supported in $[0,T+1]$ for n sufficiently large. Hence

$$\|E_n A-A\| = \|\lambda(\delta(E_n A-A))\| \leq (T+1)\|\delta(E_n A-A)\|$$

for sufficiently large n. Therefore (3.4) will follow if we prove that

$$\lim_{n\to\infty} \|\delta(E_n A-A)\| = 0. \tag{3.6}$$

Using the fact that δ is a derivation on $\mathcal{A}$ we have

$$\delta(A-E_n A) = (1-E_n)\,\delta(A)-\delta(E_n)\,A,$$

and hence

$$\|\delta(A-E_n A)\| \leq \|(1-E_n)\,\delta(A)\|+\|\delta(E_n)A\|.$$

The first term on the right tends to zero because $\delta(A)$ is compact and $1-E_n$ tends to zero in the strong operator topology. In order to estimate the second term, we claim first that for every $t>0$ and every finite rank projection F supported in $[0,t]$, we have

$$\|FA\| \leq t\|P_t\,\delta(A)\|. \tag{3.7}$$

Indeed, since $FA=FP_tA$, we see from Proposition 2.12(ii) that

$$P_tA = P_t\lambda(\delta(A)) = P_t\lambda(P_t\delta(A)) = \int_0^t P_t\alpha_s(P_t\delta(A))\,ds,$$

and thus we obtain (3.7)

$$\|FA\| \leq \|P_tA\| \leq t\|P_t\delta(A)\|.$$

Now since $\delta(E_n)=t_n^{-1}F_n$, we see from (3.7) that $\|\delta(E_n)A\|$ is bounded above by $\|P_{t_n}\delta(A)\|$. Again, because $\delta(A)$ is compact and since $P_t=1-\alpha_t(1)$ tends strongly to zero as $t\to 0+$, we have the desired conclusion

$$\overline{\lim_{n\to\infty}}\,\|\delta(E_n)A\| \leq \lim_{n\to\infty}\|P_{t_n}\delta(A)\| = 0. \qquad \square$$

4. States and decreasing weights

Let $\alpha=\{\alpha_t: t\geq 0\}$ be a singular semigroup of endomorphisms acting on $\mathcal{B}(H)$ and let $P_t=1-\alpha_t(1)$ for every $t\geq 0$. We will consider certain unbounded positive linear functionals defined on the the *-subalgebra

$$\mathcal{B} = \bigcup_{t>0} P_t\,\mathcal{B}(H)\,P_t \tag{4.1}$$

of $\mathcal{B}(H)$ consisting of all operators having bounded support (relative to α). Note that the union in (4.1) is unaffected if we restrict it to integral values of t. A linear functional ω on $\mathcal{B}$ is called a *locally normal weight* if for every $n\geq 1$, the restriction of ω to $P_n\mathcal{B}(H)P_n$ is a normal positive linear functional. With every such ω we can associate a sequence Ω_n of "density operators" as follows. For each $n\geq 1$, there is a unique positive trace-class operator Ω_n with the properties

(i) $\Omega_n = P_n\Omega_nP_n$,

(ii) $\omega(B) = \mathrm{trace}(\Omega_nB), B\in P_n\mathcal{B}(H)P_n$,

and this sequence is *coherent* in that

(iii) $P_n\Omega_{n+1}P_n = \Omega_n, n\geq 1$.

Conversely, if we start with a sequence $\{\Omega_n: n \geq 1\}$ of positive trace-class operators satisfying (i) and (iii), then we can define a locally normal weight ω on $\mathcal{B}$ as follows

$$\omega(B) = \lim_{n \to \infty} \operatorname{trace}(\Omega_n B), \quad B \in \mathcal{B}.$$

Locally normal weights are a slight generalization of normal weights, and we digress momentarily in order to discuss this significant point. Let ω be a normal weight of $\mathcal{B}(H)$ [10]. This means that ω is a function defined on the positive operators in $\mathcal{B}(H)$, which takes values in the extended interval $[0, +\infty]$, which is linear insofar as that property makes sense, and preserves the limits of bounded monotone increasing nets of positive operators. If we assume further that $\omega(P_n)$ is finite for every $n \geq 1$, then ω can be restricted to the positive part of $\mathcal{B}$, and this restriction extends uniquely by linearity to a locally normal weight of $\mathcal{B}$.

On the other hand, we want to point out that not every locally normal weight on $\mathcal{B}$ is obtained in this way from a normal weight ω of $\mathcal{B}(H)$ satisfying $\omega(P_n) < \infty$ for every n. Appendix A contains some simple examples of locally normal weights of $\mathcal{B}$ which cannot be extended to normal weights of $\mathcal{B}(H)$.

We may conclude that in this sense, locally normal weights are more general than normal weights. On the other hand, locally normal weights are susceptible to the same kind of GNS construction as are normal weights, and this construction gives rise to a normal representation of $\mathcal{B}(H)$. Indeed, if ω is a locally normal weight of $\mathcal{B}$, then we may complete $\mathcal{B}$ relative to the positive semidefinite inner product $\langle A, B \rangle = \omega(B^*A)$, $A, B \in \mathcal{B}$, to obtain a (separable) Hilbert space H_ω and a linear mapping $\Omega: \mathcal{B} \to H_\omega$ satisfying

$$\langle \Omega(A), \Omega(B) \rangle = \omega(B^*A), \quad \text{and} \tag{4.2i}$$

$$\Omega(\mathcal{B})^{\sim} = H_\omega. \tag{4.2ii}$$

There is a unique *-representation $\pi: \mathcal{B} \to \mathcal{B}(H_\omega)$ defined by $\pi(B)\Omega(C) = \Omega(BC)$ for all B, C in $\mathcal{B}$. It is rather easy to show that π is nondegenerate, and *extends uniquely to a normal *-representation of $\mathcal{B}(H)$ on H_ω*. In particular, *every representation of $\mathcal{B}$ obtained in this way from a locally normal weight is unitarily equivalent to a multiple of the identity representation of $\mathcal{B}$*. We omit the proof since this result is an elementary one that is peripheral to the main discussion. But we may conclude from these remarks that, while locally normal weights of $\mathcal{B}$ are more general than normal weights, they are always *quasi-equivalent* to normal weights in the sense that their associated representations are quasi-equivalent to the representations associated with normal weights.

Every endomorphism α_t, $t \geq 0$, leaves the *-algebra $\mathscr{B}$ invariant, and thus we can compose a locally normal weight ω with α_t to obtain another locally normal weight.

Definition 4.3. A locally normal weight ω is called decreasing if for all $B \in \mathscr{B}$, and $t \geq 0$ we have

$$\omega(\alpha_t(B^*B)) \leq \omega(B^*B).$$

We will refer to an ω satisfying Definition 4.3 simply as a *decreasing weight*. The set of all decreasing weights is a cone of linear functionals on $\mathscr{B}$, which is partially ordered by the relation $\omega_1 \leq \omega_2$ iff $\omega_2 - \omega_1$ is a decreasing weight.

Let $\mathscr{A}$ be the *-algebra $\delta^{-1}(\mathscr{K} \cap \mathscr{B})$ introduced in section 3, and let ω be a locally normal weight. Since δ maps $\mathscr{A}$ into the domain of ω, we can define a linear functional $d\omega$ on $\mathscr{A}$ as follows:

$$d\omega(A) = \omega(\delta(A)), \quad A \in \mathscr{A}. \tag{4.4}$$

$d\omega$ can be interpreted as the derivative of ω in the direction of the flow of the semigroup α. Typically, both ω and $d\omega$ are unbounded. In this section we will characterize the set of all locally normal weights ω for which $d\omega$ extends (necessarily uniquely) to a positive bounded linear functional on the C^*-algebra obtained by closing $\mathscr{A}$ in the norm topology. Indeed, we will obtain in this way a rather explicit description of the set of all states of this C^*-algebra (Theorem 4.15).

As it happens, the key issue is to determine when $d\omega$ is positive in the sense that $d\omega(A^*A) \geq 0$, $A \in \mathscr{A}$, and the following result gives the relevant criterion.

THEOREM 4.5. *Let ω be a locally normal weight on $\mathscr{B}(H)$ and let $d\omega$ be the linear functional on $\mathscr{A}$ defined by* (4.4). *The following are equivalent.*

(i) $d\omega(A^*A) \geq 0$, *for every* $A \in \mathscr{A}$.
(ii) ω *is decreasing.*

Proof. (ii)⇒(i). Let ω be a decreasing weight. Letting $\lambda = \delta^{-1}$ be the inverse map of δ defined in Theorem 2.2 and setting $K = \delta(A)$, the desired inequality Theorem 4.5(i) becomes

$$\omega(K^*\lambda(K) + \lambda(K)^*K) = 2\mathrm{Re}\, \omega(K^*\lambda(K)) \geq 0, \tag{4.6}$$

$\mathrm{Re}\, z$ denoting the real part of the complex number z, and we must prove (4.6) for every compact operator K in $\mathscr{B}$. Let H_ω be the Hilbert space obtained by completing $\mathscr{B}$ relative to the positive semidefinite inner product $(K,L)\to\omega(L^*K)$. Then we have a natural linear mapping $\Omega: \mathscr{B}\to H_\omega$ with the property that $\Omega(\mathscr{B})$ is dense in H_ω and

$$\langle \Omega(K), \Omega(L)\rangle = \omega(L^*K), \quad K, L\in\mathscr{B}.$$

Now for each $t\geq 0$ and every K in $\mathscr{B}$, we have

$$\begin{aligned}\|\Omega(\alpha_t(K))\|^2 &= \omega(\alpha_t(K)^*\alpha_t(K)) = \omega(\alpha_t(K^*K))\\ &\leq \omega(K^*K) = \|\Omega(K)\|^2,\end{aligned}$$

and hence we can define a contraction operator $A(t)$ on H_ω by

$$A(t): \Omega(K)\to\Omega(\alpha_t(K)), \quad K\in\mathscr{B}.$$

It is obvious that $\{A(t): t\geq 0\}$ is a contraction semigroup for which $A(0)=1$, and it is strongly continuous because for each $K\in\mathscr{B}$ we have

$$\begin{aligned}\|A(t)\,\Omega(K)-\Omega(K)\|^2) &= \omega((\alpha_t(K)-K)^*(\alpha_t(K)-K))\\ &= \omega(\alpha_t(K^*K))+\omega(K^*K)-2\operatorname{Re}\omega(K^*\alpha_t(K))\\ &\leq 2\operatorname{Re}\{\omega(K^*K)-\omega(K^*\alpha_t(K))\},\end{aligned}$$

and the right side tends to zero as $t\to 0+$.

Let $U=\{U_t: t\in\mathbf{R}\}$ be a unitary dilation of the contraction semigroup $\{A(t): t\geq 0\}$. This is to say that U is a strongly continuous one parameter unitary group which acts on a Hilbert space containing H_ω in such a way that

$$\langle A(t)\,\Omega(K), \Omega(L)\rangle = \langle U_t\Omega(K), \Omega(L)\rangle, \quad K, L\in\mathscr{B},\ t\geq 0. \tag{4.7}$$

In order to prove the inequality (4.6), choose $K\in\mathscr{B}$ and suppose that K is supported in the interval $[0, T]$ in the sense that $K=P_TKP_T$. Define a complex valued function ϕ on the real line $\mathbf{R}$ by

$$\phi(t)=\begin{cases}\omega(K^*\alpha_t(K)), & \text{if } t\geq 0\\ \bar{\phi}(-t), & \text{if } t<0.\end{cases} \tag{4.8}$$

ϕ is continuous, and vanishes outside the interval $[-T, +T]$. Moreover, by formula (2.3) and the fact that $K^*\alpha_s(K)=K^*P_T\alpha_s(K)=0$ for $s>T$, we have

$$\omega(K^*\lambda(K)) = \int_0^T \omega(K^*\alpha_t(K))\,dt = \int_0^T \phi(t)\,dt,$$

and hence

$$2\,\mathrm{Re}\,\omega(K^*\lambda(K)) = \int_{-\infty}^{\infty} \phi(t)\,dt$$

is the value of the Fourier transform of ϕ at 0. Thus it suffices to show that the Fourier transform of ϕ is nonnegative. The desired inequality (4.6) follows.

But by (4.7), we have a representation of ϕ

$$\phi(t) = \langle U_t \Omega(K), \Omega(K) \rangle$$

as a coordinate function of a unitary representation of the additive group $\mathbf{R}$. Hence ϕ is a *positive definite* function of compact support, which implies that the Fourier transform of ϕ is nonnegative. The desired inequality (4.6) follows.

In order to prove the implication (i)⇒(ii), we require a bit of lore associated with dilation theory. Since we lack an appropriate reference, we have included a sketch of the proof of Lemma 4.9 in Appendix B.

LEMMA 4.9. *Let* $A: \mathbf{R} \to \mathcal{B}(H)$ *be an operator function such that for every vector* ξ *in a dense subset of* H, *the scalar function* $\langle A(t)\xi, \xi \rangle$ *is positive definite in* t, *and which satisfies* $A(0)=1$. *Then* $\|A(t)\| \leq 1$ *for all* t.

Proof of Theorem 4.5, (i)⇒(ii). In order to reverse the above argument, we introduce the one parameter unitary group associated with the resolution of the identity defined by $P_t = 1 - \alpha_t(1)$, and we observe that these unitiaries are "eigenvalues" of the semigroup α (see (4.10) below). More precisely, let P be the unique spectral measure on $[0, +\infty)$ defined by the requirement

$$P([0,t)) = 1 - \alpha_t(1), \quad t \geq 0,$$

and let $U = \{U_\tau : \tau \in \mathbf{R}\}$ be the one parameter unitary group

$$U_\tau = \int_0^\infty e^{it\tau} P(dt).$$

Since $P([a,b)) = \alpha_a(1) - \alpha_b(1)$ for $0 \leq a \leq b < +\infty$, we have $\alpha_t(P[a,b)) = P[a+t, b+t)$ for all $t \geq 0$. It follows that

$$\alpha_t(U_\tau) = \int_0^\infty e^{is\tau} P(t+ds) = e^{-it\tau} \int_t^\infty e^{is\tau} P(ds) = e^{-it\tau} \alpha_t(1)\, U_\tau.$$

Thus for any operator B we have

(4.10) $$(U_\tau B)^*\alpha_\tau(U_\tau B)=e^{-i\tau t}B^*\alpha_t(B).$$

Assume now that ω is a locally normal weight satisfying Theorem 4.5(i):

$$d\omega(A^*A)\geq 0,\quad A\in\mathscr{A}.$$

As in the preceding argument, this is equivalent to the assertion

$$\omega(K^*\lambda(K)+\lambda(K)^*K)\geq 0$$

for every compact operator K in $\mathscr{B}$. As before, we fix such a K and consider the function $\phi(t)$ defined in (4.8). Again, ϕ is a continuous function having compact support, which satisfies

(4.11) $$\int_{-\infty}^{\infty}\phi(t)\,dt=\omega(K^*\lambda(K)+\lambda(K)^*K)\geq 0.$$

Notice now that the Fourier transform of ϕ is nonnegative. For if we replace K with $U_\tau K$ for some $\tau\in\mathbf{R}$, then we see from (4.10) that the function ϕ_τ associated with $U_\tau K$ is given by

$$\phi_\tau(t)=\omega((U_\tau K)^*\alpha_t(U_\tau K))=e^{-i\tau t}\omega(K^*\alpha_t(K))=e^{-i\tau t}\phi(t).$$

Replacing K with $U_\tau K$ in (4.11) gives

$$\int_{-\infty}^{\infty}e^{-i\tau t}\phi(t)\,dt=\int_{-\infty}^{\infty}\phi_\tau(t)\,dt=\omega((U_\tau K)^*\lambda(U_\tau K)+\lambda(U_\tau K)^*(U_\tau K))\geq 0$$

for all τ, as asserted.

We conclude from the preceding paragraph that ϕ is a function of positive type. Therefore $|\phi(t)|\leq\phi(0)$ for every real t, and hence

(4.12) $$|\omega(K^*\alpha_t(K))|\leq\omega(K^*K)$$

for every $t\geq 0$ and every compact operator K in $\mathscr{B}$. Because the restriction of ω to every von Neumann subalgebra of $\mathscr{B}$ of the form $P_T\mathscr{B}(H)P_T$ is normal, the inequality (4.12) persists for all $K\in\mathscr{B}$.

Now by a familiar polarization argument ([9], Theorem 2, p. 33), the inequality (4.12) implies that

$$|\omega(L^*\alpha_t(K))| \leq 2\sqrt{\omega(K^*K)\,\omega(L^*L)}, \tag{4.13}$$

for all K, L in $\mathcal{B}$.

We show next that the constant 2 in (4.13) can be replaced with 1. To see this, let H_ω be the Hilbert space associated with ω as in the first part of the proof and let $\Omega: \mathcal{B} \to \mathcal{B}(H_\omega)$ be the natural map. For each $t \geq 0$, (4.13) implies that

$$\|\Omega(\alpha_t(K))\| \leq 2\|\Omega(K)\|, \quad K \in \mathcal{B},$$

and hence there is a unique operator $A(t)$ on H_ω satisfying

$$A(t): \Omega(K) \to \Omega(\alpha_t(K)), \quad K \in \mathcal{B}.$$

Obviously, $\|A(t)\| \leq 2$. As in the previous arguments, $\{A(t): t \geq 0\}$ is a strongly continuous semigroup. For $t<0$, put $A(t)=A(-t)^*$.

Fixing K in $\mathcal{B}$ and letting ϕ be the positive definite function associated to K as in (4.8), then we have $\phi(t)=\langle A(t)\Omega(K), \Omega(K)\rangle$ for all real t. Since $\Omega(\mathcal{B})$ is dense in H_ω, the family of operators $\{A(t): t \in \mathbf{R}\}$ satisfies the hypothesis of Lemma 4.9, and hence $\|A(t)\| \leq 1$ for every t.

It follows that ω is a decreasing weight, for if $K \in \mathcal{B}$ and $t \geq 0$,

$$\begin{aligned}\omega(\alpha_t(K^*K)) &= \omega(\alpha_t(K)^*\alpha_t(K)) = \|A(t)\,\Omega(K)\|^2 \\ &\leq \|\Omega(K)\|^2 = \omega(K^*K),\end{aligned}$$

and the proof of Theorem 4.5 is complete. □

Remark. If one is willing to assume that $d\omega$ is a *bounded* linear functional on $\mathcal{A}$, then it is possible to give a shorter proof of the implication (i)⇒(ii). This is based on the observation that for $t \geq 0$ and for every positive operator K in $\mathcal{B}$,

$$\lambda(K-\alpha_t(K)) = \lambda(K) - \alpha_t(\lambda(K)) = \int_0^t \alpha_s(K)\,ds$$

is a positive operator in the multiplier algebra of the norm closure of $\mathcal{A}$, and hence

$$\omega(K-\alpha_t(K)) = d\omega[\lambda(K-\alpha_t(K))] \geq 0.$$

In order to determine which decreasing weights ω give rise to *bounded* linear functionals $d\omega$ on $\mathcal{A}$, we require the following elementary result about positive linear functionals on normed *-algebras. By a *normed *-algebra* we mean a complex normed algebra $\mathcal{A}$ which is endowed with an isometric involution $a \to a^*$.

LEMMA 4.14. *Let $\mathscr{A}$ be a normed *-algebra and let ϱ be a linear functional on A satisfying $\varrho(a^*a)\geq 0$ for every $a\in\mathscr{A}$. Assume further that*

(i) $\overline{\lim}_{n\to\infty} |\varrho(x^n)|^{1/n} \leq \|x\|$, *for every* $x=x^*\in\mathscr{A}$,

and that there is a sequence $\{e_n: n\geq 1\}$ of self adjoint elements of $\mathscr{A}$ satisfying $\varrho(e_n^2)\leq M<+\infty$ and such that for every $x\in\mathscr{A}$ we have

(ii) $\varrho(x)=\lim_{n\to\infty}\varrho(e_n x)$.

Then $\|\varrho\|^2\leq M$.

Remark. Perhaps it is worth pointing out that some very elementary examples show that the essential conclusion of Lemma 4.14 (i.e., that ϱ is bounded) fails if one deletes either of the hypotheses (i) or (ii).

Proof of Lemma 4.14. By the Schwarz inequality, we have for each n

$$|\varrho(e_n x)|^2 \leq \varrho(e_n^2)\,\varrho(x^*x) \leq M\varrho(x^*x),$$

and therefore

$$|\varrho(x)| = \lim_{n\to\infty} |\varrho(e_n x)| \leq M^{1/2}\varrho(x^*x)^{1/2}.$$

Iterating the latter inequality n times, we obtain

$$\begin{aligned}|\varrho(x)| &\leq M^{1/2+1/4+\ldots+1/2^n}\varrho((x^*x)^{2^{n-1}})^{1/2^n}\\ &\leq M\varrho((x^*x)^{2^{n-1}})^{1/2^n}.\end{aligned}$$

Taking the limit on the right side and using Lemma 4.14(i), we obtain

$$|\varrho(x)| \leq M\|x^*x\|^{1/2} \leq M\|x\|,$$

as required. □

We are now in position to describe the cone of positive linear functionals on the C^*-algebra $\mathscr{A}^-$ obtained by closing $\mathscr{A}$ in the operator norm.

THEOREM 4.15. *Let $\mathcal{W}_\alpha$ be the cone of all decreasing weights ω satisfying the condition*

$$\sup_{t>0} \frac{\omega(1-\alpha_t(1))}{t} < +\infty. \tag{4.16}$$

For every ω in $\mathcal{W}_\alpha$, the linear functional $d\omega$ defined in (4.4) *is bounded, and extends uniquely to a positive linear functional on $\mathcal{A}^-$. The map $\omega \to d\omega$ is an affine order isomorphism of $\mathcal{W}_\alpha$ onto the positive part of the dual of $\mathcal{A}^-$, for which*

$$\|d\omega\| = \sup_{t>0} \frac{\omega(1-\alpha_t(1))}{t}.$$

Proof. Fix ω in $\mathcal{W}_\alpha$ and let M denote the left side of (4.16). In order to show that $d\omega$ is bounded, we will apply Lemma 4.14. Let t_n be a sequence of positive reals decreasing to zero and let E_n be an approximate unit for $\mathcal{A}$ of the type constructed in Theorem 3.3. We claim first that $d\omega(E_n^2) \leqslant M$ for every $n \geqslant 1$. Inded, since E_n has the form

$$E_n = t_n^{-1} \lambda(F_n) \tag{4.17}$$

where F_n is a finite rank projection supported in the interval $[0, t_n]$, we see that

$$d\omega(E_n) = \omega(\delta(E_n)) = t_n^{-1} \omega(F_n) \leqslant M.$$

Moreover, for every A in $\mathcal{A}$ we have $\|\delta(E_n A) - \delta(A)\| \to 0$ as $n \to \infty$, and since the restriction of ω to $P_T \mathcal{B}(H) P_T$ defines a bounded linear functional for every $T>0$, we may conclude that

$$\lim_{n\to\infty} d\omega(E_n A) = \lim_{n\to\infty} \omega(\delta(E_n A)) = \omega(\delta(A)).$$

Hence Lemma 4.14(ii) is satisifed.

Now we claim that $d\omega$ satisfies Lemma 4.14(i) for every A in $\mathcal{A}$, i.e.,

$$\overline{\lim_{n\to\infty}} |\omega(\delta(A^n))|^{1/n} \leqslant \|A\|. \tag{4.18}$$

To see this, fix A and suppose that $\delta(A) = P_T \delta(A) P_T$ for some $T>0$. By Proposition 2.12(i) we have $\delta(A^n) = P_{nT} \delta(A^n) P_{nT}$ for every $n = 1, 2, \ldots$. Since the norm of the restriction of ω to $P_s \mathcal{B}(H) P_s$ is $\omega(P_s)$ for every $s>0$, we have

$$|\omega(\delta(A^n))| \leq \omega(P_{nT})\|\delta(A^n)\| \leq M(nT)\, n\|A\|^{n-1}\|\delta(A)\|,$$

and the estimate (4.18) follows after taking nth roots and passing to the limit.

By Theorem 4.5 we know that $d\omega(A^*A) \geq 0$ for every $A \in \mathcal{A}$, and thus we may conclude from Lemma 4.14 that $d\omega$ is bounded with $\|d\omega\| \leq M$. To see that the latter inequality is actually equality, fix $t>0$, let $\{F_1, F_2, \dots\}$ be a sequence of finite rank projections which increases to P_t, and put

$$E_p = t^{-1}\lambda(F_p), \quad p \geq 1.$$

Each E_p is a positive operator in $\mathcal{A}$ of norm at most 1, hence

$$\|d\omega\| \geq \sup_p d\omega(E_p) = \sup_p t^{-1}\omega(F_p) = t^{-1}\omega(P_t).$$

Since t is arbitrary it follows that $\|d\omega\| \geq M$.

We now show that every positive linear functional ϱ on $\mathcal{A}^-$ has the form $\varrho = d\omega$ for ω as above. Fixing ϱ, define a linear functional ω_0 on $\mathcal{K} \cap \mathcal{B}$ by $\omega_0(K) = \varrho(\lambda(K))$, $K \in \mathcal{K} \cap \mathcal{B}$. Notice that ω_0 extends naturally to a locally normal weight ω of $\mathcal{B}$. Indeed, for every $t>0$ the restriction of λ to $P_t \mathcal{K} P_t$ is a (bounded) positive linear mapping, and consequently the restriction of ω_0 to $P_t \mathcal{K} P_t$ is a positive linear functional. Hence there is a unique normal positive linear functional ω_t on $P_t\, \mathcal{B}(H)\, P_t$ such that $\omega_t(K) = \omega_0(K)$ for every $K \in P_t \mathcal{K} P_t$. The family of linear functionals $\{\omega_t : t \geq 0\}$ is obviously coherent, and therefore there is a locally normal weight ω on $\mathcal{B}$ such that

$$\omega|_{P_t \mathcal{B}(H) P_t} = \omega_t$$

for every $t \geq 0$. Note that $\varrho = d\omega$. Indeed, if $A \in \mathcal{A}$ and we set $K = \delta(A)$, then K belongs to $\mathcal{K} \cap \mathcal{B}$ and we have

$$\varrho(A) = \varrho(\lambda(K)) = \omega(K) = \omega(\delta(A)) = d\omega(A).$$

Finally, since ϱ is a positive linear functional, Theorem 4.5 implies that ω is a decreasing weight.

It remains to show that for decreasing weights ω_1 and ω_2, we have $\omega_1 \leq \omega_2 \Leftrightarrow d\omega_1 \leq d\omega_2$. Assuming first that $\omega_1 \leq \omega_2$, then $\omega_2 - \omega_1$ is a decreasing weight and hence $d\omega_2 - d\omega_1 = d(\omega_2 - \omega_1)$ is a positive linear functional by what we have already proved. Using Theorem 4.5, this argument is reversible. □

5. Applications

We conclude by showing how the C^*-algebra $C^*(E)$ associated with an arbitrary product system E fits into the context of the preceding discussion, and we deduce the main results of this paper. Let E be a product system. Following the discussion of section 1 (or see [6], section 2), we realize $C^*(E)$ as a concrete C^*-algebra acting on the Hilbert space $L^2(E)$ of all square integrable sections of E,

$$C^*(E) = \operatorname{span}\{l(f)\, l(g)^*: f, g \in L^1(E)\}.$$

The properties of this C^*-algebra together with appropriate references are described in [6].

Here, we want to show how $C^*(E)$ can be defined in terms of the generator of the singular semigroup $\beta=\{\beta_t: t\geq 0\}$ associated with the regular antirepresentation $r: E\to\mathscr{B}(L^2(E))$. In more detail, β is defined as follows. For eah $t>0$, choose an orthonormal basis $\{e_1(t), e_2(t), \ldots\}$ for the Hilbert fiber space $E(t)$ over t, and define a sequence of isometries $V_1(t)$, $V_2(t), \ldots$ on $L^2(E)$ by

$$V_n(t)\,\xi(x) = \begin{cases} \xi(x-t)\, e_n(t), & \text{if} \quad x>t, \\ 0, & \text{if} \quad 0<x\leq t. \end{cases} \tag{5.1}$$

β_t is defined to be the following endomorphism of $\mathscr{B}(L^2(E))$:

$$\beta_t(A) = \sum_{n=0}^{\infty} V_n(t) A V_n(t)^*, \quad A \in \mathscr{B}(L^2(E)). \tag{5.2}$$

This definition does not depend on the choice of basis $\{e_n(t)\}$, and if we define β_0 to be the identity mapping, then $\{\beta_t: t\geq 0\}$ becomes a singular semigroup of $*$-endomorphisms of $\mathscr{B}(L^2(E))$ (see [5]) with the property that $\beta_t(1)$ is the projection of $L^2(E)$ onto the subspace consisting of all sections $\xi\in L^2(E)$ which are supported in the interval $[t, +\infty)$ (almost everywhere). We may therefore consider the generator δ of β as defined in section 2, the algebra $\mathscr{B}$ of all operators with bounded support relative to β, and the self-adjoint algebra $\mathscr{A}=\delta^{-1}(\mathscr{K}\cap\mathscr{B})=\lambda(\mathscr{K}\cap\mathscr{B})$ introduced in section 3.

PROPOSITION 5.3. *$C^*(E)$ is the norm-closure of the algebra of operators*

$$\mathscr{A} = \{A \in \operatorname{dom}(\delta): \delta(A) \in \mathscr{K}\cap\mathscr{B}\}.$$

Proof. Let H_0 denote the dense subspace of $L^2(E)$ consisting of all sections $\xi\in L^2(E)$ which are supported in some bounded subinterval of $(0,\infty)$ (notice that these are precisely the vectors having bounded support relative to the semigroup β). We may consider H_0 to be a dense subspace of the Banach space $L^1(E)$. Since every integrable section $f \in L^1(E)$ can be approximated in the L^1-norm with a section $f_0 \in H_0$, and since the left convolution mapping $l: L^1(E) \to \mathcal{B}(L^2(E))$ is a contraction, we have

$$C^*(E) = \text{span}\{l(f)\, l(g)^*: f, g \in H_0\}. \tag{5.4}$$

Now Proposition 6.4 of [4] asserts that for $f, g \in H_0$ we have

$$l(f)\, l(g)^* = \lambda(f \otimes \bar{g})$$

$f \otimes \bar{g}$ denoting the rank-one operator in $\mathcal{B}(L^2(E))$ defined by $\xi \to \langle \xi, g \rangle f$. Therefore, 5.4 implies that $C^*(E)$ is the norm-closure of the set of operators $\lambda(\mathcal{F})$ where $\mathcal{F}$ denotes the set of all finite rank operators having bounded support relative to β. The required conclusion that $C^*(E)$ is the norm closure of $\lambda(\mathcal{K} \cap \mathcal{B})$ now follows from the fact that λ is a locally bounded operator mapping of $\mathcal{K} \cap \mathcal{B}$ into $\mathcal{B}(L^2(E))$. □

For any $\omega \in \mathcal{W}_\beta$, we see from Theorem 4.15 that $d\omega$ defines a bounded linear functional on the dense subalgebra $\mathcal{A}$ of Proposition 5.3; we will use the same symbol $d\omega$ to denote the unique extension of $d\omega$ to $C^*(E)$. Moreover, Theorem 4.15 implies that the map $d: \mathcal{W}_\beta \to C^*(E)^*$ defines an affine order isomorphism of the cone $\mathcal{W}_\beta$ onto the cone of all positive linear functionals on $C^*(E)$. We must now determine which elements of $\mathcal{W}_\beta$ map to *essential* positive linear functionals on $C^*(E)$.

In [5] we introduced a semigroup $\beta^* = \{\beta^*_t: t \geq 0\}$ which acts on the dual of $C^*(E)$, and is closely related to the semigroup β of *-endomorphisms of $\mathcal{B}(L^2(E))$ encountered above. For $t > 0$, β^*_t is defined as follows. Choose an orthonormal basis $\{e_1(t), e_2(t), \dots\}$ for $E(t)$ and let $\{V_1(t), V_2(t), \dots\}$ be the sequence of isometries defined in (5.1). Letting H_0 denote the subspace of $L^2(E)$ consisting of all sections having bounded support, we see that each $V_n(t)$ leaves H_0 invariant and in particular, $V_n(t) f \in L^1(E)$ for every $f \in H_0$. For every $\varrho \in C^*(E)^*$, $\beta^*_t(\varrho)$ is defined on generators of $C^*(E)$ of the form $l(f)\, l(g)^*$, where f and g are elements of H_0, by the absolutely convergent series

$$\beta^*_t(\varrho)(l(f)\, l(g)^*) = \sum_{n=1}^{\infty} \varrho(l(V_n(t) f)\, l(V_n(t) g)^*).$$

β^*_0 is defined as the identity map of $C^*(E)^*$.

β^*_t does not depend on the particular choice of basis $\{e_n(t)\}$, and in fact this defines a contraction semigroup on $C^*(E)^*$ which has the crucial property that *a linear functional ϱ in $C^*(E)^*$ is essential iff ϱ is fixed under the action of β^** (see [5], Proposition 1.3 and Proposition 1.8). We emphasize that, while β^* does possess an appropriate continuity property ([5], Proposition 1.3), the individual mappings $\beta^*{}_t$ are not continuous in the weak* topology of $C^*(E)^*$. In particular, β^* is *not* the semigroup adjoint any semigroup of endomorphisms acting on $C^*(E)$.

In order to apply these facts about β^* in the current setting, we will show in Theorem 5.7 that for every positive t we have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}_* & \xrightarrow{\ d\ } & C^*(E)^* \\ {\scriptstyle \beta_{t*}}\downarrow & & \downarrow{\scriptstyle \beta^*{}_t} \\ \mathcal{B}_* & \longrightarrow & C^*(E)^* \end{array} \tag{5.5}$$

in the sense that for every decreasing weight $\omega \in \mathcal{W}_\beta$ one has

$$\beta^*{}_t(d\omega)(A) = d(\omega \circ \beta_t)(A), \quad \textit{for every} \quad A \in \mathcal{A}. \tag{5.6}$$

Notice that the commutative diagram (5.5) provides a sense in which the action of β^* is adjoint to the action of the semigroup β on $\mathcal{B}$, *but in a noncanonical way via the differentiation mapping* d.

THEOREM 5.7. *The diagram* (5.5) *is commutative. Moreover,* $\omega \to d\omega$ *defines an affine order isomorphism of the subcone of invariant weights in* $\mathcal{W}_\beta$ *onto the cone of essential positive linear functional in* $C^*(E)^*$.

Proof. Granting the formula (5.6) for a moment, we can readily deduce the rest of Theorem 5.7 from Theorem 4.15 and the characterization of the essential part of $C^*(E)^*$ cited above. Indeed, if ω is an invariant weight in $\mathcal{W}_\beta$ then (5.6) implies that $d\omega$ is invariant under the semigroup β^* and hence $d\omega$ is essential. Conversely, if ϱ is an essential positive linear functional on $C^*(E)$, then by Theorem 4.15 there is a decreasing weight ω such that $\varrho(A)=d\omega(A)=\omega(\delta(A))$ for every $A \in \mathcal{A}$. Since ϱ is invariant under β^*, formula (5.6) implies that for every $A \in \mathcal{A}$ and every $t \geq 0$, we have $d(\omega \circ \beta_t)(A)= d\omega(A)$, i.e.,

$$\omega(\beta_t(\delta(A))) = \omega(\delta(A)).$$

Since Proposition 3.1 implies that $\delta(\mathcal{A})=\mathcal{K} \cap \mathcal{B}$, the preceding formula asserts that $\omega(\beta_t(K))=\omega(K)$ for every compact operator K of bounded support. By local normality of the two linear functionals $\omega \circ \beta_t$ and ω it follows that $\omega \circ \beta_t=\omega$ on $\mathcal{B}$, so that ω is an invariant weight.

In order to prove (5.6), fix $\omega \in \mathcal{W}_\beta$. We have to show that for every $a>0$ and every compact operator K satisfying $K=P_a K P_a$ we have

$$\beta^*_t(d\omega)(\lambda(K)) = d\omega(\lambda(K)). \tag{5.8}$$

Notice that the right side of (5.8) is simply $\omega(\delta(\lambda(K)))=\omega(K)$. Now since the restriction of λ to $P_a \mathcal{K} P_a$ is bounded and since $P\mathcal{K}P_a$ is spanned by its rank-one operators, it suffices to prove (5.8) for K of the form $f\otimes\bar{g}$ with f, g functions in $L^2(E)$ which are supported in the interval $(0,a]$. But in this case $l(V_n(t)f)\,l(V_n(t)g)^*=\lambda(V_n(t)f\otimes V_n(t)\bar{g})$ ([4], Proposition 6.4); and using local normality of ω, we can write the left side of (5.8) as follows

$$\begin{aligned}\sum_{n=1}^{\infty} d\omega(l(V_n(t)f)\,l(V_n(t)g)^*) &= \sum_{n=1}^{\infty} \omega(\delta(\lambda(V_n(t)f\otimes V_n(t)\bar{g})))\\ &= \sum_{n=1}^{\infty} \omega(V_n(t)(f\otimes\bar{g})\,V_n(t)^*)\\ &= \omega\left(\sum_{n=1}^{\infty} \omega(V_n(t)(f\otimes\bar{g})V_n(t)^*\right).\end{aligned}$$

The last term is simply $\omega(\beta_t(f\otimes\bar{g}))$, which agrees with the right side of equation (5.8). □

We can now show that $C^*(E)$ has essential states (by a *state* of a non-unital C^*-algebra we simply mean a positive linear function of norm 1). In view of Theorem 5.7, it suffices to show that there are nonzero invariant locally normal weights on $\mathcal{B}$. The following result asserts a bit more.

THEOREM 5.9. *Let $\alpha=\{\alpha_t: t\geq 0\}$ be a semigroup of *-endomorphisms of $\mathcal{B}(H)$ such that $\alpha_t(1)\neq 1$ for every $t>0$. Then there is a normal weight of $\mathcal{B}(H)$ which is invariant under the action of α and satisifes*

$$\omega(1-\alpha_t(1)) = t, \quad \textit{for every} \quad t\geq 0. \tag{5.10}$$

Remarks. Assuming that α is a singular semigroup, (5.10) implies that the restriction of ω to the positive cone of the algebra $\mathcal{B}$ of all operators having bounded support relative of α defines (after an obvious extension by linearity) a nontrivial invariant locally normal weight on $\mathcal{B}$.

We also remark that if ω_0 is any normal weight on $\mathcal{B}(H)^+$ which is invariant under $\{\alpha_t: t\geq 0\}$ and which satisifes the condition

$$0<\omega_0(1-\alpha_T(1))<+\infty$$

for some positive T, then ω_0 can always be rescaled so as to achieve the normalization (5.10). Indeed, if we let P be the spectral measure on $[0,+\infty)$ defined by the property

$$P([a,b))=\alpha_b(1)-\alpha_a(1) \tag{5.11}$$

for every $0\leq a\leq b<+\infty$, then we can define a positive measure μ on the Borel subsets of $[0,\infty)$ by $\mu(S)=\omega_0(P(S))$. The definition (5.11) together with the semigroup property for α imply that

$$\alpha_t(P(S))=P(S+t)$$

for every $t\geq 0$ and every such Borel set S; and from the invariance of ω_0 under α we conclude that the measure μ is invariant under translations to the right in $[0,\infty)$. The hypothesis on ω_0 implies that

$$0<\mu([0,T))<+\infty$$

which, together with translation invariance, implies that μ is a nonzero measure which is positive and finite on every interval of the form $[a,b)$ with $0\leq a<b<+\infty$. It follows that μ is a nonzero multiple of Lebesgue measure, i.e., for some $c>0$ we have $\mu([0,t))=ct$, $t\geq 0$. Thus we obtain (5.10) for the invariant weight $\omega=c^{-1}\omega_0$.

Proof of Theorem 5.9. In view of the preceding remarks, it suffices to construct a normal α-invariant weight ω on $\mathcal{B}(H)^+$ with the property $0<\omega(1-\alpha_1(1))<+\infty$. Let β be the single endomorphism $\beta=\alpha_1$. Since $1-\beta(1)$ is a nonzero projection, we may find a normal state ν_0 on $\mathcal{B}(H)$ such that $\nu_0(1-\beta(1))=1$. Let V be any isometry satisfying

$$\beta(T)V=VT, \quad T\in\mathcal{B}(H), \tag{5.12}$$

and define a sequence of normal states $\nu_1,\nu_2,\ldots$ on $\mathcal{B}(H)$ by

$$\nu_n(T)=\nu_0(V^{*n}TV^n), \quad T\in\mathcal{B}(H),\ n\geq 1.$$

Since ν_0 annihilates the projection $\beta(1)$ we have $\nu_0\circ\beta=0$; and since for $n\geq 1$ the commutation relation (5.12) implies

$$V^{*n}\beta(T)V^n=V^{*n}VTV^{n-1}=V^{*n-1}TV^{n-1},$$

we have $\nu_n \circ \beta = \nu_{n-1}$. Hence

$$\nu = \sum_{n=0}^{\infty} \nu_n \tag{5.13}$$

defines a normal weight of $\mathcal{B}(H)$ which is invariant under the action of β.

Obviously $\nu_0(1-\beta(1))=1$. More generally, we claim that for each n, ν_n is supported in the projection $\beta^n(1)-\beta^{n+1}(1)$. To see that, choose $n\geq 1$. Using (5.12) again we can write

$$\begin{aligned}\nu_n(\beta^n(1)-\beta^{n+1}(1)) &= \nu_0(V^{*n}\beta^n(1)\,V^n - V^{*n}\beta^{n+1}(1)\,V^n))\\ &= \nu_0(1-\beta(1)) = 1,\end{aligned}$$

and the claim follows.

We can define a weight ω on $\mathcal{B}(H)^+$ by

$$\omega(T) = \int_0^1 \nu(\alpha_t(T))\,dt, \quad T \in \mathcal{B}(H)^+. \tag{5.14}$$

To see that ω is a normal weight, define a sequence of normal states $\{\omega_n : n \geq 1\}$ by

$$\omega_n(T) = \int_0^1 \nu_n(\alpha_t(T))\,dt, \quad T \in \mathcal{B}(H),$$

and notice that by the monotone convergence theorem, we have

$$\omega(T) = \sum_{n=0}^{\infty} \omega_n(T) \tag{5.15}$$

for every positive operator T. (5.15) implies that ω is a normal weight on $\mathcal{B}(H)$. Note too that for every positive operator $T \in \mathcal{B}(H)$, the function $f\colon [0, +\infty) \to [0, +\infty]$ defined by $f(t)=\nu(\alpha_t(T))$ is periodic with period 1, and therefore the normal weight ω defined by (5.14) is invariant under the full semigroup $\{\alpha_t : t \geq 0\}$.

It remains to show that $\omega(1-\alpha_1(1))$ is positive and finite. Now since ν_n is supported in $\alpha_n(1)-\alpha_{n+1}(1)$ for every $n\geq 1$, it follows from its definition as an integral that ω_n is supported in the projection $\alpha_{n-1}(1)-\alpha_{n+1}(1)$. Thus if we apply ω to the operator $T_0 = 1-\alpha_1(1)$, the infinite series in (5.15) reduces to just two terms

$$\omega(T_0) = \omega_0(T_0)+\omega_1(T_0) = \int_0^1 [\nu_0(\alpha_t(T_0))+\nu_1(\alpha_t(T_0))]\,dt. \tag{5.16}$$

Since $\nu_k(\alpha_t(T_0))$ is continuous in t, nonnegative, and positive at $t=0$, we see that $\omega(1-\alpha_1(1))$ is a positive number which is at most 2. That completes the proof. □

In view of the correspondence between essential states of $C^*(E)$, essential representations of E, and E_0-semigroups, we obtain

COROLLARY 5.17. *For every product system E, there is an E_0-semigroup whose canonical product system is isomorphic to E.*

Proof. Theorems 5.5 and 5.9 imply that $C^*(E)$ has an essential state, and therefore an essential representation on a separable Hilbert space H. Because of the universal property of $C^*(E)$ ([6], Theorem 2.7), there is an essential representation $\phi: E \to \mathcal{B}(H)$. This means that ϕ is a representation of E with the property that $\phi(E_t)H$ spans H for every $t>0$. Let $\alpha=\{\alpha_t: t\geq 0\}$ be the semigroup of endomorphisms of $\mathcal{B}(H)$ associated with ϕ ([1], Proposition 2.7). Since $\alpha_t(1)$ is the projection onto the subspace of H spanned by $\phi(E_t)H$, we see that $\alpha_t(1)=1$ for every $t>0$ and hence α is an E_0-semigroup. By ([1], Proposition 2.7), the product system associated to α is isomorphic to E. □

Using standard reduction theory, it is possible to obtain a stronger form of Corollary 5.17. More precisely, suppose that we are given an E_0-semigroup $\alpha=\{\alpha_t: t\geq 0\}$ acting on $\mathcal{B}(H)$. We will say that α is *ergodic* if the only operators $A\in\mathcal{B}(H)$ satisfying $\alpha_t(A)=A$ for every $t\geq 0$ are scalars. Let E be the product system associated with α and let $\phi: E\to\mathcal{B}(H)$ be the associated (essential) representation of E. It is not hard to show that α is ergodic iff the von Neumann algebra generated by $\phi(E)$ is irreducible. See [1]. In turn, because of the universal property of $C^*(E)$ it follows that α is ergodic iff the associated (essential) representation $\pi: C^*(E)\to\mathcal{B}(H)$ is an irreducible representation.

We conclude from these remarks that if one starts with an abstract product system E, then the problem of constructing all ergodic E_0-semigroups α whose canonical product systems are isomorphic to E is equivalent to the problem of constructing all essential pure states of $C^*(E)$. In particular, *there exists an ergodic E_0-semigroup α whose product system is isomorphic to E iff $C^*(E)$ has an essential pure state*. While it is possible to exhibit such states quite explicitly in certain cases (see Appendix C), our proof of the following result which establishes this fact in general is highly nonconstructive.

COROLLARY 5.18. *For every product system E, there is an ergodic E_0-semigroup α such E_α is isomorphic to E.*

Proof. Fix $t>0$, and let $e_1, e_2, \ldots$ be an orthonormal basis for E_t. We will show that there is a representation ϕ of E on a separable Hilbert space with the property that $\phi(E) \cup \phi(E)^*$ is an irreducible set of operators and

$$\sum_{n=1}^{\infty} \phi(e_n)\,\phi(e_n)^* = 1.$$

The associated semigroup α of endomorphisms must then satisfy $\alpha_t(1)=1$, and therefore $\alpha_s(1)=1$ for every $s\geq 0$, by the semigroup property. Hence α will be an E_0-semigroup with the asserted properties.

In order to accomplish this, we use Corollary 5.17 to find an essential state ϱ of $C^*(E)$. Let $\pi: C^*(E) \to \mathscr{B}(H)$ be the representation associated to ϱ by the GNS construction.

H must be a separable Hilbert space because $C^*(E)$ is separable and π is cyclic, and thus we may apply von Neumann's reduction theory in an uncomplicated way. We find a standard finite measure space (X, μ), a decomposition

$$H = \int_X^{\oplus} H_x \, d\mu(x)$$

of H into a measurable field of separable Hilbert spaces over X, and a decomposition of π

$$\pi = \int_X^{\oplus} \pi_x \, d\mu(x) \tag{5.19}$$

into a measurable field of irreducible representations $\{\pi_x : x \in X\}$.

Fix $x \in X$. Because of the universal property of $C^*(E)$, there is a representation $\phi_x: E \to \mathscr{B}(H_x)$ associated to π in a particular way (see [6], section 2), and $\phi_x(E) \cup \phi_x(E)^*$ generates $\pi_x(C^*(E))'' = \mathscr{B}(H_x)$ as a von Neumann algebra. The family $\{\phi_x : x \in X\}$ is a measurable field of representations of E, and corresponding to (5.19) we have

$$\phi = \int_X^{\oplus} \phi_x \, d\mu(x), \tag{5.20}$$

where $\phi: E \to \mathscr{B}(H)$ is the representation of E associated to π.

For each $x \in X$, let $P_n(x)$ be the projection

$$P_n(x) = \sum_{k=1}^{n} \phi_x(e_k)\,\phi_x(e_k)^*.$$

For fixed n, $P_n(\cdot)$ is a measurable field of projections; and for fixed $x \in X$, the sequence $\{P_1(x), P_2(x), \dots\}$ increases to the projection $P(x)$ of H_x onto the subspace of H_x spanned by $\phi_x(E_t) H_x$.

Because of the preceding remarks, it suffices to show that there is a point $a \in X$ for which $P(a)=1$. But by the monotone convergence theorem we have strong convergence of the integrals

$$\int_X^{\oplus} P(x)\, d\mu(x) = \lim_{n\to\infty} \int_X^{\oplus} P_n(x)\, d\mu(x);$$

and in view of (5.20) the right side is

$$\lim_n \sum_{k=1}^{n} \phi(e_k)\, \phi(e_k)^* = \sum_{n=1}^{\infty} \phi(e_k)\, \phi(e_k)^*.$$

The latter is the identity because ϕ is an essential representation of E. Hence

$$\int_X^{\oplus} P(x)\, d\mu(x) = 1.$$

It follows that $P(x)=1$ almost everywhere $(d\mu)$, and in particular there must be at least one point $a \in X$ for which $P(a)=1$. □

Finally, as we have described in ([6], Theorem 5.8), there is an important application of Corollary 5.17 which completes the proof of a result asserted by Powers and Robinson [14] about the existence of extensions of E_0-semigroups to one parameter automorphism groups.

COROLLARY 5.21. *Let $\alpha=\{\alpha_t: t\geq 0\}$ be an E_0-semigroup acting on a type I_∞ factor M having a separable predual. Then there is a faithful normal nondegenerate representation π of M on a separable Hilbert space H, and a strongly continuous one parameter unitary group $U=\{U_t: t\in \mathbf{R}\}$ acting on H such that*

$$\pi(\alpha_t(A)) = U_t \pi(A)\, U_t^*, \quad A \in M,\ t \geq 0.$$

The argument required to deduce this from Corollary 5.17 can be found in ([6], Theorem 5.8).

Added in proof. In a personal communication, Akitaka Kishimoto has recently given a more direct proof of Corollary 5.21 which does not make use of Corollary 5.17.

Appendix A

Let E be a product system, and let $\mathscr{B}$ be the *-algebra of all operators on $L^2(E)$ having bounded support (see action 1). We will describe a class of examples of locally normal weights defined on $\mathscr{B}$ which cannot be extended to normal weights of $\mathscr{B}(L^2(E))^+$.

For every $t>0$, let P_t be the projection of $L^2(E)$ onto the subspace of all L^2 sections which are supported in the interval $(0,t]$. Choose any measurable section $t\in(0,\infty)\to e(t)\in E_t$ of E satisfying $\|e(t)\|=1$ for every t, and define a family of vectors $\{\varepsilon_t: t>0\}$ in $L^2(E)$ as follows

$$\varepsilon_t(x)=\begin{cases} e(x), & \text{if } 0<x\leq t,\\ 0, & \text{if } x>t.\end{cases}$$

PROPOSITION A.1. *The linear functional ω_e defined on $\mathscr{B}$ by*

$$\omega_e(B)=\lim_{t\to\infty}\langle B\varepsilon_t,\varepsilon_t\rangle \tag{A.2}$$

is a locally normal weight satisfying $w_e(P_t)=t$ for every $t>0$. ω_e cannot be extended to a normal weight on $\mathscr{B}(L^2(E))^+$.

Proof. It is clear that the family of vectors $\{\varepsilon_t: t>0\}$ is coherent in the sense that $P_s\varepsilon_t=\varepsilon_s$ for every $0<s\leq t$. This implies that for every $B\in\mathscr{B}$ the function $t\to\langle B\varepsilon_t,\varepsilon_t\rangle$ stabilizes for large t, and hence (A.2) defines a linear functional on $\mathscr{B}$. Clearly

$$\omega_e(P_t)=\int_0^t\|e(x)\|^2dx=t,$$

and the restriction of ω_e to $P_t\mathscr{B}P_t$ is given by

$$\omega_e(B)=\langle B\varepsilon_t,\varepsilon_t\rangle,\quad B\in P_t\mathscr{B}P_t. \tag{A.3}$$

In order to show that ω_e cannot be extended to a normal weight of $\mathscr{B}(L^2(E))$, we will show that there is no nonzero vector $\xi\in L^2(E)$ such that

$$\|B\xi\|^2\leq\omega_e(B^*B),\quad B\in\mathscr{B}. \tag{A.4}$$

Granting that for a moment, it follows that ω_e cannot be so extended. Indeed, if there were a normal weight ω' on $\mathscr{B}(L^2(E))$ which extends ω_e, then by a rather special case of Haagerup's theorem for normal weights on von Neumann algebras ([10], 5.1.8) ω' would have a representation

$$\omega'(T)=\sum_{n=1}^{\infty}\langle T\xi_n,\xi_n\rangle,\quad T\in\mathcal{B}(L^2(E))^+,$$

for some sequence ξ_n in $L^2(E)$. Thus for each n we have

$$\|B\xi_n\|^2\leqslant\omega_e(B^*B),\quad B\in\mathcal{B}.$$

The assertion implies that each ξ_n is zero, hence $\omega'=0$, and we have the contradiction $t=\omega_e(P_t)=\omega'(P_t)=0$.

In order to prove the assertion of the previous paragraph, choose a vector $\xi\in L^2(E)$ which satisfies (A.4) and fix $t>0$. We claim first that there is a complex number $\lambda(t)$ such that

(A.5) $$P_t\xi=\lambda(t)\,\varepsilon_t.$$

Indeed, the inequality

$$\|BP_t\xi\|^2\leqslant\omega_e(P_tB^*BP_t)=\|B\varepsilon_t\|^2$$

implies that there is a unique contraction operator L on $P_tL^2(E)$ which satisfies $L(B\varepsilon_t)=BP_t\xi$, for every bounded operator B on $P_tL^2(E)$. Since L commutes with every operator on $P_tL^2(E)$, it must be a scalar multiple $\lambda(t)1$ of the identity. (A.5) follows.

Notice next that $\lambda(t)$ does not depend on t. For if $0<s\leqslant t$, then we can write $P_s(\lambda(t)\,\varepsilon_t)=\lambda(t)P_s\varepsilon_t=\lambda(t)\,\varepsilon_s$. Using (A.5), the left side of the preceding equation is $P_sP_t\xi=P_s\xi=\lambda(s)\,\varepsilon_s$, and hence $\lambda(s)\,\varepsilon_s=\lambda(t)\,\varepsilon_s$. Thus $\lambda(s)=\lambda(t)$.

This proves that there is a complex number λ such that $P_t\xi=\lambda\varepsilon_t$ for every $t>0$. In particular, for every $t>0$ we have

$$|\lambda|t=\|\lambda\varepsilon_t\|=\|P_t\xi\|\leqslant\|\xi\|,$$

which implies that $\lambda=0$. Hence $P_t\xi=0$ for every $t>0$, and we have the desired conclusion

$$\xi=\lim_{t\to\infty}P_t\xi=0.$$ □

Appendix B

Proof of Lemma 4.9. Let H_0 be the given dense subset of H, and fix $t\in\mathbf{R}$. For every $n\in\mathbf{Z}$, put $B(n)=A(nt)$. We have to show that $\|B(1)\|\leqslant 1$.

For every $\xi \in H_0$, the sequence of complex numbers $\{c_n : n \in \mathbf{Z}\}$ defined by $c_n = \langle B(n)\xi, \xi\rangle$ is positive definite and satisfies $c_0 = \|\xi\|^2$. Hence there is a positive measure μ_ξ on the unit circle $\mathbf{T}$ such that

$$\text{(B.1)} \qquad \int_{\mathbf{T}} z^n d\mu_\xi(z) = \langle B(n)\xi, \xi\rangle, \quad n \in \mathbf{Z}.$$

Notice that (B.1) implies that $\mu_\xi(T)=1$. We may define a linear mapping ϕ on the space $\mathcal{P}$ of trigonometric polynomials on $\mathbf{T}$ into $\mathcal{B}(H)$ by

$$\phi\left(\sum_n a_n z^n\right) = \sum_n a_n B(n).$$

Clearly $\phi(1)=1$, and (B.1) implies that for every $\xi \in H_0$ and every $f \in \mathcal{P}$,

$$\text{(B.2)} \qquad \langle \phi(f)\xi, \xi\rangle = \int_{\mathbf{T}} f(z)\, d\mu_\xi(z).$$

(B.2) implies that ϕ is a positive linear map in the sense that for every $f \in \mathcal{P}$ we have

$$\text{(B.3)} \qquad f(z) \geq 0 \textit{ for every } z \in \mathbf{T} \Rightarrow \phi(f) \geq 0.$$

Indeed, if $f \geq 0$ then (B.2) implies that $\langle \phi(f)\xi, \xi\rangle \geq 0$ for every $\xi \in H_0$, and hence $\phi(f) \geq 0$ because H_0 is dense in H.

From (B.3) it follows that ϕ is bounded, and that ϕ extends uniquely to a unital positive (in fact, completely positive) linear map of the commutative C^*-algebra $C(\mathbf{T})$ into $\mathcal{B}(H)$. A standard dilation-theoretic argument now shows that ϕ is a contraction (indeed, a complete contraction). In particular,

$$\|B(1)\| = \|\phi(z)\| \leq \|z\|_\infty = 1. \qquad \square$$

Appendix C

Let E be a product system which has a unit $e = \{e(t) : t > 0\}$. This means that e is a measurable section of $p: E \to (0, \infty)$ which satisfies $e(s+t) = e(s)e(t)$ and is not the trivial section $e \equiv 0$. We will show how one can write down an essential pure state of $C^*(E)$ in terms of e.

By rescaling e if necessary, we may assume that e is normalized in the sense that and $\|e(t)\| = 1$ for every t (see [1]). We will exhibit a locally normal weight ω which is invariant under the action of the semigroup $\beta=\{\beta_t: t\geq 0\}$ discussed in section 5, which satisfies $\omega(P_t)=t$ for every $t\geq 0$, and which defines an extreme ray in the partially ordered cone $\mathcal{W}_\beta$ in the sense that the only elements $\mu\in\mathcal{W}_\beta$ satisfying $0\leq\mu\leq\omega$ are of the form $\mu=c\omega$ for some $c\in[0,1]$. Granting that for a moment, Theorem 4.15 implies that $d\omega$ is an essential pure state of $C^*(E)$.

ω is defined as follows. For every $t>0$, define $\varepsilon_t\in L^2(E)$ by $\varepsilon_t(x)=e(x)$ if $0<x<t$, and $\varepsilon_t(x)=0$ if $x\geq t$. Clearly $\|\varepsilon_t\|^2=t$, and the family of vectors $\{\varepsilon_t: t>0\}$ is coherent in the sense that we have $P_s\varepsilon_t=\varepsilon_s$ for every $0<s\leq t$. Hence there is a unique locally normal weight ω on $\mathcal{B}$ satisfying

$$\omega(B)=\langle B\varepsilon_t,\varepsilon_t\rangle,\quad B\in P_t\,\mathcal{B}(L^2(E))\,P_t,$$

for every $t>0$. One has $\omega(P_t)=t$ for every $t>0$. Moreover, because e satisfies $e(s+t)=e(s)\,e(t)$, a straightforward computation (which we omit) shows that $\omega\circ\beta_s=\omega$ for every $s\geq 0$. Hence ω is an invariant weight in $\mathcal{W}_\beta$.

The restriction of $t^{-1}\omega$ to the von Neumann algebra $P_t\,\mathcal{B}(L^2(E))\,P_t$ is a pure normal state, and the required extremal property of ω follows.

Remark. Not every essential pure state of $C^*(E)$ is obtained from the above construction. Indeed, it follows from the work of Powers [12] that there exist E_0-semigroups α with the property that their corresponding product systems E^α have no units whatsoever. In such a situation, every essential pure state of $C^*(E^\alpha)$ (whose existence is guaranteed by Corollary 5.18) is an example of an essential pure state which is not associated with a unit of E^α.

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