Acta Math., 175 (1995), 151–164

On the Dirichlet problem for Hessian equations

by

NEIL S. TRUDINGER(1)

The Australian National University
Canberra, Australia

1. Introduction

In this paper we consider the classical solvability of the Dirichlet problem for nonlinear, second-order elliptic partial differential equations of the form,

$$F(D^2u) \equiv f(\lambda[D^2u]) = \psi(x,u,Du), \tag{1.1}$$

in domains Ω in Euclidean n-space, $\mathbf{R}^n$, where f is a given symmetric function on $\mathbf{R}^n$, λ denotes the eigenvalues $\lambda_1, ..., \lambda_n$ of the Hessian matrix of second derivatives D^2u and ψ is a given function in $\Omega\times\mathbf{R}\times\mathbf{R}^n$. Equations of this type were treated by Caffarelli, Nirenberg and Spruck [2], for the case $\psi\equiv\psi(x)$, who demonstrated the existence of classical solutions for the Dirichlet problem, under various hypotheses on the function f and the domain Ω. Their results extended their previous work [1], and that of Krylov [13], Ivochkina [8] and others, on equations of Monge–Ampère type,

$$F(D^2u) = \det D^2u = \psi(x,u,Du). \tag{1.2}$$

Typical cases, embraced by [2] and treated as well by Ivochkina [9], are the elementary symmetric functions,

$$f(\lambda) = S_k(\lambda) = \sum_{i_1<i_2<...<i_k} \lambda_{i_1} ... \lambda_{i_k}, \tag{1.3}$$

$k=1, ..., n$. Note that the case $k=1$ corresponds to Poisson's equation, while for $k=n$, we have the Monge–Ampère equation (1.2). If the function $\psi(x)$, boundary $\partial\Omega$ and boundary function ϕ are sufficiently smooth and ψ is uniformly positive in Ω, the classical Dirichlet problem,

$$\begin{aligned} F(D^2u) = S_k(\lambda[D^2u]) &= \psi \quad \text{in } \Omega, \\ u &= \phi \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

(1) Research supported by the Australian Research Council.

11-950852 *Acta Mathematica* 175. Imprimé le 21 décembre 1995

is uniquely solvable, in the class of admissible functions, provided the domain Ω is uniformly $(k-1)$-convex. A function $u\in C^2(\Omega)$ is called *admissible* with respect to the operator F if

$$F(D^2u+\eta)\geqslant F(D^2u) \tag{1.5}$$

for all $\eta\geqslant 0$, $\eta\in\mathbf{S}^n$, where $\mathbf{S}^n$ denotes the space of $n\times n$ symmetric real matrices. For the operators (1.4) (with $k>1$), the function u is admissible if and only if $\lambda[D^2u]\in\bar{\Gamma}_k$, where Γ_k is the cone in $\mathbf{R}^n$ given by

$$\Gamma_k=\{\lambda\in\mathbf{R}^n \mid S_j(\lambda)>0,\, j=1,\ldots,k\}. \tag{1.6}$$

The domain Ω is called *m-convex* (*uniformly m-convex*), $m=1,\ldots,n-1$, if the principal curvatures of the boundary $\partial\Omega$, $\varkappa=(\varkappa_1,\ldots,\varkappa_{n-1})\in\bar{\Gamma}_m$ (Γ_m). In the Monge–Ampère case, $k=n$, the above condition with $m=n-1$ reduces to the uniform convexity of Ω. Recently Guan and Spruck [6] in the Monge–Ampère case, and Guan [5] in the general case (1.1), showed that these geometric conditions could be replaced by the more general assumption of existence of a *strict* subsolution.

The hypotheses on the function f in the papers [2], [5], [14] (except for the case of constant ψ in [2]), include the requirement that

$$f(\lambda_1,\ldots,\lambda_{n-1},\lambda_n+R)\to\infty \tag{1.7}$$

as $R\to\infty$, for each admissible λ with $f(\lambda)>0$, which precludes the important examples of quotients of elementary symmetric functions,

$$f(\lambda)=S_{k,l}(\lambda)=\frac{S_k(\lambda)}{S_l(\lambda)},\quad n\geqslant k>l\geqslant 1. \tag{1.8}$$

Specifically, condition (1.7) is used in [2], [5] and [14] to estimate the double normal derivative $D_{nn}u$ of admissible solutions at the boundary. On the other hand, Lipschitz viscosity solutions of the Dirichlet problem for these cases are readily deduced by the methods of [18]. In this paper, we present a new technique for estimation of the double normal second derivatives, which covers the situation when (1.7) does not hold. We also show how the same technique can be used to provide alternative proofs in the presence of (1.7), including, in particular, a new proof to that of Ivochkina [7] in the Monge–Ampère case. The same technique is applied to curvature quotient equations in [16].

To illustrate our results, we formulate two existence theorems that will follow from our estimates. The first concerns the special case of quotients of elementary symmetric functions. For completeness, we define $S_0(\lambda)\equiv 1$.

THEOREM 1.1. *Let* $0\leqslant l<k\leqslant n$ *and* Ω *be a bounded, uniformly* $(k-1)$*-convex domain in* $\mathbf{R}^n$, *with* $\partial\Omega\in C^{3,1}$, $\phi\in C^{3,1}(\partial\Omega)$ *and let* ψ *be a positive function in* $C^{1,1}(\bar{\Omega})$. *Then the Dirichlet problem,*

$$\begin{aligned} F(D^2u)=S_{k,l}(\lambda[D^2u])=\psi \quad &\text{in } \Omega,\\ u=\phi \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

is uniquely solvable for admissible $u\in C^{3,\alpha}(\bar{\Omega})$ *for any* $0<\alpha<1$.

More generally, let us assume that the symmetric function f is defined on an open convex symmetric cone Γ in $\mathbf{R}^n$, with vertex at the origin. The function $f\in C^2(\Gamma)$ is assumed to also satisfy:

$$D_i f>0 \ \text{ in } \Gamma, \quad i=1,\dots,n; \tag{1.10}$$

$$f \text{ is concave in } \Gamma; \tag{1.11}$$

$$\limsup_{\lambda\to\lambda_0} f(\lambda)\leqslant 0 \quad \text{for every } \lambda_0\in\partial\Gamma; \tag{1.12}$$

$$f(R\lambda)\to\infty \ \text{ as } R\to\infty \quad \text{for every } \lambda\in\Gamma. \tag{1.13}$$

THEOREM 1.2. *Let* Ω *be a bounded domain in* $\mathbf{R}^n$, *with* $\partial\Omega\in C^{3,1}$, $\phi\in C^{3,1}(\partial\Omega)$ *and* ψ *a positive function in* $C^{1,1}(\bar{\Omega})$. *Then the Dirichlet problem,*

$$\begin{aligned} F(D^2u)=f(\lambda[D^2u])=\psi \quad &\text{in } \Omega,\\ u=\phi \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.14}$$

is uniquely solvable for admissible $u\in C^{3,\alpha}(\bar{\Omega})$, $0<\alpha<1$, *provided the curvatures of* $\partial\Omega$, $\varkappa_1,\dots,\varkappa_{n-1}$, *satisfy* $(\varkappa_1,\dots,\varkappa_{n-1},R)\in\Gamma$ *for some* $R>0$.

Note that the above properties of the cone Γ ensure that an admissible solution of (1.14) satisfies $\lambda[D^2u]\in\Gamma$. The basic properties of quotients of elementary symmetric functions (see for example [18]) show that Theorem 1.1 corresponds to the special case of Theorem 1.2 when

$$f(\lambda)=(S_{k,l}(\lambda))^{1/(k-l)}, \quad \Gamma=\Gamma_k. \tag{1.15}$$

The geometric conditions on Ω in Theorems 1.1 and 1.2 are necessary when the boundary function ϕ is constant. In general we may, as in Guan [5], replace them through the existence of an admissible strict subsolution $\mathbf{u}$ taking the same boundary values ϕ on $\partial\Omega$. We shall also address this more general version below.

The plan of this paper is as follows. In the next section we derive the fundamental double normal second derivative estimate for the cases typified by the example (1.8), thereby proving Theorems 1.1 and 1.2 in these cases. In §3, we show how our techniques provide an alternative and considerably shorter proof of the remaining cases when (1.7) holds to that given by Caffarelli, Nirenberg and Spruck in [2]. In the last section we treat various extensions to degenerate problems, general domains Ω and inhomogeneous terms ψ, as in [5], more general functions f and related curvature equations. Our notation, unless otherwise specified, follows the book [4].

Finally, it is a pleasure to acknowledge the hospitality of the Department of Mathematical Sciences, University of Tokyo, where this research was undertaken in the Spring of 1993.

2. Second derivative estimates in the bounded case

Let Γ_∞ be the projection of the cone Γ (in the hypotheses of Theorem 1.2) on $\mathbf{R}^{n-1}$ and suppose that for some $\lambda'=(\lambda_1, \ldots, \lambda_{n-1})\in\Gamma_\infty$, we have

$$f_\infty(\lambda') = \lim_{\lambda_n\to\infty} f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) < \infty. \tag{2.1}$$

It follows, by the concavity and positivity of f on Γ, that f_∞ is finite on all of Γ_∞, and we shall refer to this case as the *bounded case*. Typical examples are given by

$$\begin{aligned} f(\lambda) &= (S_{k,l}(\lambda))^{1/(k-l)}, \quad \Gamma=\Gamma_k,\ 1\leqslant l<k\leqslant n, \\ f_\infty(\lambda') &= (S_{k-1,l-1}(\lambda'))^{1/(k-l)}, \quad \Gamma_\infty=\Gamma_{k-1}. \end{aligned} \tag{2.2}$$

As is well known (see, for example, [4], [2]), the solvability of the Dirichlet problem (1.14), by the method of continuity, depends upon the establishment of *a priori* estimates for derivatives, up to the second order, of prospective solutions. All of these estimates are covered by Caffarelli, Nirenberg and Spruck [2], under our hypotheses (1.10) and (1.13), except for the double normal derivative at the boundary, where they assume, in addition, (1.7). In this section, we treat this estimate in the bounded case. Accordingly, let us assume that the asymptotic limit f_∞ is well defined on Γ_∞ and that $u\in C^3(\bar{\Omega})$ is an admissible solution of the Dirichlet problem (1.14) with estimates

$$\begin{aligned} |u|+|Du| &\leqslant M_1 \quad \text{in } \Omega, \\ |D_{ij}u(y)| &\leqslant M_2', \quad i+j<2n,\ y\in\partial\Omega, \end{aligned} \tag{2.3}$$

already under control. In (2.3), the coordinate system is chosen so that the positive x_n-axis is directed along the inner normal at the point $y\in\partial\Omega$. To complete the estimation of second derivatives, it therefore remains to estimate the double normal derivative $D_{nn}u(y)$.

We first illustrate our procedure for the special case when f is given by (2.2) and $\phi\equiv 0$. Letting γ denote the unit outer normal vector field on $\partial\Omega$ we then have, with respect to a principal coordinate system at any point $y\in\partial\Omega$ ([4, §14.6]),

$$D_{ij}u=(D_\gamma u)\varkappa_i\delta_{ij},\quad i,j=1,\ldots,n-1,\ D_\gamma u=-D_n u, \tag{2.4}$$

where $\varkappa=\varkappa_1,\ldots,\varkappa_{n-1}$ denotes the principal curvatures of $\partial\Omega$ at y. Consequently we have the formulae

$$\begin{aligned} S_k(\lambda[D^2u]) &= (D_\gamma u)^{k-1}S_{k-1}(\varkappa)D_{nn}u+(D_\gamma u)^k S_k(\varkappa)\\ &\quad -(D_\gamma u)^{k-2}\sum_{i=1}^{n-1}S_{k-2;i}(\varkappa)(D_{in}u)^2 \\ &\equiv A_k D_{nn}u+B_k, \end{aligned} \tag{2.5}$$

where $S_{k-2;i}(\varkappa)=S_{k-2}(\varkappa)|_{\varkappa_i=0}$. Accordingly we can write equation (1.14) in the form

$$S_{k,l}(\lambda[D^2u])=\frac{A_kD_{nn}u+B_k}{A_lD_{nn}u+B_l}=\psi^{k-l}. \tag{2.6}$$

Since u is admissible, we must have

$$\psi\leqslant\left(\frac{A_k}{A_l}\right)^{1/(k-l)}=f_\infty(\varkappa)D_\gamma u. \tag{2.7}$$

Note that since $\Delta u>0$ for $\lambda\in\Gamma$, we must have $D_\gamma u>0$ and also, by virtue of our geometric assumption in Theorems 1.1 and 1.2, $f_\infty(\varkappa)>0$. Our trick is to consider a point $y\in\partial\Omega$, where the quantity

$$g=D_\gamma u-f_\infty^{-1}\psi \tag{2.8}$$

is minimized, that is, y satisfies

$$D_\gamma u(x)\geqslant D_\gamma u(y)+f_\infty^{-1}\psi(x)-f_\infty^{-1}\psi(y) \tag{2.9}$$

for all $x\in\partial\Omega$. With respect to a principal coordinate system at y, we then have

$$D_nu(x)\leqslant-\gamma_n(x)D_nu(y)+\gamma_n(x)[f_\infty^{-1}\psi(y)-f_\infty^{-1}\psi(x)] \tag{2.10}$$

for $x\in\mathcal{N}\cap\partial\Omega$, where $\mathcal{N}$ is some neighbourhood of y such that $\gamma_n<0$ in $\mathcal{N}\cap\partial\Omega$. By differentiation of equation (1.14), we have

$$F^{ij}(D^2u)D_{ijn}u=D_n\psi\quad\text{in }\Omega, \tag{2.11}$$

where

$$F^{ij}(r)=\frac{\partial}{\partial r_{ij}}F(r),\quad r\in\mathbf{S}^n,$$

so that with the aid of the barrier constructions in [2], we infer a one-sided estimate

$$D_{nn}u(y)\leqslant C, \tag{2.12}$$

with constant C depending on n, $\partial\Omega$, M_1 and $|\psi|_2$. Observe that our assumptions $\partial\Omega\in C^{3,1}$, $\psi\in C^{1,1}$ guarantee that the function on the right hand side of (2.10) belongs to $C^{1,1}(\partial\Omega)$. From (2.3), (2.12) and since $\Delta u\geqslant 0$ for admissible u, we then have a full second derivative estimate at y, that is,

$$|D^2u(y)|\leqslant C \tag{2.13}$$

with C depending on n, $\partial\Omega$, M_1, M_2' and $|\psi|_2$. Consequently, if

$$\psi(y)\geqslant\psi_0>0, \tag{2.14}$$

we obtain, from (1.10) (and [2, Lemma 1.2]),

$$D_\gamma u f_\infty(\varkappa)=\lim_{t\to\infty} f(D^2u+t\gamma\otimes\gamma)\geqslant\psi+\delta \tag{2.15}$$

for some positive constant δ, depending on ψ_0, M_1, M_2', n, $\partial\Omega$ and $|\psi|_2$. An estimate for the double normal derivative on the whole boundary then follows from (2.6) and we are done.

Let us now move on to the general case. In place of formula (2.4) we have

$$D_{ij}u=D_\gamma(u-\phi)\varkappa_i\delta_{ij}+D_{ij}\phi=D_\gamma u\partial_i\gamma_j+\partial_i\partial_j\phi,\quad i,j=1,\dots,n-1, \tag{2.16}$$

where

$$\partial=D-\gamma(\gamma\cdot D) \tag{2.17}$$

denotes the tangential gradient in $\partial\Omega$. Furthermore, from equation (1.14), we have

$$\psi\leqslant\lim_{t\to\infty}F(D^2u+t\gamma\otimes\gamma)=f_\infty(\lambda'[D^2u]), \tag{2.18}$$

where $\lambda'=\lambda_y'=(\lambda_1,\dots,\lambda_{n-1})$ are the eigenvalues of $[D_{ij}u]_{i,j=1,\dots,n-1}$, with respect to the principal coordinate system at y. Following our argument above, for the special case (2.2), we now fix a point $y\in\partial\Omega$, where the function g given by

$$g(x)=f_\infty(\lambda_x'[D^2u])-\psi(x) \tag{2.19}$$

is minimized. To proceed further, we need to express the function g in terms of a fixed orthonormal frame. Fixing a principal coordinate system at the point y and a corresponding neighbourhood $\mathcal{N}$ of y with $\gamma_n<0$ in $\mathcal{N}\cap\partial\Omega$, we let $\xi^{(1)},\dots,\xi^{(n-1)}\in C^{2,1}(\mathcal{N}\cap\partial\Omega)$ be an orthonormal vector field which is tangential, i.e. $\xi^{(j)}\cdot\gamma=0$, $j=1,\dots,n-1$, and which agrees with our coordinate system at y, i.e. $\xi_i^{(j)}(y)=\delta_{ij}$, $i,j=1,\dots,n-1$. Writing

$$\nabla_i u=\xi_k^{(i)}D_k u, \tag{2.20}$$

in place of (2.17), we then have

$$\begin{aligned}\nabla_i\nabla_j u&=\xi_l^{(i)}D_l(\xi_k^{(j)}D_k u)\\&=\xi_l^{(i)}\xi_k^{(j)}D_{kl}u+\xi_l^{(i)}(D_l\xi_k^{(j)})D_k u\\&=\xi_l^{(i)}\xi_k^{(j)}D_{kl}u+\xi_l^{(i)}(D_l\xi_k^{(j)})\gamma_k D_\gamma u+\xi_l^{(i)}(D_l\xi_k^{(j)})\xi_k^{(r)}\nabla_r u\\&=\xi_l^{(i)}\xi_k^{(j)}(D_{kl}u-D_l\gamma_k D_\gamma u)+\xi_l^{(i)}(D_l\xi_k^{(j)})\xi_k^{(r)}\nabla_r u.\end{aligned} \tag{2.21}$$

Consequently, writing

$$\begin{aligned}\nabla_{ij}u=\xi_l^{(i)}\xi_k^{(j)}D_{kl}u,\quad \mathcal{C}_{ij}=\xi_l^{(i)}\xi_k^{(j)}D_l\gamma_k,\\ \nabla^2 u=[\nabla_{ij}u],\quad \mathcal{C}=[\mathcal{C}_{ij}],\end{aligned} \tag{2.22}$$

we have, for $x\in\partial\Omega\cap\mathcal{N}$,

$$\lambda_x'[D^2u]=\lambda'[\nabla^2u](x) \tag{2.23}$$

and, moreover, from the boundary condition $u=\phi$ on $\partial\Omega$,

$$\nabla^2u=(D_\gamma u)\mathcal{C}+\nabla^2\phi-(D_\gamma\phi)\mathcal{C}, \tag{2.24}$$

which, of course, agrees with (2.16) at the point y. For any matrix $r\in\mathbf{S}^{n-1}$, with eigenvalues $\lambda_1,\dots,\lambda_{n-1}$, let us now define

$$\begin{aligned}G(r)&=f_\infty(\lambda_1,\dots,\lambda_{n-1}),\\ G^{ij}&=\frac{\partial G}{\partial r_{ij}},\quad G_0^{ij}=G^{ij}(\nabla^2u(y)).\end{aligned} \tag{2.25}$$

Clearly the limit function f_∞ is non-decreasing and concave in the cone Γ_∞ and hence so also is the function G for matrix arguments having eigenvalues $\lambda'\in\Gamma_\infty$ (see [2]). We thus have, from (2.19), (2.23), (2.25),

$$\begin{aligned}G_0^{ij}\{D_\gamma u\mathcal{C}_{ij}(x)+\nabla_{ij}\phi(x)-D_\gamma\phi\mathcal{C}_{ij}(x)-D_\gamma u\mathcal{C}_{ij}(y)-\nabla_{ij}\phi(y)+D_\gamma\phi\mathcal{C}_{ij}(y)\}\\ \geqslant\psi(x)-\psi(y),\end{aligned} \tag{2.26}$$

and hence

$$\begin{aligned}G_0^{ij}\mathcal{C}_{ij}(y)D_\gamma u(x)\geqslant G_0^{ij}\{&(D_\gamma u(x)-D_\gamma u(y))(\mathcal{C}_{ij}(y)-\mathcal{C}_{ij}(x))+D_\gamma u(y)\mathcal{C}_{ij}(y)\\&+D_\gamma u(y)(\mathcal{C}_{ij}(y)-\mathcal{C}_{ij}(x))+\nabla_{ij}\phi(y)-\nabla_{ij}\phi(x)\\&+D_\gamma\phi\mathcal{C}_{ij}(x)-D_\gamma\phi\mathcal{C}_{ij}(y)\}+\psi(x)-\psi(y).\end{aligned} \tag{2.27}$$

Now, with respect to the principal coordinate system at y, we have

$$\mathcal{C}_{ij}=\partial_i\gamma_j=\varkappa_i\delta_{ij}$$

so that for $\delta>0$,

$$G_0^{ij}\mathcal{C}_{ij}=\varkappa_i G^{ii}=(\varkappa_i-\delta)G^{ii}+\delta G^{ii}. \tag{2.28}$$

Since $\varkappa\in\Gamma_\infty$ by hypothesis, we also have

$$(\varkappa_1-\delta,\varkappa_2-\delta,\dots,\varkappa_{n-1}-\delta)\in\Gamma_\infty$$

for some positive δ, depending on $\partial\Omega$. Consequently, by the concavity of G, we have for $A>0$,

$$(\varkappa_i-\delta)G^{ii}\geqslant\frac{1}{A}\{f_\infty(A(\varkappa_i-\delta))-f_\infty(\lambda'(D^2u))\}\geqslant\delta_1>0 \tag{2.29}$$

if A is chosen sufficiently large, where δ_1 depends on $\partial\Omega$ and $\max\psi$. Note that we can assume that $f_\infty(\lambda'(D^2u)\leqslant\psi(y)+1$. If the function f_∞ is homogeneous of degree one, we can simply estimate

$$(\varkappa_i-\delta)G^{ii}\geqslant f_\infty(\varkappa_1-\delta,\dots,\varkappa_{n-1}-\delta)\geqslant\delta_1. \tag{2.30}$$

From (2.27), (2.28) and (2.29), we thus obtain

$$\begin{aligned}D_\gamma u(x)-D_\gamma u(y)\geqslant a^{ij}\{&D_\gamma u(y)(\mathcal{C}_{ij}(y)-\mathcal{C}_{ij}(x))\\&+\nabla_{ij}\phi(y)-\nabla_{ij}\phi(x)+D_\gamma\phi\mathcal{C}_{ij}(x)-D_\gamma\phi\mathcal{C}_{ij}(y)\}\\&+a(\psi(x)-\psi(y))-\frac{CM_2'}{\delta}|x-y|^2\end{aligned} \tag{2.31}$$

for $x\in\partial\Omega$, where C is a constant depending on n, $\partial\Omega$ and a^{ij}, a, $i,j=1,\dots,n-1$, are constants satisfying

$$|a^{ij}|\leqslant\frac{1}{\delta},\quad 0<a\leqslant\frac{1}{\delta_1}. \tag{2.32}$$

Finally, we deduce, in place of (2.10),

$$\begin{aligned}D_nu(x)\leqslant-\gamma_n(x)D_nu(y)+\partial_n\phi(x)-\gamma_n(x)\Big[&a^{ij}\{D_\gamma u(y)(\mathcal{C}_{ij}(x)-\mathcal{C}_{ij}(y))\\&+\nabla_{ij}\phi(x)-\nabla_{ij}\phi(y)+D_\gamma\phi\mathcal{C}_{ij}(y)-D_\gamma\phi\mathcal{C}_{ij}(x)\}\\&+a(\psi(y)-\psi(x))+\frac{CM_2'}{\delta}|x-y|^2\Big].\end{aligned} \tag{2.33}$$

As in our previous case (2.2), we then conclude an estimate of the form (2.12), where the constant C depends in addition on M_2' and $|\phi|_4$. Subsequently, by the same argument as before, if $\psi\geqslant\psi_0>0$, we obtain a lower bound $g\geqslant\delta_0$ for the function g in (2.19), where δ_0 is a positive constant depending on n, $\partial\Omega$, M_1, M_2', $|\phi|_4$, $|\psi|_2$ and ψ_0. Since the convergence in (2.15) is uniform for $\lambda[D^2u]$ in a compact subset of Γ, we finally obtain an estimate for $D_{nn}u$ on all of $\partial\Omega$. Taking account of our remarks at the beginning of this section, we thus complete the proof of Theorem 1.2 in the bounded case (2.1).

3. A new proof in the unbounded case

In the unbounded case, condition (1.7) holds and all the necessary second derivative estimates to establish Theorem 1.2 are covered by Caffarelli, Nirenberg and Spruck [2]. However, our approach to the double normal second derivatives in the preceding section extends simply to the unbounded case, thereby providing an alternative proof to that in [2]. As in the bounded case, the method is readily illustrated for the special case of elementary symmetric functions,

$$f(\lambda) = (S_k(\lambda))^{1/k}, \quad \Gamma = \Gamma_k, \quad \Gamma_\infty = \Gamma_{k-1}, \quad 1 \leqslant k \leqslant n. \tag{3.1}$$

In this case, an estimate for $D_{nn}u$ arises from the equation itself, if we bound $S_{k-1}(\lambda'[D^2u])$ from below on $\partial\Omega$, so that in place of (2.8), (2.9) we may minimize the function

$$g = \{S_{k-1}(\lambda'[D^2u])\}^{1/(k-1)}. \tag{3.2}$$

At a minimum point $y \in \partial\Omega$, we infer an estimate of the form (2.12), as before, except that the details are simpler here because ψ is not present in g. If $\psi(y) \geqslant \psi_0 > 0$, we deduce an estimate from below for g on $\partial\Omega$ and then from equation (1.14) the desired estimate for $D_{nn}u$ on all of $\partial\Omega$.

For the general case, we can still work with a concave function g by proceeding as follows. Fixing ψ_0 as before, we choose $R_0 > 0$ by the formula,

$$R_0 = \inf\{R' > R \mid (\lambda', R') \in \Gamma, \, f(\lambda', R') > \psi_0 \text{ on } \partial\Omega\}, \tag{3.3}$$

where R is as given in the hypothesis of Theorem 1.2, and then define

$$g = f(\lambda'[D^2u], R_0) \equiv G(\nabla^2 u) \tag{3.4}$$

in place of (2.19). Again, we obtain an estimate of the form (2.12) at a minimum point $y \in \partial\Omega$. From this estimate, we infer an upper bound for R_0, namely

$$R_0 \leqslant C, \tag{3.5}$$

where C depends on n, $\partial\Omega$, M_1, M_2', $|\phi|_4$, $|\psi|_1$ and ψ_0. We thus have

$$f(\lambda'[D^2u], C) \geqslant \psi_0, \quad f(\lambda[D^2u]) \leqslant \psi \quad \text{on } \partial\Omega \tag{3.6}$$

and an estimate for $D_{nn}u$ on all of $\partial\Omega$, follows by virtue of (1.7) and Lemma 1.2 of [2]. Note that the convergence in (1.7) will be uniform on compact subsets of Γ. Furthermore, we can extend this argument to embrace the bounded case without explicit use of the limit function f_∞.

4. Other results

In this section, we treat various extensions of Theorems 1.1 and 1.2.

(i) *Degenerate equations.* Under our hypotheses (1.10)–(1.13) on the function f, the equation (1.1) will be only degenerate elliptic when $\psi=0$. This situation is covered by the following existence theorem.

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbf{R}^n$, with $\partial\Omega\in C^{3,1}$ and $\psi\geqslant 0$, $\psi\in C^{1,1}(\bar\Omega)$. Then the Dirichlet problem,*

$$\begin{aligned} F(D^2u) &= \psi \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

is uniquely solvable for admissible $u\in C^{1,1}(\bar\Omega)$ ($u\in C^{3,\alpha}(\Omega)\cap C^{1,1}(\bar\Omega)$ if $\psi>0$ in Ω), provided the curvatures of $\partial\Omega$, $\varkappa=(\varkappa_1,\dots,\varkappa_{n-1})$, satisfy $(\varkappa,R)\in\Gamma$ for some $R>0$.

Proof. By replacement of ψ by $\psi+\varepsilon$ for constant $\varepsilon>0$, it suffices to obtain second derivative estimates for solutions that are independent of $\inf\psi$. As in the non-degenerate case, we only need to check the double normal second derivatives on the boundary $\partial\Omega$, as the remaining estimation follows by direct extension of estimates in [2]. Returning to our proofs in §§ 2 and 3, we see that we need to establish a lower bound for the functions g, given by (2.19), (3.4) at points y where ψ vanishes. From the representation (2.4), it therefore suffices, in all cases, to bound

$$g=f(\varkappa D_\gamma u, R) \tag{4.2}$$

from below for sufficiently large R, which follows, by virtue of our geometric assumption on $\partial\Omega$, from a lower positive bound for $D_\gamma u$ on $\partial\Omega$. If $\psi(x_0)\geqslant\psi_0>0$ for some $x_0\in\Omega$, then clearly $\Delta u(x_0)\geqslant\delta_0$ for some δ_0 depending on ψ_0 and our particular function f. From this observation we infer a bound

$$D_\gamma u\geqslant\delta \tag{4.3}$$

on $\partial\Omega$, for some positive constant δ depending on $|\psi|_1$ and Ω, provided $\psi\not\equiv 0$ in Ω. The uniqueness assertion in Theorem 4.1 follows from the Aleksandrov maximum principle, [4].

(ii) *General domains.* As indicated in the introduction, the geometric assumption on the boundary $\partial\Omega$ in Theorems 1.1 and 1.2 can be replaced by the existence of a strict subsolution, as shown by Guan [5] for the case when (1.7) holds.

THEOREM 4.2. *Let Ω be a bounded domain in $\mathbf{R}^n$, with $\partial\Omega\in C^{3,1}$, $\phi\in C^{3,1}(\partial\Omega)$ and ψ a positive function in $C^{1,1}(\bar{\Omega})$. Then the Dirichlet problem* (1.14) *is uniquely solvable for admissible $u\in C^{3,\alpha}(\bar{\Omega})$, $0<\alpha<1$, provided there exists an admissible function, $u_0\in C^{3,1}(\bar{\Omega})$ satisfying*

$$\begin{aligned} F(D^2u_0) &\geqslant \psi+\delta_0 \quad \text{in } \Omega, \\ u_0 &= \phi \quad \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

for some positive constant δ_0.

Proof. From [5], we need only estimate the double normal second derivatives on the boundary $\partial\Omega$, and this estimation is readily accomplished by modification of the previous arguments. Observe that, by the maximum principle, $u>u_0$ in Ω, $D_\gamma u<D_\gamma u_0$ on $\partial\Omega$. Substituting u_0 for ϕ, we then have, instead of (2.29),

$$(\varkappa_i-\delta)G^{ii}\geqslant\frac{1}{D_\gamma(u_0-u)}\{G(\nabla^2u_0-\delta D_\gamma(u_0-u)I)-G(\nabla^2u)\} \tag{4.5}$$

where $\delta>0$ is chosen sufficiently small to ensure that

$$\begin{aligned} &\lambda'[\nabla^2u_0-\delta D_\gamma(u_0-u)I]\in\Gamma_\infty, \\ &G(\nabla^2u_0-\delta D_\gamma(u_0-u)I)\geqslant\psi+\tfrac{1}{3}\delta_0. \end{aligned} \tag{4.6}$$

But then, if $G(\nabla^2u)-\psi(y)\leqslant\frac{1}{3}\delta_0$, we obtain

$$(\varkappa_i-\delta)G^{ii}\geqslant\frac{\delta_0}{3(D_\gamma(u_0-u))}\geqslant\delta_1>0, \tag{4.7}$$

where δ_1 depends on u_0 and M_1. The unbounded case is similarly modified.

Condition (4.4) is clearly also a necessary condition for existence of an admissible solution in $C^2(\bar{\Omega})$. While it is more general than the hypothesis, $(\varkappa,R)\in\Gamma$, in Theorem 1.2, the two conditions coincide for constant boundary values [2].

(iii) *General inhomogeneous terms.* The general equation (1.2) is treated by Guan [5] under condition (4.4) and our techniques also permit elimination of condition (1.7) in his results, although the special cases (1.8) are excluded by further conditions imposed to obtain global second derivative bounds. To obtain second derivative boundary estimates, we assume that the function $\psi\in C^{1,1}(\bar{\Omega}\times\mathbf{R}\times\mathbf{R}^n)$ is positive and convex with respect to the gradient variables. Retaining our previous conditions (1.10)–(1.13) on the function f and Ω, namely $\partial\Omega\in C^{3,1}$, we then have the following second derivative estimate.

THEOREM 4.3. *Let $u\in C^3(\bar{\Omega})$ be an admissible solution of equation* (1.2) *in Ω with $u=u_0$ on $\partial\Omega$ where $u_0\in C^{3,1}(\bar{\Omega})$ is an admissible function satisfying* (4.4). *Then we have the estimate*

$$\max_{\partial\Omega}|D^2u|\leqslant C, \tag{4.8}$$

where C depends on n, Ω, ψ, u_0 and $|u|_1$.

Proof. The mixed tangential-normal derivatives are controlled by [5], with the function $w=u-u_0$ serving as a barrier ([5, Lemma 3.1]). The estimation of double normal second derivatives is achieved by replacing $\psi(x)$ by the composite function $\psi(x,u_0(x),Du(x))$ in the proof of Theorem 4.2 and replacing the coefficient $G_0^{ij}c_{ij}(y)$ in (2.27) by

$$G_0^{ij}c_{ij}(y)-D_{p_\gamma}\psi(y) \tag{4.9}$$

which is estimated from below, as in the proof of Theorem 4.2, by invoking the convexity of $\psi(x,z,p)$ with respect to p.

As we remarked above, we would have to impose strong restrictions on ψ to conclude an existence theorem from Theorem 4.3, without further assumptions on f. The case $\psi=\psi(x,z)$ with $\partial\psi/\partial z\geqslant 0$ is however readily embraced by Theorems 1.1, 1.2, 4.1, 4.2; also the case where ψ is independent of x,z is permissible in Theorem 4.2.

(iv) *More general functions.* We can permit more general symmetric functions f in the hypotheses of our preceding theorems. In particular, the cone Γ may be replaced by any convex symmetric open set $\Gamma\subset\mathbf{R}^n$ ($\neq\mathbf{R}^n$) satisfying $\Gamma+K^+\subset\Gamma$, $a\Gamma\subset\Gamma$ for all $a\geqslant 1$, where $K^+=K_n$ denotes the positive cone. That is, Γ is closed under addition of the positive cone and scalar multiplication by $a\geqslant 1$. Moreover, we can by approximation, relax the condition $f\in C^2(\Gamma)$ to $f\in C^{0,1}(\Gamma)$ with $\inf_K D_i f>0$ on compact subsets of Γ, $i=1,\dots,n$. In this situation we can only infer our solutions $u\in C^{2,\alpha}(\bar\Omega)$ for some $\alpha>0$ ($C^{2,\alpha}(\Omega)\cap C^{1,1}(\bar\Omega)$ in the case of Theorem 4.1).

(v) *Curvature problems.* Associated curvature problems are obtained by replacing the eigenvalues of the Hessian in (1.2) by the principal curvatures of the graph of u. The technique of this paper is applied in [16] to treat the special case (1.8), although here a much finer structure of the elementary symmetric function is necessary for the barrier arguments [15]. We may also adapt our technique here to the general equations studied by Caffarelli, Nirenberg and Spruck [3], thereby eliminating their condition (6).

Hessian and curvature equations involving elementary symmetric functions are linked to properties of Minkowski quermassintegrals. An application of Theorem 1.1, in the case $\phi\equiv 0$, to isoperimetric inequalities is given in [19] (see also [17]).

Further remarks (January 4, 1995). By means of a completely different approach, which avoids independent estimation of boundary second derivatives, Krylov established existence theorems, which include Theorems 1.1, 1.2, 4.1, in his Lipschitz lectures, University of Bonn, 1993. However, the estimations of this paper, in particular Theorems 4.2, 4.3, do not appear to be obtainable by his approach.

References

[1] CAFFARELLI, L., NIRENBERG, L. & SPRUCK, J., The Dirichlet problem for nonlinear second order equations, I: Monge–Ampère equations. *Comm. Pure Appl. Math.*, 37 (1984), 369–402.

[2] — The Dirichlet problem for nonlinear second order equations, III: Functions of the eigenvalues of the Hessian. *Acta Math.*, 155 (1985), 261–301.

[3] — Nonlinear second order elliptic equations, V. The Dirichlet problem for Weingarten hypersurfaces. *Comm. Pure Appl. Math.*, 41 (1988), 47–70.

[4] GILBARG, D. & TRUDINGER, N. S., *Elliptic Partial Differential Equations of Second Order*, 2nd edition. Springer-Verlag, Berlin–New York, 1983.

[5] GUAN, B., The Dirichlet problem for a class of fully nonlinear elliptic equations. *Comm. Partial Differential Equations*, 19 (1994), 399–416.

[6] GUAN, B. & SPRUCK, J., Boundary value problems on S^n for surfaces of constant Gauss curvature. *Ann. of Math.*, 138 (1993), 601–624.

[7] IVOCHKINA, N. N., An a priori estimate of $\|u\|_{C^2(\Omega)}$ for convex solutions of Monge–Ampère equations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*), 96 (1980), 69–79 (Russian).

[8] — Classical solvability of the Dirichlet problem for the Monge–Ampère equation. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*), 131 (1983), 72–79 (Russian).

[9] — Solution of the Dirichlet problem for some equations of Monge–Ampère type. *Mat. Sb.*, 128 (1985), 403–415; English translation in *Math. USSR-Sb.*, 56 (1987), 403–415.

[10] — Solution of the Dirichlet problem for curvature equations of order m. *Mat. Sb.*, 180 (1989), 867–887; English translation in *Math. USSR-Sb.*, 67 (1990), 317–339.

[11] — The Dirichlet problem for the equations of curvature of order m. *Leningrad Math. J.*, 2 (1991), 631–654.

[12] KRYLOV, N. V., *Nonlinear Elliptic and Parabolic Equations of the Second Order*. D. Reidel Publishing Co., Dordrecht–Boston, MA, 1987.

[13] — On degenerate nonlinear elliptic equations. *Mat. Sb.*, 121 (1983), 301–330; English translation in *Math. USSR-Sb.*, 48 (1984), 307–326.

[14] LI, Y. Y., Some existence results for fully nonlinear elliptic equations of Monge–Ampère type. *Comm. Pure Appl. Math.*, 43 (1990), 233–271.

[15] LIN, M. & TRUDINGER, N. S., On some inequalities for elementary symmetric functions. *Bull. Austral. Math. Soc.*, 50 (1994), 317–326.

[16] — The Dirichlet problem for the prescribed curvature quotient equations. *Topol. Methods Nonlinear Anal.*, 3 (1994), 307–323.

[17] TRUDINGER, N. S., On degenerate fully nonlinear equations in balls. *Bull. Austral. Math. Soc.*, 35 (1987), 299–307.

[18] — The Dirichlet problem for the prescribed curvature equations. *Arch. Rational Mech. Anal.*, 111 (1990), 153–170.

[19] — Isoperimetric inequalities for quermassintegrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11 (1994), 411–425.

NEIL S. TRUDINGER
Centre for Mathematics and its Applications
The Australian National University
Canberra ACT 0200
Australia

Received May 24, 1994
Received in revised form January 4, 1995