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Relative K-theory and topological cyclic homology

by

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0. Introduction

Let $f: A \to B$ be a map of rings up to homotopy (or "FSP's", see e.g. [Bö1], [Bö3] or Definition 3.1 below). When is it possible to give a good description of the relative algebraic K-theory? Generally, K-theory is hard to calculate, so it is of special importance to be able to measure the effect of a change of input.

Special instances of the case where f induces an epimorphism $\pi_0(A) \to \pi_0(B)$ with nilpotent kernel have been studied by several authors. The first general result in this direction was Goodwillie's theorem [G1], that in the case of simplicial rings, relative K-theory is rationally given by the corresponding relative negative cyclic homology. Recently, McCarthy has complemented this by giving a short and beautiful proof [Mc] showing that at a given prime p, the relative K-theory is given by the corresponding relative topological cyclic homology.

Although of great interest, simplicial rings do not cover all the important cases. In algebraic K-theory of spaces (Waldhausen's A-theory) the same question has been given considerable attention. If $X \to *$ is a 2-connected map (corresponding to a 1-connected map of FSP's), it is shown in [Bö2] that at a given prime p the relative K-theory is given by the topological cyclic homology. In [G5, p. 621] Goodwillie announced the statement for general 2-connected maps $X \to Y$ (this will also follow from the main theorem below).

This paper stemmed from a desire to understand the linearization $A(BG) \to K(\mathbf{Z}[G])$ (which corresponds to a 1-connected map of FSP's); that is, the connection between the algebraic K-theory of spaces and the algebraic K-theory of rings, each of which has theorems of the desired sort. Waldhausen has shown that this map is a rational equivalence, but torsion information has so far been out of reach.

In this paper we prove the conjecture of Goodwillie, posed at the International Congress of Mathematicians in Kyoto, 1990 [G5, p. 628].

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Main theorem. *Let $f\colon A\to B$ be a map of FSP's inducing an epimorphism with nilpotent kernel $\pi_0(A)\to\pi_0(B)$, and let p be a prime. Then*

$$\begin{array}{ccc} K(A)^{\wedge}_p & \longrightarrow & TC(A)^{\wedge}_p \\ \downarrow & & \downarrow \\ K(B)^{\wedge}_p & \longrightarrow & TC(B)^{\wedge}_p \end{array}$$

is homotopy Cartesian.

The conjecture as stated in [G5] was integral, but only about 1-connected maps of FSP's. Using ideas from [G4] the main theorem can be extended to an integral statement; this will appear in [DGM].

Corollary. *If X is a connected space then*

$$\begin{array}{ccc} A(X)^{\wedge}_p & \longrightarrow & TC(X)^{\wedge}_p \\ \downarrow & & \downarrow \\ K(\mathbf{Z}[\pi_1(X)])^{\wedge}_p & \longrightarrow & TC(\mathbf{Z}[\pi_1(X)])^{\wedge}_p \end{array}$$

is homotopy Cartesian.

The homotopy types of $TC(*)^{\wedge}_p$ and $TC(\mathbf{Z})^{\wedge}_p$ have been calculated by Bökstedt, Hsiang, Madsen and Rognes (for $TC(*)^{\wedge}_p$ see [Bö3], for $TC(\mathbf{Z})^{\wedge}_p$ see [Bö4]/[Bö5] when p is an odd prime, and [R] when $p=2$), and so one could hope to gain some access to the linearization $A(*)\to K(\mathbf{Z})$. The map is a rational equivalence, and both spaces have finitely generated homotopy groups (see [Dw] and [Q1]), and so we have a homotopy-Cartesian diagram

$$\begin{array}{ccc} A(*) & \longrightarrow & \prod_{p\text{ prime}} TC(*)^{\wedge}_p \\ \downarrow & & \downarrow \\ K(\mathbf{Z}) & \longrightarrow & \prod_{p\text{ prime}} TC(\mathbf{Z})^{\wedge}_p. \end{array}$$

Thus one should be in position to compare number-theoretical ideas in $K(\mathbf{Z})$ with geometrical information in $A(*)=QS^0\times \mathrm{Wh}^{\mathrm{Diff}}(*)$. Using the diagram above, John Klein and Rognes have calculated the homotopy groups of the fiber of $A(*)^{\wedge}_p\to K(\mathbf{Z})^{\wedge}_p$ in dimension less than roughly p^3 [KR]. See also the conjecture of Madsen [M1, p. 120].

For general X, if one wants to calculate $A(X)$ one should expect that the least accessible term in the Cartesian square of the corollary is $K(\mathbf{Z}[\pi_1(X)])$. However, for some torsion-free groups $\pi_1(X)$, we know by the work of Farrell, Jones, Waldhausen and others that the assembly map $B\pi_1(X)_+ \wedge K(\mathbf{Z}) \to K(\mathbf{Z}[\pi_1(X)])$ is an equivalence. This list includes free groups [W2]; and so $A(S^1)$ "only" depends on our understanding of $K(\mathbf{Z})$ and TC. In [M2] Madsen points out that this has the consequence that $\mathrm{Wh}^{\mathrm{Top}}(S^1)$ fits in a cofiber sequence where the other terms are the cofibers of the corresponding assembly maps in TC. Hence one may hope to be able to calculate $\mathrm{Wh}^{\mathrm{Top}}(S^1)$, which by the work of Farrel and Jones [FJ] determines $\mathrm{Wh}^{\mathrm{Top}}$ for negatively curved manifolds; and hence by Igusa [I], the homotopy groups of the space of pseudoisotopies in a range of dimensions. Christian Slichtcrull is currently pursuing these ideas.

One should also note that the proof of the main theorem gives as a spinoff the expected affirmative answer to the older conjecture of Goodwillie:

THEOREM. *Let A be an FSP and P an A-bimodule. Then $K^S(A,P) \simeq THH(A,P)$.*

The proof is purely homotopy theoretic, and depends only on the corresponding theorem for simplicial rings, [DM], and not on knowledge of TC, or of the manifold models for the algebraic K-theory of spaces. An outline for a different proof of the theorem is given by Schwänzl, Staffeldt and Waldhausen in [Sä]. Their outset is that the theorem is already known by [W2] in the "initial" case: $A^S(*) \simeq \lim_{k\to\infty} \Omega^k S^k$ ("the vanishing of the mystery homology theory") and that this should determine the behavior on all other rings up to homotopy.

Another idea for the proof, which seemed plausible after the ring case was established, and which I owe to Goodwillie, is the following. Try to "resolve rings up to homotopy by simplicial rings" by means of co-Cartesian diagrams and recover the general result by means the calculus of functors. It is an interesting question whether this approach can be carried out literally.

The approach chosen in this paper is to look at the cosimplicial resolution coming from the triple defining the integral completion. This resolution gives rise to cubes, just as in the calculus setup, except that they are not "pushout cubes" in any category.

The main point is that these cubes are in a sense uniformly (depending on size) close to being both pushouts and pullbacks. Using this uniformity to translate back and forth between pushouts and pullbacks whenever desirable, the theorem is reduced to elementary connectivity book keeping. In particular, we do not use the formal machinery of calculus of functors; the only trace of the relationship is the use of "Blakers–Massey-type" results (more precisely Theorems 2.5 and 2.6 in [G3]) to handle highly (co-)Cartesian cubes.

Plan. In the first section we show that the main theorem follows from two approximation theorems, one for topological cyclic homology, and one for algebraic K-theory. In the second section we prove some technicalities regarding cubical diagrams. The third section contains the actual construction of the resolution used in the first section. Then, in the fourth and fifth sections we prove the two theorems listed in §1, and we are done. In the last section we provide the promised proof of $K^S \simeq THH$.

For our purposes, good references on K, THH and TC are either one of [Bö3], [G4] and [HM]. The survey article [M1] is particularly recommended for a general overview of the subject.

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1. A reduction

Let $K \to TC$ be the cyclotomic trace of [G4, 7] or [Bö2]. This is a natural transformation of functors from FSP's (in [G4] it is only a chain of natural transformations, with some equivalences pointing in the "wrong" direction, but this can be rectified functorially to give a natural transformation). Let $f\colon A \to B$ be any map of FSP's inducing an epimorphism $\pi_0(A) \to \pi_0(B)$ with nilpotent kernel. Then the main theorem is true for f if it is true for all the other maps in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \pi_0(A) & \longrightarrow & \pi_0(B). \end{array}$$

To see this, apply $K \to TC$ to the square to get a cube, where by assumption three faces are Cartesian. By inspection we see that this forces the face corresponding to the main theorem to be Cartesian too.

By McCarthy's theorem in [Mc], relative K-theory and topological cyclic homology agree on the lower horizontal map, so we have

REDUCTION I. *The main theorem is true if it is true for $A \to \pi_0(A)$ for all FSP's A.*

Let $\mathcal{P}$ be the category whose objects are the finite subsets of the natural numbers, and with the inclusions as morphisms. In §3 we construct a functor from $\mathcal{P}$ to FSP's, $S \mapsto (A)_S$, satisfying the following:

(1) for $S \neq \varnothing$, $(A)_S$ is (equivalent to an FSP coming from) a simplicial ring, and

(2) all maps in $\mathcal{P}$ induce 1-connected maps of FSP's, and hence $\pi_0(A) \cong \pi_0(A_S)$ for all S.

If the cardinality of S is n, then $(A)_S(X) \cong \widetilde{\mathbf{Z}}^n(A(X))$, the "reduced integral homology" functor applied n times to the space $A(X)$ (see Example 2.1 for notation), and the maps connecting these spaces are induced by the Hurewicz map.

In §4 we prove

THEOREM (TC). $TC(A)\hat{_p} \to \underset{S \in \mathcal{P}-\varnothing}{\underleftarrow{\text{holim}}}\ TC((A)_S)\hat{_p}$ *is an equivalence,*

and in §5 we prove

THEOREM (K). $K(A) \to \underset{S \in \mathcal{P}-\varnothing}{\underleftarrow{\text{holim}}}\ K((A)_S)$ *is an equivalence.*

REDUCTION II. *The main theorem follows from Theorems* (TC) *and* (K), *and from the properties* (1) *and* (2) *of* $S \mapsto (A)_S$.

Proof. By Reduction I we have to show that

$$\begin{array}{ccc} K(A)\hat{_p} & \longrightarrow & TC(A)\hat{_p} \\ \downarrow & & \downarrow \\ K(\pi_0 A)\hat{_p} & \longrightarrow & TC(\pi_0 A)\hat{_p} \end{array}$$

is Cartesian, but by Theorems (TC) and (K), and property (2), this is equivalent to the square

$$\begin{array}{ccc} \underset{S \in \mathcal{P}-\varnothing}{\underleftarrow{\text{holim}}}\ K((A)_S)\hat{_p} & \longrightarrow & \underset{S \in \mathcal{P}-\varnothing}{\underleftarrow{\text{holim}}}\ TC((A)_S)\hat{_p} \\ \downarrow & & \downarrow \\ \underset{S \in \mathcal{P}-\varnothing}{\underleftarrow{\text{holim}}}\ K(\pi_0(A)_S)\hat{_p} & \longrightarrow & \underset{S \in \mathcal{P}-\varnothing}{\underleftarrow{\text{holim}}}\ TC(\pi_0(A)_S)\hat{_p}. \end{array}$$

This square is Cartesian if for each $S \in \mathcal{P}-\varnothing$,

$$\begin{array}{ccc} K((A)_S)\hat{_p} & \longrightarrow & TC((A)_S)\hat{_p} \\ \downarrow & & \downarrow \\ K(\pi_0(A)_S)\hat{_p} & \longrightarrow & TC(\pi_0(A)_S)\hat{_p} \end{array}$$

is, which follows from [Mc] and the property (1). □

Note that, for all this to make sense, it is important that we have chosen some model for K and TC such that there actually *is* a natural transformation $K \to TC$ of functors from FSP's. However, when proving Theorem (K) and Theorem (TC) we are free to choose the models for either functor which serve us best.

On our way we will also prove

THEOREM (THH). *$THH(A) \to \underset{S\in\mathcal{P}-\varnothing}{\underleftarrow{\operatorname{holim}}}\, THH((A)_S)$ is an equivalence.*

This together with Theorem (K) and [DM] gives $K^S \simeq THH$. This is carried out in §6.

2. On cubes and limits

To fix notation, we recall a few facts regarding homotopy limits. When we write $\underleftarrow{\operatorname{holim}}$ we shall mean a functorial model for what [BK, XI.8.8] calls "the total right derived functor of $\underleftarrow{\operatorname{holim}}$". For spaces this brings nothing new, but if $\mathcal{X}$ is a functor from a small category I to pointed simplicial sets we may for instance choose

$$\underset{I}{\underleftarrow{\operatorname{holim}}}\, \mathcal{X} = \hom(N(I/-), \sin|\mathcal{X}(-)|)$$

in the notation of [BK, XI.3 and XI.4] (cf. also [G3, 0.1]). This means that $\underleftarrow{\operatorname{holim}}_I \mathcal{X} \simeq \sin \underleftarrow{\operatorname{holim}}_I |\mathcal{X}|$, and "the cube $\mathcal{X}$ of simplicial sets is k-Cartesian" then is the same as "the cube $|\mathcal{X}|$ is k-Cartesian". Homotopy colimits are treated dually, but as $|\underrightarrow{\operatorname{holim}}\, \mathcal{X}| \simeq \underrightarrow{\operatorname{holim}}\, |\mathcal{X}|$ and all simplicial sets are cofibrant, this makes no difference. In view of this, "spaces" will mean either simplicial sets or topological spaces (which in all cases will come from simplicial sets anyhow).

Recall that if $\ldots \to X_n \to X_{n-1} \to \ldots \to X_0 = *$ is a tower of fibrations, then the homotopy limit is equivalent to the categorical limit, i.e. the map

$$\underset{n\in\mathbf{N}}{\underleftarrow{\lim}}\, X_n \to \underset{n\in\mathbf{N}}{\underleftarrow{\operatorname{holim}}}\, X_n$$

is an equivalence. If $I \subseteq J$ is some subcategory, and F a functor from J to pointed simplicial sets, then the restriction $\underleftarrow{\operatorname{holim}}_J F \to \underleftarrow{\operatorname{holim}}_I F|_I$ is a fibration. If $J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots \subseteq J_n \subseteq \ldots \subseteq J = \underrightarrow{\lim}_{n\in\mathbf{N}} J_n$ is a sequence of inclusions of small categories, then

$$\underset{J}{\underleftarrow{\operatorname{holim}}}\, F \cong \underset{n\in\mathbf{N}}{\underleftarrow{\lim}}\, \underset{J_n}{\underleftarrow{\operatorname{holim}}}\, F|_{J_n} \to \underset{n\in\mathbf{N}}{\underleftarrow{\operatorname{holim}}}\, \underset{J_n}{\underleftarrow{\operatorname{holim}}}\, F|_{J_n}$$

is an equivalence, and so $\underleftarrow{\operatorname{holim}}_{J_n} F|_{J_n}$ approximates $\underleftarrow{\operatorname{holim}}_J F$.

Let S be a finite set, and let $\mathcal{P}S$ be the category whose objects are the subsets of S and with the inclusions as morphisms. An S-*cube* is a functor from $\mathcal{P}S$ to some nice category where the standard homotopic notions make sense. We let $\mathcal{I}S=\mathcal{P}S-\{S\}$ and $\mathcal{F}S=\mathcal{P}S-\varnothing$. For a finite set S, we let $|S|$ denote the cardinality, and by abuse of notation we will often write "$|S|$-cube", or "$|S|$-dimensional cube" instead of S-cube. If $d\leqslant|S|$ we will let a d-*subcube* be a ($|T|=d$)-cube formed by some inclusion $\mathcal{P}T\to\mathcal{P}S$. (Note: this is not the same as a *face* in the sense of [G3, 1.12]—not all subcubes are faces—but if the reader prefers one can use faces instead in all that follows)

We say that an S-cube $\mathcal{X}$ is k-Cartesian if $\mathcal{X}_\varnothing\to\underleftarrow{\operatorname{holim}}_{\mathcal{F}S}\mathcal{X}$ is k-connected, and k-co-Cartesian if $\underrightarrow{\operatorname{holim}}_{\mathcal{I}S}\mathcal{X}\to\mathcal{X}_S$ is k-connected. As a convention we shall say that a 0-cube is k-Cartesian (or k-co-Cartesian) if $\mathcal{X}_\varnothing$ is $(k-1)$-connected (or k-connected).

So, a 0-cube is an object $\mathcal{X}_\varnothing$, a 1-cube is a map $\mathcal{X}_\varnothing\to\mathcal{X}_{\{1\}}$, and a 1-cube is k-(co-)Cartesian if it is k-connected as a map. A 2-cube is a square

$$\begin{array}{ccc}\mathcal{X}_\varnothing & \longrightarrow & \mathcal{X}_{\{1\}}\\ \downarrow & & \downarrow\\ \mathcal{X}_{\{2\}} & \longrightarrow & \mathcal{X}_{\{1,2\}}\end{array}$$

and so on. We will regard a natural transformation of n-cubes $\mathcal{X}\to\mathcal{Y}$ as an $(n+1)$-cube. In particular, if $F\to G$ is some natural transformation of functors of simplicial sets, and $\mathcal{X}$ is an n-cube of simplicial sets, then we get an $(n+1)$-cube $F\mathcal{X}\to G\mathcal{X}$. The cubes which are important to this paper are all connected with the following example.

Example 2.1. If X is a pointed set, we can form the free Abelian group $\widetilde{\mathbf{Z}}X=\mathbf{Z}[X]/\mathbf{Z}[*]$. We extend the construction to pointed simplicial sets X, and because $\pi_q(\widetilde{\mathbf{Z}}X)\cong\widetilde{H}_q(X)$ we call $\widetilde{\mathbf{Z}}X$ the reduced (integral) homology of X. If X is $(k-1\geqslant1)$-connected then the inclusion $X\xrightarrow{h_X}\widetilde{\mathbf{Z}}X$ is $(k+1)$-connected. This follows e.g. from the Hurewicz theorem and the Leray–Serre spectral sequence for $X\to K(\pi_kX,k)$, together with the fact that $H_{k+1}(K(\pi,k))=0$ if $k>1$. Using the transformation $1\xrightarrow{h}\widetilde{\mathbf{Z}}$ on $X\to\widetilde{\mathbf{Z}}X$ we get a square

$$\begin{array}{ccc}X & \xrightarrow{h_X} & \widetilde{\mathbf{Z}}X\\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_{\widetilde{\mathbf{Z}}X}}\\ \widetilde{\mathbf{Z}}X & \xrightarrow{\widetilde{\mathbf{Z}}h_X} & \widetilde{\mathbf{Z}}\widetilde{\mathbf{Z}}X.\end{array}$$

One may check by brute force that this square is $(k+2)$-Cartesian if X is $(k-1>0)$-connected. We may continue this process to obtain arbitrarily high-dimensional cubes by repeatedly applying h (see Lemma 2.6 below).

We will be using the following generalization of the Blakers–Massey theorem of Goodwillie many times.

THEOREM 2.2 ([G3, 2.5 and 2.6]). *Let S be a finite set with $|S|=n\geqslant 1$, and let $k\colon \mathcal{P}S\to\mathbf{Z}$ be a monotone function. Set $M(k)$ to be the minimum of $\sum_\alpha k(T_\alpha)$ over all partitions $\{T_\alpha\}$ of S by nonempty sets. Let $\mathcal{X}$ be an S-cube.*

(1) If $\mathcal{X}|_T$ is $k(T)$-co-Cartesian for each nonempty $T\subseteq S$, then $\mathcal{X}$ is $(1-n+M(k))$-Cartesian.

(2) If $\mathcal{X}(-\cup(S-T))|_T$ is $(k(T))$-Cartesian for each nonempty $T\subseteq S$, then $\mathcal{X}$ is $(n-1+M(k))$-co-Cartesian. □

Definition 2.3. If f is some integral function, we say that an S-cube $\mathcal{X}$ is f-Cartesian if each d-subcube (face) of $\mathcal{X}$ is $f(d)$-Cartesian. Likewise for f-co-Cartesian.

With this definition, the following is a rather trivial corollary of Theorem 2.2, but since it is a key observation used repeatedly in the paper we list it as a lemma and prove it carefully.

LEMMA 2.4. *Let $k>0$. An S-cube of spaces is $(\mathrm{id}+k)$-Cartesian if and only if it is $(2\cdot\mathrm{id}+k-1)$-co-Cartesian. The implication Cartesian to co-Cartesian holds even if $k=0$.*

Proof. Note that it is trivially true if $|S|\leqslant 1$. Assume that it is proven for all d-cubes with $d<n$.

To prove one implication, let $\mathcal{X}$ be an $(\mathrm{id}+k)$-Cartesian $(n=|S|)$-cube. All strict subcubes are also $(\mathrm{id}+k)$-Cartesian, and so $(2\cdot\mathrm{id}+k-1)$-co-Cartesian, and the only thing we need to show is that $\mathcal{X}$ itself is $(2n+k-1)$-co-Cartesian. This follows from Theorem 2.2 (2): $\mathcal{X}$ is K-co-Cartesian where

$$K=n-1+\min\Big(\sum_\alpha(|T_\alpha|+k)\Big),$$

where the minimum is taken over all partitions $\{T_\alpha\}$ of S by nonempty sets. But this minimum is clearly attained by the trivial partition, for if we subdivide T into T_1 and T_2 then $|T|+k=|T_1|+|T_2|+k\leqslant|T_1|+k+|T_2|+k$, and so $K=(n-1)+(n+k)=2n+k-1$.

In the opposite direction, let $\mathcal{X}$ be a $(2\cdot\mathrm{id}+k-1)$-co-Cartesian $(n=|S|)$-cube. This time, all strict subcubes are by assumption $(\mathrm{id}+k)$-Cartesian, and so we are left with showing that $\mathcal{X}$ is $(n+k)$-Cartesian. Again this follows from Theorem 2.2 (1): $\mathcal{X}$ is K-Cartesian where

$$K=(1-n)+\min\Big(\sum_\alpha(2|T_\alpha|+k-1)\Big),$$

where the minimum is taken over all partitions $\{T_\alpha\}$ of S by nonempty sets. But this minimum is clearly attained by the trivial partition, for if we subdivide T into T_1 and T_2 then $2|T|+k-1=2|T_1|+2|T_2|+k-1\leqslant 2|T_1|+k-1+2|T_2|+k-1$, and so $K=(1-n)+(2n+k-1)=n+k$. □

Homology takes cofiber sequences to long exact sequences. This is a reflection of the well-known statement

LEMMA 2.5. *If $\mathcal{X}$ is a co-Cartesian cube, then $\widetilde{\mathbf{Z}}\mathcal{X}$ is Cartesian.* □

We will need the following generalization of the Hurewicz theorem (cf. Example 2.1).

LEMMA 2.6. *Let $k>1$. If $\mathcal{X}$ is an $(\mathrm{id}+k)$-Cartesian cube of simplicial sets, then so is $\mathcal{X}\to\widetilde{\mathbf{Z}}\mathcal{X}$.*

Proof. To fix notation, let $\mathcal{X}$ be an $(n=|S|)$-cube with iterated fiber F and iterated cofiber C. Let $\mathcal{C}$ be the S-cube which sends S to C, and all strict subsets to $*$. Then the $(|S|+1)$-cube $\mathcal{X}\to\mathcal{C}$ is co-Cartesian.

As $\mathcal{X}$ is $(\mathrm{id}+k)$-Cartesian, it is $(2\cdot\mathrm{id}+k-1)$-co-Cartesian, and in particular, C is $(2n+k-1)$-connected. Furthermore, if $\mathcal{X}|_T$ is some d-subcube of $\mathcal{X}$ where $\{S\}\notin T$, then $\mathcal{X}|_T$ is $(2d+k-1)$-co-Cartesian, and so $\mathcal{X}|_T\to\mathcal{C}|_T=*$ is $(2d+k)$-co-Cartesian. Also, if $\mathcal{X}|_T$ is some strict subcube with $\{S\}\in T$, then $\mathcal{X}|_T\to\mathcal{C}|_T$ is still $(2d+k)$-co-Cartesian because C is $(2n+k-1)$-connected, and $d<n$. Thus $\mathcal{X}\to\mathcal{C}$ is $(2\cdot\mathrm{id}+k-2)$-co-Cartesian, and co-Cartesian. Using Theorem 2.2 (1) again, we see that $\mathcal{X}\to\mathcal{C}$ is $(1-n+2(n+1+k-2)=n+2k-1)$-Cartesian as the minimal partition is obtained by partitioning $S\cup\{n+1\}$ in two.

This implies that the map of iterated fibers $F\to\Omega^nC$ is $(n+2k-1)$-connected. We note that $n+2k-1\geqslant n+k+1$ as $k>1$. Furthermore, as C is $(2n+k-1)$-connected, $\Omega^nC\to\Omega^n\widetilde{\mathbf{Z}}C$ is $(n+k+1)$-connected.

But Lemma 2.5 implies that

$$\widetilde{\mathbf{Z}}\mathcal{X}\to\widetilde{\mathbf{Z}}\mathcal{C}$$

is Cartesian. Hence the iterated fiber of $\widetilde{\mathbf{Z}}\mathcal{X}$ is $\Omega^n\widetilde{\mathbf{Z}}C$, and we have shown that the map from the iterated fiber of $\mathcal{X}$ is $(n+1+k)$-connected. Doing this also on all subcubes gives the result. □

Note that Quillen's plus construction can be made functorial: choose for instance the partial integral completion functor of [BK, p. 219]. Then we have a natural transformation $X\to X^+$ which for pointed connected spaces is an acyclic cofibration killing the maximal perfect subgroup of the fundamental group of X.

LEMMA 2.7. *If $\mathcal{X}$ is an* (id+1)*-Cartesian S-cube of spaces, then so is $\mathcal{X}^+$.*

Proof. As all maps are 2-connected, the fundamental groups are equal for all spaces in $\mathcal{X}$ (call it π), and hence also in $\mathcal{X}^+$ (it is π/P where $P\subseteq\pi$ is the maximal perfect subgroup). The homology spectral sequences for the homotopy colimits ([BK, p. 340])

$$E^2_{pq}=\underset{\mathcal{I}S}{\underrightarrow{\lim}}{}_{(p)}\, H_q(\mathcal{X},\mathbf{Z}[\pi/P]) \;\Rightarrow\; H_{p+q}\big(\underset{\mathcal{I}S}{\underrightarrow{\operatorname{holim}}}\,\mathcal{X},\mathbf{Z}[\pi/P]\big)$$

(the coefficients are pulled back) and

$$E^2_{pq}=\underset{\mathcal{I}S}{\underrightarrow{\lim}}{}_{(p)}\, H_q(\mathcal{X}^+,\mathbf{Z}[\pi/P]) \;\Rightarrow\; H_{p+q}\big(\underset{\mathcal{I}S}{\underrightarrow{\operatorname{holim}}}\,\mathcal{X}^+,\mathbf{Z}[\pi/P]\big)$$

coincide. Hence the map $\underrightarrow{\operatorname{holim}}_{\mathcal{I}S}\,\mathcal{X}\to\underrightarrow{\operatorname{holim}}_{\mathcal{I}S}\,\mathcal{X}^+$ is acyclic and kills the maximal perfect subgroup of the fundamental group, and so $\underrightarrow{\operatorname{holim}}_{\mathcal{I}S}\,\mathcal{X}^+\simeq(\underrightarrow{\operatorname{holim}}_{\mathcal{I}S}\,\mathcal{X})^+$. By Lemma 2.4 $\mathcal{X}$ is (2·id)-co-Cartesian, and so $(\underrightarrow{\operatorname{holim}}_{\mathcal{I}S}\,\mathcal{X})^+\to\mathcal{X}^+_S$ is $2n$-connected. Thus $\mathcal{X}^+$ is $2n$-co-Cartesian. We argue likewise for every subcube and see that $\mathcal{X}^+$ is in fact (2·id)-co-Cartesian, which is the same as (id+1)-Cartesian. □

3. The FSP resolution

Recall the definition of an FSP (functor with smash product: [Böl]). We use a simplicial version partially following M. Lydakis [L] and S. Schwede [Se]. Let Γ be the category of finite pointed sets, and $\mathcal{S}_*$ the category of pointed simplicial sets. A Γ-space is a functor $F:\Gamma\to\mathcal{S}_*$. It is important to notice that the obvious maps

$$\begin{aligned}\Gamma(X\wedge Y\wedge Z, X\wedge Y\wedge Z)&\cong\Gamma(X\wedge Z,\Gamma(Y,X\wedge Y\wedge Z))\\ &\to\mathcal{S}_*(X\wedge Z,\mathcal{S}_*(F(Y),F(X\wedge Y\wedge Z)))\\ &\cong\mathcal{S}_*(X\wedge F(Y)\wedge Z,F(X\wedge Y\wedge Z))\end{aligned}\tag{3.0}$$

give rise to a natural transformation $X\wedge F(Y)\wedge Z\to F(X\wedge Y\wedge Z)$. By applying F degreewise to a simplicial finite set and taking diagonal, we may regard F as a pointed functor from simplicial pointed finite sets to $\mathcal{S}_*$, and [BF, 4] imply that it has good stabilization properties, i.e. it defines a functor with stabilization in the sense of Bökstedt.

Note that this gives rise to a connective spectrum $\{k\mapsto F(S^k)\}$, and we will say that a map (natural transformation) $F\to G$ is a stable equivalence if the map of associated spectra induces an isomorphism on homotopy groups. We will call a map $F\to G$ simply an equivalence if it induces a weak equivalence $F(X)\to G(X)$ for every X.

Definition 3.1 (cf. [Böl] or [Se]). An FSP A, is a Γ-space with a strictly associative multiplication and unit. More precisely we have natural transformations

$$\mu_{X,Y}\colon A(X)\wedge A(Y)\to A(X\wedge Y)$$

and $\mathbf{1}\colon \mathrm{id}\to A$, such that for all X, Y and Z the diagrams

$$\begin{array}{ccc} A(X)\wedge A(Y)\wedge A(Z) & \xrightarrow{\mu_{X,Y}\wedge \mathrm{id}} & A(X\wedge Y)\wedge A(Z) \\ \Big\downarrow{\scriptstyle \mathrm{id}\wedge\mu_{Y,Z}} & & \Big\downarrow{\scriptstyle \mu_{X\wedge Y,Z}} \\ A(X)\wedge A(Y\wedge Z) & \xrightarrow{\mu_{X,Y\wedge Z}} & A(X\wedge Y\wedge Z) \end{array}$$

and

$$\begin{array}{ccccc} X\wedge A(Y) & \xrightarrow{\mathbf{1}_X\wedge\mathrm{id}} & A(X)\wedge A(Y) & \xleftarrow{\mathrm{id}\wedge\mathbf{1}_Y} & A(X)\wedge Y \\ & \searrow & \Big\downarrow{\scriptstyle \mu_{X,Y}} & \swarrow & \\ & & A(X\wedge Y) & & \end{array}$$

commute, where the unlabeled diagonal maps are the natural transformations coming from (3.0).

I am grateful to the referee for pointing out that I had a redundant axiom $\mathbf{1}\cdot\mathbf{1}=\mathbf{1}$. The FSP's defined above serve as a nice model for "rings up to homotopy" (which we perhaps could call Γ-rings? They are called DFSP's in [Se] where they are defined as monoids in the closed symmetric monoidal category structure on the category of Γ-spaces displayed in [L]). Translation to and from the competing theories is left to the reader.

Let I be the category whose objects are the natural numbers (including zero), considered as sets $\{1,\dots,n\}$ for $n\geqslant 0$, and whose morphisms are all injective maps. This is the category Bökstedt used to define topological Hochschild homology. It is not filtering, but the homotopy colimits over I have the "right" homotopy properties (see the approximation lemma of [Böl] or [M1, p. 210]). This is needed in the general procedure for replacing an FSP A by a stably equivalent one whose associated spectrum is an Ω-spectrum. We set

$$QA(X)=\underset{k\in I}{\underrightarrow{\mathrm{holim}}}\,\Omega^k A(S^k\wedge X),$$

and the approximation property of I tells us that $A\to QA$ is a stable equivalence. Note that $\Omega^k(-)$ here means $\{[q]\mapsto \mathcal{S}_*(S^k\wedge\Delta[q]_+,\sin|-|)\}$.

If A is an FSP we can think of the functor $S\mapsto (A)_S$ of §1 as follows. The composition $X\mapsto\widetilde{\mathbf{Z}}A(X)$ (see Example 2.1) is a new FSP, and the Hurewicz map $A\to\widetilde{\mathbf{Z}}A$ is a map of FSP's. In fact, $\widetilde{\mathbf{Z}}$ is a functor from FSP's to FSP's and the Hurewicz map is a natural transformation from the identity. The functor $S\mapsto A_S$ can be formed iteratively by starting with the 0-cube $X \mapsto A(X)$, and applying $\widetilde{\mathbf{Z}}$, until we have the desired dimension, just as suggested in Example 2.1. That is, if $\mathcal{X}$ is the d-cube we reached at one stage, then the $(d+1)$-cube is $\mathcal{X} \to \widetilde{\mathbf{Z}}\mathcal{X}$, where the map is the Hurewicz map. In particular, $A_{\{1,\dots,n\}}X = \widetilde{\mathbf{Z}}^n(A(X))$—the n-fold iteration of $\widetilde{\mathbf{Z}}$ applied to the space $A(X)$.

A compact codification is the following. Recall that the integral completion functor of [BK] is defined by means of the cosimplicial space $\underset{\sim}{\mathbf{Z}}X$ coming from the free/forgetful adjoint pair connecting pointed sets and Abelian groups (stated by means of the associated triple in [BK, I.2]). Applying this to the underlying Γ-space of A, we get a cosimplicial Γ-space $\underset{\sim}{\mathbf{Z}}A$, i.e. a functor from the category Δ of nonempty ordered finite subsets of the natural numbers to Γ-spaces. We augment this by adding an initial element $\varnothing$, and declaring that $\underset{\sim}{\mathbf{Z}}A(\varnothing) = A$ with the obvious maps induced by the Hurewicz maps.

Definition 3.2. The composite

$$\mathcal{P} \subset \Delta \cup \varnothing \xrightarrow{\underset{\sim}{\mathbf{Z}}A} \{\Gamma\text{-spaces}\}$$

is what we in §1 called $S \mapsto (A)_S$. This is a functor to FSP's as the only maps involved are induced by Hurewicz maps. For n a natural number, let $\mathbf{n} = \{1, \dots, n\}$ and $\mathcal{P}\mathbf{n} \subset \mathcal{P}$, the category of subsets of $\mathbf{n}$ under inclusions. Let $S \mapsto (A)^n_S$ be the cube given by restricting $S \mapsto (A)_S$ to $\mathcal{P}\mathbf{n}$.

We now must prove the properties (1) and (2). (2) follows from Proposition 3.3 and (1) follows from Proposition 3.5.

We say that a cube of FSP's is k-(co-)Cartesian if the underlying cube of spectra is.

PROPOSITION 3.3. *$S \mapsto (A)^n_S$ is* id*-Cartesian.*

Proof. For each $k > 1$, $S \mapsto (A)^n_S(S^k)$ is $(\mathrm{id}+k)$-Cartesian by Lemma 2.6, which is stronger than $S \mapsto (A)^n_S$ being id-Cartesian as a spectrum. □

Aside. Taking the homotopy limit of a cosimplicial object under the composition $\mathcal{P}\mathbf{n} - \varnothing \subset \mathcal{P} - \varnothing \subset \Delta$ is closely related, [G2], to taking the $(n-1)$st total. So, saying that $S \mapsto (A)^n_S$ is so and so Cartesian is just the same as saying that the $n-1$ total spaces calculate the homotopy groups up to a certain point. If one looks at the spectral sequence [BK] coming from the cosimplicial resolution, this corresponds to saying that there is a "vanishing line" in the E^1-term, and the one found is just the classical for the E^2-term of the Adams spectral sequence [A].

Notation 3.4. The homotopy colimit and loop construction can equally well be performed in simplicial Abelian groups. More precisely, for M a simplicial Abelian group let $\Omega^k_{\mathcal{A}b}M$ be the simplicial Abelian group $\{[q]\mapsto s\mathcal{A}b(\tilde{\mathbf{Z}}(S^k\wedge\Delta[q]_+),M)\}$, and observe that we have a natural equivalence $\Omega^k M\leftarrow\Omega^k_{\mathcal{A}b}M$ (since all simplicial groups are fibrant as simplicial sets). The homotopy colimit is the diagonal of the simplicial replacement [BK, XII.5] in the category of pointed simplicial sets. If we perform this on $M\colon I\to s\mathcal{A}b$ in the category of simplicial Abelian groups instead we get

$$\underset{I}{\underrightarrow{\operatorname{holim}}}^{\mathcal{A}b}M=\operatorname{diag}\{[q]\mapsto\bigoplus_{i_0\leftarrow i_1\leftarrow\ldots\leftarrow i_q\in N_qI}M(i_q)\}.$$

The inclusion of wedges in direct sums induce a map $\underrightarrow{\operatorname{holim}}_I M\to\underrightarrow{\operatorname{holim}}_I^{\mathcal{A}b}M$, and if M takes m-connected values, then Blakers–Massey gives that this map is $(2m+1)$-connected.

PROPOSITION 3.5. *For every $S\neq\varnothing$, the FSP $(A)^n_S$ is equivalent to a simplicial ring, or more precisely, $(A)^n_S$ is stably equivalent to an FSP coming from a simplicial ring R in the usual way $X\mapsto\tilde{\mathbf{Z}}X\otimes_{\mathbf{Z}}R$.*

Proof. As we started with an arbitrary FSP A, this follows once we have shown that $\tilde{\mathbf{Z}}A$ is of the desired form.

Consider the transformations of FSP's

$$\tilde{\mathbf{Z}}A(X)\to Q\tilde{\mathbf{Z}}A(X)\leftarrow\underset{k\in I}{\underrightarrow{\operatorname{holim}}}\,\Omega^k\tilde{\mathbf{Z}}(X\wedge A(S^k))\leftarrow\underset{k\in I}{\underrightarrow{\operatorname{holim}}}\,\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}(X\wedge A(S^k)).$$

The first transformation is the stable equivalence making $\tilde{\mathbf{Z}}A$ an Ω-spectrum. The second is an equivalence since it is induced from the structure map (3.0), $X\wedge A(S^k)\to A(X\wedge S^k)$, which by the stability of A [BF, 4.1] is a stable equivalence. The third is an equivalence since $\Omega^k_{\mathcal{A}b}\to\Omega^k$ is.

Consider also the following transformations of FSP's:

$$\underset{k\in I}{\underrightarrow{\operatorname{holim}}}^{\mathcal{A}b}\tilde{\mathbf{Z}}(X)\otimes\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}A(S^k)\to\underset{k\in I}{\underrightarrow{\operatorname{holim}}}^{\mathcal{A}b}\Omega^k_{\mathcal{A}b}(\tilde{\mathbf{Z}}(X)\otimes\tilde{\mathbf{Z}}A(S^k))\leftarrow\underset{k\in I}{\underrightarrow{\operatorname{holim}}}\,\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}(X\wedge A(S^k)).$$

We have just seen that the last FSP is stably equivalent to $\tilde{\mathbf{Z}}A$. The first transformation is an equivalence by [Q2, II.6] since $\tilde{\mathbf{Z}}A(S^k)$ is $(k-1)$-connected. That the second is a stable equivalence follows by the last line of Notation 3.4 since $\Omega^k_{\mathcal{A}b}(\tilde{\mathbf{Z}}(X)\otimes\tilde{\mathbf{Z}}A(S^k))\cong\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}(X\wedge A(S^k))$ is as connected as X is.

Furthermore, tensor product commutes with homotopy colimits and we get that

$$\underset{k\in I}{\underrightarrow{\operatorname{holim}}}^{\mathcal{A}b}(\tilde{\mathbf{Z}}(X)\otimes\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}A(S^k))\cong\tilde{\mathbf{Z}}(X)\otimes\underset{k\in I}{\underrightarrow{\operatorname{holim}}}^{\mathcal{A}b}\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}A(S^k),$$

and by the following Lemma 3.6 we are done. □

Lemma 3.6. *Let*

$$R = \underset{\overrightarrow{k\in I}}{\text{holim}}^{\mathcal{A}b}\Omega^k_{\mathcal{A}b}\tilde{\mathbf{Z}}A(S^k)\in s\mathcal{A}b.$$

The multiplicative structure inherited from A, makes R a simplicial ring (integral spectrum homology of A), which in dimension q is

$$R_q = \bigoplus_{k_0\leftarrow\ldots\leftarrow k_q\in N_qI} s\mathcal{A}b(\tilde{\mathbf{Z}}(S^{k_q}\wedge\Delta[q]_+),\tilde{\mathbf{Z}}A(S^{k_q})).$$

Proof. The unit element is in the $(0=\ldots=0)$-summand and is given by the unit $S^0\to A(S^0)$, and multiplication is given as follows. If $f\colon\tilde{\mathbf{Z}}(S^{k_q}\wedge\Delta[q]_+)\to\tilde{\mathbf{Z}}A(S^{k_q})$ is in the $(k_0\leftarrow\ldots\leftarrow k_q)$-summand and $g\colon\tilde{\mathbf{Z}}(S^{l_q}\wedge\Delta[q]_+)\to\tilde{\mathbf{Z}}A(S^{l_q})$ is in the $(l_0\leftarrow\ldots\leftarrow l_q)$-summand, then $f\cdot g$ is the composite

$$\begin{aligned}\tilde{\mathbf{Z}}(S^{k_q}\wedge S^{l_q}\wedge\Delta[q]_+)&\to\tilde{\mathbf{Z}}(S^{k_q}\wedge\Delta[q]_+)\otimes\tilde{\mathbf{Z}}(S^{l_q}\wedge\Delta[q]_+)\\&\xrightarrow{f\otimes g}\tilde{\mathbf{Z}}A(S^{k_q})\otimes\tilde{\mathbf{Z}}A(S^{l_q})\to\tilde{\mathbf{Z}}A(S^{k_q}\wedge S^{l_q})\end{aligned}$$

in the $(k_0\sqcup l_0\leftarrow\ldots\leftarrow k_q\sqcup l_q)$-summand. The first map is induced by the diagonal $\Delta[q]\to\Delta[q]\times\Delta[q]$ followed by a twist, and the last is induced by the multiplication in A. The axioms for the FSP A now show that R is in fact a simplicial ring, and we leave the checking to the reader. □

4. Proof of Theorem (*TC*)

Let X be a space, and let $THH(A,X)_q$ be the q-simplicies in the topological Hochschild homology of A. More precisely,

$$THH(A,X)_q = \underset{\overrightarrow{\mathbf{x}\in I^{q+1}}}{\text{holim}}\,\Omega^{\sqcup\mathbf{x}}(X\wedge\bigwedge_{0\leqslant i\leqslant q}A(S^{x_i})),$$

where $\mathbf{x}=(x_0,\ldots,x_q)$ and $\sqcup\colon I^{q+1}\to I$ is concatenation.

We shall need that for each $q\geqslant 0$, $S\mapsto THH_q((A)^n_S)$ is id-Cartesian. Here $THH_q((A)^n_S)$ is regarded as a spectrum in the trivial way: $k\mapsto THH_q((A)^n_S,S^k)$.

The case $q=0$ follows from Theorem 3.3 above. For $q>0$ we must do some rewriting. Note that $THH((A)^n_S)_q$ essentially is a $(q+1)$-fold smash of $(A)^n_S$, and so we must study closer what happens when we smash this cube with itself. Another thing worth noting is that we never use the FSP structure when we are just looking at one simplicial dimension at the time, so we may just as well look at it on the space level.

The augmented cosimplicial object $\underset{\sim}{\mathbf{Z}}$ may be subdivided r times. Formally we compose with the edgewise subdivision $\text{sd}_r\colon\Delta\cup\varnothing\to\Delta\cup\varnothing$, $x\mapsto\coprod_r x$ [Bö3], but when looking at the corresponding cubes $(\mathcal{Z}^r)^n$, they are easier to understand as the cubes we get by iterating $\mathcal{X}\mapsto\{\mathcal{X}\xrightarrow{h_{\widetilde{\mathbf{Z}}^{r-1}\mathcal{X}}\cdots h_{\widetilde{\mathbf{Z}}\mathcal{X}}h_{\mathcal{X}}}\widetilde{\mathbf{Z}}^r\mathcal{X}\}$ as in Example 2.1. For instance, if $n=r=2$ we have

$$(\mathcal{Z}^2)^2X=\begin{array}{ccc} X & \xrightarrow{h_{\widetilde{\mathbf{Z}}X}h_X} & \widetilde{\mathbf{Z}}^2X \\ \downarrow{\scriptstyle h_{\widetilde{\mathbf{Z}}X}h_X} & & \downarrow{\scriptstyle h_{\widetilde{\mathbf{Z}}^3X}h_{\widetilde{\mathbf{Z}}^2X}} \\ \widetilde{\mathbf{Z}}^2X & \xrightarrow{\widetilde{\mathbf{Z}}^2h_{\widetilde{\mathbf{Z}}X}h_X} & \widetilde{\mathbf{Z}}^4X. \end{array}$$

LEMMA 4.1. *For $q\geqslant 0$ the cube $THH((A)^n_S,S^k)$ is equivalent to*

$$\operatorname*{holim}_{\overrightarrow{\mathbf{x}\in I^{q+1}}}\Omega^{\sqcup\mathbf{x}}(\mathcal{Z}^{q+1})^n\big(S^k\wedge\bigwedge_{0\leqslant i\leqslant q}A(S^{x_i})\big).$$

Proof. Let X_i be n_i-connected. The map $\widetilde{\mathbf{Z}}X_0\wedge\widetilde{\mathbf{Z}}X_1\to\widetilde{\mathbf{Z}}^2(X_0\wedge X_1)$ given by $(\sum n_x x)\wedge(\sum m_y y)\mapsto\sum n_x m_y(x\wedge y)$ is $(n_0+n_1+1+\min(n_0,n_1))$-connected. Using this repeatedly we get a natural transformation from

$$\mathcal{S}_*^{q+1}\xrightarrow{\widetilde{\mathbf{Z}}}\mathcal{S}_*^{q+1}\xrightarrow{\wedge}\mathcal{S}_*,\quad (X_i)\mapsto(\widetilde{\mathbf{Z}}X_i)\mapsto\bigwedge_{0\leqslant i\leqslant q}\widetilde{\mathbf{Z}}X_i$$

to

$$\mathcal{S}_*^{q+1}\xrightarrow{\wedge}\mathcal{S}_*\xrightarrow{\widetilde{\mathbf{Z}}^{q+1}}\mathcal{S}_*,\quad (X_i)\mapsto\bigwedge_{0\leqslant i\leqslant q}X_i\mapsto\widetilde{\mathbf{Z}}^{q+1}\bigwedge_{0\leqslant i\leqslant q}X_i,$$

which is $\big(\sum_{i=0}^q n_i+1+\min\{n_i\}\big)$-connected if the X_i are n_i-connected. Letting $X_i=A(S^{x_i})$ we get a $\big(\sum_{i=0}^q x_i-q+\min\{x_i-1\}+k\big)$-connected map

$$S^k\wedge\bigwedge_{0\leqslant i\leqslant q}\widetilde{\mathbf{Z}}A(S^{x_i})\to\widetilde{\mathbf{Z}}^{q+1}\big(S^k\wedge\bigwedge_{0\leqslant i\leqslant q}A(S^{x_i})\big)$$

inducing a map of cubes

$$\{S\mapsto\Omega^{\sqcup\mathbf{x}}\big(S^k\wedge\bigwedge_{0\leqslant i\leqslant q}(A)^n_S(S^{x_i})\big)\}\to\{S\mapsto\Omega^{\sqcup\mathbf{x}}\big((\mathcal{Z}^{q+1})^n_S\big(S^k\wedge\bigwedge_{0\leqslant i\leqslant q}A(S^{x_i})\big)\big)\},$$

which is $(-q+\min\{x_i-1\}+k)$-connected at every vertex. Taking the homotopy colimit, and using the approximation property of I we get an equivalence of cubes. □

PROPOSITION 4.2. *$S\mapsto THH((A)^n_S)$ is* id*-Cartesian (meaning $S\mapsto THH((A)^n_S,S^k)$ is* $(\mathrm{id}+k)$*-Cartesian for big k), and so Theorem (THH) follows.*

Proof. We see that the cube representing THH_q is exactly as the cube representing THH_0, except that we start with $S^k\wedge\bigwedge_{0\leqslant i\leqslant q}A(S^{x_i})$, and that the inductive step is not "if $\mathcal{X}$ is $(\mathrm{id}+K)$-Cartesian, then so is $\widetilde{\mathbf{Z}}\mathcal{X}$", but "if $\mathcal{X}$ is $(\mathrm{id}+K)$-Cartesian so is

$\mathcal{X}\to\tilde{\mathbf{Z}}^{q+1}\mathcal{X}$" (with $K=k+\sum_{0\leqslant i\leqslant q}x_i$). But this follows from [G3, 1.8 (i)] as it is the composite $\mathcal{X}\to\tilde{\mathbf{Z}}\mathcal{X}\to\ldots\to\tilde{\mathbf{Z}}^{q+1}\mathcal{X}$, each of which is $(\mathrm{id}+K)$-Cartesian.

This means that for each q, $THH_q((A)^n_S, S^k)$ is $(2\cdot\mathrm{id}+k-1)$-co-Cartesian, and as realization commutes with homotopy colimits we get that $THH((A)^n_S, S^k)$ is $(2\cdot\mathrm{id}+k-1)$-co-Cartesian, or equivalently $(\mathrm{id}+k)$-Cartesian. □

Let p be a prime, A an FSP and X a space. Recall the restriction map

$$R\colon \mathrm{sd}_{p^l}\, THH(A,X)^{C_{p^l}} \to \mathrm{sd}_{p^{l-1}}\, THH(A,X)^{C_{p^{l-1}}},$$

whose fiber is naturally equivalent to

$$\underset{m\in\mathbf{N}}{\underrightarrow{\mathrm{holim}}}\,\Omega^m\, THH(A, S^m\wedge X)_{hC_{p^l}}$$

(see [HM, 2] or [G4, 11.1]—where by a historical quirk, the restriction map R is called ϕ, and the Frobenius F is called i). Set

$$TR(A,X;p)=\underset{R}{\underleftarrow{\mathrm{holim}}}\,\mathrm{sd}_{p^l}\, THH(A,X)^{C_{p^l}}$$

and let $TR(A;p)$ be the associated spectrum.

LEMMA 4.3. *Assume that $\mathcal{A}$ is a cube of FSP's such that $THH(\mathcal{A})$ is* id-*Cartesian. Then $TR(\mathcal{A};p)$ is also* id-*Cartesian.*

Proof. Choose a big k such that $THH(\mathcal{A}, S^k)$ is $(\mathrm{id}+k)$-Cartesian. Let $\mathcal{X}$ be any m-subcube and $\mathcal{X}^l=\mathrm{sd}_{p^l}\mathcal{X}^{C_{p^l}}$. We are done if we can show that $\underleftarrow{\mathrm{holim}}_R\,\mathcal{X}^l$ is $(m+k)$-Cartesian. Let Z^l be the iterated fiber of $\mathcal{X}^l$ (i.e. the homotopy fiber of $\mathcal{X}^l_\varnothing\to\underleftarrow{\mathrm{holim}}_{S\neq\varnothing}\mathcal{X}^l_S$). Then $Z=\underleftarrow{\mathrm{holim}}_R\, Z^l$ is the iterated fiber of $\underleftarrow{\mathrm{holim}}_R\,\mathcal{X}^l$, and we must show that Z is $(m+k-1)$-connected. Since homotopy orbits preserve connectivity and homotopy colimits, $THH(\mathcal{A}, S^k)_{hC_{p^l}}$ must be $(\mathrm{id}+k)$-Cartesian, and so the fiber of $R\colon\mathcal{X}^l\to\mathcal{X}^{l-1}$ is $(\mathrm{id}+k)$-Cartesian. Hence $\pi_qZ^l\to\pi_qZ^{l-1}$ is surjective for $q=m+k$ and an isomorphism for $q<m+k$, and so

$$\pi_qZ\cong\underset{R}{\underleftarrow{\lim}}{}^{(1)}\pi_{q+1}Z^l\times\underset{R}{\underleftarrow{\lim}}\,\pi_qZ^l=0\quad\text{for } q<m+k. \qquad \square$$

PROPOSITION 4.4. *Let p be a prime. Assume that $\mathcal{A}$ is a cube of FSP's such that $THH(\mathcal{A})$ is* id-*Cartesian. Then $TC(\mathcal{A};p)$ is* $(\mathrm{id}-1)$-*Cartesian. In particular, $S\mapsto TC((A)^n_S;p)$ is* $(\mathrm{id}-1)$-*Cartesian, and so Theorem (TC) follows.*

Proof. The first statement follows from Lemma 4.3 and the fiber sequence

$$TC((A)^n_S, S^k;p)\to TR((A)^n_S, S^k;p)\xrightarrow{1-F} TR((A)^n_S, S^k;p),$$

where F is the Frobenius map. The second statement then follows by Proposition 4.2. □

5. Proof of Theorem (K)

Let A be an FSP. The $(m\times m)$-matrix FSP M_mA is given by

$$X\mapsto M_mA(X)=\mathcal{S}_*(\{1,...,m\}_+,\{1,...,m\}_+\wedge A(X))$$

("matrices with only one entry in each column") with the obvious multiplication and unit. The FSP structure makes $QM_mA(S^0)=\underrightarrow{\mathrm{holim}}_{x\in I}\,\Omega^xM_mA(S^x)$ a simplicial monoid. Let $\widehat{GL}_m(A)$ be the grouplike monoid formed by pullback in

$$\begin{array}{ccc}\widehat{GL}_m(A) & \longrightarrow & QM_mA\\ \downarrow & & \downarrow\\ GL_m(\pi_0A) & \longrightarrow & M_m(\pi_0A)\end{array}$$

and let $\widehat{GL}(A)=\underrightarrow{\lim}_m\widehat{GL}_m(A)$ (see [Bö3, p. 494]). The connected cover of the infinite loop space of $K(A)$ is naturally equivalent to $B\widehat{GL}(A)^+$.

PROPOSITION 5.1. *Let $\mathcal{A}$ be an* id*-Cartesian n-cube of FSP's, $n>0$. Then $K(\mathcal{A})$ is $(n+1)$-Cartesian.*

Proof. Let $M_m\mathcal{A}$ be the cube given by the $(m\times m)$-matrices in $\mathcal{A}$. This is id-Cartesian, and so $QM_m\mathcal{A}$ is an id-Cartesian cube of simplicial monoids. As all maps are 1-connected, we get $T\mapsto\widehat{GL}_m(\mathcal{A}_T)$ as the pullback in

$$\begin{array}{ccc}\widehat{GL}_m(\mathcal{A}_T) & \longrightarrow & QM_m\mathcal{A}_T\\ \downarrow & & \downarrow\\ GL_m(\pi_0\mathcal{A}_\varnothing) & \longrightarrow & M_m(\pi_0\mathcal{A}_\varnothing)\end{array}$$

for all $T\subset\mathbf{n}$. Hence $\widehat{GL}_m(\mathcal{A})$ is id-Cartesian, and so $B\widehat{GL}_m(\mathcal{A})$ is (id+1)-Cartesian. Using Lemma 2.7 we get that also $B\widehat{GL}(\mathcal{A})^+=\underrightarrow{\lim}_m B\widehat{GL}_m(\mathcal{A})^+$ is (id+1)-Cartesian. As $\pi_0K(\mathcal{A})$ is the constant cube (either $K_0(\pi_0\mathcal{A}_\varnothing)$ or $\mathbf{Z}$, depending on your choice of model), we see that $K(\mathcal{A})$ is $(n+1)$-Cartesian. (It is not (id+1)-Cartesian because its vertices are not connected). □

Theorem (K) now follows from Propositions 3.3 and 5.1.

6. Stable K-theory and topological Hochschild homology

Given the result in [DM], Theorems (*THH*) and (K) (4.2 and 5.1) also give a quick proof of the older conjecture "stable K-theory is topological Hochschild homology" for

FSP's in general (see the program of [Sä]). In particular, it gives a non-manifold proof of "the vanishing of the mystery homology theory" of Waldhausen [W1].

Let A be an FSP and P an A-bimodule in the sense of [PW]. We get a new FSP $X \mapsto (A\vee P)(X) = A(X)\vee P(X)$ by demanding the product in P to be trivial. This is in analogy with the classical case where A is a simplicial ring and P is an A-bimodule: $A \ltimes P$ is the square zero extension of A by P, and the corresponding FSP is $X \mapsto \widetilde{A\ltimes P}[X]$, which by Blakers–Massey is stably equivalent to the FSP $A\vee P$.

If P is an A-bimodule, so is $X \mapsto \widetilde{P}[S^m](X) = S^m \wedge P(X)$. We define

$$K^S(A,P) = \underset{\overrightarrow{k}}{\mathrm{holim}}\, \Omega^k \,\mathrm{fiber}\{K(A\vee \widetilde{P}[S^{k-1}]) \to K(A)\}.$$

The trace map induces a map to

$$\underset{\overrightarrow{k}}{\mathrm{holim}}\, \Omega^k \,\mathrm{fiber}\{THH(A\vee \widetilde{P}[S^{k-1}]) \to THH(A)\}.$$

We have an equivalence to this space from $\underrightarrow{\mathrm{holim}}_k\, \Omega^k(S^1_+ \wedge THH(A, \widetilde{P}[S^{k-1}]))$ given by the inclusion and the cyclic action. If we compose with the projection down to

$$\underset{\overrightarrow{k}}{\mathrm{holim}}\, \Omega^k(S^1 \wedge THH(A,\widetilde{P}[S^{k-1}])) \xrightarrow{\simeq} \underset{\overrightarrow{k}}{\mathrm{holim}}\, \Omega^k(THH(A,\widetilde{P}[S^k])) \xleftarrow{\simeq} THH(A,P)$$

we get a map on the homotopy groups, which for rings is equal to the map given in [DM], see [Mc, 4].

THEOREM 6.0. *Let A be an FSP and P an A-bimodule. Then*

$$K^S(A,P) \simeq THH(A,P).$$

Proof. The functor $S \mapsto (A)^n_S$ displayed in §3, can clearly be applied to A-bimodules as well, and $S \mapsto (P)^n_S$ will be a cube of $S \mapsto (A)^n_S$-bimodules, which ultimately gives us a cube $S \mapsto (A)^n_S \vee (P)^n_S$ of FSP's. There is a stable equivalence $(A)^n_S \vee (P)^n_S \to (A\vee P)^n_S$, consisting of repeated applications of the $2k$-connected map

$$\widetilde{\mathbf{Z}}[A(S^k)] \vee \widetilde{\mathbf{Z}}[P(S^k)] \to \widetilde{\mathbf{Z}}[A(S^k)] \oplus \widetilde{\mathbf{Z}}[P(S^k)] \cong \widetilde{\mathbf{Z}}[A(S^k)\vee P(S^k)].$$

The noninitial nodes in these cubes are of a sort taken care of by [DM], and all we need to know is that

$$K(A\vee P) \to \underset{S\neq\varnothing}{\underleftarrow{\mathrm{holim}}}\, K((A)_S \vee (P)_S),$$

$$THH(A\vee P) \to \underset{S\neq\varnothing}{\underleftarrow{\mathrm{holim}}}\, THH((A)_S \vee (P)_S)$$

and

$$THH(A,P) \to \underleftarrow{\operatorname{holim}}_{S\neq\varnothing} THH((A)_S,(P)_S)$$

are equivalences. This follows from Propositions 5.1 and (for the last statement: a very slight variation of) 4.2. □

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