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On equiresolution and a question of Zariski

by

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A mi padre

1. Introduction

Fix $x \in Y \subset V \subset W$ where x is a closed point, W is smooth over over the field $\mathbf{C}$ of complex numbers, V is a reduced hypersurface in W, and Y is an irreducible subvariety of V. Zariski proposes a notion of equisingularity intended to decide if the singularity at $x \in V$ is in some sense equivalent to that at $y \in V$, where y denotes the generic point of Y. In case the condition holds, we say that $x \in V$ and $y \in V$ are equisingular, or that V is equisingular along Y locally at x.

Zariski's notion relies and is characterized by two elementary properties, say (A) and (B).

(A) If $x \in V$ and $y \in V$ are equisingular, then $x \in V$ is regular if and only if $y \in V$ is regular.

Zariski formulates the second property in the algebroid context, namely at the completion of the local ring $\mathcal{O}_{W,x}$, say $R = \mathbf{C}[[x_1, ..., x_n]]$, a ring of formal power series over $\mathbf{C}$, and $n = \dim \mathcal{O}_{W,x}$. Assume for simplicity that Y is analytically irreducible at x (e.g. that Y is regular at x), and let y denote again the generic point of Y at R. By the Weierstrass preparation theorem one can define a formally smooth morphism

$$\pi: U_1 = \operatorname{Spec}(\mathbf{C}[[x_1, ..., x_n]]) \to U_2 = \operatorname{Spec}(\mathbf{C}[[x_1, ..., x_{n-1}]])$$

so that π induces a finite morphism $\pi: V \to U_2$. In such case let $D_\pi \in \mathbf{C}[[x_1, ..., x_{n-1}]]$ be the discriminant. Let $\Sigma_\pi = V(D_\pi) \subset U_2$ be the reduced hypersurface in U_2 defined by D_π (reduced discriminant). Note now that $\dim U_2 = \dim V = n-1$, and V is unramified over $U_2 - \Sigma_\pi$; so $\pi(y) \in \Sigma_\pi$ if V is singular at y.

(B) If $y \in V$ is a singular point, then x and y are equisingular points in V if and only if, for all sufficiently general formally smooth morphisms π, $\pi(x)$ and $\pi(y)$ are equisingular as points of the reduced hypersurface Σ_π.

Let us recall why these two naive properties characterize this notion of equisingularity. So set $x \in Y \subset V \subset W$, and y as before. If $\dim(V)-\dim(Y)=0$, then $y \in V$ is a regular point. In fact, recall that V is a reduced hypersurface and note that in this case y must be the generic point of an irreducible component of V. So in this case property (A) enables us to decide if x and y are equisingular points of V, namely if and only if x is a regular point of V.

If $\dim(V)-\dim(Y)>0$ we may choose π sufficiently general so that

$$(\dim(\Sigma_\pi)-\dim(\pi(Y)))=(\dim(V)-\dim(Y))-1,$$

and now property (B) and induction on the codimension of $Y \subset V$ finally assert that these two elementary properties characterize this notion of equisingularity.

The notion of equisingularity of $x \in Y \subset V \subset W$ (of V at Y locally at x) implies the smoothness of Y at x and equimultiplicity of V along Y at x. Passing from the algebroid to the algebraic context was possible by [H2]. The outcome is that there is a partition of the variety V in finitely many locally closed subvarieties $\{Y_i\}$, such that at each closed point $x \in V$, if say $x \in Y_{i_0}$:

(1) V is equisingular along Y_{i_0} locally at x;

(2) Y_{i_0} is smooth and V is equimultiple along Y_{i_0} locally at x;

(3) if $x \in Y' \subset V$, then V is equisingular along Y' locally at x, if and only if $x \in Y' \subset Y_{i_0}$ (in particular, V is also equimultiple along Y' locally at x).

This partition is called the Zariski stratification of V, and each Y_i is called a Zariski stratum (see [Za5, pp. 643–647]).

We are not mentioning here the development done in the study of the Zariski stratification in relation with other properties on stratifications that arise from topological or differentiable classifications of singularities.

With fixed $x \in Y \subset V \subset W$ and Y regular, there are natural ways to define a notion of *equiresolution* of V along Y, locally at the point x. We focus here on the equisingular stratification from the point of view of equiresolution. The question we address here is:

Question. If Y is the Zariski stratum that contains x, is V equiresolvable along Y locally at x?

In [Li2] Lipman proposed an inductive strategy which reduces the problem to finding an embedded resolution process for quasi-ordinary singularities which depends only on their characteristic monomials.

Characteristic monomials are data attached to quasi-ordinary singularities (see [Li3] and [Li4]). They are, together with properties of constructive desingularization presented in [V2], the main ingredients of this work.

The concept of equiresolution is largely studied and clarified in [T1] within the context of non-embedded resolutions. As for the notion of *embedded* equiresolution, we follow here condition ER (of embedded equiresolution) proposed in [Li2]. This concept has been recently applied to the classification of embedded curves in [N]. In this paper we study the compatibility of condition ER with Zariski's properties (A) and (B).

Any notion of equiresolution involves a retraction and an embedded resolution. We begin this paper by discussing this concept with an example. Set $x \in Y \subset V \subset W$, with Y smooth at x. At a suitable étale neighborhood of x we fix a retraction of W on Y. Note that the retraction defines smooth fibers $W_{x'}$, and that $V_{x'} = V \cap W_{x'}$ is a family of transversal sections. We require that the fibers $V_{x'} \subset W_{x'}$ define a family of reduced hypersurfaces (for each closed point $x' \in Y$ in a neighborhood of x). Now we require that there be an embedded resolution of $V \subset W$ which induces naturally and simultaneously an embedded resolution of each $V_{x'} \subset W_{x'}$.

The pullback of V by an embedded resolution is a union of smooth hypersurfaces having only normal crossings. One can define a natural stratification of this pullback by taking intersections of these smooth hypersurfaces. Equiresolution along Y (locally at x) imposes that each such stratum be evenly spread over Y for any $x' \in Y$ in a neighborhood of x (see Definition 2.14 and particularly Remark 2.12).

If V is regular, we may take $Y = V$, and the identity map of W will fulfill the requirement of equiresolution.

If the smooth subscheme Y is of codimension 2 in W (i.e. of codimension 1 at V), then each $W_{x'}$ is a smooth surface, $V_{x'}$ is a reduced curve, and equiresolution entails equivalence of $x' \in V_{x'} \subset W_{x'}$ as singularities of plane curves. In case that an equisingular stratum Y is of codimension 2 in W (i.e. in case $\dim \Sigma_\pi - \dim \pi(Y) = 0$), Zariski proves that $V \subset W$ is equiresolvable along Y (see also [A1]).

The embedded resolution of $V \subset W$ that Zariski uses in that proof, is a sequence of monoidal transformations along regular equimultiple centers of the hypersurface. An example of Luengo shows that if Y is an equisingular stratum of codimension >2, then equiresolution along Y will not be achieved if we only take into consideration those embedded resolutions defined by successive blowing-ups along regular equimultiple centers (see [Lu2]).

Our development goes in a different line: we shall achieve embedded resolution, but not as a sequence of monoidal transformations. Note first that property (B) is formulated in terms of general, formally smooth morphisms of relative dimension 1 over the completion of the local ring $\mathcal{O}_{W,x}$; we replace here completion by henselization. Set $x \in Y \subset V \subset W$, and Y regular at x. After suitable restriction to an étale neighborhood of $x \in W$, we may define $\pi: W \to W_1$ smooth, of relative dimension 1. By taking π sufficiently general, the restriction $\pi: V \to W_1$ is finite, and hence $\dim Y = \dim \pi(Y)$. Note also that for π sufficiently general, $\pi(Y)$ is regular at $\pi(x)$.

Our question is formulated for V equisingular along Y locally at x; in particular, V is also equimultiple along Y. In general $(\dim \Sigma_\pi - \dim \pi(Y)) = (\dim V - \dim Y) - 1$, and thus, by property (B) and induction we may assume that Σ_π is equiresolvable along $\pi(Y)$ locally at $\pi(x)$. Now Theorem 3.2 answers our Question, at least at a suitable étale neighborhood of $x \in Y \subset V \subset W$.

So our Theorem 3.2 proves that a Zariski stratum Y, and moreover, any $Y \subset V$ equisingular and regular containing a closed point x, is also equiresolvable, after suitable restriction of $V \subset W$ to an étale neighborhood of x.

In Theorem 3.2 we assume that $\Sigma_\pi \subset W_1$ is equiresolvable along $\pi(Y)$ locally at $\pi(x)$. Induction provides a nice stratification defined by the embedded resolution of the discriminant, with each strata evenly spread over $\pi(Y)$. The fiber product of this embedded resolution with $\pi: W \to W_1$ defines a stratification over the original hypersurface, but now each stratum is defined by quasi-ordinary singularities. The strategy that we follow in our proof, clearly formulated in [Li2, §4], is to try to carry on from there, by defining a procedure of embedded resolution of quasi-ordinary singularities which relies entirely on the characteristic monomials.

Our proof also requires embedded desingularization in the form of constructive desingularization. In fact it relies on properties of compatibility of constructive desingularization with étale and smooth maps studied in [V2] and [EV1].

I profited from discussions with Encinas, Luengo and Nobile.

2. On equiresolution

2.1. Motivation and definitions

Consider the (reduced) hypersurface $V_1 \subset \mathbf{C}^3$ defined by the polynomial $x^2 + yz^2$ (in $\mathbf{C}[x,y,z]$). The projection on the y-axis defines a smooth morphism $\mathbf{C}^3 \to \mathbf{C}$, and hence, for each point $t \in \mathbf{C}$ the fiber is a smooth surface. By taking intersections of these fibers with V_1 we obtain a family of plane curves. For $t \in \mathbf{C} - \mathbf{0}$ we get a family of reduced plane curves.

Set $Z = \mathbf{C} - \mathbf{0}$, let $W \subset \mathbf{C}^3$ be the pullback of Z, and set $V = V_1 \cap W$. Now the projection $\pi: W \to Z$ defines a family of reduced curves. Let Y be the singular locus of V. Note that we may view Y as the image of a section of $\pi: W \to Z$.

Let $f: W_1 \to W$ be the monoidal transformation with center Y. This is a very elementary example in which we have resolved the full family of embedded curves. But let us draw attention to some facts in order to motivate our development:

(1) Set $H = f^{-1}(Y)\,(\subset W_1)$. Then $W_1 - H$ is isomorphic to $W - Y$, and the Jacobian of $f: W_1 \to W$ vanishes along H.

(2) Note that π maps Y smoothly to Z, and the composition $\pi.f: W_1 \to Z$ is smooth.

(3) Fix a point $t \in Z$. Then the natural map induced by f, from the smooth fiber $(\pi.f)^{-1}(t)$ to the fiber $\pi^{-1}(t)$, is the blow-up of a smooth surface at a closed point.

(4) f defines an embedded resolution of the surface V, and f together with the smooth morphism in (2) define an embedded resolution of the family of plane curves.

(5) The strict transform of V together with H are regular hypersurfaces in W_1 having only normal crossings. One can naturally stratify the union of the two reduced hypersurfaces by multiplicity.

(6) $f^{-1}(Y) \subset W_1$ is a union of strata in (5), and the birational morphism f maps each such stratum smoothly on Y.

In general, we shall say that V is equiresolvable along Y by considering the conditions in (4), (5) and (6).

Note that (6) requires smoothness. We will achieve this by "lifting" the smooth map $W \to Z$ to a smooth morphism $W_1 \to Z$ (as in (2)) which maps each stratum smoothly on Z. (6) is, however, a condition on the way that each strata maps on Y, but then we shall make use of the section of $W \to Z$ with image Y.

We shall stress on *local conditions* along points at the pullback of Y in order to characterize the conditions in (4), (5) and (6).

2.1.1. Here k will denote an algebraically closed field of characteristic zero. Let W, W' be two smooth schemes over k, both irreducible of finite type and of the same dimension. If $f: W' \to W$ is a morphism over k, which is generically étale, we define a hypersurface of W', say $J(f,k)$, in terms of the principal sheaf of ideals, say $j(f,k)$, locally generated by the Jacobian. So $J(f,k)$ consists of the points of W' where f is not étale.

2.1.2. If $f: W' \to W$ is a birational morphism of smooth schemes, then $J(f,k)$ will be the set of points $x \in W'$ where f does not define a local isomorphism. A morphism $f: W' \to W$ is said to be a *modification* of smooth schemes if it is birational and proper. In this case, $J(f,k)$ will be called the *exceptional locus* of f. Here f will denote a modification, and we shall *always assume* that $J(f,k)$ is a union of regular hypersurfaces $\{H_1, ..., H_s\}$ having only normal crossings (that the invertible sheaf of ideal $j(f,k)$ is locally monomial).

Definition 2.1.3. Let $V \subset W$ be a reduced hypersurface. We shall say that a modification $f: W' \to W$ defines an *embedded resolution* of V if the hypersurface $f^{-1}(V) \cup J(f,k)$ has only normal crossings (is locally monomial). We do not require that the induced morphism $V' \to V$ be an isomorphism over the open subset $V - \mathrm{Sing}(V)$. Since V is a reduced hypersurface, however, it defines an isomorphism over a dense open subset of $V - \mathrm{Sing}(V)$.

Fix a modification of smooth schemes, $f: W' \to W$, such that $J(f,k)$ has normal crossings. Let E denote the set of hypersurfaces that are irreducible components of the exceptional locus, say $E = \{H_1, \dots, H_s\}$. Since each H_i is regular, E is sometimes said to have strict normal crossings (e.g. in the complex-analytic context). We shall use the fact that, under these conditions, an intersection, say $H_{i_1} \cap \dots \cap H_{i_r}$, is either empty or regular of pure codimension r.

Definition 2.1.4. Fix a smooth morphism of regular schemes $\pi: W \to Z$ of relative dimension m, and a modification $f: W' \to W$. We will say that π and f define a family of modifications if

(a) the composition $\pi . f: W' \to Z$ is smooth of relative dimension m, and

(b) for any subset $\{H_{i_1}, \dots, H_{i_r}\} \subset E$, setting

$$F(i_1, \dots, i_r) = H_{i_1} \cap \dots \cap H_{i_r},$$

either $F(i_1, \dots, i_r)$ is empty, or the induced morphism $F(i_1, \dots, i_r) \to Z$ is also smooth of relative dimension $m - r$.

Remark 2.1.5. Let $f: W' \to W$ and $\pi: W \to Z$ define a family of modifications. Fix a closed point $t \in Z$, and let $W(t), W'(t)$ be the fibers over t. Condition (a) says that $W'(t) \subset W'$ is smooth of dimension m. Condition (b) says that $W'(t)$ has normal crossings with the components of the exceptional locus, and that $W'(t)$ is not included in any component H_i. In particular, there is a dense open set in $W'(t)$ where the restriction, say $f_t: W'(t) \to W(t)$, defines a local isomorphism; hence $W'(t)$ is the strict transform of $W(t)$, and f_t is a modification.

Note also that the exceptional hypersurface $J(f_t, k)$ is included in a hypersurface with normal crossings, namely $J(f,k) \cap W'(t)$, so $J(f_t, k)$ also has normal crossings in $W'(t)$.

Remark 2.1.6. We now introduce a local criterion for (a) and (b). Fix a point $q \in W'$ and a regular system of parameters $\{t_1, \dots, t_r\}$ at the local ring at $t = \pi f(q) \in Z$, $\mathcal{O}_{Z,t}$. We require that

(a′) $\{t_1, \dots, t_r\}$ can be extended to a regular system of parameters, let us say $\{y_1, \dots, y_m, t_1, \dots, t_r\}$, at the local ring $\mathcal{O}_{W',q}$, and (b′) the principal ideal $j(f,k)$ can be generated by a monomial involving only the variables $\{y_1,\dots,y_m\}$.

Remark 2.1.7 (not used elsewhere). Let $f\colon W'\to W$ and $\pi\colon W\to Z$ define a family of modifications such that for any $t\in Z$ there is a modification $f_t\colon W'(t)\to W(t)$. A local computation shows that $j(f_t,k)$ is the restriction of $j(f,k)$ to the fiber $W'(t)$. This shows that relative and absolute Jacobians coincide ($j(f,k)=j(f,Z)$). In fact, if $q\in W'(t)$ maps to $x\in W(t)$, then $\{t_1,\dots,t_r\}$ extend to a regular system of parameters at both rings; it suffices now to define the square Jacobian matrix in terms of these two extended regular systems of parameters. This argument also shows that if Kf is the dualizing sheaf of f, it induces the family of dualizing sheaves over the fibers of π.

2.2. Families of resolutions and stratifications

Definition 2.2.1. We say that a reduced hypersurface V in W, and a smooth morphism $\pi\colon W\to Z$, define a *family of embedded hypersurfaces* if, for each $t\in Z$, $V(t)=V\cap W(t)$ is a *reduced* hypersurface in $W(t)$ (fiber over t).

Definition 2.2.2. Suppose that

(i) $V\subset W$ and $\pi\colon W\to Z$ define a family of hypersurfaces, and

(ii) $f\colon W'\to W$ defines an embedded resolution of V.

We say that π and f define a *family of embedded resolutions* if they define a family of modifications, and for any $t\in Z$ the modification $f_t\colon W'(t)\to W(t)$ is an embedded resolution of the reduced hypersurface $V(t)$ in $W(t)$.

Remark 2.2.3. Fix $V\subset W$, and let π and f be as above in (i) and (ii). We now introduce the following local criterion at points of W' in order to decide if they define a family of embedded resolutions.

Fix a point $q\in W'$ and let $\{t_1,\dots,t_r\}$ be a regular system of parameters at the local ring at $t=\pi f(q)\in Z$ (at $\mathcal{O}_{Z,t}$). Then

(1) $\{t_1,\dots,t_r\}$ extends to a regular system of parameters $\{t_1,\dots,t_r,y_1,\dots,y_m\}$ at $\mathcal{O}_{W',q}$;

(2) $j(f,k)$ is generated by a monomial in $\{y_1,\dots,y_s\}$;

(3) the ideal of the pullback or total transform of V at W', say $I(V)\mathcal{O}_{W'}$, is generated by a monomial in the variables $\{y_1,\dots,y_s\}$ (at $\mathcal{O}_{W',q}$).

Recall that we assume that $V(t)$ is a reduced hypersurface for each $t\in Z$. Note that conditions (2) and (3) say that the modification f_t defines an embedded resolution of $V(t)$.

Fix now $x \in Y \subset V$ where Y is regular and included in the hypersurface V in W. Whenever we say that V is *equiresolvable along* Y, we want to express that there exists a resolution of V which is, in some sense, evenly spread along Y. This notion is made precise via some natural stratification defined by a family of embedded resolutions (see [Li2, §4] or Remark 2.2.5 below). This will impose conditions on the pullback of Y in a desingularization of V.

Suppose that

(a) $\pi: W \to Z$ and $V \subset W$ define a family of embedded hypersurfaces, and

(b) locally at $x \in Y$ the restriction $\pi: Y \to Z$ is étale.

At a suitable étale neighborhood of $\pi(x) \in Z$, we can define Y as the image of a section σ of π; so for any $t \in Z$, the section defines a point $\sigma(t) = Y \cap W(t) \in V(t)$.

PROPOSITION 2.2.4. *Fix $x \in Y \subset V$ and $\pi: W \to Z$ so that* (a) *and* (b) *hold. Suppose now that $f: W' \to W$ is a modification with exceptional hypersurfaces $\{H_1, \dots, H_s\}$. Assume that*

(A) *f, π and $V \subset W$ define a family of embedded resolutions, and*

(B) *$(f^{-1}(Y))_{\text{red}} = H_1 \cup \dots \cup H_r$ $(r \leqslant s)$ is the union of some of the exceptional hypersurfaces.*

Then for any index $1 \leqslant i \leqslant s$,

$$Y \subseteq f(H_i).$$

Proof. Here Y is of codimension at least 2, and the requirement in (B) is that $f^{-1}(Y)$ be of pure codimension 1 (a hypersurface). Note first that $f(H_i) \subseteq Y$ for an index $i \leqslant r$. Since f is proper, $f(H_i)$ is closed for any index i. After suitable restriction at x we may assume that $x \in f(H_i)$ for each index $1 \leqslant i \leqslant s$, so that

$$f^{-1}(x) \cap H_i \neq \varnothing.$$

Note that $f^{-1}(x) \subset f^{-1}(Y) = H_1 \cup \dots \cup H_r$. Thus, for any index i, there is an index j, $1 \leqslant j \leqslant r$, such that

$$F(i,j) = H_i \cap H_j \neq \varnothing,$$

and clearly $f(F(i,j)) \subset f(H_j) \subset Y$.

Since the induced morphism $\pi f: F(i,j)) \to Z$ is smooth (Definition 2.1.4), it follows that $\pi f(F(i,j)) = Z$. Finally, since Y is étale over Z, and $f(F(i,j)) \subset Y$, it follows that

$$f(F(i,j)) = Y.$$

In particular, $Y \subset f(H_i)$ as was to be shown.

Remark 2.2.5 (on stratifications). Later in Definition 2.3.2 we shall formulate a definition for V to be *equiresolvable* along Y. We shall, however, present there a formulation of *equiresolution* in terms of properties that can be checked locally (properties characterized by local criteria). Our definition will be formulated so as to be suitable for our proof.

Let us discuss here what equiresolution will mean in terms of stratifications. So fix $x \in Y \subset V$ where Y is regular and included in the hypersurface V in W. We want to define a proper birational morphism $W' \to W$ such that

(i) $f^{-1}(V)$ is a union of smooth irreducible hypersurfaces, say $E_1 = \{H_1, \ldots, H_N\}$, in W' having only normal crossings.

Now set $G(i_1, \ldots, i_l)$ as the open set in $H_{i_1} \cap \ldots \cap H_{i_1}$ of points that belong to no other hypersurface of E.

(ii) We require now that $f^{-1}(Y)$ be a union of strata $G(i_1, \ldots, i_l)$, and that f map each such stratum in $f^{-1}(Y)$ smoothly onto Y.

We will come to these conditions as follows. We begin with $x \in Y \subset V \subset W$ and a smooth morphism $\pi: W \to Z$, together with a section $\sigma: Z \to Y$. If $f: W' \to W$ is defined so that conditions (1), (2) and (3) in Remark 2.2.3 hold, then one can check that $f^{-1}(V) \cup J(f, k)$ is a union of regular hypersurfaces, say $E_1 = \{H_1, \ldots, H_N\}$, having only normal crossings, and that, setting $G(i_1, \ldots, i_l)$ as before, $\pi.f$ maps $G(i_1, \ldots, i_l)$ smoothly on Z. We want to prove that they map smoothly on Y.

We claim now that given the smooth morphism $\pi: W \to Z$, together with a section $\sigma: Z \to Y$, and if conditions (A) and (B) of Proposition 2.2.4 hold for f, then both (i) and (ii) will hold.

Let $E_2 = \{H_1, \ldots, H_r\}$ be as in Proposition 2.2.4 (B). Clearly $E_2 \subset E_1$; note that $(f^{-1}(Y))_{\mathrm{red}}$ is a union of those $G(i_1, \ldots, i_l)$ with at least one $H_{i_j} \in E_2$. Now $(f^{-1}(Y))_{\mathrm{red}}$ can be naturally stratified by those $G(i_1, \ldots, i_l)$, and any such stratum is mapped smoothly to Z by $\pi.f$.

Finally note that $\sigma\pi: W \to W$ is the identity map along points of Y, and hence $f = \sigma\pi f$ when restricted to $f^{-1}(Y)$. In particular, $f: G(i_1, \ldots, i_l) \to Y$ is smooth for those $G(i_1, \ldots, i_l) \subset (f^{-1}(Y))_{\mathrm{red}}$, and hence (ii) also holds.

So our formulation of equiresolution along Y will be provided by a smooth morphism $\pi: W \to Z$, together with a section $\sigma: Z \to Y$. We will also require that there be a proper birational morphism $f: W' \to W$ such that conditions (A) and (B) of Proposition 2.2.4 hold. Note here that π and σ define a retraction of W on Y. Our definition includes resolution on each transversal section. The intersection of Y with the section is a closed point, and the pullback of that closed point in the resolution of the section can also be stratified. We will require that the stratification of the resolution of each section relate to the stratification in (ii): each such stratum arises as fibers of the smooth morphisms $G(i_1, ..., i_l) \to Z$. We check this condition on the example discussed at the beginning of §2. There $f^{-1}(V)$ is the union of H (exceptional hypersurface) and V_1 (the strict transform of V). The stratification is defined by $G(1)=H-V_1$, $G(2)=V_1-H$ and $G(1,2)=H\cap V_1$. Here

$$f^{-1}(Y)=G(1)\cup G(1,2).$$

Let us note:

(a) A fiber of the smooth map $W\to Z$ over a point $t\in Z$ cuts the hypersurface in a curve $V_t\subset W_t$. Note also that the section $Z\to Y$ defines the singular point of the curve.

(b) The blow-up at the singular point of the curve $V_t\subset W_t$ in (a) is also defined by a fiber over $t\in Z$ of the smooth morphism $W_1\to Z$.

(c) The curve in (a) has been desingularized. The pullback of the singular point is of pure codimension 1 (the union of the exceptional curve with the strict transform of the singular curve), and, as a union of hypersurfaces with only normal crossings, it can be naturally stratified.

(d) The stratification in (c) is defined as fibers of the smooth morphisms $G(1)\to Z$, $G(2)\to Z$ and $G(1,2)\to Z$.

2.3. On equiresolution

2.3.1. Consider inclusions of schemes, say $Y\subset V$ and $V\subset W$, where Y is regular and V is a reduced hypersurface in W.

Definition 2.3.2. We say now that condition $\mathrm{ER}(x, V\subset W, Y)$ holds at a given point $x\in Y$ if, after W is replaced by a suitable neighborhood of x:

(i) there is a smooth morphism $\pi: W\to Z$ such that π and $V\subset W$ define a family of reduced hypersurfaces;

(ii) there is an embedded resolution $f: W'\to W$ of $V\subset W$ such that π and f define a family of embedded resolutions;

(iii) the restriction $\pi: Y\to Z$ is étale, and $(f^{-1}(Y))_{\mathrm{red}}=H_1\cup ...\cup H_r$ for some $r\leqslant s$ (i.e. $(f^{-1}(Y))_{\mathrm{red}}$ is a hypersurface in W' and a union of exceptional components, for s and r as in Proposition 2.2.4).

Remark 2.3.3 (condition ER and the local criterion). In the setting of the proof of our main theorem, we will start with a family of embedded hypersurfaces in Definition 2.3.2 (i) and with suitable section. We will construct a modification $f: W'\to W$ defining an embedded resolution of the hypersurface. At that point we will make use of the local conditions (1), (2) and (3) in Remark 2.2.3 in order to show that f and π define a family of embedded resolutions (i.e. for (i) and (ii) in Definition 2.3.2 to hold).

As for Definition 2.3.2 (iii), we note that V is reduced, so if Y is an equimultiple center in the singular locus, then the codimension of Y in W is at least 2. In particular, $f^{-1}(Y)$ is of pure codimension 1 in W' if and only if it is a union of exceptional components.

Once (ii) is proved, then the discussion in Remark 2.2.5 says that a nice stratification of $(f^{-1}(Y))_{\mathrm{red}}$ (mapping smoothly to Y) is guaranteed if we only require (iii) in Definition 2.3.2. So it suffices to prove that $f^{-1}(Y)$ is of pure codimension 1 in W', which is also a condition of local nature. In fact, we will check this condition locally, by showing that given any closed point $z \in f^{-1}(Y))_{\mathrm{red}} \subset W'$, there is (locally) an invertible ideal supported on $(f^{-1}(Y))_{\mathrm{red}}$.

3. Formulation of the theorem

3.1. Fix again $x \in Y$, $Y \subset V$ and $V \subset W$ as before. The validity of condition $\mathrm{ER}(x, V \subset W, Y)$ depends on the existence of the morphisms f and π in Definition 2.3.2.

Suppose now that $\beta: W \to W_1$ is a smooth morphism of relative dimension 1. Set $x_1 = \beta(x) \in W_1$, and let $\{y_1, ..., y_d\}$ be a regular system of parameters at $\mathcal{O}_{W_1, x_1}$. We say that β is transversal to V at x if there is a regular system of parameters $\{z, y_1, ..., y_d\}$ at $\mathcal{O}_{W,x}$ such that at the completion (at the henselization) of this local regular ring, the ideal $I(V)$ is defined by a polynomial equation on z, say

$$z^n + a_1 z^{n-1} + ... + a_n,$$

where n denotes here the multiplicity of the reduced hypersurface V at x. If we assume that $x \in Y \subset V$, and V is equimultiple along the regular subvariety Y, we may also assume that locally at x,

(i) the induced morphism $\beta: Y \to \beta(Y)$ is étale (see Remark 3.5), and

(ii) the restriction $\beta: V \to W_1$ is finite and transversal.

Here Σ will denote the reduced discriminant of this branched covering. Recall that V is reduced, so that Σ is a hypersurface in the regular scheme W_1. We now state

THEOREM 3.2. *Set β and Y as above. If the reduced hypersurface V is equimultiple along Y, and condition*

$$\mathrm{ER}(\beta(x), \Sigma \subset W_1, \beta(Y))$$

holds, then also $\mathrm{ER}(x, V \subset W, Y)$ *holds.*

Remark 3.3. We are mainly interested in applying the theorem for Y an equisingular stratum. Note that the theorem makes use of a particular β, whereas property (B) is a condition on all sufficiently general projections. A weaker form of equisingularity, once considered, but ultimately abandoned, by Zariski, required that property (B) hold just for one "transversal projection" β as in §3.1.

Zariski also proves that if $\beta: V \to W_1$ is simply finite, and Σ is equimultiple along $\beta(Y)$ locally at $\beta(x)$, then conditions (i) and (ii) of §3.1 hold, and V is equimultiple along Y locally at $x \in Y$ (see [Za5, p. 525] for a proof in the algebroid context).

Remark 3.4. In the formulation of $\mathrm{ER}(\beta(x), \Sigma \subset W_1, \beta(Y))$ there is a smooth morphism, say $\pi_1: W_1 \to Z$, which defines together with Σ a family of embedded hypersurfaces. Note that the composition of π_1 with β, say $\pi = \pi_1 \beta$, defines a family of embedded hypersurfaces over $V \subset W$ (Definition 2.2.1). This follows from the smoothness of $\beta: W \to W_1$ and the fact that the restriction $\beta: V \to W_1$ is finite and étale over $W_1 - \Sigma$.

Remark 3.5. The conditions of transversality and equimultiplicity lead to the following observation. Let R denote, as before, the completion (the henselization) of the local ring $\mathcal{O}_{W,x}$. We may modify z so that the ideal $I(V)R$ is defined by

$$F(z) = z^n + a_2 z^{n-2} + \dots + a_n$$

(i.e. so that $a_1 = 0$). Here $a_i \in S$ where S denotes the completion (the henselization) of the regular ring at $\beta(x) \in W'$. Let $P \subset R$ be the ideal of Y at R, and $Q \subset S$ the ideal of $\beta(Y) \subset W_1$. Note that $a_1 = 0$ implies that $z \in P$; in fact, $\partial_z^{n-1}(F(z)) = n!.z$ must be an element of the regular prime ideal P. The condition of equimultiplicity along Y also implies that each a_i has order at least i at the local regular ring S_Q.

Since the henselization of $\mathcal{O}_{W,x}$ is a direct limit of étale neighborhoods, this already proves (i) in §3.1, at least in our context (for the hypersurface case). The result is more general and relates to the "generic non-splitting" (see Lemma (4.4) in [Li5]).

3.6 (on the general strategy of the proof of Theorem 3.2). Suppose now that $f_1: W_1' \to W_1$ defines, together with $\pi_1: W_1 \to Z$, all the conditions required in Definition 2.3.2, so that $\mathrm{ER}(\beta(x), \Sigma \subset W_1, \beta(Y))$ holds.

(A) Define $\pi: W \to Z$, setting $\pi = \pi_1 \beta$ (π_1 and β as before). We will show that $\mathrm{ER}(x, V \subset W, Y)$ holds by using this particular morphism. Of course we must construct a morphism f as in Definition 2.3.2. But our construction of f will be such that $\mathrm{ER}(x, V \subset W, Y)$ will hold with this particular π.

(B) Set again $f_1: W_1' \to W_1$ as above (in terms of $\mathrm{ER}(\beta(x), \Sigma \subset W_1, \beta(Y))$ (Definition 2.3.2)). Let $f': W' \to W$ be the fiber product of $f_1: W_1' \to W_1$ with $\beta: W \to W_1$, and finally let F denote the fiber of f' over the point $x \in Y$ $(F=(f')^{-1}(x) \subset W')$. Set

$$\begin{array}{ccc} W' & \xrightarrow{\beta'} & W_1' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f_1} \\ W & \xrightarrow{\beta} & W_1 \end{array} \tag{1}$$

Note that the morphism $\beta': W' \to W_1'$ is smooth, of relative dimension 1, and that $J(f', k)$ is the pullback of $J(f_1, k)$.

Set π as in (A). It follows from the square diagram that

$$\pi' = \pi f' : W' \to Z \tag{2}$$

is smooth, and defines, together with f', a family of modifications (Definition 2.3.2).

We set $V' \subset W'$ as the pullback of V in W. Locally at any point $q \in F$, the smooth morphism $\beta': W' \to W_1'$ induces $V' \to W_1'$ which is finite, with a reduced discriminant having normal crossings at $\beta'(q)$. In these conditions the point $q \in V'$ is said to be a *quasi-ordinary singularity* (see [Li3], [Li4]).

We shall see now that V' is singular; in particular, the family of modifications defined by π' and f' is not a family of embedded resolutions (Definition 2.2.2). The result we present says that there is enough local information on quasi-ordinary singularities so as to define a proper birational morphism, say $g: W''' \to W'$, such that the compositions of both π' and f' with g define a family of embedded resolutions. Furthermore, g will be defined so that the total pullback of Y is of pure codimension 1.

(C) Set $q \in F \subset V'$ as before. Following §5 in [Li2], we first note that

$$\beta'(q) \in (f_1)^{-1}(\beta(Y)) = H_1 \cup \ldots \cup H_r \subset W_1' \tag{3}$$

is a union of regular hypersurfaces in W_1'. Set $d = \dim W' = \dim W$, so that $\dim W_1 = \dim W_1' = d-1$, and a hypersurface in W_1' is of dimension $d-2$.

Since V is equimultiple along Y, we may assume that Y is étale over $\beta(Y)$ (see Remark 3.5). Since f' is defined as a fiber product,

$$((f')^{-1}(Y))_{\mathrm{red}} = C_1 \cup \ldots \cup C_r \subset W' \tag{4}$$

is a union of regular varieties of dimension $d-2$ (of codimension 2), in one-to-one correspondence with the components H_i of $(f_1)^{-1}(\beta(Y))$. Actually, each C_i is étale over H_i, so $((f')^{-1}(Y))_{\mathrm{red}}$ is a union of regular components having normal crossings.

By assumption, V is equimultiple along Y, say of multiplicity n. We claim now that $V' = (f')^{-1}(V)$ is equimultiple with multiplicity n along each C_i. In fact, let

$k\{\{z, y_1, ..., y_{d-1}\}\}$ be the completion of the local ring $\mathcal{O}_{W,x}$, so that $k\{\{y_1, ..., y_{d-1}\}\}$ is the completion of $\mathcal{O}_{W_1,\beta(x)}$. Let, as in Remark 3.5,

$$z^n + a_2 z^{n-2} + ... + a_n \in k\{\{z, y_1, ..., y_{d-1}\}\}$$

be an equation defining $I(V)$; now

$$I(Y) = \langle z, y_1, ..., y_s \rangle \in k\{\{z, y_1, ..., y_{d-1}\}\}$$

and

$$I(\beta(Y)) = \langle y_1, ..., y_s \rangle \in k\{\{y_1, ..., y_{d-1}\}\}.$$

Since V is equimultiple along Y, each a_i has order at least i at the localization at $\langle y_1, ..., y_s \rangle$. Now locally at $q \in F = (f')^{-1}(x) \subset V'$ we may extend $\{z\}$ to a regular system of coordinates so that

$$z^n + a_2 z^{n-2} + ... + a_n \in k\{\{z, y'_1, ..., y'_{d-1}\}\}$$

where $(k\{\{y_1, ..., y_{d-1}\}\} \subset)\ k\{\{y'_1, ..., y'_{d-1}\}\}$ is the completion of the local ring of $\mathcal{O}_{W'_1,\beta'(q)}$. It follows now that any such point q is an n-fold point of the hypersurface.

On the other hand, if C_1 in (4) is in correspondence with the component, say $H_1 \subset W'_1$, and y_1 is an equation defining H_1 locally at $\beta'(q)$, it follows from Remark 3.5 that

$$C_1 = V(\langle z, y_1 \rangle) \tag{5}$$

is an n-fold subscheme of dimension $d-2$.

(D) We will finally define an embedded resolution of $V' \subset W'$, as mentioned at the end of part (B). This will be done in two steps:

(D1) The first step, developed in §4, is the construction of a proper birational morphism, defined entirely in terms of *characteristic monomials* as local data of quasi-ordinary singularities. We recall these concepts in §4 as developed in [Li3], stressing on local properties. In fact we first construct this morphism in a neighborhood of a point $q \in F$ (the fiber over x), and then we show that these locally defined morphisms patch so as to define a morphism locally at F.

More will be said on the general strategy of this first step in §4.4.

(D2) A second step, developed in §5, will be required to prove Theorem 3.2. Note that the first step (in (D1)) introduces singularities on the ambient space. Our study in §4 will also indicate how to overcome this difficulty, by showing that these singularities will be provided with a toroidal structure. This will lead to a nice resolution of these singularities, and we finally prove that this resolution together with the smooth morphism $\pi: W \to Z$ mentioned in (A) will fulfill the conditions in Definition 2.3.2.

We remark that both steps (D1) and (D2) rely entirely on properties of quasi-ordinary singularities; in fact, both the construction of the morphism and also the toroidal structure of singularities introduced in (D1) are encoded in the characteristic monomials of the quasi-ordinary singularities in (C). In §4 we show that there is a finite and smooth Galois extension where the pullback of the hypersurface with n-fold quasi-ordinary singularity becomes a union of n different smooth hypersurfaces. We then show that the embedded desingularization of this hypersurface is totally determined by the characteristic monomials of the quasi-ordinary singularities. Furthermore, this embedded desingularization is equivariant with the action of the Galois group. Factoring out the group action at this embedded resolution we lose smoothness, but we end up with a toroidal structure totally determined by the characteristic monomials. This line of proof was suggested in 5.1 of [Li2].

3.7. We end this section by introducing some concepts, or notation, which we shall use in §4. Let W be a smooth scheme over a field k. We consider data

$$(W,(J,b),E)$$

where $J \subset \mathcal{O}_W$ is a sheaf of ideals in W, b is a positive integer, and $E=\{H_1,\dots,H_s\}$ denotes a set of smooth hypersurfaces in W having only normal crossings. We introduce a few definitions:

(i) Let $\mathrm{Sing}(J,b) \subset W$ be the closed set of points x in W such that J has order at least b at the local regular ring $\mathcal{O}_{W,x}$.

(ii) (transformations) We denote by

$$(W,(J,b),E) \leftarrow (W_1,(J_1,b),E_1)$$

a transformation of the data, where $W_1 \to W$ is a monoidal transformation with center Y, $Y \subset \mathrm{Sing}(J,b)$ is closed and regular in W, and has normal crossings with all hypersurfaces of E. We set E_1 as the hypersurfaces $\{H'_1,\dots,H'_s,H_{s+1}\}$, where H'_i denotes the strict transform of H_i, and H_{s+1} is the exceptional locus of $W_1 \to W$, the monoidal transformation with center Y. Finally, J_1 is obtained from the total transform of J to W' where we factor out the ideal $I(H_{s+1})$ to the power b (i.e. $J\mathcal{O}_{W,x}=I(H_{s+1})^b.J_1$).

(iii) We define a resolution of $(W,(J,b),E)$ to be a sequence of transformations (as in (ii))

$$(W,(J,b),E) \leftarrow (W_1,(J_1,b),E_1) \leftarrow \dots \leftarrow (W_s,(J_s,b),E_s)$$

such that $\mathrm{Sing}(J_s,b)$ is empty.

3.8. Let V be a hypersurface in W.

(a) If V is regular and $J=I(V)$, then $\mathrm{Sing}(J,1)=V$. Furthermore, if

$$(W,(J,1),E)\leftarrow(W_1,(J_1,1),E_1)$$

is as in §3.7 (ii), then $J_1=I(V')$ where V' is the strict transform of V. In particular,

$$V'=\mathrm{Sing}(J_1,1).$$

(b) Set:

max-mult$(V)=n$ if n is the highest multiplicity of the hypersurface at its points;

Max-mult(V) as the closed set of points where V has multiplicity n (with $n=$ max-mult(V)).

Suppose now that the reduced hypersurface V is a union of regular hypersurfaces D_j, say

$$V=D_1\cup\ldots\cup D_s.$$

Then max-mult(V) is the biggest integer n such that there is a subset, say $\{D_{i_1},\ldots,D_{i_n}\}$, with non-empty intersection.

(c) Fix n as in (b), so that Max-mult$(V)=\mathrm{Sing}(I(V),n)$, and let $I(D_i)\subset\mathcal{O}_W$ be the sheaf of ideals defining D_i. Then

$$\mathrm{Sing}(I(V),n)=\mathrm{Sing}((I(D_{i_1}),\ldots,I(D_{i_n})),1)$$

where

$$(I(D_{i_1}),\ldots,I(D_{i_n}))=I(D_{i_1})+\ldots+I(D_{i_n})\subset\mathcal{O}_{W,x},$$

and these equalities are stable by transformations defined in §3.7 (ii) (replacing V by its strict transform V', and each D_i by its strict transform D'_i). In fact, if $Y\subset\mathrm{Sing}(I(V),n)$ is closed and regular, then $Y\subset\mathrm{Sing}(I(D_{i_j},1)$ for each index i_j above. So Y is included in each regular hypersurface D_{i_j}, and hence D'_{i_j} is also regular (see (a)).

(d) (maximal contact) With the setting as in (c), we have

$$\mathrm{Sing}((I(D_{i_1}),\ldots,I(D_{i_n})),1)\subset D_{i_1}\subset W$$

and

$$\mathrm{Sing}((I(D_{i_1}),\ldots,I(D_{i_n})),1)=\mathrm{Sing}((J(D_{i_2}),\ldots,J(D_{i_n})),1)\subset D_{i_1}$$

where now $J(D_{i_j})\subset\mathcal{O}_{D_{i_1},x}$ denotes the trace of the sheaf of ideals $I(D_{i_j})\subset\mathcal{O}_W$ on $\mathcal{O}_{D_{i_1}}$.

These inclusions and equalities are stable by transformations as in §3.7 (ii). In particular, a resolution of $(D_{i_1},((J(D_{i_2}),\ldots,J(D_{i_n})),1),E)$ induces a sequence of transformations

$$V\subset W\leftarrow V_1\subset W_1\leftarrow\ldots\leftarrow V_s\subset W_s$$

where V_{i+1} denotes the strict transform of V_i, so that

(1) $n=$ max-mult$(V)=$ max-mult$(V_1)=\ldots=$ max-mult(V_{s-1}), and

(2) max-mult$(V_s)<n$.

4. A canonical proper morphism over embedded quasi-ordinary singularities

4.1. Fix notation as in §3.6 (A) and (B) where we defined, along points of $F=(f')^{-1}(x)\subset W'$, a smooth morphism $\beta': W'\to W_1'$ of relative dimension 1. Recall, from diagram (1) in §3.6, that $f_1: W_1'\to W_1$ was defined in terms of $\mathrm{ER}(\beta(x), \Sigma\subset W_1, \beta(Y))$, so that $J(f_1,k)\cup f_1^{-1}(\Sigma)$ is a union of hypersurfaces with normal crossings. In particular, the total transform of the discriminant is a union of smooth hypersurfaces having strict normal crossings at W_1', and thus $V'=(f')^{-1}(V)$ has only quasi-ordinary singularities.

Now we fix, for once and for all, an order on these hypersurfaces. A proper birational morphism will be constructed, and this morphism will depend on this particular order. Since β' is smooth, the pullbacks of the components of $J(f_1,k)\cup f_1^{-1}(\Sigma)$ are regular hypersurfaces in W' having only normal crossings, say

$$E=\{H_1,\dots,H_s\}. \tag{6}$$

Note that the Jacobian of $W'\to W$ is a union of some of these components (other components arise from the pullback of the strict transform of the discriminant).

So locally at a closed point $q\in V'\subset W'$ we have

(i) the smooth morphism $\beta': W'\to (W_1)'$;

(ii) an induced finite morphism $\beta': V'\to W_1'$, with discriminant with normal crossings;

(iii) an order at the irreducible components of the discriminant locally at $\beta'(q)$ defined by the order given above in (6).

At the completion of $\mathcal{O}_{W',q}$, the data in (i) and (ii) will provide us with a new and main invariant at q, namely,

(iv) the characteristic exponents.

A central part of this work is a canonical construction of a proper birational morphism $W''\to W'$, defined entirely in terms of these data, in a neighborhood of $F\subset W'$. This defines $W''\to W$ in a neighborhood of $x\in W$. Here the scheme W'' is not necessarily smooth.

We formulate now the strategy for this construction which is finally developed in §4.9.

(I) We shall first construct a morphism

$$W''\to W', \tag{7}$$

but only locally at q ($\in F$), and depending entirely on the formal data at the point, particularly on the characteristic monomials. We want this morphism to globalize along points of F.

We will note that our local construction of the morphism will have the following natural properties of compatibility with restriction to étale neighborhoods:

Suppose that $(U', q') \to (W', q)$ is an étale neighborhood. Take $V' \subset U'$ as the pullback of $V' \subset W'$, and set $\beta' : U' \to (W_1)'$ by composition (see (i) above). The completion of local rings at $q' \in U'$ and $q \in W'$ are naturally identified, and so are the local data (i), (ii), (iii) and (iv) (replacing smoothness by formal smoothness in (i)). In fact, characteristic monomials will be defined at the completion of the local rings (see (18) in §4.5).

Our local construction of (7) defines two morphisms $W'' \to W'$ and $U'' \to U'$ locally at q and q'. We shall show that one can naturally define a square diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ W'' & \longrightarrow & W' \end{array} \tag{8}$$

where $U' \to W'$ means $(U', q') \to (W', q)$. We shall furthermore see that

(a) $U'' \to W''$ induces a natural correspondence or bijection from points of the fiber over q' to points of the fiber over q, and

(b) for two points in correspondence as above, there is a natural identification of the completions and also of the local data at such points.

Note that (b) says that the morphism $U'' \to W''$ is unramified over points at the fiber of q.

(II) We show that points in W'' mapping to q in (7) are provided with natural toroidal structure, and that the completions of such local rings are quotient singularities.

Toroidal singularities are normal; in particular, W'' is normal at such points. This asserts that $U'' \to W''$ is étale since unramified morphisms over normal rings are étale ([M, Theorem 3.20, p. 29]).

(III) We shall make use of the properties in (I) to show that the locally defined morphisms in (7) patch as q varies along points of F, so that $W'' \to W'$ is finally defined as a morphism in a neighborhood of F.

4.2. In what follows, n will denote the order of V at $x \in V \subset W$. We fix notation as in §3.6, and a closed point $q \in F = (f')^{-1}(x)$. There is an inclusion of local regular rings $\mathcal{O}_{W'_1, \beta'(q)} \subset \mathcal{O}_{W', q}$. We now choose a regular element, say $y_i \in \mathcal{O}_{W'_1, \beta'(q)}$, defining the ideal of a component H_i of the discriminant (see (6)); recall that some of those H_i are components of the Jacobian of $f' : W' \to W$.

- Let A be the ring of an affine neighborhood of W' at q, and A_q the local ring $\mathcal{O}_{W', q}$.
- Let A_1 be the ring of an affine neighborhood of W'_1 at $\beta'(q)$, and $(A_1)_q$ the local ring $\mathcal{O}_{W'_1, \beta'(q)}$.

• Let R be the finite extension of A defined by the adjunction of the $(n!)$th roots of each y_i above, defining a component of the discriminant. R_q will denote a localization at a point. R_q is a local ring dominating and also finite over A_q.

• Let R_1 be the finite extension of A_1 obtained by adjunction of the $(n!)$th roots of each y_i as before, and $(R_1)_q$ the local ring dominating and finite over $(A_1)_q$.

By taking a suitable choice of neighborhoods we may assume that

(1) all rings A, A_1, R and R_1 are smooth over k;

(2) there is a finite group G acting on R so that A is the subring of G-invariants;

(3) the group in (2) also acts in R_1, and A_1 is the subring of G-invariants;

(4) the group acts on the local rings R_q and $(R_1)_q$ (i.e. the corresponding maximal ideals are fixed by G).

So now we have

$$\begin{array}{ccc} A & \subset & R \\ \cup & & \cup \\ A_1 & \subset & R_1. \end{array} \tag{9}$$

Note that R and R_1 are of finite type over the field k. They are, however, constructed in terms of a particular choice of the equation y_i defining a hypersurface H_i. One can check that for two different choices, the rings, say R and R', patch in the étale topology so as to define a natural identification of their completions; furthermore, the local data (i), (ii), (iii) and (iv) of §4.1, defined at the completion of such rings, are the same.

Note that R_q is the only local ring dominating A_q; in fact, it is the integral closure of A_q, and A_q is the subring of G-invariants of R_q. Note also that the completion of R_q is a ring of formal power series. There is a natural action of G at such a completion, and the subring of G-invariants is the completion of A_q (see §4.4).

In our setting, the group G acts faithfully by multiplication by roots of unity on a regular system of parameters at R_q, so G is an abelian group.

4.3. We will define a proper birational morphism over the regular ring R. So $U'=\operatorname{Spec}(R)$ is smooth over the field k, and there is a group G acting on U', with quotient

$$\operatorname{Spec}(R)=U'\to\operatorname{Spec}(A)\subset W'.$$

The hypersurface V' ($\subset W'$) defines by pullback a hypersurface in U', which we call V' again, and G also acts on the hypersurface V'. Finally set $E=\{H_1,\dots,H_s\}$, each H_i being the pullback at U' of H_i in (6). So E is a set of smooth hypersurfaces with only normal crossings, and the group G acts on each H_i. Now set

$$U_1'=U',\quad V_1=V',\quad E_1=E.$$

We will define

$$\begin{array}{cccccc} \mathrm{Spec}(R)=U'_1 & \longleftarrow & U'_2 & \longleftarrow \; \dots \; \longleftarrow & U'_s \\ V_1 & & V_2 & & V_s \\ Y_1 & & Y_2 & & Y_s \\ E_1 & & E_2 & & E_s, \end{array} \tag{10}$$

a sequence of monoidal transformations on centers Y_i, each Y_i regular, included in the hypersurface V_i (actually the hypersurface will be equimultiple along Y_i), where V_{i+1} denotes the strict transform of V_i. Here E_i is defined as the hypersurfaces which are strict transforms of those of E_{i-1} together with the hypersurface in U'_i defined by the pullback of Y_{i-1} (exceptional hypersurface). We shall require that

(1) each Y_i have normal crossings with E_i;

(2) each center Y_{i-1} be G-invariant so that G acts on U_i, on V_i and also on each hypersurface of E_i;

(3) the sequence be an embedded desingularization of V_1 ($\subset U_1$).

Note that G acts on $V_i \subset U_i$, so G also acts on the closed set $\underline{\mathrm{Max}}\text{-mult}(V_i)$ (§3.8 (b)). Recall that G acts on any hypersurface in E_i. Now we require that

(4)

$$Y_i = \underline{\mathrm{Max}}\text{-mult}(V_i) \cap H_i$$

where i is the *smallest index* so that the intersection has codimension 2 in U'_i.

This last condition will play an important role, and we discuss it below.

4.4. Centers of monoidal transformations are closed and regular, and requirement (4) in §4.3 is saying that such an intersection is regular, which will be easy to check locally. It is also defining Y_i as an intersection of two étale and G-invariant formulas. So (4) is intended for globalization, and it was in order to make this globalization possible that we fixed an order on $E=\{H_1, \dots, H_s\}$.

With these conditions fulfilled we will define, in terms of the characteristic monomials (see (18)), a proper birational morphism over a neighborhood, say $W'(q)$, of q in W',

$$W'(q) = \mathrm{Spec}(A) \leftarrow W'', \tag{11}$$

and we shall do this by taking the sheaves of G-invariant functions both on U'_1 and on U'_s of (10).

Note that V_s defines naturally a subscheme, say $V'' \subset W''$, which is the strict transform of V'. We shall see that, in general, both W'' and V'' are not smooth over k. If $q_1 \in W''$ maps to $q \in W'(q)$, the completion of the local ring $\mathcal{O}_{W'',q_1}$ will be defined in terms of the completion of the local ring of U'_s at a point mapping to q_1. But U'_s is regular, so the completion of $\mathcal{O}_{W'',q_1}$ is the subring of G'-invariants in a ring of formal power series, where G' is the *decomposition group* of a point in U_s' (a subgroup of G). The construction of (10) together with the description of decomposition groups will be treated in §4.9. In §4.5, §4.6, Proposition 4.7 and Lemma 4.8 we study actions of our decomposition subgroups on a ring of formal power series. Let us first recall here some facts on decomposition groups.

Note that there is a finite morphism from the regular scheme U_s' to W''. Let K be the total quotient field of W'' (and of A), and K' that of U_s' (and of R). G is the Galois group of K' over K. Fix a point in W'' mapping to q via (11). We will study the completion of the local ring at such a point.

Our development will show that U_s' can be covered by affine and G-invariant charts. So let S be a smooth k-algebra with total quotient field K', and assume that G acts on S and that $\mathrm{Spec}(S)\subset U_s'$. Set $B=S\cap K$. A theorem of Noether says that $S^G\subset B\subset S$ is a finite extension of finitely generated k-algebras.

If we fix a maximal ideal $M\subset B$, then the group G acts on the finite maximal ideals in S dominating M. We may assume that B is a local ring, and that S is semi-local and finite over B.

Let $\mathcal{H}$ denote a localization of S at a maximal ideal. Let G^s denote the decomposition group of $\mathcal{H}$ (the subgroup of automorphisms that fix $\mathcal{H}$). Let B^* and $\mathcal{H}^*$ denote the completions of B and $\mathcal{H}$, and let E, E^* denote their quotient fields.

Let us check that $E\subset E^*$ is Galois with group G^s. Since the group acts on the extension, it suffices to show that the order of the group is the same as the order of the field extension. In fact, note that $[G]=[K',K]$ is the rank of the B-module S at the generic point. The ranks of finitely generated modules are defined by Fitting ideals, and are therefore stable by passing from B to B^*. Let S^* be the completion of S; then S^* is a direct sum of complete regular local rings,

$$S^* = \mathcal{H}_1^* + \mathcal{H}_2^* + \ldots + \mathcal{H}_r^*,$$

where

$$r = [G^s, G]$$

is the number of localizations of S dominating B. This shows that $[G^s]=[E^*,E]$. So G^s is the Galois group of E^*/E.

This defines B^* as the subring of G^s-invariants in the local regular ring $\mathcal{H}^*$ (as a quotient singularity). In fact, by Abhyankar's development of ramification theory ([A3]) one can naturally identify the finite abelian group G^s with a quotient of lattices $\Sigma_1\subset\Sigma_2$ defining $B^*\subset\mathcal{H}^*$ (and $E\subset E^*$) as the subring of G^s-invariants of the ring of formal power series $\mathcal{H}^*$. Furthermore, any normal ring $B^*\subset B'\subset\mathcal{H}^*$ (any subextension $E\subset E'\subset E^*$) is naturally identified with a lattice Σ', $\Sigma_1 \subset \Sigma' \subset \Sigma_2$, so that $B' \subset H^*$ ($E' \subset E^*$) is the subring (the subfield) of (Σ'/Σ_1)-invariants (see [F, pp. 33–34] and [Gi, p. 58]).

4.5. Fix $q \in W'$, R_q and $(R_1)_q$ as in §4.2. Set $t = \pi'(q) \in Z$ and π' as in (2). We call R^* the completion of R_q, R_1^* the completion of $(R_1)_q$, and let T^* denote the completion of $T = \mathcal{O}_{Z,t}$. Here R^*, $(R_1)^*$ and T^* are rings of formal power series over k, say

$$\begin{aligned} &R^* = k\{\{z, v_1, ..., v_s, t_1, ..., t_r\}\} \supset (R_1)^* = k\{\{v_1, ..., v_s, t_1, ..., t_r\}\}, \\ &T^* = k\{\{t_1, ..., t_r\}\}, \end{aligned} \tag{12}$$

and we will assume that

(1) z is a non-unit in A_q, and transversal to the fiber of β' (§4.2);

(2) either v_i is an $(n!)$th root of y_i, or $v_i = y_i$; and all $y_i \in (A_1)_q$ are chosen as in §4.2;

(3) $\{t_1, ..., t_r\}$ are local coordinates at $\mathcal{O}_{Z,t}$.

Note that G acts both on R^* and R_1^*, that G fixes the variable z, the variables t_i and some of the variables v_i, and that G acts via multiplication by roots of the unit on those v_i which are not fixed. Here the completion of A_q is the subring

$$k\{\{z, y_1, ..., y_s, t_1, ..., t_r\}\},$$

which is the subring of G-invariants of R^*. And the completion of $(A_1)_q$ is

$$k\{\{y_1, ..., y_s, t_1, ..., t_r\}\},$$

which is the subring of G-invariants of R_1^*.

Let us indicate now that

(a) v_i is an $(n!)$th root of y_i if and only if y_i describes a component H_i of the discriminant (see §4.2), and

(b) the discriminant is (locally) a monomial in the variables $\{y_1, ..., y_s\}$ (and hence G-invariant).

At R^* there is a monic polynomial of degree n, say

$$P(z) = z^n + a_1 z^{n-1} + ... + a_n, \quad a_i \in R_1^*, \tag{13}$$

which generates the ideal of the pullback of the hypersurface.

Since (b) holds, the Abhyankar–Jung theorem says that

$$P(z) = (z - g_1)(z - g_2)...(z - g_n) \tag{14}$$

with $g_i \in R_1^*$ (see also [Lu2] and [Zu]). So at Spec(R^*), $V_1 = V(P(z)) = D_1 \cup ... \cup D_n$ is the union of n regular hypersurfaces, where

$$D_i = V(z - g_i). \tag{15}$$

The projection defined by the inclusion in (12) induces an isomorphism of D_i with $\mathrm{Spec}(R_1^*)$.

Here $P(z)$ is a product of n regular elements of order one; in particular, it has order n at the local regular ring R^*. Define now

$$N(i,j) = g_i - g_j, \tag{16}$$

which is a factor of the discriminant; in particular,

$$N(i,j) = M(i,j).U(i,j) \tag{17}$$

at R_1^*, where $M(i,j)$ is a monomial in the variables $\{v_1, ..., v_s\}$, and $U(i,j)$ a unit in R_1^*.

From (16) it follows that

$$N(i,j) + N(j,l) = N(i,l)$$

for any three different indices i, j, l, and one can check that in the set of three monomials, $\{M(i,j), M(j,l), M(i,l)\}$, at least two of them are equal, say to M, and that the third monomial is divisible by M. In particular, if we fix an index i_0, it turns out that all monomials $M(i_0, j)$ are totally ordered by division in R_1^*. Set

$$\{M(i_0,j) : j = 1, ..., n\} = \{M_1, M_2, ..., M_e\} \quad \text{(characteristic monomials)}, \tag{18}$$

where the terms at the right are the different monomials ordered so that M_i divides M_{i+1} (see [Li3, p. 166] or [Za4, p. 538]).

We define a filtration of the group G, say

$$G \supset G_1 \supset G_2 \supset ... \supset G_e, \tag{19}$$

where G_i consists of elements of the group G that fix $M_1, M_2, ..., M_{i+1}$, the first $i+1$ monomials in (18).

So we fix one index, say i_0, and argue as in §3.8. Recall from (15) the natural identification of $R^*/I(D_{i_0})$ with R_1^*; the ideal $J(D_j)$ (in §3.8 (d)) is the principal ideal defined by $M(i_0, j)$. So $(J(D_2), ..., J(D_n))$ is the ideal in R_1^* generated by M_1 in (18) since this monomial divides all other characteristic monomials.

Note also that

$$\mathrm{Sing}(I(V'), n) = \mathrm{Sing}((J(D_2), ..., J(D_n)), 1) = \mathrm{Sing}(M_1, 1) \tag{20}$$

as a closed subset of D_{i_0} (notation as in §3.8 (d)), where $\mathrm{Sing}(I(V'), n)$ is the set of n-fold points of V'.

4.6. We begin by studying the action of the group G on R^* as defined before. Consider the development of g_{i_0} ((14)) at the ring of formal power series R_1^*. Suppose that M is a monomial arising in such an expression with non-zero coefficient, say a, and that M is not divisible by M_1 ((18)). Since G acts on R^* and on the hypersurface $V(P(z))$, such a monomial M must appear in the expression of any g_i with the same coefficient a. In particular, aM must be invariant (fixed) by the action of G.

We may therefore replace z (by adding to z all those aM as before) so that

(1) z is G-invariant in R^*;

(2) g_{i_0} is divisible by M_1;

(3) all g_j are divisible by M_1.

After blowing up a suitable G-invariant center we will need a slightly more general form of this result, in which z is not necessarily fixed by the group:

PROPOSITION 4.7. *Fix an inclusion of formal power series rings*

$$R^* = k\{\{z, v_1, \dots, v_s, t_1, \dots, t_r\}\} \supset R_1^* = k\{\{v_1, \dots, v_s, t_1, \dots, t_r\}\} \tag{21}$$

and a hypersurface V defined by $I(V) = \langle P(z) \rangle$,

$$P(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

for all $a_i \in R_1^*$. Assume that the discriminant, defined in terms of the inclusion (21), *is a monomial in the variables $\{v_1, \dots, v_s\}$. Set $\{M_1, M_2, \dots, M_e\}$ as in* (18), *so that M_1 denotes the first characteristic monomial.*

Let G be a group acting on both R^* and R_1^*, and assume that the group and coordinates are such that:

(i) *G acts on each coordinate by multiplication by roots of unity.*

(ii) *G acts on V, where $I(V) = \langle P(z) \rangle$. Here*

$$P(z) = (z - g_1)(z - g_2) \dots (z - g_n)$$

for all $g_i \in R_1^* = k\{\{v_1, \dots, v_s, t_1, \dots, t_r\}\}$.

(iii) *The group G acts trivially on each coordinate t_i.*

We claim now that by changing z by $z' = z - (1/n)a_1$ we may assume that (i), (ii) *and* (iii) *still hold, and, in addition, that*

(iv)
$$\begin{aligned} P(z) &= (z - g_1)(z - g_2) \dots (z - g_n) \\ &= (z' - g_1')(z' - g_2') \dots (z' - g_n') = (z')^n + b_2 (z')^{n-2} + \dots + b_n, \end{aligned}$$

where now all g_i' are divisible by M_1 (the first characteristic exponent); and that $P(z) = z'$ if $n = 1$.

Proof. Fix an index i_0. (ii) ensures that for any $h \in G$,

$$h(z-g_{i_0}) = u(z-g_j)$$

for some unit $u \in R^*$ and some index j.

For each $h \in G$, the quotient $h(z)/z$ is a root of unity, say δ. Since $h(z)=\delta z$, we have $h(z-g_{i_0})=\delta(z-f_{i_0})$ for $f_{i_0}=h(g_{i_0})/\delta$.

In particular,

$$\delta(z-f_{i_0}) = u(z-g_j),$$

and therefore $\delta=u$ and $f_{i_0}=g_j$ for some index j in (14) (see Lemma 4.8 below). Hence

$$h(z-g_{i_0}) = \delta(z-g_j)$$

for some index j, where $g_j=h(g_{i_0})/\delta$ and $\delta=h(z)/z$.

So for any index i_0, $h(g_{i_0})=\delta g_j$ for some index j. It follows that

$$h(g_1+g_2+...+g_n) = \delta(g_1+g_2+...+g_n) = \delta(-a_1),$$

and thus $h(z')=\delta z'$ for $z'=z-(1/n)a_1$.

Assertion (iv) can be reformulated by saying that, if $a_1=0$, then any monomial M arising in the expression of any g_i must be divisible by M_1. In fact, if such a monomial M appears with non-zero coefficient, say $a \in k$, in g_{i_0}, then it must appear in the development of any g_j with the same coefficient a; in contradiction with $a_1=g_1+g_2+...+g_n=0$.

LEMMA 4.8. *Let $k\{\{z, y_1, ..., y_n\}\}$ be a ring of formal power series. Fix $g(y_1, ..., y_n)$, $h(y_1, ..., y_n)$ in the subring $k\{\{y_1, ..., y_n\}\}$, and $u(z, y_1, ..., y_n) \in k\{\{z, y_1, ..., y_n\}\}$, so that*

(1) $h(0, ..., 0)=g(0, ..., 0)=0$, and

(2) $u(z, y_1, ..., y_n)$ is a unit in $k\{\{z, y_1, ..., y_n\}\}$.

We claim that if $z-g=u.(z-h)$, then $g=h$ and $u=1$.

Proof. Express $u=a_0+a_1z+...+a_rz^r+...$, $a_i \in k\{\{y_1, ..., y_n\}\}$. Then

$$\begin{aligned} u(z-h) &= uz-uh \\ &= z(a_0+a_1z+...+a_rz^r+...)-(ha_0+ha_1z+...+ha_rz^r+...) \\ &= -ha_0+(a_0-ha_1)z+(a_1-ha_2)z^2+.... \end{aligned}$$

The equality states that

$$-ha_0 = -g, \quad a_0-ha_1 = 1, \quad a_1-ha_2 = 0,$$
$$a_2-ha_3 = 0, \quad ..., \quad a_r-ha_{r+1} = 0, \quad ... .$$

Since the statement is clear if $h=0$, we may assume that h is not zero. Note that if a_{r+1} is not zero, then a_1 is not zero and it is divisible by h to the power r. Since a_1 is in the unique factorization domain $k\{\{y_1,...,y_n\}\}$, and since h is not a unit, it follows that u must be a polynomial in z. Finally, in this context, the lemma can be easily checked.

4.9. We now proceed to define the embedded desingularization of the hypersurface $V_1 \subset \mathrm{Spec}(R)$ of (10). This desingularization of V_1 will be achieved by a sequence of monoidal transformations, where one first reduces points of multiplicity n, then points of multiplicity $n-1$ (if any), and so on. Eventually we reach a stage where all points of the final strict transform are of order 1.

We fix notation and assumptions as in §4.5: for coordinates, for the group G and also for the hypersurface.

As before, the completion of R_q is denoted by R^*, which is a ring of formal power series.

Recall that at the formally smooth scheme $\mathrm{Spec}(R^*)$ the hypersurface V_1 has order n (at the closed point), and V_1 is the union, say $D_1 \cup ... \cup D_n$, where $D_i = V(z-g_i)$. Furthermore,

$$\mathrm{Sing}(I(V_1),n) = \mathrm{Sing}(M_1,1)$$

as closed subsets of D_1 (or of any D_i), where M_1 is the first characteristic exponent ((20)).

This shows that the closed set $\mathrm{Sing}(I(V'),n)$ is a union of smooth closed sets of codimension 2 in $\mathrm{Spec}(R^*)$, and hence in a neighborhood of $q \in \mathrm{Spec}(R)$. In fact, it is the union of the intersections of D_1 with H_j, where this union is taken over all index j such that v_j divides the monomial M_1. Recall here that each H_i is defined by an equation v_i (§4.2 and §4.5 (2)); so D_1 is transversal with any H_i ((15)). Note also that the exponent of v_i at M_1 drops by one if we blow up at any such regular center, according to our law of transformation in §3.7.

Following §4.3 (4) take the center to be the G-invariant intersection

$$\mathrm{Sing}(I(V_1),n) \cap H_j = D_1 \cap H_j,$$

where now j is the smallest index such that such an intersection is of codimension 1 in V_1. Note that the center is defined by

$$P_1 = \langle z-g_1, v_j \rangle = \langle z, v_j \rangle$$

since we may assume that the first characteristic exponent divides all g_i (Proposition 4.7).

A resolution of

$$(D_1, (\langle J(D_2),...,J(D_n)\rangle, 1), E) = (D_1, (M_1,1), E)$$

would be achieved by this well-defined sequence of monoidal transformations at these G-invariant regular centers of codimension 1 in D_1 (of codimension 2 in the smooth ambient space).

The following assertions can be checked:

(1) The blow-up at P_1 can be covered by two affine G-invariant charts, $R[v_j/z]$ and $R[z/v_j]$. The fiber over q is isomorphic to the projective line $\mathbf{P}^1$, and G acts on this line.

(2) The strict transform of V_1 is totally included in the chart $R[z/v_j]$, and this chart is smooth over R_1.

(3) Both charts mentioned in (1) are smooth over Z via composition with $\pi': W' \to Z$ (see (2)).

It suffices to note that at points along the fiber of $q \in \operatorname{Spec}(R)$ one may choose coordinates

$$\{z/v_j - \beta, v_1, v_2, ..., v_r, t_1, ..., t_s\} \quad \text{in } R[z/v_j],$$
$$\{v_j/z - \beta, v_1, v_2, ..., v_r, t_1, ..., t_s\} \quad \text{in } R[v_j/z],$$

for some $\beta \in k$. Here $\{v_1, ..., v_r, t_1, ..., t_s\}$ is a regular system of parameters at $(R_1)_q$ (§4.2), and $\{t_1, ..., t_s\}$ a regular system of parameters at $\mathcal{O}_{Z,t}$.

Assume here that at R^* the ideal $I(V_1)$ is generated by

$$z^n + a_2 z^{n-2} + ... + a_n,$$

and let j be the smallest index such that v_j divides the first characteristic exponent M_1. Note that if $M_1' = M_1/v_j$ is not a unit, then

(i) the strict transform of V_1, say V_2, has a unique point over q;

(ii) such a point is an n-fold point of V_2;

(iii) such a point has coordinates $\{z/v_j, v_1, v_2, ..., v_r, t_1, ..., t_s\}$ and hence is a fixed point by the action of the group G.

(iv) Let now R' denote such a local ring, and $(R')^*$ the completion, so that $R_1^* \subset R^* \subset (R')^*$. Then the setting of §4.5 holds at $R_1^* \subset (R')^*$ (with the same R_1^*!), where now each g_i is replaced by g_i/v_j, and with characteristic monomials

$$\{M_1', M_2', ..., M_e'\},$$

setting $M_i' = M_i/v_j$ ((14)).

In this way we define a sequence of monoidal transformations at centers of codimension 2 which are n-fold for the hypersurface, say

$$\begin{array}{ccccccc} \operatorname{Spec}(R_q) = U_1 & \longleftarrow & U_2 & \longleftarrow & \dots & \longleftarrow & U_r \\ V_1 & & V_2 & & & & V_r \\ Y_1 & & Y_2 & & & & Y_r \\ E_1 & & E_2 & & & & E_r, \end{array} \tag{23}$$

by setting

$$Y_i = \mathrm{Sing}(I(V_i), n) \cap H_j \tag{24}$$

where j is the smallest index for which the intersection has codimension 2 in U_i.

Finally, for some index r,

$$\mathrm{Sing}(I(V_r), n) = \varnothing,$$

and hence

$$\text{max-mult}(V_1) = \text{max-mult}(V_2) = \ldots = \text{max-mult}(V_{r-1}) = n > n' = \text{max-mult}(V_r).$$

The final strict transform of V_1, namely V_r, lies in the affine chart defined by

$$R_q[z/M_1],$$

which is smooth over $(R_1)_q$ and over $T = \mathcal{O}_{Z,t}$ (see (12)). Furthermore, in all previous steps in (23), the strict transform of V_1 lies in an affine chart defined by $R[z/N]$, and contains the point with coordinates $\{z/N, v_1, \ldots, v_s, t_1, \ldots, t_r\}$ where N is a monomial in $(R_1)_q$ which divides M_1. Note that such a point is globally fixed by the full group G, and that the multiplicity of the strict transform of V_1 at such a point is n.

All affine charts introduced in (23) are G-invariant, and they are also smooth over Z. Hence, the elements $\{t_1, \ldots, t_r\}$ can be extended to a regular system of coordinates at any point in (23) mapping to q.

If n is strictly bigger than 1, then at level r, in (23), there are several closed points of V_r mapping to $q \in \mathrm{Spec}(R_q)$. At any such point the multiplicity of V_r is strictly smaller than n. Now we want to continue from the level r on, replacing n by n', so that ultimately the final strict transform of the embedded hypersurface V_1 is regular.

A point $q_2 \in V_r$ ((23)) is defined by a maximal ideal in the chart $R_q[z/M_1]$. Let L denote the local ring at q_2. Assume furthermore that q_2 maps to q, and that the strict transform of the regular hypersurface D_{i_0} contains q_2 ((14)). So R_q is dominated by L, and both are regular local rings.

Recall that D_{i_0} was defined in R^* by the equation $z - g_{i_0}$, and Proposition 4.7 asserts that we may assume that M_1 divides all monomials in the development of g_{i_0}. Let now $\mu \in k$ denote the coefficient of M_1 in the development of g_{i_0}, and note that:

(a) $z/M_1 - \mu$ is a non-unit in L, and furthermore $\{z/M_1 - \mu, v_1, \ldots, v_r, t_1, \ldots, t_s\}$ is a regular system of parameters at L.

This follows from the description of g_{i_0} given in Proposition 4.7, and the assumption that the strict transform of D_{i_0} contains q_2.

(b) (On the decomposition group of the point.)

(b1) The decomposition subgroup of q_2, say G' (elements of G that fix L), is described in terms of the element $z/M_1-\mu$. Note that $G'=G$ if $\mu=0$.

(b2) We claim now that $G'=G_1$ in (19) if $\mu\neq 0$. In fact, we look at all other regular hypersurfaces D_j at $\operatorname{Spec}(R^*)$, with a strict transform containing the point q_2.

Recall that in the development of g_j, only monomials divisible by M_1 can occur. Now the strict transform of D_j contains q_2 if and only if M_1 appears in the development of g_j, and also with coefficient μ (the same as for g_{i_0}). In other words, if and only if

$$g_{i_0}-g_j=N(i_0,j)=M(i_0,j).U(i_0,j)$$

and now

$$M(i_0,j)\in\{M_2,M_3,...,M_e\}\quad \text{in (18).}$$

In particular, all those $M(i_0,j)$ are divisible by M_2. Assume, for simplicity, that M_2 is the smallest monomial with this property.

Set $z_2=z/M_1-\mu$, so that the completion of L, say L^*, is

$$L^*=k\{\{z_2,v_1,...,v_s,t_1,...,t_r\}\},$$

on which an action is defined now by G'. Now check that all hypotheses of Proposition 4.7 hold for G', for the regular system of coordinates $\{z_2,v_1,...,v_s,t_1,...,t_r\}$, for M_2/M_1, $M_3/M_1,...,M_e/M_1$, and for

$$P_2(z)=(z_2-g_{j_1})(z_2-g_{j_2})...(z_2-g_{j_{n'}})$$

where the product is taken only over those indices for which the regular hypersurface contains q_2. Note also that $L^*=R_1^*[[z_2]]$, and hence is formally smooth over R_1^* and over T^* (the same R_1^* and T^* as in (12)!).

After a suitable change of z_2 we may assume that

$$P_2(z)=z_2^{n'}+b_2z_2^{n'-2}+...+b_{n'}$$

(i.e. that $g_{j_1}+...+g_{j_{n'}}=0$).

Let us also point out that the action of G' on L^* is as in Proposition 4.7 (replacing R^* by L^*).

Here we have studied the decomposition group at a point containing the strict transform of the reduced hypersurface V_1. If we now consider any closed point mapping to $q\in\operatorname{Spec}(R_q)$, then the decomposition group is defined as a subgroup of G fixing an element of the form $z/N-\mu$, where N is a monomial and μ is an element in the field. So for any point mapping to q, the decomposition group at the completion of such a point will act according to Proposition 4.7.

In this way we define the embedded desingularization (10) keeping track of decomposition groups.

Remark 4.10. Note that the construction of the embedded desingularization (10), defined above, is done entirely in terms of three local data at R_q, namely,

(a) the multiplicity n of the hypersurface V_1 at R_q;

(b) the order on the coordinates in $(R_1)_q$ that divide the discriminant;

(c) the characteristic monomials of the hypersurface.

Define two different rings, both in the conditions of R in (9), say

(i) R' by adjunction of $(n!)$th roots of y_i, and

(ii) R'' by adjunction of $(n!)$th roots of y'_i,

where both y_i and y'_i are elements at A_q defining the same component H_i of the discriminant as in §4.2. Let G' and G'' be the corresponding Galois groups, and note that

(1) the completion of R'_q and R''_q are naturally isomorphic, and

(2) the subring of G'-invariants in the completion of R'_q, and the subring of G''-invariants in the completion of R''_q, can be identified in the sense of (1).

In fact, $y_i = y'_i . u$ where u is a unit in the ring A_q. Note that u has an $(n!)$th root at the completion A^*_q and, since k is algebraically closed, it has all $(n!)$th roots at A^*_q. Hence any such root is both G'- and G''-invariant. Note finally that $v_i = v'_i . u'$ where u' is an $(n!)$th root of u.

Now check that all the conditions (I), (II) and (III) in §4.1 hold.

5. Proof of Theorem 3.2

Remark 5.1. The last Remark 4.10 says that the conditions in §4.1 hold; in particular, the locally defined morphisms $W'' \to W'(q) = \mathrm{Spec}(A)$ in (11) patch, so as to define a morphism, say

$$h: W'' \to W', \tag{25}$$

in a neighborhood of $F = (f')^{-1}(x)$ ($f': W' \to W$ as in §3.6 (B)).

Since W'' is defined by patching sheaves of G-invariant functions on the regular scheme U'_s, points are provided with a structure of formal quotient singularities.

Define now

$$f'': W'' \to W \tag{26}$$

in a neighborhood of $x \in V \subset W$, setting $f'' = f'h$.

Since W'' has singularities, it is clear that we need a desingularization $W''' \to W''$, and then we will consider the composition

$$W''' \to W.$$

Not every desingularization, however, will fulfill the conditions in Definition 2.3.2. Recall that we had a smooth morphism $\pi: W \to Z$ which must lift as a smooth morphism $W''' \to Z$. This is already a condition on the desingularization. We also require in Definition 2.3.2 that the total pullback of Y in W''' be of pure codimension 1.

Let us indicate how we organize our proof:

(a1) We show that the pullback of Y at W'' is of pure codimension 1 in the singular scheme W''.

(a2) We show that locally at any point of the normal scheme W'' there is an element having as associated primes exactly those height-1 prime ideals defining the reduced pullback of Y.

Note that (a2) asserts that for *any* desingularization $W''' \to W''$, the total pullback of Y (by the induced morphism $W''' \to W''$) is of pure codimension 1. We must also require that it be a union of regular hypersurfaces with only normal crossings in W'''.

(b) We must show that (i) and (ii) in Definition 2.3.2 hold for $W''' \to Z$.

(b1) We will use general properties of constructive desingularization developed in [V2]: compatibility of constructive desingularization with étale topology and with pullbacks by smooth morphisms. So ultimately we will define $W''' \to W''$ as a constructive desingularization.

(b2) We use the local conditions in Remark 2.2.3: if a point $p \in W'''$ maps to $t \in Z$, then we require that a regular system of parameters $\{t_1, \ldots, t_r\}$ at $\mathcal{O}_{Z,t}$ can be extended so as to fulfill (1), (2) and (3) in Remark 2.2.3.

We started with $V \subset W$ and a smooth morphism $W \to Z$. We fixed a point $x \in V$ and defined a proper birational morphism $W' \to W$. We then defined $W'' \to W'$, and we shall now define $W''' \to W''$.

Fix a point $y \in W'''$, assume that y maps to $p \in W''$, to $q \in W'$ and to $x \in W$. Note that (2) in Remark 2.2.3 is a condition on the Jacobian of $W''' \to W$. Since W''' will be smooth, the Jacobian will define a closed set of points in W''' where $W''' \to W$ is not a local isomorphism.

We studied the toroidal structure at $p \in W''$. This toroidal structure was described in terms of a choice of a regular system of coordinates locally at $q \in W'$:

In §4.2 we introduced the ring of an open affine neighborhood of $q \in W'$, called A, together with a choice of some coordinates $y_i \in A_q$ so that some of them defined the components of the Jacobian of $W' \to W$ locally at q. The ring R had $(n!)$th roots of those y_i. The toroidal structure at $p\in W''$ grows from the choice of the regular system of coordinates $\{z, v_1, \ldots, v_s, t_1, \ldots, t_r\}$ at R^* in (12), together with Proposition 4.7. The Jacobian of $W'\to W$ is locally described by a monomial on some variables $y_i\in A_q$, and $y_i=(v_i)^{n!}$ at R^*. So this monomial is invariant by the group acting on R^* (see (a) and (b) in §4.5).

5.2. For a better understanding of the further development let us recall that $((f')^{-1}(Y))_{\text{red}}=C_1'\cup\ldots\cup C_r'\subset W'$ is a union of components of codimension 2 in W' (see (4)). $((f'')^{-1}(Y))_{\text{red}}\subset W''$ will, however, be of pure codimension 1. In fact, recall here that each component C_i of $((f')^{-1}(Y))_{\text{red}}\subset W'$ was defined, locally at q, by the ideal $\langle z, y_1\rangle\subset A_q$ (see (5)). The element y_1 has an $(n!)$th root v_1 at R_q in §4.2. Therefore $\langle z, y_1\rangle\subset A_q$ lies under $\langle z, v_1\rangle\subset R_q$, and this is an n-fold component of the hypersurface V_1 in Spec(R). This component has codimension 2 in $U_1'=\text{Spec}(R)$ ((10)). Recall also that (10) is a sequence of monoidal transformations, defined by blowing up these equimultiple centers of the hypersurface, all of codimension 2 in the regular ambient space. In particular, the total transform of $((f')^{-1}(Y))_{\text{red}}=C_1'\cup\ldots\cup C_r'$ $(\subset W')$ to U_s' via (10) is of pure codimension 1 in U_s', and hence it is a union of exceptional hypersurfaces. So let

$$H_1',\ H_2',\ \ldots,\ H_m'\ \ (\subset U_s') \tag{27}$$

be the exceptional hypersurfaces of $U_s'\to\text{Spec}(R)$, and let

$$C_1'',\ C_2'',\ \ldots,\ C_m''\ \ (\subset W'') \tag{28}$$

be the image of each H_i' via the finite map $U_s'\to W''$. Each H_i' is a regular hypersurface, and thus each C_i'' is of pure codimension 1 in W''. It also follows that there is a subset $C_{i_1}'', C_{i_2}'', \ldots, C_{i_r}''$ such that

$$((f'')^{-1}(Y))_{\text{red}}=C_{i_1}''\cup C_{i_2}''\cup\ldots\cup C_{i_r}'', \tag{29}$$

which is hence of pure codimension 1 in W''.

Since $W''\to W'$ is birational and $V'=(f')^{-1}(V)\subset W'$, let $V''\subset W''$ be the strict transform of V'.

Since $f'\colon W'\to W$ is birational and smooth, let $J'\subset W'$ be the Jacobian hypersurface, and let $J''\subset W''$ be the reduced pullback at W''. Finally set

$$\Pi=C_1''\cup C_2''\cup\ldots\cup C_m''\cup V''\cup J''\ (\subset W''). \tag{30}$$

Note that $\Pi\subset W''$ is reduced of pure codimension 1. We claim that

(i) $((f'')^{-1}(V))_{\mathrm{red}} \subset \Pi$;

(ii) $((f'')^{-1}(Y))_{\mathrm{red}} \subset \Pi$ ((29));

(iii) the morphism $W'' \to W$ defines a local isomorphism at any point in $W'' - \Pi$.

The hypersurface $V' = (f')^{-1}(V) \subset W'$ was lifted to a hypersurface $V_1 \subset \mathrm{Spec}(R)$, and the reduced total transform, or reduced pullback, of this hypersurface in U'_s is, say,

$$\Pi_V = H'_1 \cup H'_2 \cup \ldots \cup H'_m \cup V_s \ (\subset U'_s). \tag{31}$$

On the other hand, following (29), the reduced pullback of $((f')^{-1}(Y))_{\mathrm{red}}$ to U'_s, is a hypersurface, say

$$\Pi_Y = H'_{i_1} \cup H'_{i_2} \cup \ldots \cup H'_{i_r} \ (\subset U'_s). \tag{32}$$

Both Π_V and Π_Y are defined as reduced hypersurfaces in U'_s; so Π contains the image of Π_V via $U'_s \to W''$, and $(f'')^{-1}(Y))_{\mathrm{red}}$ is the image of Π_Y.

As for (iii) note that $W'' \to W'$ is a local isomorphism at points of

$$W'' - (C''_1 \cup C''_2 \cup \ldots \cup C''_m).$$

In particular, $W'' \to W$ is a local isomorphism at points in $W'' - \Pi$.

5.3. Let p be a point in W'', and $p' \in U'_s$ a point mapping to p via the finite morphism $U'_s \to W''$. We want to study the corresponding local rings. We fix the setting as in Proposition 4.7. To simplify notation fix

- $\mathcal{R}^* = k\{\{z, v_1, \ldots, v_s, t_1, \ldots, t_r\}\}$ as the completion of the local ring of U'_s at p', and
- G as the decomposition group at p'.

We may assume that

(A) elements of G act on each coordinate $\{z, v_1, \ldots, v_s, t_1, \ldots, t_r\}$ by multiplication by $(n!)$th roots of unity, and trivially on coordinates $\{t_1, \ldots, t_r\}$;

(B) if $p' \in V_s \subset U'_s$, then $I(V_s) = \langle z \rangle$ at $\mathcal{R}^*$ (see §4.7 (iv) for the case $n=1$);

(C) each exceptional hypersurface arising in the sequence (10) is defined (locally) by a monomial in the coordinates $\{z, v_1, \ldots, v_s\}$ (see (22)).

We now focus on the subring of G-invariants of $k\{\{z, v_1, \ldots, v_s, t_1, \ldots, t_r\}\}$, which is also the completion of the local ring $\mathcal{O}_{W'',p}$.

Since G acts trivially on the coordinates $\{t_1, \ldots, t_r\}$, assumption (A) asserts that we may just consider the action of G on the subring $k\{\{z, v_1, \ldots, v_s\}\}$ (recall that k is algebraically closed of characteristic zero). The subring of G-invariants is a formal toroidal singularity, say

$$k\{\{N_1, \ldots, N_m\}\} \subset k\{\{z, v_1, \ldots, v_s\}\}.$$

By general properties of toroidal singularities, the N_j can and will be chosen as monomials in the variables $\{z, v_1, ..., v_s\}$. Clearly $(\mathcal{R}^*)^G = k\{\{N_1, ..., N_m, t_1, ..., t_r\}\}$. Now define

(D) M_1 as a monomial in $\{z, v_1, ..., v_s\}$ defining the reduced hypersurface Π_V locally at $\mathcal{R}^*$ (see (B), (C) and (31));

(E) M_2 as a monomial in $\{z, v_1, ..., v_s\}$ defining the reduced hypersurface Π_Y locally at $\mathcal{R}^*$ (see (C) and (32));

(F) N_1' as the $(n!)$th power of M_1 (so that N_1' is G-invariant in $\mathcal{R}^*$);

(G) N_2' as the $(n!)$th power of M_2 (so that N_2' is G-invariant in $\mathcal{R}^*$);

(H) N_3' as a G-invariant monomial defining J'' locally at the point (see the last lines in Remark 5.1).

Now take, within the ring of formal power series $k\{\{z, v_1, ..., v_s, t_1, ..., t_r\}\}$, the k-subalgebras $k[z, v_1, ..., v_s, t_1, ..., t_r]$. This is a polynomial ring. Now (A) asserts that G acts on this polynomial ring, on the ring $k[z, v_1, ..., v_s]$, and trivially on $k[t_1, ..., t_r]$.

So the subring of G-invariants of $k[z, v_1, ..., v_s]$ is $k[N_1, ..., N_m]$ (the same monomials N_j as before), which is the ring of an affine toric scheme. The subring of G-invariants of $k[z, v_1, ..., v_s, t_1, ..., t_r]$ is $k[N_1, ..., N_m, t_1, ..., t_r]$.

Recall that the equations y_i were chosen in the ring A (§4.2), so each y_i is G-invariant and $y_i \in k[N_1, ..., N_m]$. Finally note that

$$N_1', N_2', N_3' \in k[N_1, ..., N_m].$$

It can be checked that the hypersurface $\Pi \subset W''$ ((30)) is defined, at the completion of the local ring $\mathcal{O}_{W'',p}$, by the height-1 prime ideals containing the product of all N_i'; and that $((f'')^{-1}(Y))_{\text{red}}$ is defined by the height-1 prime ideals containing N_2'. We proceed now in two steps by defining

(a) a constructive desingularization of W'', say

$$j_1 : W_1''' \to W'',$$

and finally,

(b) an embedded desingularization of the hypersurface $(j_1^{-1}(\Pi))_{\text{red}}$, say

$$j_2 : W''' \to W_1'''.$$

Let p_1 be the closed point at $P = \text{Spec}(k[N_1, ..., N_m, t_1, ..., t_r])$ corresponding to the maximal ideal $\langle N_1, ..., N_m, t_1, ..., t_r \rangle$, so that $p \in W''$ is étale locally isomorphic to $p_1 \in P$. By a general property of constructive desingularization (compatibility with étale restrictions, see [V2, Theorem 7.6.1, p. 668]), both steps (a) and (b) above are determined, locally at p, by (a′) a constructive desingularization of P locally at p_1, followed by

(b′) an embedded desingularization of the reduced hypersurface defined by the total transform of $N'_1.N'_2.N'_3$.

Let p_2 be the closed point in $\mathrm{Spec}(k[N_1,...,N_m])$ corresponding to the maximal ideal $\langle N_1,...,N_m\rangle$. There is a natural smooth map

$$P = \mathrm{Spec}(k[N_1,...,N_m,t_1,...,t_r]) \to \mathrm{Spec}(k[N_1,...,N_m]). \tag{33}$$

By the restriction properties of constructive desingularization (compatibility with pull-back by smooth morphisms, see [V2, 4.1 (b), p. 647]), both (a′) and (b′) follow, locally at p_1, from, say,

(a″) a constructive desingularization of $\mathrm{Spec}(k[N_1,...,N_m])$ locally at p_2, followed by

(b″) an embedded desingularization of the reduced hypersurface defined by the total transform of $N'_1.N'_2.N'_3$.

The total transform of Π, say $\Pi'''\subset W'''$, is a hypersurface with normal crossings. Set

$$f = f''j_1j_2 : W''' \to W.$$

By construction, f defines an isomorphism over points in $W'''-\Pi'''$, so $J(f,k)_{\mathrm{red}}\subset\Pi'''$. In particular, $J(f,k)$ is a union of hypersurfaces having only normal crossings. Note that $(f^{-1}(Y))_{\mathrm{red}}=((j_1j_2)^{-1}((f'')^{-1}(Y)))_{\mathrm{red}}\subset\Pi'''$ is (locally) the support of a principal ideal (see (G)), so $(f^{-1}(Y))_{\mathrm{red}}$ is of pure codimension 1 and has normal crossings in W'''.

It is clear from (30) that f also defines an embedded resolution of $V\subset W$. Let us check now that all three conditions (i), (ii) and (iii) of Definition 2.3.2 are fulfilled:

(i) Set π as in §3.6 (A). Apply Remark 3.4 to show that π and $V\subset W$ define a family of embedded hypersurfaces (Definition 2.2.1).

(ii) To show that π and f define a family of embedded resolutions we check that the local conditions (1), (2) and (3) in Remark 2.10 hold. But this is straightforward from (33) and the fact that the constructive desingularization of P is the pullback of that of $\mathrm{Spec}(k[N_1,...,N_m])$ (the pullback defined by multiplication with the affine scheme $\mathrm{Spec}(k[t_1,...,t_r])$).

(iii) As mentioned above, this condition is now clear since $(f^{-1}(Y))_{\mathrm{red}}$ is (locally) the support of a principal divisor.

This proves Theorem 3.2.

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