

# Classification of negatively pinched manifolds with amenable fundamental groups

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## 1. Introduction

In this paper we study manifolds of the form  $X/\Gamma$ , where  $X$  is a simply-connected complete Riemannian manifold with sectional curvatures pinched (i.e. bounded) between two negative constants, and  $\Gamma$  is a discrete torsion-free subgroup of the isometry group of  $X$ . According to [10], if  $\Gamma$  is amenable, then either  $\Gamma$  stabilizes a biinfinite geodesic, or else  $\Gamma$  fixes a unique point  $z$  at infinity. The case when  $\Gamma$  stabilizes a biinfinite

geodesic is completely understood, namely the normal exponential map to the geodesic is a  $\Gamma$ -equivariant diffeomorphism, hence  $X/\Gamma$  is a vector bundle over  $S^1$ ; there are only two such bundles each admitting a complete hyperbolic metric.

If  $\Gamma$  fixes a unique point  $z$  at infinity (such groups are called *parabolic*), then  $\Gamma$  stabilizes horospheres centered at  $z$  and permutes geodesics asymptotic to  $z$ , so that, given a horosphere  $H$ , the manifold  $X/\Gamma$  is diffeomorphic to the product of  $H/\Gamma$  with  $\mathbf{R}$ . We refer to  $H/\Gamma$  as a *horosphere quotient*. In this case, a delicate result of B. Bowditch [6] shows that  $\Gamma$  must be finitely generated, which, by Margulis' lemma [3], implies that  $\Gamma$  is virtually nilpotent.

The main result of this paper is a diffeomorphism classification of horosphere quotients, namely we show that, up to a diffeomorphism, the classes of horosphere quotients and (possibly noncompact) infranilmanifolds coincide.

By an *infranilmanifold* we mean the quotient of a simply-connected nilpotent Lie group  $G$  by the action of a torsion-free discrete subgroup  $\Gamma$  of the semidirect product of  $G$  with a compact subgroup of  $\text{Aut}(G)$ .

**THEOREM 1.1.** *For a smooth manifold  $N$  the following are equivalent:*

- (1)  *$N$  is a horosphere quotient;*
- (2)  *$N$  is diffeomorphic to an infranilmanifold;*
- (3)  *$N$  is the total space of a flat Euclidean vector bundle over a compact infranilmanifold.*

The implication (3)  $\Rightarrow$  (2) is straightforward; (2)  $\Rightarrow$  (1) is proved by constructing an explicit warped product metric of pinched negative curvature. The proof of the implication (1)  $\Rightarrow$  (3) occupies most of the paper, and depends on the collapsing theory of J. Cheeger, K. Fukaya and M. Gromov [14].

If  $N$  is compact (in which case the conditions (2) and (3) are identical), then the implication (1)  $\Rightarrow$  (2) follows from Gromov's classification of almost flat manifolds, as improved by E. Ruh, while the implication (2)  $\Rightarrow$  (1) is new. If  $N$  is noncompact, then Theorem 1.1 is nontrivial even when  $\pi_1(N) \cong \mathbf{Z}$ , although the proof does simplify in this case. A direct algebraic proof of (2)  $\Rightarrow$  (3) was given in [39, Theorem 6], but the case when  $N$  is a nilmanifold was already treated in [25], where it is shown that any nilmanifold is diffeomorphic to the product of a compact nilmanifold and a Euclidean space.

We postpone the discussion of the proof till §2, and just mention that the proof also gives geometric information about horosphere quotients, e.g. we show that  $H/\Gamma$  is diffeomorphic to a tubular neighborhood of some orbit of an  $N$ -structure on  $H/\Gamma$ .

By Chern–Weil theory, any flat Euclidean vector bundle has zero rational Euler and Pontryagin classes. Moreover, by [38], any flat Euclidean bundle with virtually abelian

holonomy is isomorphic to a bundle with finite structure group. Thus the vector bundle in (3) becomes trivial in a finite cover, and has zero rational Euler and Pontryagin classes, and in particular, any horosphere quotient is finitely covered by the product of a compact nilmanifold and a Euclidean space.

**COROLLARY 1.2.** *A smooth manifold  $M$  with amenable fundamental group admits a complete metric of pinched negative curvature if and only if it is diffeomorphic to the Möbius band, or to the product of a line and the total space of a flat Euclidean vector bundle over a compact infranilmanifold.*

The pinched negative curvature assumption in Corollary 1.2 cannot be relaxed to  $-1 \leq \sec \leq 0$  or  $\sec \leq -1$ , e.g. because these assumptions do not force the fundamental group to be virtually nilpotent [6, §6]. More delicate examples come from the work of M. Anderson [1], who proved that each vector bundle over a closed nonpositively curved manifold (e.g. a torus) carries a complete Riemannian metric with  $-1 \leq \sec \leq 0$ . Since in each dimension there are only finitely many isomorphism classes of flat Euclidean bundles over a given compact manifold, all but finitely many vector bundles over tori admit no metrics of pinched negative curvature. Also  $-1 \leq \sec(M) \leq 0$  can be turned into  $\sec(M \times \mathbf{R}) \leq -1$  for the warped product metric on  $M \times \mathbf{R}$  with warping function  $e^t$  [5], hence Anderson's examples carry metrics with  $\sec \leq -1$  after taking product with  $\mathbf{R}$ . Specifically, if  $E$  is the total space of a vector bundle over a torus with nontrivial rational Pontryagin class, then  $M = E \times \mathbf{R}$  carries a complete metric of  $\sec \leq -1$  but not of pinched negative curvature. Finally, Anderson also showed that every vector bundle over a closed negatively curved manifold admits a complete Riemannian metric of pinched negative curvature, hence amenability of the fundamental group is indispensable.

Because an infranilmanifold with virtually abelian fundamental group is flat, Theorem 1.1 immediately implies the following result.

**COROLLARY 1.3.** *Let  $M$  be a smooth manifold with virtually abelian fundamental group. Then the following are equivalent:*

- (1)  *$M$  admits a complete metric of  $\sec \equiv -1$ ;*
- (2)  *$M$  admits a complete metric of pinched negative curvature.*

In [7] Bowditch developed several equivalent definitions of geometrical finiteness for pinched negatively curved manifolds, and conjectured the following result.

**COROLLARY 1.4.** *Any geometrically finite pinched negatively curved manifold  $X/\Gamma$  is diffeomorphic to the interior of a compact manifold with boundary.*

We believe that the main results of this paper, including Corollary 1.4, should extend to the orbifold case, i.e. when  $\Gamma$  is not assumed to be torsion free. However, working

in the orbifold category creates various technical difficulties, both mathematical and expository, and we do not attempt to treat the orbifold case in this paper.

*Acknowledgements.* It is a pleasure to thank Ilya Kapovich and Derek J. S. Robinson for Lemma A.1, Hermann Karcher for pointing us to [8], Robion C. Kirby for Lemma 7.3, Anton Petrunin for numerous helpful conversations and suggestions and particularly for Lemma D.1, and Xiaochun Rong and Kenji Fukaya for helpful discussions on collapsing. We are grateful to the referee for various editorial comments. This work was partially supported by the NSF grants DMS-0352576 (Belegradek) and DMS-0204187 (Kapovitch).

## 2. Sketch of the proof of (1) $\Rightarrow$ (3)

The Busemann function corresponding to  $z$  gives rise to a  $C^2$ -Riemannian submersion  $X/\Gamma \rightarrow \mathbf{R}$  whose fibers are horosphere quotients each equipped with the induced  $C^1$ -Riemannian metric  $g_t$ . By the Rauch comparison theorem, the second fundamental form of a horosphere is bounded in terms of curvature bounds of  $X$  (cf. [8]). In particular, each fiber has curvature uniformly bounded above and below in comparison sense. Let  $\sigma(t)$  be a horizontal geodesic in  $X/\Gamma$ , i.e. a geodesic that projects isometrically to  $\mathbf{R}$ . Because of the exponential convergence of geodesics in  $X$ , the manifold  $X/\Gamma$  is “collapsing” in the sense that the unit balls around  $\sigma(t) \in X/\Gamma$  form an exhaustion of  $X/\Gamma$  and have small injectivity radius for large  $t$ . Similarly, each fiber of  $X/\Gamma \rightarrow \mathbf{R}$  also collapses, and in fact  $X/\Gamma$  is noncollapsed in the direction transverse to the fibers.

There are essential difficulties in applying the collapsing theory of [14] to  $X/\Gamma$ . First, we do not know whether  $(X/\Gamma, \sigma(t))$  converges in pointed Gromov–Hausdorff topology to a single limit space. By general theory, the family  $(X/\Gamma, \sigma(t))$  is precompact and thus has many converging subsequences. While different limits might be nonisometric, one of the main steps of the proof is obtaining a uniform (i.e. independent of the subsequence) lower bound on the “injectivity radius” of the limit spaces at the base point. This is done by a comparison argument involving taking “almost square roots” of elements of  $\Gamma$ , and using the flat connection of [6] discussed below. Another complication is that the  $N$ -structure on  $(X/\Gamma, \sigma(t))$  provided by [14] may well have zero-dimensional orbits outside the unit ball around  $\sigma(t)$ , in other words a large noncompact region of  $(X/\Gamma, \sigma(t))$  may be noncollapsed, which makes it hard to control topology of the region.

However, once the “injectivity radius” bound is established, critical points of distance functions considerations yield the “product structure at infinity” for  $X/\Gamma$ , and also for  $(H/\Gamma, g_t)$  if  $t$  is large enough.

Furthermore, one can show that  $H/\Gamma$  is diffeomorphic to the normal bundle of an orbit  $O_t$  of the  $N$ -structure. The orbit corresponds to the point in a limit space given to us by the convergence, and at which we get an “injectivity radius” bound. This depends on a few results on Alexandrov spaces with curvature bounded below, with key ingredients provided by [21] and [31].

By the collapsing theory, the structure group of the normal bundle to the orbit of an  $N$ -structure is a finite extension of a torus group [14]. Of course, not every such a bundle has a flat Euclidean structure.

The flatness of the normal bundle to the orbit is proved using a remarkable flat connection discovered by B. Bowditch [6], and later in a different disguise by W. Ballmann and J. Brüning [2], who were apparently unaware of [6]. It follows from [6] or [2] that each pinched negatively curved manifold  $X/\Gamma$ , where  $\Gamma$  fixes a unique point  $z$  at infinity, admits a natural flat  $C^0$ -connection that is compatible with the metric and has nonzero torsion, and such that on short loops it is close to the Levi–Civita connection. Furthermore, the parallel transport of the connection preserves the fibration of  $X/\Gamma$  by horosphere quotients. Hence each horosphere quotient has flat tangent bundle.

In fact, we prove a finer result that the normal bundle to  $O_t$  is also flat, for suitable large  $t$ . (Purely topological considerations are useless here since there exist vector bundles without flat Euclidean structure whose total spaces have flat Euclidean tangent bundles, for example this happens for any nontrivial orientable  $\mathbf{R}^2$ -bundle over the 2-torus that has even Euler number.) It turns out that  $O_t$  sits with flat normal bundle in a totally geodesic stratum of the  $N$ -structure, so it suffices to show that the normal bundle to the stratum is flat, when restricted to  $O_t$ . Now, since the above flat connection is close to the Levi–Civita connection, the normal bundle is “almost flat”, and it can be made flat by averaging via center of mass. This completes the proof.

Throughout the proof we use the collapsing theory developed in [14]. This paper is based on the earlier extensive work of Fukaya, and Cheeger–Gromov, and many arguments in [14] are merely sketched. We suggest reading [17] for a snapshot of the state of affairs before [14], and [34], [33], [16] for a current point of view.

### 3. Topological digression

The result of Bowditch [6] that horosphere quotients have finitely generated fundamental groups actually implies that any horosphere quotient is homotopy equivalent to a compact infranilmanifold (because any torsion-free finitely generated virtually nilpotent group is the fundamental group of a compact infranilmanifold [15], and because for aspherical manifolds any  $\pi_1$ -isomorphism is induced by a homotopy equivalence).

To help appreciate the difference between this statement and Theorem 1.1, we discuss several types of examples that are allowed by Bowditch’s result, and are ruled out by Theorem 1.1. The simplest example is a vector bundle over an infranilmanifold with nonzero rational Euler or Pontryagin class: such a manifold cannot be the total space of a flat Euclidean bundle, as is easy to see using the fact that the tangent bundle to any infranilmanifold is flat.

Another example is the product of a closed infranilmanifold and a contractible manifold of dimension  $>2$  that is not simply-connected at infinity. Finally, even more sophisticated examples come from the fact that below metastable range (starting at which any homotopy equivalence is homotopic to a smooth embedding, by Haefliger’s embedding theorem) there are many smooth manifolds that are thickenings of say a torus, yet are not vector bundles over the torus. It would be interesting to see whether the “weird” topological constructions of this paragraph can be realized geometrically, even as nonpositively curved manifolds.

#### 4. Parallel transport through infinity and rotation homomorphism

Let  $X$  be a simply-connected complete pinched negatively curved  $n$ -manifold normalized so that  $-a^2 \leq \sec(X) \leq -1$ . One of the key properties of  $X$  used in this section is that any two geodesic rays in  $X$  that are asymptotic to the same point at infinity converge exponentially, i.e. for any asymptotic rays  $\gamma_1(t)$  and  $\gamma_2(t)$ , with  $\gamma_1(0)$  and  $\gamma_2(0)$  lying on the same horosphere, the function  $d(\gamma_1(t), \gamma_2(t))$  is monotonically decreasing as  $t \rightarrow \infty$ , and

$$e^{-at} c_1(a, d(\gamma_1(0), \gamma_2(0))) \leq d(\gamma_1(t), \gamma_2(t)) \leq e^{-t} c_2(a, d(\gamma_1(0), \gamma_2(0))),$$

where  $c_i(a, d)$  is linear in  $d$  for small  $d$ . This is proved by triangle comparison with spaces of constant negative curvature.

Bowditch introduced a connection on  $X$  that we now describe (see [6, §3] for details). Fix a point  $z$  at infinity of  $X$ . Let  $w_i \rightarrow z$  as  $i \rightarrow \infty$ . For any  $x, y \in X$ , consider the parallel transport map from  $x$  to  $w_i$  followed by the parallel transport from  $w_i$  to  $y$  along the shortest geodesics. This defines an isometry between the tangent spaces at  $x$  and  $y$ . By [6, Lemma 3.1], this map converges to a well-defined limit isometry  $P_{xy}^\infty: T_x M \rightarrow T_y M$  as  $i \rightarrow \infty$ . We refer to  $P_{xy}^\infty$  as the *parallel transport through infinity from  $x$  to  $y$* .

We denote the Levi–Civita parallel transport from  $x$  to  $y$  along the shortest geodesic by  $P_{xy}$ ; clearly, if  $x$  and  $y$  lie on a geodesic ray that ends at  $z$ , then  $P_{xy}^\infty = P_{xy}$ . A key feature of  $P^\infty$  is that it approximates the Levi–Civita parallel transport on short geodesic segments (see [6, Lemma 3.2]; more details can be found in [11, §6]). This is because any geodesic triangle in  $X$  spans a “ruled” surface of area at most the area of the comparison

triangle in the hyperbolic plane of  $\sec = -1$ . By exponential convergence of geodesics, the area of the comparison triangle is bounded above by a constant times the shortest side of the triangle. As the holonomy around the circumference of the triangle is bounded by the integral of the curvature over its interior, we conclude that  $|P_{xy} - P_{xy}^\infty| \leq q(a) d(xy)$ , where  $q(a)$  is a constant depending only on  $a$ .

Given  $x \in X$ , fix an isometry  $\mathbf{R}^n \rightarrow T_x X$  and translate it around  $X$  using  $P^\infty$ . This defines a  $P^\infty$ -invariant trivialization of the tangent bundle to  $X$ . Let  $\text{Iso}_z(X)$  be the group of isometries of  $X$  that fixes  $z$ . For any point  $y \in X$  look at the map  $\text{Iso}_z(X) \rightarrow O(n)$  given by  $\gamma \mapsto P_{\gamma(y)y}^\infty \circ d\gamma$ . It turns out that this map is a homomorphism independent of  $y$ . We call it the *rotation homomorphism*. Starting with a different base point  $x \in X$  or a different isometry  $\mathbf{R}^n \rightarrow T_x X$  has the effect of replacing the rotation homomorphism by its conjugate.

Now, if  $\Gamma$  is a discrete torsion-free subgroup of  $\text{Iso}_z(X)$ , then, since the rotation homomorphism is independent of  $y$ ,  $P^\infty$  gives rise to a flat connection on  $X/\Gamma$  with holonomy given by the rotation homomorphism. By the above discussion of  $P^\infty$ , this is a  $C^0$ -flat connection that is compatible with the metric and close to the Levi-Civita connection on short loops. Of course, this connection has torsion.

*Remark 4.1.* The above discussion is easily seen to be valid if  $X$  is a simply-connected complete  $C^1$ -Riemannian manifold of pinched negative curvature in the comparison sense. This is because any such  $C^1$ -metric can be approximated uniformly in  $C^1$ -topology by smooth Riemannian metrics of pinched negative curvature [28], perhaps with slightly larger pinching. Then the distance functions and Levi-Civita connections converge uniformly in  $C^0$ -topology, and we recover all the statements above.

*Remark 4.2.* The connection of Bowditch, that was described above, was reinvented later in a different disguise by Ballmann and Brüning [2, §3]. The connection in [2] is defined by an explicit local formula in terms of the curvature tensor and the Levi-Civita connection of  $X$ . Actually, [2] only discusses the case of compact horosphere quotients; however, all the arguments there are local, hence they apply to any horosphere quotient. The only feature which is special for compact horosphere quotients is that in that case the connection has finite holonomy group [2], as follows from estimates in [11]. For noncompact horosphere quotients, the holonomy need not be finite as seen by looking at a glide rotation with irrational angle in  $\mathbf{R}^3$ , thought of as a horosphere in the hyperbolic 4-space. We never have to use [2] in this paper; however, for completeness, we discuss their construction in Appendix C, where we also show that the connections of [6] and [2] coincide.

### 5. Passing to the limit

Let  $X$  be a simply-connected complete pinched negatively curved  $n$ -manifold normalized so that  $-a^2 \leq \sec(X) \leq -1$ , let  $c(t)$  be a biinfinite geodesic in  $X$ , and let  $\Gamma$  be a closed subgroup of  $\text{Iso}(X)$  that fixes the point  $c(\infty)$  at infinity. We refer to the gradient flow  $b_t$  of a Busemann function for  $c(t)$  as *Busemann flow*. Following Bowditch, we sometimes use the notation  $x+t:=b_t(x)$ .

Since  $X$  has bounded curvature and infinite injectivity radius, the family  $(X, c(t), \Gamma)$  has a subsequence  $(X, c(t_i), \Gamma)$  that converges to  $(X_\infty, p, G)$  in the equivariant pointed  $C^{1,\alpha}$ -topology [32, Chapter 10]. Here  $X_\infty$  is a smooth manifold with  $C^{1,\alpha}$ -Riemannian metric that has infinite injectivity radius and the same curvature bounds as  $X$  in the comparison sense, and  $G$  is a closed subgroup of  $\text{Iso}(X_\infty)$ . Note that  $\text{Iso}(X_\infty)$  is a Lie group that acts on  $X$  by  $C^3$ -diffeomorphisms (this last fact is probably known but for a lack of reference we give a simple proof in Appendix B).

Furthermore, geodesic rays in  $X$  that start at uniformly bounded distance from  $c(t_i)$  converge to rays in  $X_\infty$ . In particular, the rays  $c(t+t_i)$  starting at  $c(t_i)$  converge to a ray  $c_\infty(t)$  in  $X_\infty$  that starts at  $p$ , and the corresponding Busemann functions also converge. Since the Busemann functions on  $X$  are  $C^2$  [19], they converge to a  $C^1$  Busemann function on  $X_\infty$ . Thus the horosphere passing through  $p$  is a  $C^1$ -submanifold of  $X_\infty$ , and is the limit of horospheres passing through  $c(t_i)$ . The Busemann flow is  $C^1$  on  $X$ , and  $C^0$  on  $X_\infty$ . Since the horospheres in  $X$  and  $X_\infty$  have the same dimension, the sequence of horospheres passing through  $c(t_i)$  does not collapse, and, more generally, each horosphere centered at  $c_\infty(\infty)$  is the limit of a noncollapsing sequence of horospheres in  $X$ .

It is easy to see that the group  $G$  fixes  $c_\infty(\infty)$ , i.e. any  $\gamma \in G$  takes  $c_\infty$  to a ray asymptotic to  $c_\infty$ . Furthermore,  $G$  leaves the horospheres corresponding to  $c_\infty(\infty)$  invariant.

Thus, one can define the rotation homomorphism  $\phi_\infty: G \rightarrow O(n)$  corresponding to the point  $c_\infty(\infty)$ . The point only determines  $\phi_\infty$  up to conjugacy, so we also need to fix an isometry  $L: \mathbf{R}^n \rightarrow T_p X_\infty$ . Similarly, a choice of an isometry  $L_i: \mathbf{R}^n \rightarrow T_{c(t_i)} X$  specifies the rotation homomorphism  $\phi_i: \Gamma \rightarrow O(n)$  corresponding to the point  $c(\infty)$ . We can assume that  $\phi_i = \phi_0$  for each  $i$ , by choosing  $L_i$  equal to  $L_0$  followed by the parallel transport  $P_{c(t_0), c(t_i)}^\infty = P_{c(t_0), c(t_i)}$ . Henceforth we denote  $\phi_0$  by  $\phi$ . Also it is convenient to choose  $L$  as follows.

**LEMMA 5.1.** *After passing to a subsequence of  $(X, c(t_i), \Gamma)$ , there exists  $L$  such that if  $\gamma_i \rightarrow \gamma$ , then  $\phi(\gamma_i) \rightarrow \phi_\infty(\gamma)$ .*

*Proof.* Since  $(X, c(t_i), \Gamma) \rightarrow (X_\infty, p, G)$  in pointed equivariant  $C^{1,\alpha}$ -topology, we can



find the corresponding  $C^{1,\alpha}$  approximations  $f_i: B(c(t_i), 1) \rightarrow B(p, 1)$ . We may assume that  $df_i: T_{c(t_i)}X \rightarrow T_pX_\infty$  is an isometry for all  $i$ . By compactness of  $O(n)$ ,  $df_i \circ L_i$  subconverge to an isometry  $L: \mathbf{R}^n \rightarrow T_pX$ , so, by modifying  $f_i$  slightly, we can assume that  $df_i \circ L_i = L$ . This  $L$  is then used to define  $\phi_\infty$ , and it remains to show that if  $\gamma_i \rightarrow \gamma$ , then  $\phi(\gamma_i) \rightarrow \phi_\infty(\gamma)$ . For the rest of the proof we suppress  $L_i$  and  $L$ .

Since  $d\gamma_i \rightarrow d\gamma$ , it is enough to show that parallel transports through infinity from  $c(t_i)$  to  $\gamma_i(c(t_i))$  converge to the parallel transport through infinity from  $p$  to  $\gamma(p)$ .

For  $x \in X$  and  $c(t)$  that lie on the same horosphere, and for any  $s > t$ , denote by  $P_{x,s,c(t)}: T_{c(t)}M \rightarrow T_xM$  the parallel transport along the piecewise geodesic path  $x, x+s, c(t)+s, c(t)$ . Here  $c(t)+s = c(t+s)$  and  $x+s$  also lie on the same horosphere. Now  $|P_{xc(t)}^\infty - P_{x,s,c(t)}|$  can be estimated as

$$|P_{x+s,c(t)+s}^\infty - P_{x,s,c(t)+s}| \leq q(a)d(x+s, c(t)+s) \leq q(a)e^{-s}c_2(a, d(x, c(t))).$$

By Remark 4.1, the same estimate holds for  $X_\infty$  and  $c_\infty$ .

Fix  $\varepsilon > 0$  and pick  $R > 0$  such that  $d(c(t_i), \gamma_i(c(t_i))) \leq R$  for all  $i$ . Take a large enough  $s$  so that  $q(a)c_2(a, R)e^{-s} < \varepsilon$ . Since  $B(c(t_i), R+s)$  converges to  $B(p, R+s)$  in  $C^{1,\alpha}$ -topology, and  $\gamma_i \rightarrow \gamma$ , we conclude that  $P_{\gamma_i(c(t_i)),s,c(t_i)} \rightarrow P_{\gamma(p),s,p}$  in  $C^0$ -topology, or more formally,

$$|df_i \circ P_{\gamma_i(c(t_i)),s,c(t_i)} \circ d\gamma_i - P_{\gamma(p),s,p} \circ d\gamma| < \varepsilon$$

for large  $i$ , where  $f_i$  is the  $C^{1,\alpha}$ -approximation. By the estimate in the previous paragraph,  $|P_{\gamma_i(c(t_i)),s,c(t_i)} - P_{\gamma_i(c(t_i)),c(t_i)}^\infty| < \varepsilon$  and  $|P_{\gamma(p),s,p} - P_{\gamma(p),p}^\infty| < \varepsilon$ , so the triangle inequality implies that

$$|df_i \circ P_{\gamma_i(c(t_i)),c(t_i)}^\infty \circ d\gamma_i - P_{\gamma(p),p}^\infty \circ d\gamma| < 3\varepsilon$$

for large  $i$ . Hence,  $|\phi(\gamma_i) - \phi_\infty(\gamma)| < 3\varepsilon$  for all large  $i$ , and, since  $\varepsilon > 0$  is arbitrary, it follows that  $\phi(\gamma_i) \rightarrow \phi_\infty(\gamma)$  as  $i \rightarrow \infty$ . □

**PROPOSITION 5.2.** *Let  $K = \ker \phi_\infty$  and let  $G_p$  be the isotropy subgroup of  $p$  in  $G$ . Then*

- (1)  $\overline{\phi(\Gamma)} = \phi_\infty(G_p) = \phi_\infty(G)$ ;
- (2)  $K$  acts freely on  $X$ , in particular  $K \cap G_p = \{\text{id}\}$ ;
- (3) the short exact sequence  $1 \rightarrow K \rightarrow G \xrightarrow{\phi_\infty} \phi_\infty(G) \rightarrow 1$  splits with the splitting given by  $\phi_\infty(G) \simeq G_p \hookrightarrow G$ . In particular,  $G$  is a semidirect product of  $K$  and  $G_p$ .

*Proof.* (1) For each  $\gamma \in \Gamma$  we have  $d(\gamma(c(t_i)), c(t_i)) \rightarrow 0$  as  $i \rightarrow \infty$ , so the constant sequence  $\gamma$  converges to some  $g \in G_p$ . Lemma 5.1 yields  $\phi(\gamma) \rightarrow \phi_\infty(g)$ , which means  $\phi(\gamma) = \phi_\infty(g)$ . Thus,  $\phi(\Gamma) \subset \phi_\infty(G_p)$ . Now,  $G_p$  is compact, so  $\phi_\infty(G_p)$  is closed, and

therefore  $\overline{\phi(\Gamma)} \subset \phi_\infty(G_p)$ . Since  $\phi_\infty(G_p) \subset \phi_\infty(G)$ , it remains to show that  $\phi_\infty(G) \subset \overline{\phi(\Gamma)}$ . Given  $\gamma \in G$ , we find  $\gamma_i \in \Gamma$  with  $\gamma_i \rightarrow \gamma$ . By Lemma 5.1,  $\phi_\infty(\gamma)$  is the limit of  $\phi(\gamma_i) \in \phi(\Gamma)$ , so  $\phi_\infty(\gamma) \in \overline{\phi(\Gamma)}$ .

(2) If  $k \in K$  fixes a point  $x$ , then  $1 = \phi_\infty(k) = P_{k(x)x}^\infty \circ dk = P_{xx}^\infty \circ dk = dk$ . Since  $k$  is an isometry,  $k = \text{id}$ .

(3) This is a formal consequence of (1) and (2). □

*Remark 5.3.* Since  $G$  is the semidirect product of  $K$  and  $G_p$ , any  $\gamma \in G$  can be uniquely written as  $kg$  with  $k \in K$  and  $g \in G_p$ . We refer to  $k$  and  $g$ , respectively, as the *translational part* and the *rotational part* of  $\gamma$ .

*Remark 5.4.* If  $\Gamma$  is discrete, then by Margulis' lemma any finitely generated subgroup of  $\Gamma$  has a nilpotent subgroup whose index  $i$  and the degree of nilpotency  $d$  are bounded above by a constant depending only on  $n$ . (Of course, by [6],  $\Gamma$  itself is finitely generated, as is any subgroup of  $\Gamma$ , but we do not need this harder fact here.) The same then holds for  $G$ . Indeed, take finitely many elements  $g_l$  of  $G$  and approximate them by  $\gamma_{j,l} \in \Gamma$ , so they generate a finitely generated subgroup of  $\Gamma$ . Then  $\gamma_{j,l}^i$  approximate  $g_l^i$ , and, by above,  $g_l^i$  lie in a nilpotent subgroup of  $\Gamma$ . Hence a  $d$ -fold iterated commutator in the  $g_{j,l}^i$ 's is trivial for all  $j$ , and then so is the corresponding commutator in the  $g_l^i$ 's. Hence  $G$  is nilpotent by [35, Lemma VIII.8.17].

### 6. Controlling injectivity radius

We continue working with the notation of §5, except now we also assume that  $\Gamma$  is discrete. The family  $(X, c(t), \Gamma)$  may have many converging subsequences with limits of the form  $(X_\infty, p, G)$ . We denote by  $K(p)$  the  $K$ -orbit of  $p$ , where  $K$  is the kernel of the rotation homomorphism  $G \rightarrow O(n)$ . The goal of this section is to find a common lower bound, on the normal injectivity radii of the  $K(p)$ 's.

**PROPOSITION 6.1.** *There exists a constant  $f(a)$  such that, for each  $x \in K(p)$ , the norm of the second fundamental form  $\Pi_x$  of  $K(p)$  at  $x$  is bounded above by  $f(a)$ .*

*Proof.* Since  $K$  acts by isometries,  $|\Pi_x| = |\Pi_p|$  for any  $x \in K(p)$ , so we can assume  $x = p$ . Let  $X, Y \in T_p K(p)$  be unit tangent vectors. Extend  $Y$  to a left-invariant vector field on  $K(p)$ , and let  $\alpha(t) = \exp(tX)(p)$  be the orbit of  $p$  under the one-parameter subgroup generated by  $X$ . Since  $\Pi_p(X, Y)$  is the normal component of  $\nabla_X Y(p)$ , it suffices to show that  $|\nabla_X Y| \leq f(a)$ . Let  $P_{p, \alpha(t)}^\alpha$  be the parallel transport from  $p$  to  $\alpha(t)$  along  $\alpha$ . By §4, we have  $|P_{p, \alpha(t)} - P_{p, \alpha(t)}^\infty| \leq q(a)d(p, \alpha(t)) \leq 2q(a)t$  for all small  $t$ .

A similar argument shows that  $|P_{p, \alpha(t)}^\alpha - P_{p, \alpha(t)}^\alpha| \leq 2q(a)t$  for all small  $t$ . Indeed, look at the "ruled" surface obtained by joining  $p$  to the points of  $\alpha$  near  $p$ . If we approximate

$\alpha$  by a piecewise geodesic curve  $p\alpha(t_1) \dots \alpha(t_k)$ , where  $\alpha(t_k)=q$  is some fixed point near  $p$ , then the area of the surface can be computed as the limit as  $k \rightarrow \infty$  of the sum of the areas of geodesic triangles  $p\alpha(t_i)\alpha(t_{i+1})$ . The area of each triangle is bounded above by  $d(\alpha(t_i)\alpha(t_{i+1}))$ , so the area of the ruled surface is bounded above by the length of  $\alpha$  from  $p$  to  $q$ , which is at most  $2t$ , for small  $t$ .

Therefore,  $|P_{p,\alpha(t)}^\infty - P_{p,\alpha(t)}^\alpha| \leq 4q(a)t = f(a)t$  by the triangle inequality, so

$$|P_{p,\alpha(t)}^\infty Y - P_{p,\alpha(t)}^\alpha Y| \leq f(a)t.$$

On the other hand,  $P_{p,\alpha(t)}^\infty Y = Y(p)$  because  $Y$  is left-invariant, and since elements of  $K$  have trivial rotational parts. Thus,  $|P_{p,\alpha(t)}^\alpha Y - Y(p)| \leq f(a)t$ , which by definition of covariant derivative implies that  $|\nabla_X Y(p)| \leq f(a)$ . □

**COROLLARY 6.2.** (i) *There exists  $r(a) > 0$  such that, if  $C$  is the connected component of  $K(p) \cap B_{r(a)}(p)$  that contains  $p$ , and if  $x \in B_{r(a)}(p)$  is the endpoint of the geodesic segment  $[x, p]$  that is perpendicular to  $C$  at  $p$ , then  $d(x, c) > d(x, p)$  for any  $c \in C \setminus \{p\}$ .*

(ii) *If there exists  $s < r(a)$  such that  $K(p) \cap B_s(p)$  is connected, then the normal injectivity radius of  $K(p)$  is  $\geq s/3$ .*

*Proof.* (i) The metric on  $X_\infty$  can be approximated in  $C^1$ -topology by smooth metrics with almost the same two-sided negative curvature bounds and infinite injectivity radius [28]. Also  $C^1$ -closeness of metrics implies  $C^0$ -closeness of Levi-Civita connections, and hence almost the same bounds on the second fundamental forms of  $C$ . Now, for the smooth metrics as above the assertion of (i) is well-known, and, after choosing a slightly smaller  $r(a)$ , it passes to the limits, so we also get it for  $X_\infty$ .

(ii) Consider two arbitrary geodesic segments of equal length  $\leq s/3$  that start at  $K(p)$ , are normal to  $K(p)$  and have the same endpoint. Since  $K$  acts isometrically on  $X_\infty$  and transitively on  $K(p)$ , we can assume that one of the segments starts at  $p$ . By the triangle inequality, the other segment starts at a point of  $C$ , so by part (i) the segments have to coincide. □

*Remark 6.3.* The proof that  $K(p) \cap B_s(p)$  is connected, for some  $s < r(a)$  independent of the converging subsequence  $(X, c(t_i), \Gamma)$ , occupies the rest of this section, and this is the only place in the paper where we use Bowditch’s theorem [6] that  $\Gamma$  is finitely generated. Other key ingredients are the existence of approximate square roots in finitely generated nilpotent groups (see Appendix A), and the following comparison lemma that relates the displacement of an element of  $\Gamma$  to the displacement of its square root.

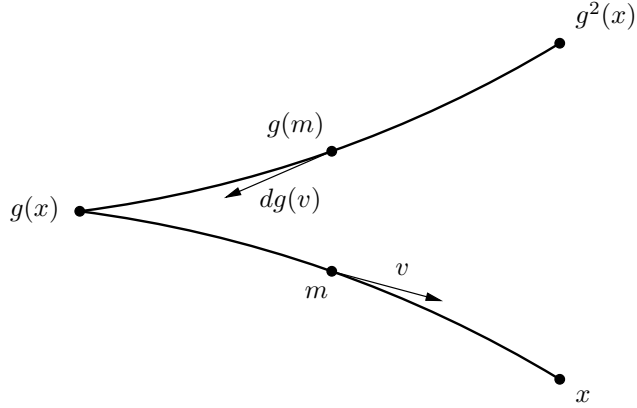


Figure 1.

LEMMA 6.4. *Let  $U$  be the neighborhood of  $1 \in O(n)$  that consists of all  $A \in O(n)$  satisfying  $|Av - v| < 1$  for any unit vector  $v \in \mathbf{R}^n$ . Then there exists a function  $f: (0, \infty) \rightarrow (0, \infty)$  such that  $f(r) \rightarrow 0$  as  $r \rightarrow 0$  and  $d(g(x), x) \leq f(d(g^2(x), x))$ , for any  $x \in X$  and any  $g \in \Gamma$  with  $\phi(g) \in U$ .*

*Proof.* Define  $f(r)$  to be the supremum of  $d(g(x), x)$  over all  $x \in X$  and  $g \in \Gamma$  with  $\phi(g) \in U$  satisfying  $r = d(x, g^2(x))$ .

To see that  $f(r) < \infty$ , take arbitrary  $x \in X$  and  $g \in \Gamma$  with  $r = d(x, g^2(x))$ , and let  $R = d(x, g(x))$ . Look at the geodesic triangle in  $X$  with vertices  $x$ ,  $g(x)$  and  $g^2(x)$  (see Figure 1). Arguing by contradiction, assume that by choosing  $g$  and  $x$  one can make  $R$  arbitrarily large while keeping  $r$  fixed. The geodesic triangle then becomes very long and thin. Let  $m$  be the midpoint of the geodesic segment  $[x, g(x)]$ , so that  $g(m)$  is the midpoint of the geodesic segment  $[g(x), g^2(x)]$ . By exponential convergence of geodesics and comparison with the hyperbolic plane of  $\text{sec} = -1$ , we get  $d(m, g(m)) \leq C(r)e^{-R/2}$ , which is small since  $R$  is large. So  $P_{g(m)m}^\infty$  is close to  $P_{g(m)m}$ , which in turn is close to  $P_{g(x)m} \circ P_{g(m)g(x)}$ , since the geodesic triangle with vertices  $m$ ,  $g(m)$  and  $g(x)$  has small area. Thus,  $\phi(g)$  is close to  $P_{g(x)m} \circ P_{g(m)g(x)} \circ dg$ . Let  $v$  be the unit vector tangent to  $[x, g(x)]$  at  $m$  and pointing towards  $x$ . Then  $dg(v)$  is tangent to  $[g(x), g^2(x)]$  at  $g(m)$  and is pointing towards  $g(x)$ . Since the geodesic triangle with vertices  $x$ ,  $g(x)$  and  $g^2(x)$  has small angle at  $g(x)$ , the map  $P_{g(x)m} \circ P_{g(m)g(x)}$  takes  $dg(v)$  to a vector that is close to  $-v$ . This gives a contradiction since  $|\phi(g)(v) - v| \leq 1$ .

A similar argument yields  $f(r) \rightarrow 0$  as  $r \rightarrow 0$ . Namely, if one can make  $d(g^2(x), x)$  arbitrarily small while keeping  $d(g(x), x)$  bounded below, then the geodesic triangle with vertices  $x$ ,  $g(x)$  and  $g^2(x)$  becomes thin, and we get a contradiction exactly as above.  $\square$

PROPOSITION 6.5. *Let  $r(a)$  be the constant of Corollary 6.2. Then there exists a positive  $s \leq r(a)$ , depending only on  $X$ ,  $c$  and  $\Gamma$ , such that for any converging sequence  $(X, c(t_i), \Gamma) \rightarrow (X_\infty, p, G)$ , the normal injectivity radius of  $K(p)$  is  $\geq s$ .*

*Proof.* By Corollary 6.2, it suffices to find a universal  $s$  such that  $K^s := K(p) \cap B_s(p)$  is connected. Let  $K_0^s$  be the component of  $K^s$  containing  $p$ .

By [6]  $\Gamma$  is finitely generated, hence, by Margulis' lemma [3],  $\Gamma$  contains a normal nilpotent subgroup  $\tilde{\Gamma}$  of index  $i \leq i(a, n)$ . Therefore  $H = \overline{\phi(\Gamma)}$  is virtually nilpotent. Hence, its identity component  $H_0$  is abelian, since compact connected nilpotent Lie groups are abelian. Then  $\phi^{-1}(H_0)$  is a subgroup of finite index in  $\Gamma$ . Let

$$\Gamma' = \tilde{\Gamma} \cap \phi^{-1}(H_0).$$

Clearly,  $[\Gamma : \Gamma'] = k = k(\phi)$  is also finite.

We first give a proof for the case  $\Gamma = \Gamma'$ . Arguing by contradiction, suppose that for any  $s > 0$  there exists a sequence  $(X, c(t_i), \Gamma) \rightarrow (X_\infty, p, G)$  and a point  $\gamma(p) \in K^s \setminus K_0^s$  with  $d(p, \gamma(p)) < s$ . By possibly making  $d(p, \gamma(p))$  smaller, we can choose  $\gamma(p)$  so that  $d(p, \gamma(p))$  is the distance from  $p$  to  $K^s \setminus K_0^s$ . By the first variation formula, the geodesic segment  $[p, \gamma(p)]$  is perpendicular to  $K_0^s$ . The next goal is to construct the square root of  $\gamma$  with no rotational part and displacement bounded by  $f(d(p, \gamma(p)))$ , where  $f$  is the function of Lemma 6.4.

Take  $\gamma_i \in \Gamma$  converging to  $\gamma$ . Since  $\Gamma$  is finitely generated, we can apply Lemma A.1 to find an (independent of  $i$ ) finite set  $F \subset \Gamma$  such that each  $\gamma_i$  can be written as  $\gamma_i = g_i^2 f_i$  with  $f_i \in F$ . We can further write each  $g_i$  as the product  $g_i = x_i r_i$ , where  $x_i$  has a small rotational part and  $r_i$  is close to a rotation, namely we let  $r_i$  be an element of  $\Gamma$  that is close to  $\phi(g_i)$  and let  $x_i = g_i r_i^{-1}$ . Thus  $\gamma_i = (x_i r_i)^2 f_i$  and

$$\gamma_i = x_i r_i x_i r_i f_i = x_i^2 [x_i^{-1} r_i] r_i^2 f_i.$$

Applying Lemma A.2 to  $x_i^2 [x_i^{-1} r_i]$  we see that  $\gamma_i$  can be written as  $(x_i h_i)^2 f_i' r_i^2 f_i$  with  $h_i \in [\Gamma, \Gamma]$  and  $f_i' \in F'$ . Since  $\phi(\Gamma)$  is abelian, we have  $\phi(h_i) = 1$  and hence  $\phi(x_i h_i) = \phi(x_i)$  is small. Since  $F$  and  $F'$  are independent of  $i$ , each element of  $F$  and of  $F'$  is close to a rotation for large  $i$ . So  $f_i' r_i^2 f_i$  is close to a rotation and since both  $\phi(\gamma_i)$  and  $\phi(x_i h_i)$  are small,  $f_i' r_i^2 f_i$  subconverges to the identity. Thus,  $(x_i h_i)^2$  subconverges to  $\gamma$ , and we might as well assume that  $\gamma_i = (x_i h_i)^2$  in the beginning. By Lemma 6.4,

$$d(c(t_i), x_i h_i(c(t_i))) \leq f(d(c(t_i), (x_i h_i)^2(c(t_i)))) ,$$

where the right-hand side converges to  $f(d(p, \gamma(p)))$ . Hence  $x_i h_i$  subconverges to  $w \in K$  such that  $w^2 = \gamma$  and  $d(p, w(p)) \leq f(d(p, \gamma(p)))$ .

Since  $w$  has no rotational part,  $P_{pw(p)}^\infty = dw_p$ . By assumption,  $s$  can be taken arbitrarily small, so we can assume that  $f(d(p, \gamma(p)))$  is small, in particular  $w(p) \in K^s$ . So  $P_{pw(p)}$  is close to  $dw_p$ . Hence, if  $v$  is the unit vector tangent to  $[p, w(p)]$  at  $p$  and pointing towards  $w(p)$ , then  $P_{pw(p)}(v)$  is close to  $dw_p(v)$ . Therefore,  $w(p)$  is close to the midpoint of  $[p, w^2(p)] = [p, \gamma(p)]$ . Since  $[p, \gamma(p)]$  is perpendicular to  $K_0^s$  and  $d(p, \gamma(p)) < r(a)$ , it is clear that  $w(p) \notin K_0^s$ . This contradicts the minimality of  $d(p, \gamma(p))$  and completes the proof in the case  $\Gamma = \Gamma'$ .

We now turn to the general case. Let  $G'$  be the subset of  $G$  that consists of limits of elements of  $\Gamma'$  under the convergence  $(X, c(t_i), \Gamma) \rightarrow (X_\infty, p, G)$ . It is straightforward to check that  $G'$  is a closed subgroup of  $G$  of index  $\leq k$ . Thus the limit of any converging subsequence of  $(X, c(t_i), \Gamma')$  has to be equal to  $(X_\infty, p, G')$ , therefore, in fact,  $(X, c(t_i), \Gamma')$  converges to  $(X_\infty, p, G')$ . Since the rotation homomorphism of  $G$  restricts to the rotation homomorphism of  $G'$ , the translational part  $K'$  of  $G'$  is  $G' \cap K$ . In particular,  $|K:K'| \leq k$ , hence the identity components of  $K$  and  $K'$  coincide. Using the first part of the proof, we fix  $s$  such that  $K'(p) \cap B_s(p)$  is connected. Thus  $K'(p) \cap B_s(p) = K_0^s$ .

Now let  $\gamma(p) \in K^s \setminus K_0^s$  be such that  $d(p, \gamma(p))$  is the distance from  $p$  to  $K^s \setminus K_0^s$ . Then the geodesic segment  $[p, \gamma(p)]$  is perpendicular to  $K_0^s$  by the first variation formula. Arguing by contradiction, suppose that  $d(p, \gamma(p))$  can be arbitrarily small. Then, by the triangle inequality,  $\gamma^j(p)$  is close to  $p$  for  $j=1, \dots, k$ . Also,  $\gamma^k \in K'$ , so in fact  $\gamma^k(p) \in K_0^s$  because  $K'(p) \cap B_s(p) = K_0^s$ .

On the other hand, since the  $\gamma^j$ 's have no rotational part, the argument used above to prove that  $w(p)$  is close to the midpoint of  $[p, w^2(p)]$  shows that the points  $\gamma^j(p)$  almost lie on a geodesic segment  $[p, \gamma^k(p)]$ . Then the segments  $[p, \gamma^k(p)]$  and  $[p, \gamma(p)]$  have almost the same direction, so  $[p, \gamma^k(p)]$  is almost perpendicular to  $K_0^s$ . Hence by Corollary 6.2, if  $d(p, \gamma(p))$  is small enough, then  $\gamma^k(p) \notin K_0^s$ , which is a contradiction.  $\square$

*Remark 6.6.* Although it is not needed for the proof of Theorem 1.1, note that all possible limits of  $(X/\Gamma, \sigma(t_i))$  have the same dimension independent of the sequence  $t_i \rightarrow \infty$ . Consider all possible limits with fixed  $\dim(K)$ , and look at the space  $X_\infty/K$ . It has a lower bound on the injectivity radius for points near the projection  $\bar{p}$  of  $p$ , and hence  $\text{vol}(B(\bar{p}, 1)) > c > 0$  in all such spaces, where  $c$  is independent of the converging sequence. By Proposition 5.2, the isotropy group  $G_p$  is the same for all possible limits and, moreover, the  $G_p$ -actions on  $T_p X_\infty$  are all equivalent. Also, by Lemma A.3, the identity component  $G_p^{\text{id}}$  of  $G_p$  commutes with the identity component of  $K$ , hence  $G_p^{\text{id}}$  fixes pointwise the component of  $K(p)$  containing  $p$ . Thus the  $G_p^{\text{id}}$ -actions on  $T_{\bar{p}} X_\infty/K$  are all equivalent. This implies that a unit ball in  $X_\infty/G$  has volume  $> c' > 0$  with  $c'$  only depending on  $\dim(K)$ . Hence all limits of the same dimension form a closed subset among all limits; therefore, the space of all limits is the union of these closed sets. On

the other hand, the space of all limits is connected by Lemma 6.7 below, thus all the limits have the same dimension.

LEMMA 6.7. *If  $\gamma: [0, \infty) \rightarrow Z$  is a continuous precompact curve in a metric space  $Z$ , then the space  $\text{Lim}(\gamma)$  of all possible subsequential limits  $\lim_{t_i \rightarrow \infty} \gamma(t_i)$  is connected.*

*Proof.* If  $\text{Lim}(\gamma)$  is not connected, then we can write it as a disjoint union of closed (and hence compact) sets  $\text{Lim}(\gamma) = A \sqcup B$ . Then  $U_\varepsilon(A) \cap U_\varepsilon(B) = \emptyset$  for some  $\varepsilon > 0$ , where  $U_\varepsilon(S)$  denotes the  $\varepsilon$ -neighborhood of  $S$ . Let  $\gamma(t_i) \rightarrow a \in A$  and  $\gamma(t'_i) \rightarrow b \in B$ . Arguing by contradiction, we see that the curve  $\gamma|_{[t_i, t'_i]}$  lies in  $U_\varepsilon(\text{Lim}(\gamma)) = U_\varepsilon(A) \cup U_\varepsilon(B)$  for all large  $i$ . Clearly,  $\gamma(t_i) \in U_\varepsilon(A)$  and  $\gamma(t'_i) \in U_\varepsilon(B)$  for all large  $i$ , which contradicts  $U_\varepsilon(A) \cap U_\varepsilon(B) = \emptyset$ .  $\square$

### 7. Product structure at infinity

In the next two sections we apply the critical point theory for distance functions to show the following result.

THEOREM 7.1. *For each large  $t$ , the horosphere quotient  $H_t/\Gamma$  is diffeomorphic to the normal bundle of an orbit of an  $N$ -structure on  $H_t/\Gamma$ .*

*Proof.* Let  $\sigma(t)$  be the projection of  $c(t)$  to  $X/\Gamma$ . Given a converging sequence  $(X, c(t_i), \Gamma) \rightarrow (X_\infty, p, G)$ , the sequence of pointed Riemannian manifolds  $(X/\Gamma, \sigma(t_i))$  converges in pointed Gromov–Hausdorff topology to a pointed Alexandrov space  $(Y, q) := (X_\infty/G, q)$  with curvature bounded below by  $-a^2$ .

The identity component  $K_{\text{id}}$  of  $K$  is normal in  $K$ , hence its  $p$ -orbit  $K_{\text{id}}(p)$  is invariant under the action of  $G_p$ . Let  $r \ll s$  be a positive constant to be determined later, where  $s$  comes from Proposition 6.5. The  $3r$ -tubular neighborhood of  $K_{\text{id}}(p)$  is also  $G_p$ -invariant, so the ball  $B_{3r}(q)$  is isometric to the  $G$ -quotient of this tubular neighborhood. By Proposition 6.5, any  $x \in B_{3r}(q)$  can be joined to  $q$  by a unique shortest geodesic segment  $[q, x] \subset B_{3r}(q)$ .

Recall that in general a distance function  $d(\cdot, q)$  on an Alexandrov space is called *regular at the point  $x$*  if there exists a segment emanating from  $x$  that forms an angle  $> \pi/2$  with any shortest segment joining  $x$  to  $q$ .

In our case, the function  $d(\cdot, q)$  is regular at any  $x \in B_{3r}(q) \setminus \{q\}$ .

Let  $w(a) > 1$  be a constant depending only on  $a$  that will be specified later. By angle comparison, the function  $d(\cdot, \sigma(t))$  on

$$A_r(\sigma(t)) = \{x \in B_s(\sigma(t)) : d(x, \sigma(t)) \in [r/w(a), w(a)r]\}$$

is regular provided the Gromov–Hausdorff distance between  $B_s(q)$  and  $B_s(\sigma(t))$  is  $\ll r/w(a)$ . Because the family  $\{B_s(\sigma(t))\}$  is precompact in the Gromov–Hausdorff topology, Proposition 6.5 implies that the function  $d(\cdot, q)$  is regular on  $A_r(\sigma(t))$  for all  $t \geq t_0$  with sufficiently large  $t_0$ .

We denote by  $H_t$  the horosphere centered at  $c(\infty)$  that contains  $c(t)$ . Since the second fundamental form of  $H_t$  is bounded in terms of  $a$ , any short segment joining nearby points of  $H_t/\Gamma$  is almost tangent to  $H_t/\Gamma$ . Hence, by taking  $r$  sufficiently small, we can assume that for all  $t \geq t_0$  and all  $x \in A_r(\sigma(t))$  there exists a unit vector  $\lambda \in T_x(H_t/\Gamma)$  that forms an angle  $\alpha_{\lambda, [x, \sigma(t)]} \in [\frac{2}{3}\pi, \pi]$  with any shortest segment  $[x, \sigma(t)]$ . By the first variation formula, the derivative of  $d(\cdot, q)$  in the direction of  $\lambda$  equals the minimum of  $-\cos \alpha_{\lambda, [x, \sigma(t)]}$ , over all shortest segments  $[x, \sigma(t)]$ , and by above it lies in  $[\frac{1}{2}, 1]$ .

The distance function on  $X/\Gamma$  need not be smooth, and for what follows it is convenient to replace  $d(\cdot, \sigma(t))$  by its average over a small ball  $B_\delta(\sigma(t))$  as follows. Given  $\delta \ll r$ , define  $f: X/\Gamma \rightarrow \mathbf{R}$  by

$$f(x) = \frac{1}{\text{vol}B_\delta(\sigma(t))} \int_{B_\delta(\sigma(t))} d(x, y) dy,$$

where  $x \in H_t/\Gamma$  (i.e.  $t=b(x)$ ). Now,  $f$  is a  $C^1$  1-Lipschitz function with

$$|f(x) - d(x, \sigma(t))| \leq \delta$$

for any  $x \in A_r(\sigma(t))$ . Observe that, for any  $\eta \in T_x(X/\Gamma)$ ,

$$df_x(\eta) = \frac{1}{\text{vol}B_\delta(\sigma(t))} \int_{B_\delta(\sigma(t))} (-\cos \alpha_{\eta, [x, y]}) dy. \tag{7.2}$$

Also note that, since  $\delta \ll r$  and  $\sec(X/\Gamma) \geq -a^2$ , for all large  $t$ , if  $x \in A_r(\sigma(t))$  and  $y \in B_\delta(\sigma(t))$ , then there is a point  $z$  such that  $d(z, x) \approx d(x, y)$  and  $d(z, y) \approx 2d(x, y)$ . Therefore, the angle corresponding to  $x$  in the comparison triangle in the space of  $\sec \equiv -a^2$  is almost  $\pi$ . By Toponogov comparison, the angle at  $x$  in any geodesic triangle  $\triangle xyz$  is almost  $\pi$ . By (7.2), this implies that if  $\eta$  is a direction of any shortest segment connecting  $x$  to  $z$ , then  $df_x(\eta) \in [\frac{1}{2}, 1]$  provided  $\delta$  is small enough.

By gluing  $\lambda$ 's via a partition of unity, we obtain a  $C^1$  unit vector field  $\Lambda$  that is tangent to  $H_t/\Gamma$  and defined for all  $t \geq t_0$  and  $x \in A_r(\sigma(t))$ , and such that  $df_x(\Lambda) \in [\frac{1}{4}, 1]$  if  $\delta$  is sufficiently small. Then,  $S_r = \{x \in X/\Gamma : f(x) = r\}$  is a properly embedded  $C^1$ -hypersurface in  $X/\Gamma$  that is transverse to  $\Lambda$ . Also, the compact submanifold  $S_t(t) := S_r \cap H_t/\Gamma$  is  $\delta$ -close to the metric  $r$ -sphere in  $H_t/\Gamma$  centered at  $\sigma(t)$ . Furthermore,  $A(r, t) := A_r(\sigma(t)) \cap H_t/\Gamma$  is  $C^1$ -diffeomorphic to the product  $S_r(t) \times [r/w(a), w(a)r]$ .



Here we are only interested in the part of  $X/\Gamma$  with  $t \geq t_0$ . There, the Busemann function  $X/\Gamma \rightarrow \mathbf{R}$  restricts to a  $C^1$ -submersion  $S_r \rightarrow [t_0, \infty)$ , because otherwise at some point the tangent spaces of  $S_r$  and  $H_t/\Gamma$  coincide by dimension reasons, so that  $S_r$  cannot be transverse to  $\Lambda \in T(H_t/\Gamma)$ . By construction, the submersion is proper, hence it is a  $C^1$ -fiber bundle, which is  $C^1$ -trivial by the covering homotopy theorem. The trivialization defines a  $C^1$ -isotopy  $F: S_r(t_0) \times [t_0, \infty) \rightarrow X/\Gamma$  such that  $S_r(t_0) \times \{t\}$  is mapped onto  $S_r(t)$ .

We push this isotopy along the Busemann flow back into  $H_{t_0}/\Gamma$  by setting

$$G(t, x) = b_{t_0-t}(F(x, t))$$

for any  $t \geq t_0$  and  $x \in S_r(t_0)$ , to get the  $C^1$ -isotopy  $G: S_r(t_0) \times [t_0, \infty) \rightarrow H_{t_0}/\Gamma$ .

The Busemann flow induces a  $C^1$ -diffeomorphism  $H_t/\Gamma \rightarrow H_{t_0}/\Gamma$ , so around each submanifold  $b_{t_0-t}(S_r(t))$  there is a ‘‘tubular neighborhood’’  $b_{t_0-t}(A(r, t))$ .

By the exponential convergence of geodesics, one can choose  $w(a)$  in the definition of  $A_r(\sigma(t))$  so that, for any  $t \geq t_0$ , there exists  $t' \geq t+1$  such that  $b_{t_0-t'}(S_r(t'))$  is contained in  $b_{t_0-t}(A(r, t))$  and is disjoint from  $b_{t_0-t}(S_r(t))$ . By the following elementary lemma, the region between  $b_{t_0-t}(S_r(t))$  and  $b_{t_0-t'}(S_r(t'))$  is  $C^1$ -diffeomorphic to  $S_r(t) \times [0, 1]$ .

**LEMMA 7.3.** *Let  $M$  be a closed smooth manifold and  $F_t: M \rightarrow M \times \mathbf{R}$  be a  $C^1$ -isotopy with  $F_0(M) = M \times \{0\}$ . If  $F_0(M)$  and  $F_s(M)$  are disjoint for some  $s$ , then the region between  $F_0(M)$  and  $F_s(M)$  is diffeomorphic to  $M \times [0, 1]$ .*

*Proof.* By the isotopy extension theorem [12, p. 293], we can extend the isotopy  $F_t$  to an ambient  $C^1$ -isotopy which is the identity outside a compact subset of  $M \times \mathbf{R}$ . Assume, without loss of generality, that  $s < 0$ , and then take  $n > 0$  so large that the isotopy is the identity on  $M \times \{n\}$ . Then, by restricting the ambient isotopy to the region between  $M \times \{n\}$  and  $M \times \{0\}$ , we get a diffeomorphism of the region between  $M \times \{n\}$  and  $M \times \{0\}$  onto the region between  $M \times \{n\}$  and  $F_s(M)$ . The former region is the product, so is the latter. But the latter region is diffeomorphic to the region between  $M \times \{0\}$  and  $F_s(M)$ , because the region between  $M \times \{n\}$  and  $M \times \{0\}$  is  $M \times [0, 1]$ .  $\square$

By gluing a countable number of such diffeomorphisms together we conclude that, for all sufficiently large  $t$ ,

$$(H_t/\Gamma) \setminus U(r, t) \text{ is } C^1\text{-diffeomorphic to } [t, \infty) \times S_r(t), \tag{7.4}$$

where  $U(r, t) = \{x \in H_t/\Gamma : f(x) < r\}$ .

**8. Tubular neighborhood of an orbit**

It remains to understand the topology of  $U(r, t)$ , and we do so for large enough  $t$  and small enough  $r$ . The proof involves the collapsing theory developed in [14] and the geometry of Alexandrov spaces (for which we refer to [9] and Appendix D).

Let us look at a converging sequence  $(H_{t_i}/\Gamma, \sigma(t_i)) \rightarrow (H, q)$ . First, we replace the metric on  $H_{t_i}/\Gamma$  with an invariant Riemannian metric which is  $\varepsilon_i$  close to  $H_{t_i}/\Gamma$  in  $C^1$ -topology and is  $A(\varepsilon_i)$ -regular [37], [14], [28], where  $\varepsilon_i \rightarrow 0$  as  $i \rightarrow \infty$ . Also, the spaces  $H_{t_i}/\Gamma$  with the new metrics have uniform curvature bound  $|\text{sec}| \leq C'$  [37] that depends only on the original curvature bound of  $H_{t_i}/\Gamma$ . The collapsing theory [14] yields, for each  $i$ , the following commutative diagram given by the invariant metric  $h_i$  on  $H_{t_i}/\Gamma$ :

$$\begin{array}{ccc} FB_i & \xrightarrow{\eta_i} & Y_i \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{\bar{\eta}_i} & X_i. \end{array}$$

Here  $B_i$  is the ball  $B(\sigma(t_i), 1)$  in the metric  $h_i$ , and  $FB_i$  is the frame bundle of  $B_i$ . The vertical arrows are quotient maps under isometric  $O(n)$ -actions and  $\eta_i$  is a Riemannian submersion given by the  $N$ -structure on  $FB_i$ . Clearly the induced map  $\bar{\eta}_i$  is a submetry (see Appendix D for background on submetries). By Lemma D.3, the Toponogov comparison with  $\text{curv} \geq -C'$  holds for any triangle with vertices in  $B(\bar{\eta}_i(\sigma(t_i)), \frac{1}{4})$  for all large  $i$ .

Since we will only be interested in the geometry of  $X_i$  inside the  $\frac{1}{8}$ -neighborhood of  $\bar{\eta}_i(\sigma(t_i))$ , we will treat the  $X_i$ 's as Alexandrov spaces.

Note that  $X_i \xrightarrow{\text{G-H}} \bar{X} = B(q, 1)$  and  $\dim X_i = \dim \bar{X}$  for all large  $i$ . We claim that there exists an  $R > 0$  and a sequence  $q_i \in X_i$  converging to  $q$  such that  $d(\cdot, q_i)$  has no critical points in  $B(q_i, R)$  for all large  $i$ . Consider two cases depending on whether  $q$  lies on the boundary of the Alexandrov space  $\bar{X}$ .

*Case 1.* Suppose  $q \notin \partial \bar{X}$ . Since  $\bar{X}$  has  $\text{curv} \geq -C'$  in comparison sense, by [21], there exists a strictly concave function  $u$  on a  $B(q, R)$ , for some  $R \ll 1$ , such that it has a maximum at  $q$ , and the superlevel sets are compact. By possibly making  $R$  smaller, we can assume that  $R < d(q, \partial \bar{X})$ . This function is constructed by taking averages and minima of distance functions. Therefore, it naturally lifts to a function  $u_i$  on  $X_i$  such that  $u_i$  converges uniformly to  $u$ . By [21, Lemma 4.2], the lifts  $u_i$  are strictly concave on  $B(\bar{\eta}_i(\sigma(t_i)), R/2)$  for all large  $i$ . Let  $q_i$  be the point of maximum of  $u_i$ . By uniqueness of the maximum,  $q_i \rightarrow q$  as  $i \rightarrow \infty$ . By Lemma D.1,  $d(\cdot, q_i)$  has no critical points in  $B(q_i, R/3)$  for all large  $i$ .

*Case 2.* Suppose now that  $q \in \partial \bar{X}$ . Denote by  $D\bar{X}$  and  $DX_i$  the doubles of  $\bar{X}$  and  $X_i$  along the boundary and let  $\iota$  be the canonical involution. By [30], the doubles are also Alexandrov spaces with  $\text{curv} \geq -C'$ . It is clear that  $DX_i \xrightarrow{\text{G-H}} D\bar{X}$ . By construction, we can choose  $u$  and  $u_i$  to be  $\iota$ -invariant. As before, let  $q_i$  be the point of maximum of  $u_i$ . Since  $u_i(q_i) = u_i(\iota(q_i))$ , by uniqueness of maxima of strictly concave functions, we see that  $q_i$  must lie on  $\partial X_i$ . Again by Lemma D.1,  $d(\cdot, q_i)$  has no critical points in  $B(q_i, R/3)$  in  $DX_i$  for all large  $i$ , and hence the same is true for  $d(\cdot, q_i)$  in  $B(q_i, R/3)$ .

This immediately implies that the distance function  $d(\cdot, O_i)$  to the orbit  $O_i$  over  $q_i$  has no critical points in the  $R/3$ -neighborhood  $U_{R/3}(O_i)$  of  $O_i$  for all large  $i$ . Indeed, let  $x \in U_{R/3}(O_i) \setminus O_i$  and let  $\gamma(t)$  be a geodesic starting at  $\bar{\eta}_i(x)$  such that  $\frac{d}{dt}d(\gamma(\cdot), q_i)'|_{t=0} > 0$ . Since  $\bar{\eta}_i: (H(t_i)/\Gamma, h_i) \rightarrow X_i$  is a submetry, there exists a horizontal lift  $\tilde{\gamma}$  of  $\gamma$  starting at  $x$ . Then  $d(\tilde{\gamma}(t), O_i) = d(\gamma(t), q_i)$ , and hence  $d(\tilde{\gamma}(t), O_i)'|_{t=0} = d(\gamma(t), q_i)'|_{t=0} > 0$ .

Therefore  $U_r(O_i)$  is diffeomorphic to the total space of the normal bundle to  $O_i$  in  $H_{t_i}/\Gamma$  for any  $r \leq R/3$ . Since  $O_i$  is Hausdorff close to  $\sigma(t_i)$ , the same is true for  $U(r, t_i)$ .

Combining this with (7.4), we conclude that  $H_{t_i}/\Gamma$  is diffeomorphic to the total space of the normal bundle to  $O_i$  in  $H_{t_i}/\Gamma$  for all sufficiently large  $i$ . Finally, since the above proof works for any sequence  $t_i \rightarrow \infty$ , arguing by contradiction we conclude that  $H_t/\Gamma$  is diffeomorphic to the normal bundle of an orbit of an  $N$ -structure for all sufficiently large  $t$ . This completes the proof of Theorem 7.1. □

*Remark 8.1.* The reader may be wondering why we work with the Alexandrov spaces  $X_i$  instead of the Riemannian manifolds  $Y_i$ . This is because the curvature of  $Y_i$  may tend to  $\pm\infty$  as  $i \rightarrow \infty$ , which makes it hard to control the geometry of the  $Y_i$ 's. If instead of  $\varepsilon_i \rightarrow 0$ , we take  $\varepsilon_i$  equal to a small positive constant  $\varepsilon$ , then  $|\text{sec}(Y_i)| \leq C(\varepsilon)$ ; but then it may happen that the injectivity radius of the Gromov–Hausdorff limit of the  $Y_i$ 's is  $\ll \varepsilon$ , so we cannot translate the lower bound on the injectivity radius from  $Y_i$  to  $H_{t_i}/\Gamma$ .

*Remark 8.2.* By Theorem 7.1, each orbit  $O_{q_i}$  as above is homotopy equivalent to  $X/\Gamma$ . Thus all the  $O_{q_i}$ 's are homotopy equivalent, hence they are all affinely diffeomorphic (see e.g. [39, Theorem 2]).

### 9. The normal bundle is flat

**THEOREM 9.1.** *For each large  $t$ , the horosphere quotient  $H_t/\Gamma$  admits an  $N$ -structure that has an orbit  $O_t$  such that the normal bundle to  $O_t$  is a flat Euclidean vector bundle with total space diffeomorphic to  $H_t/\Gamma$ .*

Arguing by contradiction, it suffices to prove the theorem for any sequence  $t_i \rightarrow \infty$  such that  $H_{t_i}/\Gamma$  converges in pointed Gromov–Hausdorff topology. We fix such a sequence and assume for the rest of the proof that  $t$  belongs to the sequence.

We denote by  $g_t$  the  $C^1$ -Riemannian metric on the horosphere quotient  $H_t/\Gamma$  induced by the ambient metric  $(M, g)$ . Fix a small positive  $\varepsilon > 0$  to be determined later; this constant will only depend on  $(M, g)$ . By Theorem 7.1, [14] and [28], for all large  $t$  there exists an  $N$ -structure on  $H_t/\Gamma$  with an orbit  $O_t$  such that the normal bundle to  $O_t$  is diffeomorphic to  $H_t/\Gamma$ . Also all orbits of the  $N$ -structure have diameter  $< \varepsilon$  with respect to an invariant metric  $h_t$  that is  $\varepsilon$ -close to  $g_t$  in uniform  $C^1$ -topology, i.e.  $|g_t - h_t| < \varepsilon$  and  $|\nabla_{g_t} - \nabla_{h_t}| < \varepsilon$ . It remains to show that for all large  $t$ , the normal bundle to  $O_t$  in  $H_t/\Gamma$  is flat Euclidean. The proof breaks into two independent parts.

In §9.1 we find a stratum  $F_t$  of the  $N$ -structure on  $H_t/\Gamma$  such that  $F_t$  is an  $h_t$ -totally-geodesic closed submanifold that contains  $O_t$  with flat normal bundle. This uses only general properties of  $N$ -structures.

In §9.2 we show that the restriction to  $O_t$  of the normal bundle of  $F_t$  in  $H_t/\Gamma$  is flat, for all large  $t$ . This uses the flat connection of §4 and the fact that  $O_t \rightarrow H_t/\Gamma$  is a homotopy equivalence.

### 9.1. The normal bundle in a stratum is flat

Throughout §9.1 we suppress the index  $t$ , writing  $O$  in place of  $O_t$ , etc. Let  $V$  be the tubular neighborhood of  $O$  that is sufficiently small so that all orbits in  $V$  have dimension  $\geq \dim(O)$ . Let  $\tilde{O}$  and  $\tilde{V}$  be their universal covers. According to [14, pp.364–365], the group  $\text{Iso}(\tilde{V})$  contains a connected (but not necessarily simply-connected) nilpotent subgroup  $N$  that stabilizes  $\tilde{O}$ , acts transitively on  $\tilde{O}$ , and also  $\Lambda = N \cap \pi_1(V)$  is a finite index subgroup of  $\pi_1(V)$  and is a lattice in  $N$ . The following lemma is implicit in [14].

**LEMMA 9.2.** *The subgroup  $H$  of  $\text{Iso}(\tilde{V})$  generated by  $N$  and  $\pi_1(V)$  is closed,  $N$  is the identity component in  $H$ , and the index of  $N$  in  $H$  is finite.*

*Proof.*  $\Lambda$  is a cocompact discrete subgroup of  $N$  and also of its closure  $\bar{N}$  in  $\text{Iso}(\tilde{V})$ . Since  $\dim(N)$  and  $\dim(\bar{N})$  are both equal to the cohomological dimension of  $\Lambda$ , we get  $N = \bar{N}$ . Now let  $\Lambda_0$  be a maximal finite index *normal* subgroup of  $\pi_1(V)$  that is contained in  $\Lambda$ . If  $\gamma \in \pi_1(V)$ , then  $N \cap \gamma N \gamma^{-1}$  contains  $\Lambda_0$  as a cocompact discrete subgroup, thus, as before,  $\dim(N)$  and  $\dim(N \cap \gamma N \gamma^{-1})$  are both equal to the cohomological dimension of  $\Lambda_0$ , so  $N = N \cap \gamma N \gamma^{-1}$ , and  $N$  is normalized by  $\pi_1(V)$ . Thus  $N$  is normal in  $H$ , and  $\Lambda = \Lambda_0$ . Since  $N$  is connected, it remains to show that  $|H:N|$  is finite. Since  $N$  and  $\pi_1(V)$  generate  $H$ , the finite subgroup  $\pi_1(V)/\Lambda$  of  $H/N$  generates  $H/N$ , hence  $\pi_1(V)/\Lambda = H/N$ .  $\square$

Let  $I$  be the intersection of the isotropy subgroups of  $H$  of the points of  $\tilde{O}$ . Since  $\tilde{O}$  is  $H$ -invariant,  $I$  is normal in  $H$ . The fixed point set of  $I$  is a totally geodesic submanifold  $\tilde{F}$  of  $\tilde{V}$ . The  $H$ -action on  $\tilde{F}$  descends to an  $H/I$ -action on  $\tilde{F}$ . Since  $\pi_1(V)$  is torsion free and discrete,  $\pi_1(V) \cap I$  is trivial, and we identify  $\pi_1(V)$  with its image in  $H/I$ . Denote the projection of  $\tilde{F}$  into  $V$  by  $F$ .

LEMMA 9.3. *The normal bundle to  $O$  in  $F$  is flat.*

*Proof.* The group  $N$  acts transitively on  $\tilde{O}$ , so all isotropy subgroups for the  $N$ -action on  $\tilde{O}$  are conjugate. Since they are also compact, they lie in the center of  $N$  [20]; in particular, all the isotropy subgroups are equal, and hence each of them is equal to  $I \cap N$ . In particular,  $N/(I \cap N)$  acts freely and transitively on  $\tilde{O}$ . Since  $\tilde{O}$  is simply-connected, so is  $N/(I \cap N)$ . Thus,  $I \cap N$  is the maximal compact subgroup of  $N$ ; hence, by Lemma A.3,  $I \cap N$  is a torus, which we denote by  $T$ . The torus is the identity component of the compact group  $I$ , because  $|I \cap N| \leq |H : N| < \infty$ . Since  $N/T$  acts freely and transitively on  $\tilde{O}$ , we can choose a trivialization of  $\nu$ , the normal bundle of  $F$  in  $\tilde{V}$  that is invariant under the left translations by  $N/T$ . Namely, let  $e \in \tilde{O}$  be the point corresponding to  $1 \in N/T$  under the diffeomorphism  $N/T \cong \tilde{O}$ . Fix an isomorphism  $\phi: \nu_e \rightarrow \{e\} \times \mathbf{R}^k$ , and then extend it to the  $N/T$ -left-invariant isomorphism  $\nu \cong \tilde{O} \times \mathbf{R}^k$ . Now, take  $\gamma \in \pi_1(V)$  and  $x \in \tilde{O}$ . Using the above trivialization we define the *rotational part* of  $\gamma$  as the automorphism of  $\{e\} \times \mathbf{R}^k$  given by

$$\phi \circ dL_{\gamma(x)^{-1}} \circ d\gamma \circ dL_x \circ \phi^{-1},$$

where  $dL_x$  is the differential of the left translation by  $x \in N/T$ . Since  $O$  is an infranilmanifold,  $\pi_1(V)$  acts on  $\tilde{O}$  by affine transformations, that is if  $y \in \tilde{O}$ , then  $\gamma(y) = n_\gamma \cdot A_\gamma(y)$ , where  $n_\gamma \in N/T$  and  $A_\gamma$  is a Lie group automorphism of  $N/T$ . Hence, for  $z \in \tilde{O}$ , we get

$$(L_{\gamma(x)^{-1}} \circ \gamma \circ L_x)(z) = \gamma(x)^{-1} \cdot \gamma(xz) = A_\gamma(x)^{-1} \cdot n_\gamma^{-1} \cdot n_\gamma \cdot A_\gamma(xz) = A_\gamma(z) = L_{n_\gamma^{-1}} \circ \gamma(z),$$

where the third equality holds as  $A$  is an automorphism and  $N/T = \tilde{O}$ . This establishes the above equality only on  $\tilde{O}$ , not on  $\tilde{F}$ , but both sides of the equality make sense as elements of  $H/I$ , and since  $\tilde{F}$  is the fixed point set of  $I$ , any two elements of  $H/I$  that coincide on  $\tilde{O}$  must coincide on  $\tilde{F}$ . Now the right-hand side is independent of  $x$ , which implies that the rotational part of  $\gamma$  is independent of  $x$ . This means that the bundle  $(\tilde{O} \times \mathbf{R}^k) / \pi_1(V)$  is a flat  $O(k)$ -bundle, and hence so is the normal bundle of  $F$  in  $V$ .  $\square$

### 9.2. The normal bundle to a stratum is flat

Let  $F_t$  be the stratum from §9.1.

LEMMA 9.4. *For all large  $t$ , the restriction to  $O_t$  of the normal bundle to  $F_t$  in  $H_t/\Gamma$  is flat.*

*Proof.* Let  $\nu_t$  be the restriction to  $O_t$  of the normal bundle to  $F_t$  in  $H_t/\Gamma$ . By an obvious contradiction argument, it suffices to show that any subsequence of  $\{t_i\}_i$  has a subsequence for which the  $\nu_t$ 's are flat; thus passing to subsequences during the proof causes no loss of generality. Since  $k_t = \dim(\nu_t)$  can take only finitely many values, we pass to a subsequence for which  $k_t$  is constant; we then denote  $k_t$  by  $k$ . We look at the Grassmanian  $G^k(TM)$  of  $k$ -planes in  $TM$  with the metric induced by  $g$ . Of course, for  $k$ -planes tangent to  $H_t/\Gamma$  the metric is also induced by  $g_t$ . We fix a point  $o_t \in O_t$  and denote by  $G_t^k$  the fiber of  $G^k(TM)$  over  $o_t$ . Let  $l_t$  be the fiber of  $\nu_t$  over  $o_t$ .

The fibers of  $G^k(TM)$  are (noncanonically) pairwise isometric via the Levi-Civita parallel transport of  $(M, g)$ . Let  $\varrho \ll \text{diam}(G_t^k)$  be such that the center of mass of [22] is defined in any  $4\varrho$ -ball of  $G_t^k$ . For large enough  $t$ , the ball  $B(l_t, 4\varrho) \subset G_t^k$  contains no  $k$ -plane tangent to  $F_t$  because  $\varrho \ll \text{diam}(G_t^k)$ , whereas  $l_t$  and  $TF_t$  are  $h_t$ -orthogonal, and  $h_t$  and  $g_t$  are  $\varepsilon$ -close, so that the distance between  $l_t$  and any  $k$ -plane in  $TF_t$  is within  $\varepsilon$  of  $\text{diam}(G_t^k)$ .

Denote by  $\nabla^\infty$  the connection from §4 associated with the parallel transport  $P^\infty$ . Also denote by  $\nabla_t$  the Levi-Civita connection of  $h_t$ , and let  $P_t$  be its parallel transport. The second fundamental form of  $H_t$  is uniformly bounded, hence  $P^\infty$  and  $P_t$  are close over any short loop in  $H_t/\Gamma$ .

Since  $\nabla^\infty$  is flat and compatible with the metric  $g$ , it defines the holonomy homomorphism  $\phi_t: \Gamma \rightarrow \text{Iso}(T_{o_t}M)$ . Let  $R_t$  be the closure of  $\phi_t(\Gamma)$  in  $\text{Iso}(T_{o_t}M)$ . Since  $R_t$  is compact, there exists a finite subset  $S_t \subset \Gamma$  so that for each  $r_t \in R_t$  there exists  $s \in S_t$  such that  $r_t(l_t)$  and  $\phi_t(s)(l_t)$  are  $\varrho$ -close in  $G_t^k$ . The direction orthogonal to  $H_t/\Gamma$  is  $\nabla^\infty$ -parallel, so the  $R_t$ -action preserves the subspace  $TH_t/\Gamma \subset TM$ . Also  $P^\infty$  defines an isometry between the  $G_t^k$ 's that is  $\phi_t$ -equivariant and  $R_t$ -equivariant. Thus  $S_t$  can be chosen independently of  $t$ , and we denote  $S_t$  by  $S$ .

If  $t$  is sufficiently large, then, for every  $s \in S$ , the  $k$ -planes  $\phi_t(s)(l_t)$  and  $l_t$  are  $\varrho$ -close in  $G_t^k$ . Indeed, by the exponential convergence of geodesics, if  $t$  is large, then  $s$  can be represented by a short loop based at  $o_t$ . Since the inclusion  $O_t \hookrightarrow H_t/\Gamma$  induces a homotopy equivalence, the loop can be assumed to lie on  $O_t \subset F_t$ . Since  $F_t$  is  $h_t$ -totally geodesic and  $l_t$  is orthogonal to  $F_t$  and has complementary dimension,  $l_t$  is fixed by  $P_t$  along any loop in  $F_t$  based at  $o_t$ . Hence the same is almost true for  $P^\infty$ , provided the loop is short enough, which proves the claim since  $S$  is finite.

Thus, by the triangle inequality in  $G_t^k$ , the  $k$ -planes  $r_t(l_t)$  and  $l_t$  are  $2\varrho$ -close for all  $r_t \in R_t$ . Now let  $\bar{l}_t$  be the center of mass of all  $r_t(l_t)$ 's with  $r_t \in R_t$ . Clearly,  $\bar{l}_t$  is  $2\varrho$ -close to  $l_t$ , hence  $\bar{l}_t$  is transverse to  $TF_t$ . Since  $P^\infty$  preserves  $TH_t/\Gamma$ , each  $r_t(l_t)$ ,

and hence  $\bar{l}_t$  is tangent to  $H_t/\Gamma$ . Now we translate  $\bar{l}_t$  around  $O_t$  using  $P^\infty$  along paths of length  $\leq \text{diam}(O_t) < \varepsilon$ . Since  $\bar{l}_t$  is  $R_t$ -invariant and  $\nabla^\infty$  is flat, this gives a well-defined  $k$ -dimensional flat  $C^0$  subbundle  $\bar{\nu}_t$  of the restriction of  $TH_t/\Gamma$  to  $O_t \subset F_t$ .

Note that  $P_t$  takes any  $k$ -plane tangent to  $F_t$  to a  $k$ -plane tangent to  $F_t$ , since  $F_t$  is totally geodesic. On the other hand,  $\bar{l}_t$  is a uniform distance away from any  $k$ -plane in  $TF_t$ , so its  $P_t$ -image remains far away from any  $k$ -plane in  $TF_t$ . Since on short paths  $P_t$  is close to  $P^\infty$ , we conclude that  $\bar{\nu}_t$  is transverse to  $TF_t$  for large  $t$ . Thus  $\bar{\nu}_t$  is  $C^0$ -isomorphic to  $\nu_t$ , so that  $\nu_t$  carries a  $C^0$ -flat connection.

In fact,  $\nu_t$  also carries a smooth flat connection. Indeed, in the universal cover  $\tilde{O}_t$ , the  $C^0$ -flat connection defines a  $\Gamma$ -equivariant  $C^0$ -isomorphism of the pullback of  $\nu_t$  to  $\tilde{O}_t$  onto  $\tilde{O}_t \times \mathbf{R}^k$ , where  $\Gamma$  acts as the covering group on the  $\tilde{O}_t$ -factor and via a holonomy homomorphism on the  $\mathbf{R}^k$ -factor. This is a smooth action, so the quotient  $(\tilde{O}_t \times \mathbf{R}^k)/\Gamma$  is a smooth flat vector bundle that is  $C^0$ -isomorphic to  $\nu_t$ . But any  $C^0$ -isomorphic bundles are smoothly isomorphic because the continuous homotopy of classifying maps can be approximated by a smooth homotopy. □

### 10. Infrnilmanifolds are horosphere quotients

Z. Shen constructed in [36] a pinched negatively curved warped product metric on the product of an arbitrary infrnilmanifold and  $(0, \infty)$  so that the metric is complete near the  $\infty$ -end, but is incomplete at the 0-end. Here we modify Shen’s construction to produce a complete pinched negatively curved metric on the product of any infrnilmanifold with  $\mathbf{R}$ .

Let  $G$  be a simply-connected nilpotent Lie group acting on itself by left translations, and let  $K$  be a compact subgroup of  $\text{Aut}(G)$ , so that the semidirect product  $G \rtimes K$  acts on  $G$  by affine transformations. Taking product with the trivial  $G \rtimes K$ -action on  $\mathbf{R}$ , we get a  $G \rtimes K$ -action on  $G \times \mathbf{R}$  for which we prove the following result.

**THEOREM 10.1.**  *$G \times \mathbf{R}$  admits a complete  $G \rtimes K$ -invariant Riemannian metric of pinched negative curvature. In particular, if  $N$  is an infrnilmanifold, then  $N \times \mathbf{R}$  carries a complete metric of pinched negative curvature.*

*Proof.* The Lie algebra  $L(G)$  can be written as

$$L(G) = L_0 \supset L_1 \supset \dots \supset L_k \supset L_{k+1} = 0,$$

where  $L_{i+1} = [L_0, L_i]$ . Note that  $[L_i, L_j] \subset L_{i+j+1}$ . Indeed, assume  $i \leq j$  and argue by induction on  $i$ . The case  $i=0$  is obvious, and the induction step follows from the Jacobi

identity and the induction hypothesis, because  $[L_i, L_j] = [[L_0, L_{i-1}], L_j]$  lies in

$$\text{span}([L_{i-1}, L_j], L_0, [[L_0, L_j], L_{i-1}]) \subset \text{span}([L_{i+j}, L_0], [L_{j+1}, L_{i-1}]) = L_{i+j+1}.$$

The group  $K$  preserves each  $L_i$ , so we can choose a  $K$ -invariant inner product  $\langle \cdot, \cdot \rangle_0$  on  $L$ . Let

$$F_i = \{X \in L_i : \langle X, Y \rangle_0 = 0 \text{ for } Y \in L_{i+i}\}.$$

Then  $L = F_0 \oplus \dots \oplus F_k$ . Define a new  $K$ -invariant inner product  $\langle \cdot, \cdot \rangle_r$  on  $L$  by  $\langle X, Y \rangle_r = h_i(r)^2 \langle X, Y \rangle_0$  for  $X, Y \in F_i$ , and  $\langle X, Y \rangle_r = 0$  if  $X \in F_i$  and  $Y \in F_j$ , for  $i \neq j$ , where the  $h_i$ 's are some positive functions defined below. This defines a  $G \rtimes K$ -invariant Riemannian metric  $g_r$  on  $G$ .

Let  $\alpha_i = i + 1$  with  $i = 0, \dots, k$  and  $a = k + 1$ . Now define the warping function  $h_i$  to be a positive, smooth, strictly convex, decreasing function that is equal to  $e^{-\alpha_i r}$  if  $r \geq 1$ , and is equal to  $e^{-ar}$  if  $r \leq -1$ ; such a function exists since  $a \geq \alpha_i$  for each  $i$ . Thus  $h_i' < 0 < h_i''$ , and the functions  $h_i'/h_i$  and  $h_i''/h_i$  are uniformly bounded away from 0 and  $\infty$ .

Define the warped product metric on  $G \times \mathbf{R}$  by  $g = s^2 g_r + dr^2$ , where  $s > 0$  is a constant; clearly  $g$  is a complete  $G \rtimes K$ -invariant metric. A straightforward tedious computation (mostly done e.g. in [4]) yields, for  $g$ -orthonormal vector fields  $Y_s \in F_s$ , that

$$\begin{aligned} \langle R_g(Y_i, Y_j)Y_j, Y_i \rangle_g &= \frac{1}{s^2} \langle R_{g_r}(Y_i, Y_j)Y_j, Y_i \rangle_{g_r} - \frac{h_i' h_j'}{h_i h_j}, \\ \langle R_g(Y_i, Y_j)Y_l, Y_m \rangle_g &= \frac{1}{s^2} \langle R_{g_r}(Y_i, Y_j)Y_l, Y_m \rangle_{g_r}, & \text{if } \{i, j\} \neq \{l, m\}, \\ \langle R_g(Y_i, \frac{\partial}{\partial r}) \frac{\partial}{\partial r}, Y_i \rangle_g &= -\frac{h_i''}{h_i}, \\ \langle R_g(Y_i, \frac{\partial}{\partial r}) \frac{\partial}{\partial r}, Y_j \rangle_g &= 0, & \text{if } i \neq j, \\ \langle R_g(\frac{\partial}{\partial r}, Y_i)Y_j, Y_l \rangle_g &= \left( \frac{h_j'}{2h_j} + \frac{h_l'}{2h_l} \right) (\langle [Y_j, Y_i], Y_l \rangle_g + \langle [Y_i, Y_l], Y_j \rangle_g + \langle [Y_j, Y_l], Y_i \rangle_g). \end{aligned}$$

Since  $[L_i, L_j] \subset L_{i+j+1}$ , we have, for  $Z = \sum_{i=0}^k Z_i$  and  $W = \sum_{j=0}^k W_j$  with  $Z_i, W_i \in F_i$ , that

$$|[Z, W]|_{g_r} \leq \sum_{i,j} |[Z_i, W_j]|_{g_r} \leq \sum_{i,j} \sum_{s>i+j} h_s |[Z_i, W_j]|_{g_0}.$$

The above choice of the  $a_i$ 's implies that if  $r \geq 1$ , then  $\sum_{s>i+j} h_s \leq k h_i h_j$ . Also,

$$|[Z_i, W_j]|_{g_0} \leq C |Z_i|_{g_0} |W_j|_{g_0},$$

where  $C$  only depends on the structure constants of  $L$ , so that we conclude

$$|[Z, W]|_{g_r} \leq Ck |Z|_{g_0} |W|_{g_0} \sum_{i,j} h_i h_j \leq Ck(k+1) |Z|_{g_r} |W|_{g_r}.$$



It follows that if  $r \geq 1$ , then the norm of the curvature tensor of  $g_r$  is bounded in terms of  $C$  and  $k$  [13, Proposition 3.18]. The same conclusions trivially hold for  $r \leq -1$ , because then  $g_r$  is the rescaling of  $g_0$  by a constant  $e^{-ar} > 1$ , and also for  $r \in [-1, 1]$  by compactness, since  $g_r$  is left-invariant and depends continuously of  $r$ . Hence  $\langle R_g(Y_i, Y_j)Y_l, Y_m \rangle_g \rightarrow 0$  as  $s \rightarrow \infty$ .

Also,  $\langle R_g(\frac{\partial}{\partial r}, Y_i)Y_j, Y_l \rangle_g \rightarrow 0$  as  $s \rightarrow \infty$ , since

$$|\langle [Y_j, Y_i], Y_l \rangle_g| = s^2 |\langle [Y_j, Y_i], Y_l \rangle_{g_r}| \leq s^2 C(k+1) |Y_j|_{g_r} |Y_i|_{g_r} |Y_l|_{g_r} \leq \frac{C(k+1)}{s},$$

where the last inequality holds since  $s|Y|_{g_r} = 1$  for any  $g$ -unit vector  $Y$ . It follows that, as  $s \rightarrow \infty$ ,  $R_g$  uniformly converges to a tensor  $\bar{R}$  whose nonzero components are

$$\bar{R}(Y_i, Y_j, Y_j, Y_i) = -\frac{h'_i h'_j}{h_i h_j} \quad \text{and} \quad \bar{R}(Y_i, \frac{\partial}{\partial r}, \frac{\partial}{\partial r}, Y_i) = -\frac{h''_i}{h_i}.$$

Thus  $g$  has pinched negative curvature for all large  $s$ . □

**COROLLARY 10.2.** *Let  $E$  be the total space of a flat Euclidean vector bundle over an infranilmanifold  $I$ . Then  $E$  is infranil, in particular  $E \times \mathbf{R}$  admits a complete Riemannian metric of pinched negative curvature.*

*Proof.* Fix a flat Euclidean  $\mathbf{R}^k$ -bundle over the infranilmanifold  $I$ , and write  $I$  as  $G_0/\Gamma$ , where  $G_0$  is a simply-connected nilpotent Lie group and  $\Gamma$  is a discrete cocompact group of affine transformations of  $G_0$  that acts freely. Look at the nilpotent group  $G = G_0 \times \mathbf{R}^k$ , and let  $\Gamma$  act on the  $\mathbf{R}^k$ -factor via the holonomy of the flat bundle  $\Gamma \cong \pi_1(I) \rightarrow O(k)$ . Then the infranilmanifold  $G/\Gamma$  is diffeomorphic to the total space of the flat bundle we started with. By Theorem 10.1,  $G/\Gamma \times \mathbf{R}$  carries a complete metric of pinched negative curvature. □

### 11. On geometrically finite manifolds

*Proof of Corollary 1.4.* Let  $X/\Gamma$  be a geometrically finite pinched negatively curved manifold, let  $\Omega$  be the domain of discontinuity and  $\Lambda$  be the limit set for the  $\Gamma$ -action at infinity. Let  $C_\varepsilon$  be the  $\varepsilon$ -neighborhood of the convex hull of  $L$ . Then  $C_\varepsilon/\Gamma$  is a codimension zero  $C^1$  submanifold of  $X/\Gamma$  that is homeomorphic to  $(X \cup \Omega)/\Gamma$  by pushing along geodesic rays orthogonal to  $\partial C_\varepsilon/\Gamma$ . This homeomorphism restricts to a diffeomorphism on the interiors  $X/\Gamma \rightarrow \text{Int}(C_\varepsilon)/\Gamma$ .

By the discussion in [7, pp.263–264], for each end of  $(X \cup \Omega)/\Gamma$  there is a parabolic subgroup  $\Gamma_z \leq \Gamma$  stabilizing a point  $z \in \partial_\infty X$  such that the end has a neighborhood

homeomorphic to a neighborhood of the unique end of  $(X \cup \partial_\infty X \setminus \{z\})/\Gamma_z$ . Again, this homeomorphism restricts to a diffeomorphism on the interiors.

By pushing along trajectories of the Busemann flow,  $(X \cup \partial_\infty X \setminus \{z\})/\Gamma_z$  is homeomorphic to the  $\Gamma_z$ -quotient of a closed horoball  $H_z$  centered at  $z$ . Note that  $H_z/\Gamma_z$  is a  $C^2$  submanifold of  $X/\Gamma_z$ , and  $H_z/\Gamma_z$  is  $C^1$ -diffeomorphic to the product of  $[0, \infty)$  and a horosphere quotient, which, by Theorem 1.1, is diffeomorphic to the interior of a compact manifold  $L_z$ . So  $H_z/\Gamma_z$  is diffeomorphic to the interior of  $L_z \times [0, 1]$ , in which we smooth corners. Compactifying each  $z$ -end of  $C_\varepsilon/\Gamma$  with  $L_z \times [0, 1]$ , we get a compact  $C^1$  manifold whose interior is diffeomorphic to  $X/\Gamma$ .  $\square$

### Appendix A. Lemmas on nilpotent groups

LEMMA A.1. *Given a finitely generated nilpotent group  $\Gamma$  and a positive integer  $n$ , there exists a finite subset  $F \subset \Gamma$  such that for any  $g \in \Gamma$  there are  $f \in F$  and  $x \in \Gamma$  with  $gf = x^n$ .*

*Proof.* We argue by induction on the nilpotency degree of  $\Gamma$ . If  $\Gamma$  is abelian, then the  $n$ th powers of elements of  $\Gamma$  form a finite index subgroup, and we can take  $F$  as the set of coset representatives of this subgroup. In general, if  $Z$  denotes the center of  $\Gamma$ , then by induction the result is true for  $\Gamma/Z$  for the finite subset  $\{aZ : a \in F_1\}$  of  $G/Z$ , where  $F_1$  is some finite subset of  $\Gamma$ . Thus, an arbitrary  $g \in \Gamma$  satisfies  $gf_1 = x^n z$  for some  $f_1 \in F_1$ ,  $x \in \Gamma$  and  $z \in Z$ . Again, since  $Z$  is abelian, the set  $Z^n$  of  $n$ th powers is a finite index subgroup of  $Z$ . Let  $F_2$  be a set of coset representatives of  $Z^n$  in  $Z$  so that  $z = y^n f_2$  for some  $y \in Z$  and  $f_2 \in F_2$ . Then  $gf_1 = x^n y^n f_2 = (xy)^n f_2$ , so  $gf_1 f_2^{-1} = (xy)^n$ , and the assertion holds for  $\Gamma$  with  $F = F_1 F_2^{-1}$ .  $\square$

LEMMA A.2. *Let  $\Gamma$  be a finitely generated nilpotent group. Then, there exists a finite set  $F \subset \Gamma$  such that for any  $x \in \Gamma$  and  $g \in [\Gamma, \Gamma]$  there are  $h \in [\Gamma, \Gamma]$  and  $f \in F$  with  $x^2 g = (xh)^2 f$ .*

*Proof.* As usual, we write  $\Gamma_1 = \Gamma$  and  $\Gamma_{i+1} = [\Gamma, \Gamma_i]$ , so that the nilpotency degree  $k$  of  $\Gamma$  is the largest integer for which  $\Gamma_k$  is nontrivial. Since  $[\Gamma, \Gamma_k]$  is trivial,  $\Gamma_k$  lies in the center of  $\Gamma$ . We argue by induction on  $k$ . The case  $k=1$ , i.e. when  $\Gamma$  is abelian, is obvious for  $F = \{1\}$ . If  $\Gamma$  is of nilpotency degree  $k > 1$ , then by induction the statement is true in  $\Gamma/\Gamma_k$ . Let  $F_1 \subset \Gamma$  be the set of coset representatives of the corresponding finite set for  $\Gamma/\Gamma_k$ , so that given  $x \in \Gamma$ , there exist  $z \in \Gamma_k$ ,  $g, h \in [\Gamma, \Gamma]$  and  $f_1 \in F_1$  with  $x^2 g = (xh)^2 f_1 z$ . By Lemma A.1 applied to  $\Gamma_k$ , we get  $z = y^2 f_2$  for some  $y \in \Gamma_k$  and  $f_2 \in F_2$ , where  $F_2$  is a finite subset of  $\Gamma_k$ . Then  $x^2 g = (xh)^2 f_1 z = (xyh)^2 f_1 f_2$ , and since  $yh \in [\Gamma, \Gamma]$  the proof is complete.  $\square$

LEMMA A.3. *Let  $C$  be a maximal compact subgroup of a connected nilpotent Lie group  $N$ . Then  $C$  is equal to the unique maximal compact subgroup of the center of  $N$ , in particular  $C$  is a torus.*

*Proof.* Any maximal compact subgroup  $C$  of  $N$  lies in the center  $Z$  of  $N$  [20]. Hence  $C$  is also maximal compact in  $Z$ . A maximal compact subgroup is homotopy equivalent to the ambient group, hence, since  $N$  is connected, so are  $Z$  and  $C$ . Now  $Z$  is connected abelian, hence  $Z$  is isomorphic to the product of a real vector space and a torus, so  $C$  equals to the torus.  $\square$

### Appendix B. Isometries are smooth

PROPOSITION B.1. *Let  $X$  be a smooth manifold equipped with a complete  $C^0$ -Riemannian metric of curvature bounded above and below in the comparison sense. Then the isometry group acts on  $X$  by  $C^3$ -diffeomorphisms.*

*Proof.* The isometry group of any complete locally compact metric space is locally compact [23]. By [26, Chapter 5], any locally compact subgroup of  $\text{Diffeo}^r(X)$  with  $r > 0$  is a Lie group and the action is  $C^r$ . Thus, it suffices to show that each individual isometry is  $C^3$ . The construction of harmonic coordinates in [27] starts with the  $C^0$ -distance coordinates  $(d(x, a_1), \dots, d(x, a_n))$  at  $x \in X$ , where the geodesic segments  $[x, a_i]$  are pairwise orthogonal, and then solves the Dirichlet problem in a small ball around  $x$  with values on the boundary sphere given by  $d(x, a_i)$ . The solutions are the so-called harmonic coordinates. Their transition functions are  $C^{3,\alpha}$  (and the metric tensor in this coordinates is  $C^{1,\alpha}$  even though we do not need this fact here). This construction is clearly invariant under isometries, so any isometry has the same smoothness in harmonic coordinates as the identity map, namely  $C^{3,\alpha}$ .  $\square$

### Appendix C. Local formula of Ballmann and Brüning

Let  $X$  be a simply-connected manifold of pinched negative curvature. Fix a point at infinity of  $X$ , and let  $T$  be the unit vector field tangent to the Busemann flow  $b_t(x)$  towards that point. For a curve  $\alpha(s)$  in  $X$ , let  $\alpha(0) = x$  and  $\alpha'(0) = u$ . Look at the 1-parameter family of geodesic rays  $\alpha(s, t) = b_t(\alpha(s))$  and the corresponding family of Jacobi fields  $J$ . Let  $v, w \in T_x X$  and let  $X(t, s)$  and  $Y(t, s)$  be vector fields along  $\alpha(s, t)$  such that  $X(0, 0) = v$ ,  $Y(0, 0) = w$  and  $\nabla_T X = \nabla_T Y = 0$ . Then one defines a tensor field  $\bar{S}$  by

$$\langle \bar{S}(u, v), w \rangle = - \int_0^\infty \langle R(T, J)X, Y \rangle(t, 0) dt.$$

Ballmann and Brüning [2] define a new connection  $\bar{\nabla}$  by  $\bar{\nabla} = \nabla - \bar{S}$ , and show that  $\bar{\nabla}$  is a  $C^0$ -flat connection that is compatible with the metric, and that satisfies  $\bar{\nabla}T=0$  and

$$|\bar{\nabla}_X Y - \nabla_X Y| \leq C(a)|X||Y| \quad \text{for any } X \text{ and } Y. \tag{C.1}$$

Since  $\bar{\nabla}$  is flat, for any  $x, y \in M$  we have a well-defined parallel transport with respect to  $\bar{\nabla}$  from  $x$  to  $y$  which we denote by  $P_{xy}^{\bar{\nabla}}$ .

LEMMA C.2. *For any  $x, y \in M$ , parallel transport through infinity  $P_{xy}^\infty$  coincides with  $P_{xy}^{\bar{\nabla}}$ .*

*Proof.* First, suppose that  $x$  and  $y$  lie in the same horosphere  $H_{t_0}$ . Consider the quadrangle  $xb_t(x)b_t(y)y$ . By flatness,  $P_{xy}^{\bar{\nabla}}$  is equal to the  $\bar{\nabla}$  parallel transport along three other sides of this quadrangle  $P_{b_t(y)y}^{\bar{\nabla}} \circ P_{b_t(x)b_t(y)}^{\bar{\nabla}} \circ P_{xb_t(x)}^{\bar{\nabla}}$ . Also, if  $\alpha$  is a trajectory of the Busemann flow, then  $J=T$  so that  $\bar{S}=0$ , therefore parallel transports along Busemann trajectories coincide with Levi-Civita parallel transports. Thus

$$P_{xy}^{\bar{\nabla}} = P_{b_t(y)y} \circ P_{b_t(x)b_t(y)}^{\bar{\nabla}} \circ P_{xb_t(x)}.$$

Since  $d(b_t(x)b_t(y)) \rightarrow 0$  as  $t \rightarrow \infty$ , by (C.1) we have that  $P_{b_t(x)b_t(y)}^{\bar{\nabla}}$  becomes arbitrarily close to  $P_{b_t(x)b_t(y)}$  for large  $t$ , and therefore

$$P_{xy}^{\bar{\nabla}} = \lim_{t \rightarrow \infty} P_{b_t(y)y} \circ P_{b_t(x)b_t(y)} \circ P_{xb_t(x)}. \tag{C.3}$$

Similarly, by construction,  $P^\infty$  commutes with the Busemann flow, and hence

$$P_{xy}^\infty = P_{b_t(y)y} \circ P_{b_t(x)b_t(y)}^\infty \circ P_{xb_t(x)}.$$

As before,  $P_{b_t(x)b_t(y)}^\infty$  comes arbitrarily close to  $P_{b_t(x)b_t(y)}$  for large  $t$ , and therefore

$$P_{xy}^{\bar{\nabla}} = \lim_{t \rightarrow \infty} P_{b_t(y)y} \circ P_{b_t(x)b_t(y)} \circ P_{xb_t(x)}. \tag{C.4}$$

Comparing (C.3) and (C.4), we conclude that  $P_{xy}^\infty = P_{xy}^{\bar{\nabla}}$  for any  $x$  and  $y$  in the same horosphere.

Finally, since  $\bar{\nabla}$  and  $\nabla^\infty$  are flat, and their parallel transports coincide with the Levi-Civita parallel transport along trajectories of the Busemann flow, we have  $P_{xy}^\infty = P_{xy}^{\bar{\nabla}}$  for any  $x$  and  $y$ . □

**Appendix D. Concave functions and submetries on Alexandrov spaces**

The proof of the following lemma is due to A. Petrunin.

LEMMA D.1. *Let  $X$  be an Alexandrov space of  $\text{curv} \geq k$  with  $\partial X = \emptyset$ . Let  $f: X \rightarrow \mathbf{R}$  be a Lipschitz function with a local maximum at  $q$ . Suppose that  $f$  is strictly concave on an open set  $U$  containing  $q$ . Then  $d(\cdot, q)$  has no critical points on  $U \setminus \{q\}$ .*

*Proof.* Let  $x \in U \setminus \{q\}$ . By [31],  $\nabla f(x)$  is defined to be equal to  $v \in T_x X$  if  $df(v) = |v|^2$ , and  $df(u)/|u|$  attains a positive maximum at  $v$ . Since  $x$  is not a point of maximum of  $f$ ,  $\nabla f(x) \neq 0$  and  $|\nabla f(z)| \geq c > 0$  for all  $z$  near  $x$ . Consider the gradient flow for  $f$  as defined in [31]. Consider a gradient line  $\gamma(t)$  passing through  $x$  so that  $\gamma(0) = x$ . By [31], the curve  $\gamma(t)$  is locally Lipschitz.

We claim that  $\gamma(t)$  can be extended to be a gradient line defined on  $(-\varepsilon, \infty)$  for some  $\varepsilon > 0$ . Indeed, suppose this is not true. By [31, Lemma 3.2.1 (a)], the gradient flow of any concave function is 1-Lipschitz, so it defines a deformation retraction of any superlevel set  $\{x: f(x) \geq c\}$  (shortly,  $\{f \geq c\}$ ) contained in  $U$  onto  $q$ , and hence  $\{f \geq c\}$  is contractible (for a different proof see also [21, Lemma 5.2]). Take  $c = f(x)$  and let  $\varepsilon > 0$  be small enough so that  $\{f \geq c - \varepsilon\}$  is contained in  $U$ . Since  $\gamma$  cannot be extended backwards beyond zero, the gradient flow gives a deformation retraction of  $\{f \geq c - \varepsilon\} \setminus \{x\}$  onto  $q$ . To see that this is impossible, we prove that  $H^{n-1}(\{f \geq c - \varepsilon\} \setminus \{x\}, \mathbf{Z}_2) \cong \mathbf{Z}_2$ .

Indeed, since  $\{f \geq c - \varepsilon\}$  is contractible, from the long exact cohomology sequence of a pair we see that

$$H^{n-1}(\{f \geq c - \varepsilon\} \setminus \{x\}, \mathbf{Z}_2) \cong H^n(\{f \geq c - \varepsilon\}, \{f \geq c - \varepsilon\} \setminus \{x\}, \mathbf{Z}_2).$$

By [29], for some very small  $\delta > 0$ , the ball  $B(x, \delta)$  is contractible and is contained in  $\{f \geq c - \varepsilon\}$ , and  $d(\cdot, x)$  has no critical points in  $B(x, 2\delta) \setminus \{x\}$ . Therefore, by excision,

$$H^n(\{f \geq c - \varepsilon\}, \{f \geq c - \varepsilon\} \setminus \{x\}, \mathbf{Z}_2) \cong H^n(B(x, \delta), B(x, \delta) \setminus \{x\}, \mathbf{Z}_2),$$

which is isomorphic to  $H^{n-1}(S(x, \delta), \mathbf{Z}_2) \cong \mathbf{Z}_2$ , where the last equality holds since, by [29],  $S(x, \delta)$  is homotopy equivalent to the Alexandrov space  $\Sigma_x X$  with  $\partial \Sigma_x X = \emptyset$ , and since any Alexandrov space without boundary has top-dimensional  $\mathbf{Z}_2$ -cohomology isomorphic to  $\mathbf{Z}_2$  [18].

Let  $v$  be any left tangent vector of  $\gamma$  at 0. Here, following [31], we say that  $v \in T_x X$  is a *left tangent vector* if

$$v = \lim_{i \rightarrow \infty} \frac{1}{|t_i|} \exp_x^{-1} \gamma(t_i)$$

for some sequence  $t_i \rightarrow 0^-$ . (One should think of  $v$  as  $-\nabla f(x)$ .) By above,  $f(\gamma(t)) - ct$  is nondecreasing for small  $t$ . Therefore  $v \neq 0$ . We claim that  $\angle uv \geq \pi/2$  for any direction  $u$

of a shortest geodesic from  $x$  to  $y$  with  $f(y) > f(x)$ . Indeed, by [31, Lemma 3.2.1 (a)], the gradient flow of *any* concave function is 1-Lipschitz. Consider a cutoff concave function  $\hat{f}(\cdot) = \min\{f(\cdot), f(y)\}$ . Clearly the forward gradient flow of  $\hat{f}$  fixes  $y$  and coincides with the gradient flow of  $f$  near  $x$ . Since the gradient flow of  $\hat{f}$  is 1-Lipschitz,  $d(y, \gamma(t))$  is nonincreasing near  $t=0$ . By the first variation formula, this implies that  $\angle uv \geq \pi/2$ . In particular, this is true for  $y=q$ . Since any shortest geodesic from  $x$  to  $q$  points strictly inside the convex set  $\{f \geq f(x)\}$ , this inequality is in fact strict, i.e.  $\angle uv > \pi/2$ .  $\square$

*Definition D.2.* A map  $f: X \rightarrow Y$  between two metric spaces is called a *submetry* if for any  $x \in X$  and any  $r > 0$  one has  $f(B_r(x)) = B_r(f(x))$ .

We need some properties of submetries collected below.

*LEMMA D.3.* Let  $X$  be an Alexandrov space of  $\text{curv} \geq k$  and  $f: X \rightarrow Y$  be a submetry. Then,

- (a)  $Y$  is an Alexandrov space of  $\text{curv} \geq k$  [9].
- (b) For any  $x \in X$  and  $y \in Y$  we have  $d(x, f^{-1}(y)) = d(f(x), y)$ .
- (c) For any  $x \in X$  and any shortest geodesic  $\gamma: [0, 1] \rightarrow Y$  with  $\gamma(0) = f(x)$  there exists a shortest geodesic  $\tilde{\gamma}: [0, 1] \rightarrow Y$  with  $\tilde{\gamma}(0) = x$  such that  $f(\tilde{\gamma}(t)) = \gamma(t)$ . The geodesic  $\tilde{\gamma}$  is called a *horizontal lift* of  $\gamma$ . Moreover, if  $\gamma(t)$  can be extended to a shortest geodesic  $\gamma: [-\varepsilon, 1] \rightarrow Y$  then the horizontal lift is unique.

Moreover, the statement (a) is true locally in the following sense: if  $B(p, 1) \subset U \subset X$ ,  $g: U \rightarrow Y$  is a submetry and  $\text{diam}(g^{-1}(y)) < \frac{1}{10}$  for any  $y \in Y$ , then for any triangle with vertices in  $B(g(p), \frac{1}{4})$  the Toponogov comparison with  $\text{curv} \geq k$  holds.

The proofs of (b) and (c) are elementary and left to the reader (see [24] for details).

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*Received July 14, 2005*