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Connecting rational homotopy type singularities

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1. Introduction

Let (N, g) be a closed Riemannian manifold. With the help of the Nash embedding theorem, we may assume that N is a submanifold, with the induced metric, of some Euclidian space $\mathbf{R}^l$. One then has, for any $m\in\mathbf{N}$ and $p\geqslant 1$, the space of Sobolev maps

$$W^{1,p}(\mathbf{R}^m, N) = \{u \in L^p_{\mathrm{loc}}(\mathbf{R}^m, \mathbf{R}^l) : u(x) \in N \text{ for almost every } x \in \mathbf{R}^m \text{ and } \nabla u \in L^p\}.$$

An important issue regarding the description of these nonlinear function spaces, which plays an increasing role in analysis, is the question of the density in $W^{1,p}(\mathbf{R}^m, N)$, for the $W^{1,p}$-norm, of smooth maps taking values into N.

In case $p>m$, Sobolev embedding shows that any map in $W^{1,p}(\mathbf{R}^m, N)$ is (Hölder) continuous. For such a continuous $W^{1,p}$-map u, it is not difficult to see, using standard smoothing in $W^{1,p}(\mathbf{R}^m, \mathbf{R}^l)\cap C^0$ and nearest-point projection to N, that u is strongly $W^{1,p}$-approximable by maps in $C^\infty(\mathbf{R}^m, N)$.

In case $p=m$, this continuity of a Sobolev map is no longer automatically true. Nevertheless, $C^\infty(\mathbf{R}^p, N)$ is still strongly dense in $W^{1,p}(\mathbf{R}^p, N)$ as noted by Schoen and Uhlenbeck [SU]. It follows similarly that any map $u\in W^{1,p}(S^p, N)$ admits a strong $W^{1,p}$ approximation by maps in $C^\infty(S^p, N)$. White [Wh] showed how this approximation gives a well-defined homotopy class in $\pi_p(N)$. Conversely, every homotopy class in $\pi_p(N)$ (which is, by definition, given by a continuous map) admits a smooth and hence $W^{1,p}$ representative.

In case $p<m$, the strong $W^{1,p}$-density of $C^\infty(\mathbf{R}^p, N)$ in $W^{1,p}(\mathbf{R}^p, N)$ may fail (as seen in the example $x/|x|\in W^{1,2}(B^3_1, N)$ discussed in [SU]). The general problem of strong $W^{1,p}$-approximability was considered by Bethuel in [Be] (see also more recent works and updated results on the necessary and sufficient topological conditions by Hang and Lin

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in [HL1] and [HL2]). It was shown in particular in [Be] that smooth maps are *not* dense in $W^{1,p}(\mathbf{R}^m, N)$ whenever $\pi_{[p]}(N)\neq 0$ (where $[p]$ is the integer part of p). Since results of this paper for various $p>1$ will just depend on $[p]$, we henceforth assume for notational simplicity that p *is an integer greater than* 1.

For clarity of exposition, we will also, for most of this paper, restrict to the case $m=p+1$, where the bubbling is 1-dimensional. Finally, in §5, we describe how everything generalizes for higher-dimensional bubbling.

For a map u in $W^{1,p}(\mathbf{R}^{p+1}, N)$, Fubini's theorem implies that, for each center $c\in\mathbf{R}^m$, and almost every radius $r>0$, the restriction of u to the p-sphere $\partial B_r^{p+1}(c)$ belongs to $W^{1,p}(\partial B_r(c), N)$. Thus the map

$$u_{c,r}: S^p \longrightarrow N, \quad u_{c,r}(x) = u(c+rx),$$

gives, as discussed above, an element of $\pi_p(N)$ because $p=\dim \partial B_r(c)$. The map u is strongly $W^{1,p}$-approximable by smooth maps if and only if the homotopy class of such a restriction $u_{c,r}$ is zero for almost every (c,r). A motivation of this paper is to describe, for an arbitrary map in $W^{1,p}(\mathbf{R}^{p+1}, N)$, "how big" is the obstruction to strong approximability. The idea is to try to "connect" the *topological singularities* of u. Such a singularity is recognized by seeing that the homotopy class $[u_{c,r}]$ changes as the sphere $\partial B_r^m(c)$ moves across the singularity.

In this paper we restrict to obstructions coming from the *infinite nontorsion part* $\pi_{[p]}(N)\otimes\mathbf{R}$ of $\pi_{[p]}(N)$. We do this by considering a fixed member z of the vector space

$$(\pi_p(N)\otimes\mathbf{R})^* = \operatorname{Hom}(\pi_p(N), \mathbf{R}).$$

To study the z-type topological singularities of a map $u\in W^{1,p}(\mathbf{R}^{p+1}, N)$, we consider (see [Wh]), the restriction of u to spheres with the map

$$\Phi_{z,u}: \mathbf{R}^{p+1}\times\mathbf{R}_+ \longrightarrow \mathbf{R}, \quad \Phi_{z,u}(c,r) = z([u_{c,r}]).$$

This map, which is defined for almost every (c,r) in $\mathbf{R}^{p+1}\times\mathbf{R}_+$, is, as we shall see, Lebesgue-measurable. Note that $\Phi_{z,u}(c,r)=0$, in case u is continuous on the closed ball $\overline{B_r(c)}$, because then $u_{c,r}$ is homotopic to a constant.

Recall that any countable union Γ of C^1-embedded curves admits an $\mathcal{H}^1$-*measurable* orientation, that is, a unit vectorfield $\vec{\Gamma}$ so that, at $\mathcal{H}^1$-almost every point $x\in\Gamma$, $\vec{\Gamma}(x)$ orients the approximate tangent line for Γ at x (see [Fe, Theorem 3.2.19]). We keep denoting Γ as the set of points at which $\vec{\Gamma}$ exists. Moreover, for almost every (c,r) in $\mathbf{R}^{p+1}\times\mathbf{R}_+^*$, the sphere $\partial B_r(c)$ intersects Γ transversally (see Lemma 6.1); that is,

$$\vec{\Gamma}(a)\cdot(a-c)\neq 0 \quad \text{for all } a\in\Gamma\cap\partial B_r(c).$$

We can now state our main result.

THEOREM 1.1. *Let N be a compact simply connected Riemannian manifold, $p\in\{2,3,\dots\}$, and z be an element of* $\mathrm{Hom}(\pi_p(N),\mathbf{R})$. *Then there exist a non-negative integer n_z and a positive constant C_z such that for any map u in $W^{1,p}(\mathbf{R}^{p+1},N)$ which is the weak $W^{1,p}$-limit of a sequence of smooth maps, there exists a countable union Γ of C^1-curves with measurable orientation $\vec{\Gamma}$ and a non-negative $\mathcal{H}^1$-measurable multiplicity function θ from Γ into $z(\pi_p(N))$ such that*

$$\Phi_{z,u}(c,r)=z([u_{c,r}])=\sum_{a\in\Gamma\cap\partial B_r(c)}\mathrm{sgn}(\vec{\Gamma}(a)\cdot(a-c))\theta(a) \tag{1.1}$$

for almost every $(c,r)\in\mathbf{R}^{p+1}\times\mathbf{R}_+$, $\mathcal{H}^1\{a\in\Gamma:\theta(a)\neq0\}<\infty$ and

$$\int_\Gamma|\theta|^{p/(p+n_z)}\,d\mathcal{H}^1\leqslant C_z\liminf_{n\to\infty}\int_{\mathbf{R}^{p+1}}|\nabla u_n|^p\,dx<\infty \tag{1.2}$$

for any sequence $u_n\in C^\infty(\mathbf{R}^{p+1},N)$ converging $W^{1,p}$-weakly to u. The triple $(\Gamma,\vec{\Gamma},\theta)$ is called a rectifiable Poincaré dual to $\Phi_{z,u}$.

Remark 1.2. In the previous theorem, the assumption of $W^{1,p}$-*weak* approximability by maps in $C^\infty(\mathbf{R}^{p+1},N)$ can be replaced by the assumption of $W^{1,p}$-weak approximability by maps u_n in $W^{1,p}(\mathbf{R}^{p+1},N)$ satisfying $\Phi_{z,u_n}\equiv0$. The question of the sequential weak approximability of $W^{1,p}(\mathbf{R}^m,N)$ maps by smooth maps remains an open problem for general N and integers $p\geqslant3$, even for the case $m=4$, $N=S^2$, $p=3$ studied in [HR1].

This sequential weak density of smooth maps has been established with strong topological assumptions on N depending on p in [Be] and [HL3], and with no assumptions on N but for $p=2$ in [PR] and [H]. The works [PR] and [HR2], which treat special cases of bubbling from the torsion part of the homotopy, also give such weak density.

The reason why we call $(\Gamma,\vec{\Gamma},\theta)$ a Poincaré dual to $\Phi_{z,u}$ is the following: in case u has only finitely many isolated singularities $c_1,\dots,c_I$, each homotopy class $d_i=[u_{c_i,r}]$ is then independent of the choice of radius $r<\min_{j\neq i}|c_j-c_i|$. For any p-cycle C with compact support in $M=\mathbf{R}^{p+1}\setminus\{c_1,\dots,c_I\}$, $C=\partial B$ for some unique $(p+1)$-chain B of compact support in $\mathbf{R}^{p+1}$. The chain B has constant multiplicity in each component of $M\setminus\operatorname{spt}C$, and we suppose that n_j is the multiplicity of B at c_j. Then the map Φ given by

$$\Phi(C)=z\biggl(\sum_{i=1}^I n_id_i\biggr)$$

gives a well-defined cohomology element in $H^p(M,\mathbf{R})$. It is easy to see that any choice of $(\Gamma,\vec{\Gamma},\theta)$ satisfying (1.1) is a representative of the Poincaré dual in $H_1(M,\mathbf{R})$ of Φ. In particular, for any regular value y of the restriction of u to $\mathbf{R}^{p+1}\setminus\{c_1,\dots,c_I\}$, the set $\Gamma=u^{-1}\{y\}$, with the induced orientation $\vec{\Gamma}$ and $\theta\equiv 1$, provides such a representative. Recalling that maps having isolated point singularities are dense in $W^{1,p}(\mathbf{R}^{p+1},N)$ (see [Be]), we see that this notion of Poincaré dual can be interpreted as a limit of the classical one.

Theorem 1.1 was first established in the particular case where $p=2$ and $N=S^2$ in [BCL], [BBC], [GMS1] (see the discussion in [GMS2]) and in [ABO] for $p=n$ and $N=S^n$ (with possibly higher-dimensional bubbling). In these cases, there is one generator z of

$$\mathrm{Hom}(\pi_p(S^p),\mathbf{R})\simeq\mathbf{R}$$

(the *topological degree*) and $n_z=0$. These situations where $n_z=0$ are very special and allow the bubbled object $(\Gamma,\vec{\Gamma},\theta)$ to be interpreted as a *current*. Being a limit of a mass-bounded sequence of rectifiable currents, it is also rectifiable by geometric measure theory (see [GMS2]). Then in [Ri] and [HR1] the case where $N=S^2$ for arbitrary p was considered. In that case, for $p=3$, $\mathrm{Hom}(\pi_3(S^2),\mathbf{R})\simeq\mathbf{R}$ is also generated by one element z (the *Hopf degree*), but now $n_z=1$, and any corresponding $(\Gamma,\vec{\Gamma},\theta)$ *cannot*, by specific example [HR1, §2.5], be interpreted as a current.

A critical general problem behind this work is the following question.

Question 1.3. For any homotopy invariant $z\in\mathrm{Hom}(\pi_p(N),\mathbf{R})$ and $M>0$, what is the minimum possible p-energy $\int_{S^p}|\nabla u|^p\,d\mathcal{H}^p$ necessary for a map $u\in C^\infty(S^p,N)$ to have $z([u])\geqslant M$?

For $N=S^p$ and z being the topological degree, $n_z=0$, and this minimum p-energy is precisely $p^{p/2}\mathcal{H}^p(S^p)\cdot M$. On the other hand, for $N=S^2$ with $p=3$, and z being the Hopf degree, $n_z=1$, and the minimum 3-energy is asymptotically $CM^{3/4}=CM^{p/(p+n_z)}$ by [Ri]. There are other situations where we know that the integer n_z, as defined in Proposition 3.4 (iii) and in equation (2.6), is optimal for the inequality (1.2). Precisely, we have the following result.

PROPOSITION 1.4. *Let N be a compact simply connected Riemannian manifold, p be a positive integer and z be an element of* $\mathrm{Hom}(\pi_p(N),\mathbf{R})$. *Assume that the critical exponent $p(p+n_z)^{-1}$, with n_z given in Definition* 2.6 *is optimal in the sense given by Definition* 2.14. *Then, for any $\beta>p(p+n_z)^{-1}$, there exists u in $W^{1,p}(\mathbf{R}^{p+1},N)$, a weak limit of smooth maps, such that for any Poincaré dual of $\Phi_{z,u}$, $(\Gamma,\vec{\Gamma},\theta)$, satisfying* (1.1), *one has*

$$\int_\Gamma|\theta|^\beta\,d\mathcal{H}^1=\infty.$$

From §2.5.2 we know, for instance, that the optimality assumption of this proposition is fulfilled for N being a sphere or a connected sum of $\mathbf{CP}^2$ and $S^2\times S^2$ and arbitrary

$z\neq 0$. We believe that this should be true for a large class of N (including in particular every 4-dimensional simply connected manifold). The proof of Proposition 1.4 is based on the construction corresponding to the one presented in Example 2.5 of [HR1] which deals with the case $N=S^2$ and $p=3$.

Finally we make the following observation. If $p(p+n_z)^{-1}$ is optimal, in the sense given by Definition 2.14, then the following converse of Theorem 1.1 holds: let u be an arbitrary map in $W^{1,p}(\mathbf{R}^{p+1},N)$ admitting a rectifiable Poincaré dual $(\Gamma,\vec{\Gamma},\theta)$ satisfying equation (1.1) such that

$$\int_\Gamma |\theta|^{p/(p+n_z)}\,d\mathcal{H}^1<\infty.$$

Then there exists a sequence of maps u_n in $W^{1,p}(\mathbf{R}^{p+1},N)$ satisfying $\Phi_{z,u_n}\equiv 0$ and converging weakly to u in $W^{1,p}$. The proof of this assertion is quite immediate if Γ is made of finitely many C^1-curves, but requires an approximation theorem similar to the one in [ABO, §5] for dealing with the general case.

One of the main goals of this paper is to prove Theorem 1.1. We spend some time in §2 recalling facts and establishing new tools regarding the Novikov integral representation of Sullivan's rational homotopy groups that we need to prove our main result. Generalizing known formulas for the topological degree or for the Hopf degree, we derive, for any $z\in\operatorname{Hom}(\pi_p(N),\mathbf{R})$ and $u\in C^\infty(S^p,N)$, an integral expression

$$z([u])=\int_{S^p}u^K,$$

where the p-form u^K is constructed from u pull-backs of closed forms on N by operations of wedge product and explicit (and analytically estimable) "d^{-1} integrations" using certain Gauss integrals. The combinatorial form of these operations is described by the notion of a "tree-graph" (or a finite sum of tree-graphs) associated with z, which is defined and illustrated by several specific examples, in §2.3. In §3, we discuss the z-type bubbling for a $W^{1,p}$-weakly convergent sequence $u_n\in C^\infty(S^p,N)\rightharpoonup u\in W^{1,p}(S^p,N)$. In particular, a subsequence of the p-forms u_n^K converge as Radon measures to a sum

$$u^K+\sum_{i=1}^{I}m_i\delta_{a_i},$$

where the $m_i\delta_{a_i}$ are the "bubbles". In §4, we turn to maps on $\mathbf{R}^{p+1}$, again consider $W^{1,p}$-weakly convergent sequences of smooth maps and then assemble the bubbles in a limiting 1-dimensional "scan". We describe, for a subsequence, the existence and rectifiability of this scan, and, by integration, obtain Theorem 1.1. Finally, in §5, we generalize the previous works to $W^{1,p}$-bounded smooth maps u_n from $\mathbf{R}^m$ to N with $m>p+1$. These give rise to a limiting bubble which is represented by an $(m-p)$-dimensional oriented rectifiable set with a density. The corresponding scan $\mathcal{S}$ now associates with almost every oriented p-dimensional Euclidean sphere S an atomic measure $\mathcal{S}(S)$ supported in S, and we again have the relation

$$\lim_{n'\to\infty} z([u_{n'}|_S]) = z([u|_S]) + \mathcal{S}(S)(1).$$

Note that in the particular case $m>3$, $N=S^2$ and $p=2$, this has been done by Almgren, Browder and Lieb in [ABL].

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2. Gauss forms and integral representations in rational homotopy theory

In this part we shall exhibit Gauss forms associated with the Novikov linear forms of the rational homotopy groups from a smooth compact simply connected manifold N. To that aim we need to review Sullivan's [Sul] and Novikov's results [Nov1], [Nov2], [Nov3].

2.1. Minimal models and geometric realizations

A *differential graded algebra* A over $\mathbf{R}$ is an $\mathbf{R}$-graded vector space in the form

$$A = \bigoplus_{i\geqslant 0} A^i,$$

together with a skew-commutative law

$$a\cdot b = (-1)^{\deg a \deg b}\; b\cdot a,$$

and an antiderivation of degree 1 satisfying

$$d\circ d = 0 \quad \text{and} \quad d(a\cdot b) = da\cdot b + (-1)^{\deg a} a\cdot db.$$

It is *free* when it possesses no other relation than this skew-commutative law and the associativity rule. Let $V=\operatorname{Span}\{x_1, ..., x_k\}$ be a graded $\mathbf{R}$-vector space, each x_i having a degree in $\mathbf{N}$, and let $\bigwedge(x_1, ..., x_k)$ denote the free graded (skew) commutative algebra generated by $x_1, ..., x_k$. If, for instance, $x_1, ..., x_q$ are of even degree and $x_{q+1}, ..., x_n$ are odd, then, ignoring the grading, $\bigwedge(x_1, ..., x_n)$ identifies with $S(\bigotimes_{i=1}^q x_i)\otimes\bigwedge_{j=q+1}^n x_j$. Here $\bigwedge_{j=q+1}^n x_j$ denotes the exterior algebra of $\bigotimes_{j=q+1}^n x_j$, while $S(B)$ is the symmetric algebra of B: $S(B)=B/(x\otimes y - y\otimes x)$. An element $a\in A$ is said to be *decomposable* if it is a sum of products of two elements in $A^* = \bigoplus_{i>0} A^i$. A differential graded algebra $\mathcal{M}$

is called a *minimal model* for another differential graded algebra A if $\mathcal{M}$ satisfies the following three conditions:

(i) $\mathcal{M}$ is *free*. This means that there exists a graded $\mathbf{R}$-vector space $V=\bigoplus_{i\geqslant 1} V^i$ such that $\mathcal{M}=S(V^{\text{even}})\otimes\bigwedge(V^{\text{odd}})$.

(ii) There is a morphism of differential graded algebras $\Psi\colon\mathcal{M}\to A$, called a *geometric realization* of $\mathcal{M}$, which induces an isomorphism in cohomology.

(iii) The exterior differential of a generator is either 0 or decomposable.

Since $V^0=\{0\}$, one has $\mathcal{M}^0=\mathbf{R}$. Observe that (iii) means that $dV^p\in\mathcal{M}^+\cdot\mathcal{M}^+$, where $\mathcal{M}^+$ is the maximal ideal $\mathcal{M}^+=\bigoplus_{i\geqslant 1}\mathcal{M}^i$. A minimal model $\mathcal{M}$ is said to be *simply connected* if $\mathcal{M}^1=0$. A minimal model $\mathcal{M}=S(V^{\text{even}})\otimes\bigwedge V^{\text{odd}}$ is also said to be *nilpotent* if each space V^i is finite-dimensional. A basic result is the following.

PROPOSITION 2.1. ([Sul]) *For any compact simply connected manifold N, the exterior algebra of differential forms on N, $A=\bigwedge^* N$, admits a nilpotent simply connected minimal model $\mathcal{M}_N$.*

For a proof of Proposition 2.1 see, e.g., [BT, pp. 230–231] or [GM, pp. 116–117]. The uniqueness of $\mathcal{M}_N$ (modulo isomorphism of differential graded algebras) and the uniqueness of the associated geometric realization (modulo homotopy of morphisms of differential graded algebras) is given in [GM, Theorem 10.9]. For any integer $p>1$, we have the following important identification of linear forms on $\pi_p(N)\otimes\mathbf{R}$.

THEOREM 2.2. ([Sul]) *Let $\mathcal{M}_N=S(V^{\text{even}})\otimes\bigwedge V^{\text{odd}}$ be the minimal model for the compact simply connected manifold N. The space $\operatorname{Hom}(\pi_p(N),\mathbf{R})$ is isomorphic to V^p, the vector space spanned by the generators of degree p in $\mathcal{M}_N$ (or indecomposable elements of degree p in $\mathcal{M}_N$).*

Proofs of this theorem can be found in [Sul] or [GM]. In [Sul, p. 312], an expression of this duality between V^p and $\operatorname{Hom}(\pi_p(N),\mathbf{R})$ involving some "integral expression" is explained briefly in the following way.

Let u be a map from S^p into N representing a class in $\pi_p(N)$. By pull-back, u^* induces a differential graded algebra morphism between $\bigwedge^* N$ and $\bigwedge^* S^p$. Given two geometric realizations, Ψ_N between $\mathcal{M}_N$ and $\bigwedge^* N$, and Ψ_{S^p} between $\mathcal{M}_{S^p}$ and $\bigwedge^* S^p$, one can prove that u^* lifts into a differential graded algebra morphism $\hat{u}$ between $\mathcal{M}_N$ and $\mathcal{M}_{S^p}$ such that the following diagram is commutative *modulo homotopy* (see [GM, Chapter XIV]):

$$\begin{array}{ccc} \mathcal{M}_N & \xrightarrow{\hat{u}} & \mathcal{M}_{S^p} \\ {\scriptstyle\Psi_N}\big\downarrow & & \big\downarrow{\scriptstyle\Psi_{S^p}} \\ \bigwedge^* N & \xrightarrow{u^*} & \bigwedge^* S^p. \end{array}$$

The space generated by the generators of degree p in $\mathcal{M}_{S^p}$ is isomorphic to $\mathbf{R}$. There is exactly one generator x (see the computation of $\mathcal{M}_{S^p}$ in [GM]), and this isomorphism is given by integrating $\Psi_{S^p}(x)$ over S^p. Therefore, $\hat{u}$ restricted to the space V^p generated by the generators of degree p in the model $\mathcal{M}_N = S(V^{\text{even}}) \otimes \wedge V^{\text{odd}}$ is a linear form: $\int_{S^p} \Psi_{S^p} \circ \hat{u}: V^p \to \mathbf{R}$. It is not difficult to check that it only depends on the homotopy class of u. The dual of the map

$$\begin{aligned} \pi_p(N) &\longrightarrow \operatorname{Hom}(V^p, \mathbf{R}), \\ u &\longmapsto \left(\int_{S^p} \Psi_{S^p} \circ \hat{u} \right) \bigg|_{V^p}, \end{aligned}$$

is the isomorphism between V^p and $\operatorname{Hom}(\pi_p(N), \mathbf{R})$ given by Theorem 2.2 (see [GM, Chapter XIV]).

Remark 2.3. Note that this isomorphism between V^p and $\operatorname{Hom}(\pi_p(N), \mathbf{R})$ depends on the choice of the geometric realization Ψ_N. If z is an element in V^p, we will, for simplicity, keep denoting by z the corresponding image of z in $\operatorname{Hom}(\pi_p(N), \mathbf{R})$ through this isomorphism, whenever there is no ambiguity about which geometric realization we are using.

Given a geometric realization $\Psi_N: \mathcal{M}_N \to \bigwedge^* N$, it is tempting to identify the correspondance between V^p and $\operatorname{Hom}(\pi_p(N), \mathbf{R})$ in a more tractable way—the construction of $\hat{u}$ from u^* which holds up to homotopy in differential graded algebras has to be made more explicit (see for instance this construction for $[u] \in \pi_3(N)$ in [GM, pp. 159–161]). We aim to get a procedure to construct some more concrete expression of $\int_{S^p} \Psi_{S^p} \circ \hat{u}(z)$ for the elements z in V^p involving only u and smooth differential forms in N, which will generalize the well-known integral expression of the topological degree between S^p and S^p:

$$[u] \in \pi_p(S^p) \longmapsto \int_{S^p} u^* \omega,$$

where ω generates $H^p(S^p)$, or the Hopf degree between S^{4p-1} and S^{2p}:

$$[u] \in \pi_{4p-1}(S^{2p}) \longmapsto \int_{S^{4p-1}} \eta \wedge u^* \omega,$$

where ω generates $H^{2p}(S^{2p})$ and $d\eta = \omega$.

[Nov1], [Nov2] and [Nov3] contain a relatively simple procedure to compute the linear form $\int_{S^p} \Psi_{S^p} \circ \hat{u}$ on V^p. In the next section we recall that procedure.

2.2. d-extensions of minimal models and the Hopf–Novikov integral representation of elements in $\mathrm{Hom}(\pi_p(N), \mathbf{R})$

Starting from a given geometric realization Ψ_N of the minimal model $\mathcal{M}_N$, we construct the following free extension of $\mathcal{M}_N$. Let $x_{2,1}, ..., x_{2,p_2}$ be the generators of degree 2 (i.e. $V^2 = \mathrm{Span}\{x_{2,1}, ..., x_{2,p_2}\}$). We call $C_2(\mathcal{M}_N)$ the algebra $\mathcal{M}_N$ to which we add p_2 free generators of degree 1: $y_{1,1}, ..., y_{1,p_2}$ satisfying $dy_{1,i} = x_{2,i}$ for all i. Thus

$$C_2(\mathcal{M}_N) = \mathcal{M}_N[y_{1,1}, ..., y_{1,p_2}].$$

This has the effect to kill the H^2 of $\mathcal{M}_N$, and we also have

$$H^3(C_2(\mathcal{M}_N)) \simeq V^3.$$

Indeed, for a generator $x_{3,j}$ of degree 3 of $\mathcal{M}_N$, $dx_{3,j}$ is a linear combination of wedges of degree 2:

$$dx_{3,j} = \sum_{k<l} \alpha_j^{kl} x_{2,k} \wedge x_{2,l} = \sum_{k<l} \alpha_i^{kl}\, d(y_{1,k} \wedge x_{2,l}).$$

It is straightforward to check that the family $z_{3,i} = x_{3,i} - \sum_{k<l} \alpha_i^{kl} y_{1,k} \wedge x_{2,l}$ generates $H^3(C_2(\mathcal{M}_N))$, and so we add p_3 free generators $y_{2,i}$ so that $dy_{2,i} = z_{3,i}$. We then go further in this construction until reaching the d-extension of order $p-1$ of $\mathcal{M}_N$: $C_{p-1}(\mathcal{M}_N)$. This procedure goes as follows: for $q < p-1$, $H^q(C_{q-1}(\mathcal{M}_N))$ is generated by the family of elements in the form $z_{q,i} = x_{q,i} + t_{q,i}$ for $i = 1, ..., p_q$, satisfying $dz_{q,i} = 0$, where the $x_{q,i}$ are the generators of degree q of $\mathcal{M}_N$ and the $t_{q,i}$ are elements of degree q in the ideal $I^{q-1}(C_{q-1}(\mathcal{M}_N))$ generated by the elements of degree strictly less than q in $C_{q-1}(\mathcal{M}_N)$. We pass from $C_{q-1}(\mathcal{M}_N)$ to $C_q(\mathcal{M}_N)$ by adding p_q free generators $y_{q-1,1}, ..., y_{q-1,p_q}$ satisfying

$$dy_{q-1,i} = z_{q,i} = x_{q,i} + t_{q,i}.$$

Consider then the ideal generated by the elements of degree less than or equal to q in $C_{q-1}(\mathcal{M}_N)$. It is a free graded algebra

$$I^q(C_{q-1}(\mathcal{M}_N)) = \bigwedge(y_{1,1}, ..., y_{1,p_2}, z_{2,1}, ..., z_{2,p_2}, ..., y_{q-1,1}, ..., y_{q-1,p_q}, z_{q,1}, ..., z_{q,p_q})$$

generated by elements $y_{i-1,j}$ and $z_{i,j}$ for $i = 2, ..., q$ and $j = 1, ..., p_q$, where

$$\deg y_{i-1,j} = i-1, \quad \deg z_{i,j} = i \quad \text{and} \quad dy_{i-1,j} = z_{i,j}. \tag{2.1}$$

It is then easy to verify that such a free algebra has trivial cohomology

$$H^*(I^q(C_{q-1}(\mathcal{M}_N))) = \{0\}. \tag{2.2}$$

Indeed, consider $a\in I^q(C_{q-1}(\mathcal{M}_N))$ such that $da=0$ and take $i_0\in[1,q]$ and $j_0\in[1,p_q]$ such that a contains y_{i_0-1,j_0} or z_{i_0,j_0} in its decomposition in this free algebra. Assuming for instance that i_0 is even, one has

$$a=\sum_k y_{i_0-1,j_0}\wedge z_{i_0,j_0}^k\wedge A_k+z_{i_0,j_0}^{k+1}\wedge B_k+R,$$

where A_k, B_k and R contain no y_{i_0-1,j_0} or z_{i_0,j_0} in their decompositions in linear combinations of products of generators y and z. Since $da=0$, one has

$$0=\sum_k z_{i_0,j_0}\wedge z_{i_0,j_0}^k\wedge A_k+\sum_k y_{i_0-1,j_0}\wedge z_{i_0,j_0}^k\wedge dA_k+\sum_k z_{i_0,j_0}^{k+1}\wedge dB_k+dR. \tag{2.3}$$

Because of (2.1), it is clear that dA_k, dB_k and dR contain no y_{i_0-1,j_0} or z_{i_0,j_0} in their decompositions in linear combinations of products of generators y and z. Thus, since the algebra $I^q(C_{q-1}(\mathcal{M}_N))$ is free, we have uniqueness in decompositions, and we get from (2.3) that $A_k=dB_k$. Therefore, we see that

$$a=\sum_k d(y_{i_0-1,j_0}\wedge z_{i_0,j_0}^k\wedge dB_k+z_{i_0,j_0}^{k+1}\wedge B_k)+R.$$

We may iterate this fact for R this time. After finitely many steps, we finally get that a is exact. Thus (2.2) is showed. Considering now one generator $x_{q+1,i}$ of degree $q+1$ in $C_q(\mathcal{M}_N)$, since $x_{q+1,i}$ is in $\mathcal{M}_N$, $dx_{q+1,i}$ is decomposable in $\mathcal{M}_N$ which means in particular that $dx_{q+1,i}$ is in $I^q(C_{q-1}(\mathcal{M}_N))$. Because of (2.2), there exists $t_{q+1,i}$ in $I^q(C_{q-1}(\mathcal{M}_N))$ such that $d(x_{q+1,i}+t_{q+1,i})=0$. It is moreover clear that $x_{q+1,i}+t_{q+1,i}$ is not an exact form of $C_q(\mathcal{M}_N)$. Thus, $H^{q+1}(C_q(\mathcal{M}_N))\simeq V^q$, and we have proved by induction the following lemma.

LEMMA 2.4. *With the above notation and p being a positive integer, the following spaces are isomorphic*

$$H^p(C_{p-1}(\mathcal{M}_N))\simeq V^p\simeq \mathrm{Hom}(\pi_p(N),\mathbf{R}). \tag{2.4}$$

Going back now to the question of finding a procedure for getting explicit expressions of the integral representations $\int_{S^p}\Psi_{S^p}\circ\hat{u}(z)$ for arbitrary $[u]\in\pi_p(N)$ and arbitrary $z\in V^p$, we proceed as follows. We first construct a d-continuation $\tilde{u}$ of u^* between $C_{p-1}(\mathcal{M}_N)$ and $\bigwedge^* S^p$. Contrary to the case of $\hat{u}$, where this lifting existed only modulo homotopies of differential graded algebras, there is here a procedure to get $\tilde{u}$ which goes by induction as follows. First we construct $\tilde{u}$ between $C_1(\mathcal{M}_N)=\mathcal{M}_N$ and $\bigwedge^* S^p$ by taking $\tilde{u}(x)=u^*\Psi_N(x)$. Suppose $p>2$. In order to construct $\tilde{u}$ between $C_2(\mathcal{M}_N)$ and $\bigwedge^* S^p$, we just have to define the images of the $y_{1,j}$ by $\tilde{u}$ and, in order to have a morphism of differential graded algebras, they have to verify, in particular, $d\tilde{u}(y_{1,j})=\tilde{u}(x_{2,j})=u^*\Psi_N(x_{2,j})$. We look for a specific operation d^{-1} on $\bigwedge^k S^p$ for $0<k<p$. It must satisfy $d(d^{-1}\alpha)=\alpha$ for every closed form $\alpha\in\bigwedge^k S^p$. We may, using the standard metric on S^p, take the "Coulomb gauge"

$$d^{-1}=d^*\Delta^{-1},$$

where Δ is the Hodge-Laplacian dd^*+d^*d and d^* is the Hodge adjoint differential $d^*=(-1)^{p(k+1)+1}*d*$. Here Δ is invertible because $H^k(S^p)=0$. As we will see below, other operations d^{-1} can also be very useful. We will often consider the one given by (2.12) which corresponds to the Coulomb gauge but with respect to the flat metric on $\mathbf{R}^p$ after pull-back by the inverse of the stereographic projection.

Now fixing such an operation d^{-1}, we take $\tilde{u}(y_{1,j})=d^{-1}\tilde{u}(x_{2,j})$. The construction of $\tilde{u}$ then goes further, following the inductive construction we made for $C_q(\mathcal{M}_N)$ by taking for $y_{q,j}$ $(q+1<p)$

$$\tilde{u}(y_{q,j})=d^{-1}\tilde{u}(z_{q+1,j})=d^{-1}\tilde{u}(x_{q+1,j}+t_{q+1,j})=d^{-1}u^*\Psi_N(x_{q+1,j})+d^{-1}\tilde{u}(t_{q+1,j}).$$

Once $\tilde{u}$ is completely constructed, it is then straightforward to verify that

$$\int_{S^p}\Psi_{S^p}\circ\hat{u}(z)=\int_{S^p}\tilde{u}([z]),$$

where $[z]$ is the class in $H^p(C_{p-1}(\mathcal{M}_N))$ corresponding to $z\in V^p$ via the isomorphism (2.4) constructed by induction, and the specific differential form $\tilde{u}([z])$ has been constructed by induction described just above. We have then an explicit procedure to construct the integral representation of the elements in $\mathrm{Hom}(\pi_p(N),\mathbf{R})$ starting from a geometric realization Ψ_N of the minimal model $\mathcal{M}_N$. Following the procedure, the forms $\tilde{u}([z])$ can be described with the help of graphs. Suppose that, for each i, the degree-i forms $\omega_{i,j}$ give a basis for the degree-i part of the geometric realization Ψ_N of the minimal model $\mathcal{M}_N$. Thus,

$$\mathrm{Span}_{i,j}\{\omega_{i,j}\}=\Psi_N(\mathcal{M}_N)\subset\bigwedge^* N.$$

It is straightforward to observe that $\tilde{u}([z])$ is a finite linear combination of p-forms obtained as follows.

Each p-form is obtained by first constructing a connected, simply connected, oriented planar *tree-graph* K as shown in Figure 1.

The tree-graph contains finitely many vertices in the plane and finitely many non-horizontal, vertically-oriented, segments connecting these vertices. Two vertices are connected by at most one segment, and the graph is assumed to be simply connected, that is, it contains no closed path. To each vertex A is also assigned a fixed closed element

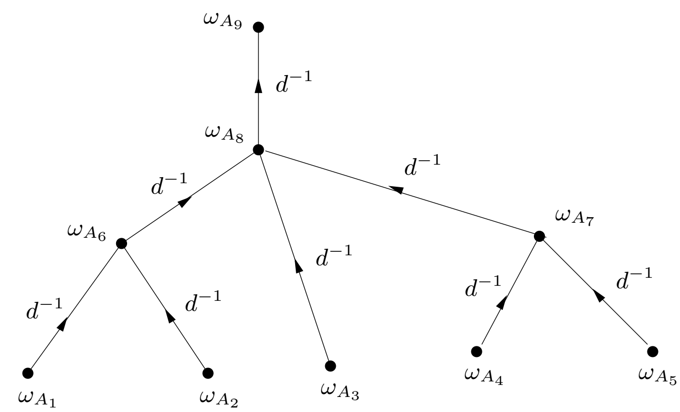

Figure 1. An example of a tree-graph arising in computing $\mathrm{Hom}(\pi_p(N), \mathbf{R})$.

ω_A of the ideal generated by $\Psi_N(\mathcal{M}_N)$. At each vertex, except one, there is exactly one segment leaving. The exceptional vertex is at the "end" or top of the graph where all the attached segments are arriving. Such a tree-graph is called a (p-dimensional) *oriented tree-graph of forms*.

There is the single important p-form u^K for a given map $u: S^p \to N$ and such a tree-graph K (associated with an element z in $\mathrm{Hom}(\pi_p(N), \mathbf{R})$). The form u^K will be constructed using the pull-backs $u^*\omega_A$ corresponding to each vertex A of K. We will obtain u^K inductively moving up the tree-graph. Each segment of K will correspond to an integration procedure d^{-1} and each connection of a segment to a vertex will correspond to taking a wedge product.

More precisely, u^K is obtained as follows. At each of the bottom vertices B_i (having no arriving and one departing segment) we start with the form $u^*\omega_{B_i}$. Any other vertex A_0 in K is connected via arriving segments to finitely many lower vertices, say $A_1, ..., A_j$. These are ordered left-to-right according to the location of their segments joining A_0. Note that by restricting K downward, we obtain sub-tree-graphs $K_0, K_1, ..., K_j$ whose top vertices are, respectively, $A_0, A_1, ..., A_j$. Assuming inductively that we have already defined the forms $u^{K_1}, ..., u^{K_j}$, we now define the form

$$u^{K_0} = u^*\omega_{A_0} \wedge \eta_{A_1} \wedge ... \wedge \eta_{A_j}, \quad \text{where } \eta_{A_1} = d^{-1}u^{K_1}, \ \ ..., \ \ \eta_{A_j} = d^{-1}u^{K_j}. \tag{2.5}$$

Continuing going up the tree-graph, we eventually get the desired form u^K when we reach the summit. For instance, the form u^K given by the graph in Figure 1 is

$$u^K = u^*\omega_{i_9,j_9} \wedge d^{-1}[u^*\omega_{i_8,j_8} \wedge d^{-1}k_1 \wedge d^{-1}(u^*\omega_{i_3,j_3}) \wedge d^{-1}k_2],$$

where

$$k_1 = u^*\omega_{i_6,j_6} \wedge d^{-1}(u^*\omega_{i_1,j_1}) \wedge d^{-1}(u^*\omega_{i_2,j_2}),$$
$$k_2 = u^*\omega_{i_7,j_7} \wedge d^{-1}(u^*\omega_{i_4,j_4}) \wedge d^{-1}(u^*\omega_{i_5,j_5}).$$

The graph has to be read from left to right: if K_1 and K_2 are two subgraphs connecting a node A, if K denotes the graph made of these two subgraphs union the node A and the two segments starting respectively from the summit of K_1 and the summit of K_2, and if K_1 is at the left of K_2, then u^K is obtained by respecting the left-right order, that is

$$u^K = u^*\omega_A \wedge d^{-1}(u^{K_1}) \wedge d^{-1}(u^{K_2}).$$

Similarly, one has a form u^L corresponding to any sub-tree-graph L of K whose summit is some vertex of K. In general

$$\deg u^L = \left(\sum_{A \text{ vertex of } L} \deg \omega_A \right) - n^K, \tag{2.6}$$

where n^K is the number of segments of L, that is one less than the number of its vertices, and $\deg u^L \leqslant p = \deg u^K$, with equality if and only if $L=K$. We have established the following result.

Proposition 2.5. *To a compact simply connected manifold N, an element z in $(\pi_p(N)\otimes\mathbf{R})^*$ and any geometric realization Ψ_N of the minimal model of N, one assigns, using the notation above, a formal real linear combination of tree-graphs $K=\sum_i \lambda_i K_i$ such that for any class $[u]$ in $\pi_p(N)$, represented by a map $u\in C^\infty(S^p, N)$, one has*

$$z([u]) = \int_{S^p} \Psi_{S^p} \circ \hat{u}(z) = \int_{S^p} \tilde{u}([z]) = \int_{S^p} u^K,$$

where $u^K = \sum_i \lambda_i u^{K_i}$. Starting from Ψ_N and z, the formal linear combination of tree-graphs $K=\sum_i \lambda_i K_i$ is given by the algorithm described above in this subsection.

Remark 2.6. For a Sobolev map $u\in W^{1,p}(S^p, N)$, the p-form u^K is defined $\mathcal{H}^p$-almost everywhere on S^p and is $\mathcal{H}^p$-integrable, and the equation

$$z([u]) = \int_{S^p} u^K$$

is still valid.

This is immediate from [Wh] and Proposition 2.5 because $\int_{S^p} u_n^K \to \int_{S^p} u^K$ whenever $u_n \in C^\infty(S^p, N) \to u$ *strongly* in $W^{1,p}$. (With only weak $W^{1,p}$ convergence, there may be additional "bubbled" limiting terms, as discussed below in §3.)

Figure 2. The two tree-graphs arising in computing $\mathrm{Hom}(\pi_2(\mathbf{CP}^2),\mathbf{R})$ and $\mathrm{Hom}(\pi_5(\mathbf{CP}^2),\mathbf{R})$.

2.3. Three examples

We give here examples of the application of the algorithm above to express elements of $\mathrm{Hom}(\pi_p(N),\mathbf{R})$ in terms of formal linear combinations of tree-graphs.

Example 1. $N=\mathbf{CP}^2$.

We first construct the minimal model of $\mathbf{CP}^2$ and a geometric realization of it. Let ω be the Kähler form on $\mathbf{CP}^2$. It is easy to check that $\mathcal{M}_{\mathbf{CP}^2}$ is generated by two elements α and β of degrees 2 and 5, respectively, satisfying

$$d\beta=\alpha^3,\quad \Psi_{\mathbf{CP}^2}(\alpha)=\omega \quad\text{and}\quad \Psi_{\mathbf{CP}^2}(\beta)=0.$$

Therefore only $\pi_2(\mathbf{CP}^2)$ and $\pi_5(\mathbf{CP}^2)$ have a nontorsion part. We have

$$C_1(\mathcal{M}_{\mathbf{CP}^2})=\mathcal{M}_{\mathbf{CP}^2} \quad\text{and}\quad C_4(\mathcal{M}_{\mathbf{CP}^2})=S(\alpha)\otimes\bigwedge[a,\beta],$$

where a is of degree 1, and satisfies $da=\alpha$. So we have that $H^2(C_1(\mathcal{M}_{\mathbf{CP}^2}))\simeq V^2$ is generated by α and $H^5(C_4(\mathcal{M}_{\mathbf{CP}^2}))\simeq V^5$ is generated by $\beta-a\wedge\alpha^2$. The tree-graphs associated with these two elements are, respectively, for α, a vertex alone with ω assigned to it and, for β, two vertices connected by one segment going from ω to ω^2 (see Figure 2). The corresponding integral expressions are

$$\alpha([u])=\int_{S^2}u^*\omega \quad\text{and}\quad \beta([u])=\int_{S^5}u^*\omega^2\wedge d^{-1}(u^*\omega).$$

Example 2. $N=S^2\times S^2$.

Let $\xi_1,\xi_2:\mathbf{R}^3\times\mathbf{R}^3\to\mathbf{R}^3$ be the projections of the three first (resp. three last) coordinates, and let $\omega_i=\xi_i^*\omega$, where ω is a given generator of $H^2(S^2)$. Easy computations give $\mathcal{M}_{S^2\times S^2}=S(\alpha_1,\alpha_2)\otimes\bigwedge[\beta_1,\beta_2]$, where the α_i's are of degree 2, whereas the β_j's are of degree 3, and the following hold:

$$d\beta_1=\alpha_1^2 \quad \text{and} \quad d\beta_2=\alpha_2^2.$$

We can chose ω_1 and ω_2 such that $\omega_1\wedge\omega_1\equiv 0$ and $\omega_2\wedge\omega_2\equiv 0$. Therefore, we have the following geometric realization:

$$\Psi_{S^2\times S^2}(\alpha_i)=\omega_i \quad \text{and} \quad \Psi_{S^2\times S^2}(\beta_i)=0.$$

Only $\pi_2(S^2\times S^2)$ and $\pi_3(S^2\times S^2)$ have nontorsion parts. One has

$$C_1(\mathcal{M}_{S^2\times S^2})=\mathcal{M}_{S^2\times S^2} \quad \text{and} \quad C_2(\mathcal{M}_{S^2\times S^2})=S(\alpha_1,\alpha_2)\otimes\bigwedge[\beta_1,\beta_2,a_1,a_2],$$

where $da_i=\alpha_i$. So $H^2(C_1(\mathcal{M}_{S^2\times S^2}))\simeq V^2$ is generated by α_1 and α_2 and

$$H^3(C_2(\mathcal{M}_{S^2\times S^2}))\simeq V^3$$

is generated by $\beta_1-a_1\alpha_1$ and $\beta_2-a_2\alpha_2$. The tree-graphs associated with these elements are, for α_1 (resp. α_2), one vertex to which ω_1 (resp. ω_2) is assigned, and, for β_1 (resp. β_2), two vertices connected by one segment going from ω_1 to ω_1 (resp. ω_2 to ω_2). The corresponding integrals are

$$\alpha_i([u])=\int_{S^2}u^*\omega_i \quad \text{and} \quad \beta_i([u])=\int_{S^3}u^*\omega\wedge d^{-1}(u^*\omega).$$

Example 3. $N=(S^2\times S^2)\#\mathbf{CP}^2$.

The manifold N is the connected sum of the two 4-manifolds that we have studied in Examples 1 and 2. Let $\mathcal{M}_N^4$ denote the ideal in $\mathcal{M}_N$ generated by the elements of degree less than or equal to 5. We shall only compute the integral expressions of the elements in $\operatorname{Hom}\big(\pi_p(\mathbf{CP}^1\times\mathbf{CP}^1)\#\mathbf{CP}^2,\mathbf{R}\big)$ for $p\leqslant 4$. After some computations one gets

$$\mathcal{M}_N^4=S(\alpha_1,\alpha_2,\alpha_3,\gamma_1,\gamma_2,\gamma_3,\gamma_{12},\gamma_{23})\otimes\bigwedge[\beta_{11},\beta_{22},\beta_{13},\beta_{23},\beta_{123}],$$

where the following relations hold:

(i) for arbitrary i and j such that β_{ij} exists, one has $d\beta_{ij}=\alpha_i\wedge\alpha_j$, and

$$d\beta_{123}=\alpha_1\wedge\alpha_2-\alpha_3^2;$$

(ii) one has

$$
\begin{aligned}
d\gamma_1 &= \alpha_3\beta_{11} - \alpha_1\beta_{13},\\
d\gamma_2 &= \alpha_3\beta_{22} - \alpha_2\beta_{23},\\
d\gamma_3 &= \alpha_1\beta_{23} - \alpha_2\beta_{13},\\
d\gamma_{13} &= \alpha_1\beta_{123} - \alpha_2\beta_{11} + \alpha_3\beta_{13},\\
d\gamma_{23} &= \alpha_2\beta_{123} - \alpha_1\beta_{22} + \alpha_3\beta_{23}.
\end{aligned}
$$

One also has

$$C_2(\mathcal{M}_N^4) = \mathcal{M}_N^4[a_1, a_2, a_3],$$

where $da_i = \alpha_i$ and $H^3(C_2(\mathcal{M}_N^5)) \simeq V^3$ is generated by $\beta_{ij} - a_i\alpha_j$ and $\beta_{123} - a_1\alpha_2 + a_3\alpha_3$. One has

$$C_3(\mathcal{M}_N^4) = C_2(\mathcal{M}_N^4)[b_{11}, b_{22}, b_{13}, b_{23}, b_{123}],$$

where $db_{ij} = \beta_{ij} - a_i\alpha_j$, $db_{123} = \beta_{123} - a_1\alpha_2 + a_3\alpha_3$ and $H^4(C_3(\mathcal{M}_N^4)) \simeq V^4$ is generated by

$$
\begin{aligned}
&\gamma_1 - \alpha_3 b_{11} + \alpha_1 b_{13},\\
&\gamma_2 - \alpha_3 b_{22} + \alpha_2 b_{23},\\
&\gamma_3 - \alpha_1 b_{23} + \alpha_2 b_{13} - a_1 a_2 \alpha_3,\\
&\gamma_{13} - \alpha_1 b_{123} + \alpha_2 b_{11} - \alpha_3 b_{13} + a_1 a_3 \alpha_3,\\
&\gamma_{23} - \alpha_2 b_{123} + \alpha_1 b_{22} - \alpha_3 b_{23} + a_2 a_3 \alpha_3.
\end{aligned}
$$

Letting $\Psi_N(\alpha_i) = \omega_i$, it is not difficult to see that we can choose ω_1, ω_2 and ω_3 representatives in $\bigwedge^2 N$ of a basis of $H^2(N)$ (generating $H^*(N)$) satisfying

$$\omega_1 \wedge \omega_1 \equiv 0, \quad \omega_2 \wedge \omega_2 \equiv 0, \quad \omega_1 \wedge \omega_3 \equiv 0, \quad \omega_2 \wedge \omega_3 \equiv 0 \quad \text{and} \quad \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_3 = 0.$$

The goal is to simplify the geometric realization Ψ_N. Let η_{123} be a form such that

$$d\eta_{123} = \omega_1 \wedge \omega_2 - \omega^3.$$

The geometric realization Ψ_N restricted to $\mathcal{M}_N^4$ is then defined by

$$\Psi_N(\alpha_i) = \omega_i, \quad \Psi_N(\beta_{ij}) = 0, \quad \Psi_N(\beta_{123}) = \eta_{123}, \quad \Psi_N(\gamma_i) = 0 \quad \text{and} \quad \Psi_N(\gamma_{ij}) = 0$$

for every β_{ij}, γ_i and γ_{ij} defined above.

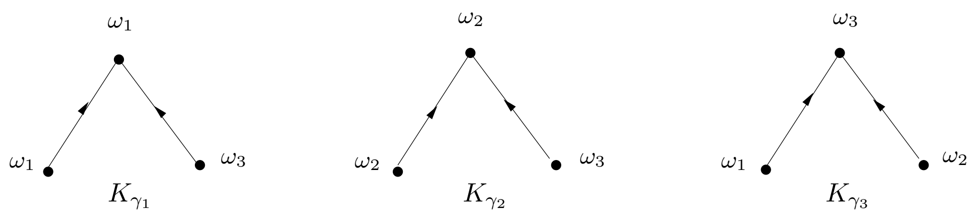

Figure 3. The three first tree-graphs K_{γ_1},K_{γ_2} and K_{γ_3}.

Given a smooth map u from S^4 into N, the d-continuation $\tilde{u}$ of u^* between $C_3(\mathcal{M}_N^4)$ into $\bigwedge^* S^4$ is defined by

$$\begin{aligned}
\tilde{u}(\cdot) &= u^*(\Psi_N(\cdot)) && \text{on } \mathcal{M}_N^4,\\
\tilde{u}(a_i) &= d^{-1}(u^*\omega_i),\\
\tilde{u}(b_{ij}) &= -d^{-1}(d^{-1}(u^*\omega_i)\wedge u^*\omega_j),\\
\tilde{u}(b_{123}) &= d^{-1}(u^*\eta_{123} - d^{-1}(u^*\omega_1)\wedge u^*\omega_2 - d^{-1}(u^*\omega_3)\wedge u^*\omega_3).
\end{aligned}$$

Thus the following forms are generating $(\pi_4((S^2\times S^2)\#\mathbf{CP}^2)\otimes\mathbf{R})^*$:

$$\begin{aligned}
\gamma_1([u]) &= \int_{S^4} u^*\omega_1\wedge d^{-1}(u^*\omega_1)\wedge d^{-1}(u^*\omega_3),\\
\gamma_2([u]) &= \int_{S^4} u^*\omega_2\wedge d^{-1}(u^*\omega_2)\wedge d^{-1}(u^*\omega_3),\\
\gamma_3([u]) &= \int_{S^4} u^*\omega_3\wedge d^{-1}(u^*\omega_1)\wedge d^{-1}(u^*\omega_2),\\
\gamma_{13}([u]) &= \int_{S^4} -u^*\eta_{123}\wedge d^{-1}(u^*\omega_2) + u^*\omega_3\wedge d^{-1}(u^*\omega_1)\wedge d^{-1}(u^*\omega_2)\\
&\quad + \int_{S^4} u^*\omega_3\wedge d^{-1}(u^*\omega_3)\wedge d^{-1}(u^*\omega_2) + u^*\omega_1\wedge d^{-1}(u^*\omega_1)\wedge d^{-1}(u^*\omega_2),\\
\gamma_{23}([u]) &= \int_{S^4} -u^*\eta_{123}\wedge d^{-1}(u^*\omega_1) + u^*\omega_3\wedge d^{-1}(u^*\omega_2)\wedge d^{-1}(u^*\omega_1)\\
&\quad + \int_{S^4} u^*\omega_3\wedge d^{-1}(u^*\omega_3)\wedge d^{-1}(u^*\omega_1) + u^*\omega_2\wedge d^{-1}(u^*\omega_2)\wedge d^{-1}(u^*\omega_1),
\end{aligned}$$

for all $u\in C^1(S^4,(S^2\times S^2)\#\mathbf{CP}^2)$.

Remark 2.7. Observe that we can restrict to graphs having only *closed* forms.

Indeed, let $\mathcal{M}_N^k$ be the minimal model at the stage k (i.e. the ideal generated by the elements of degree less than or equal to k). Suppose that an element ξ of degree $k+1$ is introduced in the minimal model in order to kill some closed polynomial expression

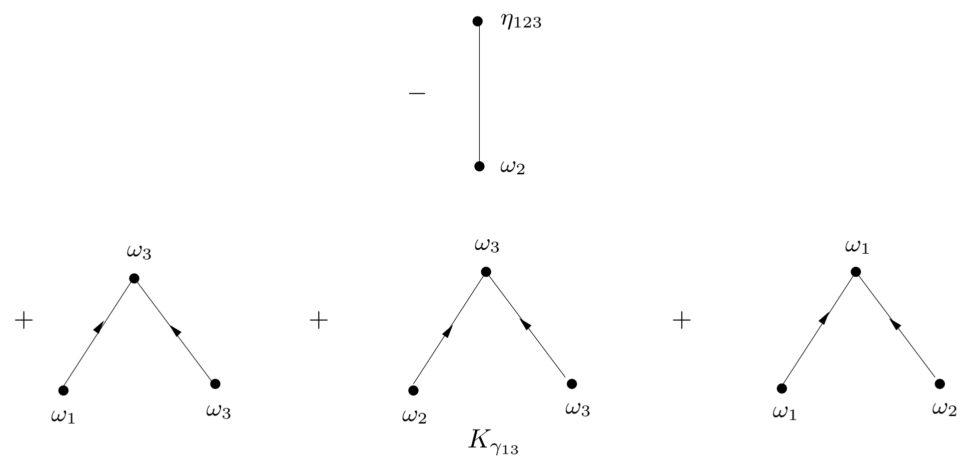

Figure 4. The linear combination of $K_{\gamma_{13}}$ arising in computing $\mathrm{Hom}(\pi_4(S^2\times S^2\#\mathbf{CP}^2),\mathbf{R})$.

$P(x_1, ..., x_l)$ of elements from $\mathcal{M}_N^k$, and ξ is not exact in $\mathcal{M}_N^k$ but $\Psi_N(\xi)$ is exact in N. Then, we can decide that $\tilde{u}(\xi)=d^{-1}(u^*\Psi_N(P(x_1, ..., x_l)))$. This modifies the graph as shown in the following example: replace for instance $u^*\eta_{123}=\tilde{u}(\beta_{123})$ by

$$d^{-1}(u^*(\omega_1\omega_2-\omega_3^2)).$$

From the graph point of view this corresponds to the change described in Figure 5.

2.4. Gauss forms associated with elements in $\mathrm{Hom}(\pi_p(N),\mathbf{R})$

To make analytic estimates, we need to have an *explicit* expression for evaluating an element of $\mathrm{Hom}(\pi_p(N),\mathbf{R})$. In this subsection we will define one specific integration operation d^{-1} by introducing certain Gauss forms associated with the tree-graphs described in §2.2.

The Gauss forms are easier to describe explicitly with formulas in $\mathbf{R}^p$ instead of S^p. In this section we will consider, in place of $C^\infty(\bigwedge^q S^p)$, the subspace $\bigwedge^q_{\mathrm{slow}}\mathbf{R}^p$ of smooth q-forms ω in $\bigwedge^q\mathbf{R}^p$ satisfying

$$(1+|x|^2)^{k+q}|\nabla\omega|(x)\leqslant C_{\omega,k}.$$

In particular, if

$$\pi: S^p\setminus\{(0, ..., 0, 1)\}\longrightarrow\mathbf{R}^p$$

denotes stereographic projection, then the pull-back by π of any form in $\bigwedge^q_{\mathrm{slow}}\mathbf{R}^p$ gives a smooth q-form on S^p.

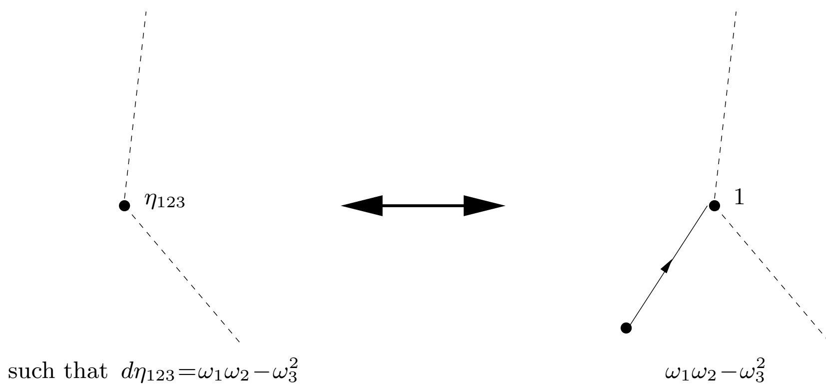

Figure 5. Replacing nonclosed forms in tree-graphs by closed ones

Let G be the Green function for the Laplacian on $\mathbf{R}^p$:

$$G(x)=C_p|x|^{2-p} \quad \text{for } p>2 \quad \text{and} \quad G(x)=C_2\log|x| \quad \text{for } p=2,$$

where $C_p=(n-2)^{-1}|S^{p-1}|^{-1}$ and $C_2=-(2\pi)^{-1}$. Given a q-form ω in $\bigwedge^q_{\text{slow}}\mathbf{R}^p$,

$$\omega=\sum_I \omega_I\, dx_I,$$

where $I=(i_1,...,i_p)$ runs over all q-tuples such that $1\leqslant i_1<i_2<...<i_q\leqslant p$, we define the operator

$$d^{-1}\omega=d^*\Delta^{-1}\omega=d^*\sum_I G\star\omega_I\, dx_I,$$

where $dx_I=dx_{i_1}\wedge...\wedge dx_{i_q}$, the first symbol $*$ is the Hodge operator for the flat metric on $\mathbf{R}^p$ and the second symbol $\star$ is the convolution operator. Observe that for $\omega\in\bigwedge^q_{\text{slow}}\mathbf{R}^p$, the convolution $\omega_I\star G$ is well defined. If ω is closed, it is clear that

$$d(d^{-1}\omega)=\omega.$$

We claim that there exists a form $\mathcal{G}^p_q$ in $(\bigwedge^{q-1}\mathbf{R}^p_x)\wedge(\bigwedge^{p-q}\mathbf{R}^p_y)$ such that for any ω in $\bigwedge^q_{\text{slow}}\mathbf{R}^p$ one has

$$[d^*\Delta^{-1}\omega](x)=\int_{y\in\mathbf{R}^p}\omega(y)\wedge\mathcal{G}^p_q(x,y). \tag{2.7}$$

Indeed, we have by definition

$$[d^*\Delta^{-1}\omega](x)=(-1)^{q(p-q)}*\sum_I[d(G\star\omega_I)\wedge *dx_I].$$

Letting I^c denote the $(p-q)$-tuple made of the complement of I ordered in such a way that $dx_{I^c}=*dx_I$ (i.e. $dx_I\wedge dx_{I^c}=\omega_{\mathbf{R}^p}=dx_1\wedge...\wedge dx_p$), we then have

$$\begin{aligned} d^*\Delta^{-1}\omega(x) &= C_p(-1)^{q(p-q)}\sum_I *\int_{y\in\mathbf{R}^p}\omega_I(y)\,d\frac{1}{|x-y|^{p-2}}\wedge dx_{I^c}\wedge\omega_{\mathbf{R}^p}(y) \\ &= C_p(-1)^{q(p-q)}(2-p)*\sum_i\sum_I\int_{\mathbf{R}^p}\omega_I(y)\frac{x_i-y_i}{|x-y|^p}\,dx_i\wedge dx_{I^c}\wedge\omega_{\mathbf{R}^p}(y). \end{aligned}$$

Also, writing $dx_{I_k}=dx_{i_1}\wedge...\wedge dx_{i_{k-1}}\wedge dx_{i_{k+1}}\wedge...\wedge dx_{i_q}$, we see that

$$*(dx_{i_k}\wedge dx_{I^c})=(-1)^{k-1}(-1)^{(q-1)(p-q)}dx_{I_k}.$$

With this notation, one has

$$\begin{aligned} d^*\Delta^{-1}\omega(x) &= C_p(-1)^{(p-q)}(2-p)\sum_I\int_{y\in\mathbf{R}^p}\omega_I(y)\omega_{\mathbf{R}^p}(y)\wedge\sum_{k=1}^q(-1)^{k-1}\frac{x_{i_k}-y_{i_k}}{|x-y|^p}\,dx_{I_k} \\ &= C_p(-1)^{(p-q)}(2-p)\int_{y\in\mathbf{R}^p}\omega(y)\wedge\sum_J\sum_{k=1}^q(-1)^{k-1}\frac{x_{j_k}-y_{j_k}}{|x-y|^p}\,dy_{J^c}\wedge dx_{J_k}. \end{aligned}$$

Thus the form

$$\mathcal{G}_q^p(x,y)=|S^{p-1}|^{-1}(-1)^{(p-q)}\sum_J\sum_{k=1}^q(-1)^{k-1}\frac{x_{j_k}-y_{j_k}}{|x-y|^p}dy_{J^c}\wedge dx_{J_k} \tag{2.8}$$

solves (2.7).

We introduce now the notion of a Gauss form associated with a tree-graph of closed forms. Let N be a compact simply connected manifold. Consider a geometric realization Ψ_N of the minimal model $\mathcal{M}_N$ from N, and consider a tree-graph K of forms from the ideal generated by $\Psi_N(\mathcal{M}_N)$. Let A_i be the vertices of the graph, and let ω_{A_i} denote the closed form assigned to A_i. Let n_K be the number of segments in the graph. Let p_i be the degree of ω_{A_i}. To each vertex, we assign two variables x^i in $\mathbf{R}^p$ and y^i in N. To each vertex A_i we assign the sub-tree-graph K_i of K whose summit is A_i and which is made of the segments and vertices "below" A_i (that is, the part of the graph connected to A_i as one follows the positively oriented paths ending at the summit of K). See Figure 6.

We denote by n_i the degree of the form obtained from this graph:

$$n_i=\Bigg(\sum_{A\text{ vertex in }K_i}\deg\omega_A\Bigg)-n_{K_i},$$

where n_{K_i} denotes, as before, the number of segments in K_i. If A_i is a "starting vertex" with no other vertex below in the graph, then K_i is just made of A_i and $n_i=p_i$. We

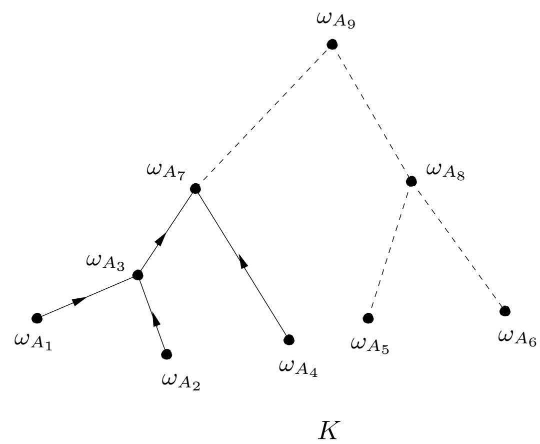

Figure 6. Plain lines correspond to the sub-tree-graph K^7 with summit A_7.

denote by $\mathcal{I}$ the set of pairs of indices (i_1, i_2) such that A_{i_1} and A_{i_2} are connected by a segment in K going from A_{i_2} to A_{i_1}. The total form of the tree-graph is a form denoted by $\omega^K \wedge \mathcal{G}^K$ in $\bigwedge_i \bigwedge^{p_i} N \bigwedge_{(i_1,i_2)\in\mathcal{I}} \bigwedge^{n_{i_2}-1} \mathbf{R}^p \wedge \bigwedge^{p-n_{i_2}} \mathbf{R}^p$, where ω^K is defined by

$$\omega^K = \bigwedge_i \omega_i(y^i), \tag{2.9}$$

and $\mathcal{G}^K$, the Gauss form associated with the tree-graph K of forms, is defined by

$$\mathcal{G}^K = \bigwedge_{(i_1,i_2)\in\mathcal{I}} \mathcal{G}^p_{n_{i_2}}(x_{i_1}, x_{i_2}). \tag{2.10}$$

With any map $u\in C^\infty(S^p, N)$, we associate the map U_K from $(\mathbf{R}^n)^{n_K+1}$ into N^{n_K+1} defined by

$$U_K(x_1, x_2, ..., x_{n_K+1}) = ((u\circ\pi^{-1})(x_1), (u\circ\pi^{-1})(x_2), ..., (u\circ\pi^{-1})(x_{n_K+1})).$$

Recall that n_K, the number of segments in a tree-graph, is one less than the number of nodes. Using the multiple "coordinates" $(y^1, ..., y^{n_K+1})$ on N^{n_K+1}, it is then easy to verify that

$$\int_{S^p} u^K = \int_{x_1} ... \int_{x_{n_K+1}} U_K^*(\omega^K) \wedge \mathcal{G}^K, \tag{2.11}$$

where the integration operation d^{-1} of a form α in $\bigwedge^k S^p$ is given by

$$d^{-1}\alpha = \pi^*\biggl(\int_{y\in\mathbf{R}^p} (\pi^{-1})^*\alpha(y) \wedge \mathcal{G}^p_k(x, y)\biggr). \tag{2.12}$$

Consider now a class z in $\mathrm{Hom}(\pi_p(N), \mathbf{R})$. Suppose that $K = \sum_l \lambda_l K^l$ is the formal linear combination of tree-graphs of closed forms associated with z (for a given geometric realization Ψ_N on $\mathcal{M}_N$), and that u^{K^l} are the associated p-forms, constructed in the previous subsection. With the p-form

$$u^K = \sum_l \lambda_l u^{K^l},$$

we now have, using the above notation,

$$z([u]) = \int_{S^p} u^K = \sum_l \lambda_l \int_{(\mathbf{R}^p)^{n_{K^l}+1}} U^*_{K^l}(\omega^{K^l}) \wedge \mathcal{G}^{K^l}. \tag{2.13}$$

We have thus succeeded in expressing the action of any element in $\mathrm{Hom}(\pi_p(N), \mathbf{R})$ on $\pi_p(N)$ as a linear combination of pull-backs by u of closed forms depending only on the class we chose in $\mathrm{Hom}(\pi_p(N), \mathbf{R})$ (modulo of course a choice of geometric realization). This was the main goal of this section. This generalizes the integral expression of the topological degree as the integral of the pull-back of a form. This will be extremely useful for analytic estimates throughout the rest of the paper.

We establish now a slightly different form of the integral expression of $z([u])$ which is of particular geometrical interest. We first make the following observation.

LEMMA 2.8. *Let $\Pi_{q,p}$ denote the canonical projection*

$$\bigwedge^{p-1}(\mathbf{R}^p_x \times \mathbf{R}^p_y) \longrightarrow \bigwedge^{q-1}\mathbf{R}^p_x \wedge \bigwedge^{p-q}\mathbf{R}^p_y.$$

The following identity holds:

$$\Pi_{p,q}\left[\left(\frac{x-y}{|x-y|}\right)^* \sum_{k=1}^{p} (-1)^{k-1} X_k (dX)^k\right] = (-1)^{(q-1)(p-q)} |S^{p-1}|^{-1} \mathcal{G}^p_q(x,y), \tag{2.14}$$

where $(dX)^k$ is the $(p-1)$-form in $\mathbf{R}^p$ given by $dX_1 \dots dX_{k-1} dX_{k+1} \dots dX_p$ and $\mathcal{G}^p_q(x,y)$ is the form given by (2.8).

Proof. A classical computation gives

$$\left(\frac{x-y}{|x-y|}\right)^* \sum_{k=1}^{p} (-1)^{k-1} X_k (dX)^k = \sum_{k=1}^{p} (-1)^{k-1} \frac{x_k - y_k}{|x-y|^p} (d(x-y))^k, \tag{2.15}$$

where $(d(x-y))^k = d(x_1-y_1) \dots d(x_{k-1}-y_{k-1}) d(x_{k+1}-y_{k+1}) \dots d(x_p-y_p)$. With the previous notation, $dx_{I^c} \wedge dx_{I_k} = (-1)^{k-1}(-1)^{(q-1)(p-q)} * dx_{i_k}$, and therefore it is clear that

$$\begin{aligned} \Pi_{p,q}&\left[\sum_{k=1}^{p} (-1)^{k-1} \frac{x_k - y_k}{|x-y|^p} (d(x-y))^k\right] \\ &= (-1)^{(q-1)(p-q)} \sum_J \sum_{k=1}^{q} (-1)^{k-1} \frac{x_{j_k} - y_{j_k}}{|x-y|^p}\, dy_{J^c} \wedge dx_{J_k}. \end{aligned} \tag{2.16}$$

The desired equality (2.14) then follows from (2.16). □

We now prove the following result.

LEMMA 2.9. *Let K be a tree-graph of forms in $\bigwedge^* N$ of dimension p. Let*

$$u\in C^\infty(S^p,N).$$

Then the following formula holds:

$$\begin{aligned}\int_{S^p} u^K &= \int_{x_1}\cdots\int_{x_{n_K}} U_K^*(\omega^K)\wedge\mathcal{G}^K\\ &= |S^{p-1}|^{-n_K}(-1)^{m_K}\int_{x_1}\cdots\int_{x_{n_K}} U_K^*(\omega^K)\wedge\bigwedge_{(i_1,i_2)\in\mathcal{I}}\left(\frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}\right)^*\Omega_{S^{p-1}},\end{aligned}\tag{2.17}$$

where $\Omega_{S^{n-1}}=\sum_{k=1}^n(-1)^{k-1}X_k(dX)^k$ and m_K is some integer depending only on K.

Proof. This can be proved by induction on the number of segments in the graph. Consider an end node i_2 in the graph (which has no segment pointing to it), and let i_1 be the node to which i_2 is connected. Let ω_{i_1} and ω_{i_2}, of degree n_{i_1} and n_{i_2}, respectively, be the forms assigned to each of these two nodes. By the previous lemma and dimensional reasons, it is clear that

$$\begin{aligned}|S^{p-1}|&\int_{x_{i_2}}(u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2})\wedge\mathcal{G}^p_{n_{i_2}}(x_{i_1},x_{i_2})\\ &=(-1)^{(n_{i_2}-1)(p-n_{i_2})}\int_{x_{i_2}}(u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2})\wedge\Pi_{p,n_{i_2}}\left[\left(\frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}\right)^*\Omega_{S^{p-1}}\right]\\ &=(-1)^{(n_{i_2}-1)(p-n_{i_2})}\int_{x_{i_2}}(u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2})\wedge\left(\frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}\right)^*\Omega_{S^{p-1}}.\end{aligned}\tag{2.18}$$

We then modify the graph by removing the node i_2 along with the branch starting from it, and by changing ω_{i_1} into the form $\omega'_{i_1}(x_{i_1})$ given by

$$(-1)^{(n_{i_2}-1)(p-n_{i_2})}|S^{p-1}|^{-1}\omega_{i_1}(x_{i_1})\wedge\int_{x_{i_2}}(u\circ\pi^{-1})^*\omega_{i_2}(x_{i_2})\wedge\left(\frac{x_{i_1}-x_{i_2}}{|x_{i_1}-x_{i_2}|}\right)^*\Omega_{S^{p-1}}.$$

We then apply the induction assumption to this new graph, having one node less, and Lemma 2.9 is proved. □

Remark 2.10. The construction of the tree-graph of forms can be described in terms of linking and "angular forms" (see [BT]).

To see this, again let π denote the stereographic projection from S^p into $\mathbf{R}^p$, singular at the north pole, and introduce the *linking map* or *segment map*

$$\begin{aligned}L\colon S^p\times S^p&\longrightarrow S^{p-1}\\ (x,y)&\longmapsto\frac{\pi(x)-\pi(y)}{|\pi(x)-\pi(y)|}.\end{aligned}\tag{2.19}$$

Let $\Omega_{S^{p-1}}$ be a $(p-1)$-form on S^{p-1} satisfying $\int_{S^{p-1}}\Omega_{S^{p-1}}=1$. It is interesting to observe that

$$d(L^*\Omega)=T_\Delta-T_{\{\mathrm{North}\}\times S^p}-T_{S^p\times\{\mathrm{North}\}},$$

where Δ denotes the diagonal $\{(x,y)\in S^p\times S^p:x=y\}$, North$=(0,...,0,1)$ is the north pole of S^p, and T_Δ, $T_{\{\mathrm{North}\}\times S^p}$ and $T_{S^p\times\{\mathrm{North}\}}$ are the currents of integration, with the appropriate orientations, along Δ, $\{\mathrm{North}\}\times S^p$ and $S^p\times\{\mathrm{North}\}$, respectively. Then, if one restricts to forms in $\bigoplus_{k=1}^p\bigwedge^k S^p\wedge\bigwedge^{p+1-k}S^p$, $L^*\Omega$ plays the role of an "angular form" (see [BT]) of the diagonal Δ in $S^p\times S^p$ (observe that it was not possible to find $\beta\in\bigwedge^{p-1}(S^p\times S^p)$, even singular, such that $d\beta=T_\Delta$ in $\mathcal{D}'\wedge\bigwedge^p(S^p\times S^p)$). Thus, in our representation by tree-graphs of forms of elements of $\mathrm{Hom}(\pi_p(N),\mathbf{R})$, two connections to one vertex by segments of two sub-tree-graphs corresponds to a connection of the two corresponding forms by wedging them together with the angular form of the diagonal Δ in $S^p\times S^p$. This implies in particular that if we replace $L^*\Omega$ everywhere by any form whose restriction to $\bigoplus_{k=1}^p\bigwedge^k S^p\wedge\bigwedge^{p+1-k}S^p$ is the angular form of the diagonal Δ in $S^p\times S^p$ (for instance $L^*_\Psi\Omega$, where Ψ is an arbitrary bilipschitz homeomorphism of S^p and $L_\Psi(x,y)=L(\Psi(x),\Psi(y))$), the computation of $\int_{S^p}u^K$ is unchanged.

2.5. Critical exponents associated with elements in $\mathrm{Hom}(\pi_p(N),\mathbf{R})$

2.5.1. Definition of the critical exponent

LEMMA 2.11. *Given a tree-graph of forms K, there exists a constant C_K such that, for any map $u\in C^\infty(S^p,N)$, the following inequality holds*

$$\left|\int_{S^p}u^K\right|^{p/(p+n_K)}\leqslant C_K\int_{S^p}|\nabla u|^p\,d\mathcal{H}^p, \tag{2.20}$$

where n_K is the number of segments in K.

Proof. The standard Sobolev inequality gives that for any q-form ω on S^p with $1<q<p$, there exists a constant C_q such that

$$\|d^*\Delta^{-1}\omega\|_{p/(q-1)}\leqslant C_q\|\omega\|_{p/q}. \tag{2.21}$$

Given a sub-tree-graph K_i, one first easily proves, by induction on the number of branches of K^i, that

$$\|u^{K_i}\|_{p/q_i}\leqslant\prod_{j\in J}\|u^*\omega_j\|_{p/p_j}, \tag{2.22}$$

where J is the set of indices of vertices in K_i, ω_j is the element of the ideal generated by $\Psi_N(\mathcal{M}_N)$ at the vertex j, p_j is its degree, and $q_i = \sum_{j\in J} p_j - n_{K_i}$ (where, as before, n_{K_i} is the number of segments in K_i). For $K_i = K$, (2.22) implies that

$$\|u^K\|_1 \leqslant C_K \prod_i \|\nabla u\|_p^{p_i}. \tag{2.23}$$

Combining now (2.6) and (2.23), we obtain (2.20) and Lemma 2.11 is proved. □

Definition 2.12. For any element $z \in \mathrm{Hom}(\pi_p(N), \mathbf{R})$, let n_z denote the *minimum* of $\max_l\{n_{K_l}\}$ among all formal linear combinations $K = \sum_l \lambda_l K^l$ of tree-graphs of *closed* forms representing z. Such a K achieving this minimum is called *optimal* for z, and, motivated by Lemma 2.11, the *critical exponent* of z is the number

$$\frac{p}{p+n_z}.$$

Remark 2.13. The constraint of restricting to tree-graphs of closed forms is important. Allowing tree-graphs of nonclosed forms could decrease the maximal number of branches (see Remark 2.7), but would not give the appropriate notion.

2.5.2. Optimality of the critical exponent

An important question is to know whether the critical exponent of a generator z of the minimal model is optimal or not: that is, given a sequence of maps u_k from S^p into N such that

$$z([u_k]) = \int_{S^p} u_k^K = \sum_l \lambda_l \int_{S^p} u_k^{K^l} = a_k \to \infty, \tag{2.24}$$

and letting

$$E_k = \inf\biggl\{ \int_{S^p} |\nabla u|^p \, d\mathcal{H}^p : [z]([u]) = a_k \biggr\},$$

does the equality

$$\lim_{k\to\infty} \frac{\log E_k}{\log a_k} = \frac{p}{p+n_z} \tag{2.25}$$

hold?

If $z([u_k]) \neq 0$, then there exists, by Lemma 2.11, a positive constant $C_{z,N}$, depending only on z and the metric on N, such that

$$|z([u])|^{p/(p+n_z)} \leqslant C_{z,N} \int_{S^p} |\nabla u|^p \, d\mathcal{H}^p$$

for all $u \in W^{1,p}(S^p, N)$. Thus the inequality

$$\liminf_{k\to\infty} \frac{\log E_k}{\log a_k} \geqslant \frac{p}{p+n_z} \tag{2.26}$$

is clear. The difficulty is whether the reverse inequality holds or not. An analogous question was raised in [Gr1] and [Gr2].

Definition 2.14. For a fixed nonzero element z in $\mathrm{Hom}(\pi_p(N),\mathbf{R})$, we say that the critical exponent $p(p+n_z)^{-1}$ is optimal if (2.25) holds for some sequence $[u_k]$ in $\pi_p(N)$ satisfying (2.24).

It was proved in [Ri] that this always holds if for instance N is a sphere (if $N\simeq S^q$, then, for $p=q$, $n_z=0$ and, for $p=2q-1$ and q even, $n_z=1$). Answering this question in general seems to be an interesting difficult open problem. We do not try to make a systematic presentation of this problem in the present work, but just illustrate this question by considering more specific N (related to the examples we exposed in the previous subsection) for which the critical exponents are always optimal. Precisely we have the following result.

PROPOSITION 2.15. *Let N be a 4-dimensional Riemannian manifold diffeomorphic to the connected sum $\#\mu\mathbf{CP}^2\#\nu\overline{\mathbf{CP}^2}\#\xi(S^2\times S^2)$, where μ, ν and ξ are three arbitrary natural numbers*, (*$\#m\,...$ denotes the connected sum of m copies of ... and $\overline{\mathbf{CP}^2}$ is $\mathbf{CP}^2$ with the opposite orientation to the standard one*). *Then*, *for any $p\in\mathbf{N}$ and nonzero $z\in\mathrm{Hom}(\pi_p(N),\mathbf{R})$*, *the critical exponent $p(p+n_z)^{-1}$ is optimal.*

Proof. Denote by $\omega_1,...,\omega_\mu\in H^2(N)$ the Poincaré duals of each of the standard $\mathbf{CP}^1$'s embedded in each of the $\mathbf{CP}^2$'s in the connected sum $\#\mu\mathbf{CP}^2\#\nu\overline{\mathbf{CP}^2}\#\xi(S^2\times S^2)$. Similarly, denote by $\bar\omega_1,...,\bar\omega_\nu\in H^2(N)$ the Poincaré duals of each of the standard $\overline{\mathbf{CP}^1}$'s embedded in each of the $\overline{\mathbf{CP}^2}$'s in the connected sum $\#\mu\mathbf{CP}^2\#\nu\overline{\mathbf{CP}^2}\#\xi(S^2\times S^2)$. Finally let $\alpha_1,...,\alpha_\xi$ and $\beta_1,...,\beta_\xi$ denote the Poincaré duals of each of the $S^2\times\{\mathrm{North}\}$ and each of the $\{\mathrm{North}\}\times S^2$, respectively, in the connected sum. For any $k\in\mathbf{Z}$, we first construct a map F_k from $N\simeq\#\mu\mathbf{CP}^2\#\nu\overline{\mathbf{CP}^2}\#\xi(S^2\times S^2)$ into itself such that

$$\begin{aligned} &F_k^*\omega_h=k\omega_h && \text{for } h=1,...,\mu,\\ &F_k^*\bar\omega_i=k\bar\omega_i && \text{for } i=1,...,\nu,\\ &F_k^*\alpha_j=k\alpha_j \text{ and } F_d^*\beta_j=d\beta_j && \text{for } j=1,...,\xi, \end{aligned} \tag{2.27}$$

and

$$\|\nabla F_k\|_\infty\leqslant C\sqrt{k} \tag{2.28}$$

with C independent of k. First, the existence of F_k, in the case where $\mu=0$, $\nu=0$ and $\xi=1$, is quite elementary to establish: it is not difficult (see [Ri]) to construct a family of maps ϕ_k from S^2 into itself such that

$$\deg\phi_k=k \quad\text{and}\quad \|\nabla^j\phi_k\|_\infty\leqslant C_j\sqrt{k^j},$$

where C_j is independent of k. Then observe that $F_k(x,y)=(\phi_d(x),\phi_d(y))$ is a solution to (2.27) and (2.28) in the case $N=S^2\times S^2$.

We now construct F_k solving (2.27) and (2.28) in the case $N=\mathbf{CP}^2$. We first split $\mathbf{CP}^2\simeq N^1\# N^2$, where $N^2\simeq B^4$, $N^1\simeq E$ and E is diffeomorphic to the *Hopf disk bundle* over S^2 which is the D^2-bundle homogeneously extending the S^1-*Hopf bundle* whose total space is diffeomorphic to S^3 (we thus have $\partial E\simeq S^3$). We first construct F_k from N^1 into N^1. Let H be the Hopf fibration from $\partial E\simeq S^3$ into S^2, and let ϕ_k be the map described above. We claim that we can lift ϕ_k to a map $\tilde\phi_k\colon S^3\to S^3$ (i.e. $H\circ\tilde\phi_k=\phi_k\circ H$) satisfying

$$\|\nabla\tilde\phi_k\|_\infty\leqslant C\sqrt{k}, \tag{2.29}$$

where C is independent of k. We follow the idea in [HR1]. Let (e_1^*,e_2^*,e_3^*) be the orthonormal coframe of $\bigwedge^1 S^3$ given by the Lie group action on S^3 starting from $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ at $(1,0,0,0)$. Classical computations give $2^{-1}de_i^*=e_{i+1}^*\wedge e_{i-1}^*$, where we use indexation in $\mathbf{Z}_3$. We get the *Coulomb Hopf lift* (see [HR1]) of ϕ_k in the following way: there exists $\tilde\phi=\tilde\phi_k$ satisfying (forgetting the subscript k)

$$\begin{aligned}
&\tilde\phi^*e_1^*(x)=(dH_{\tilde\phi(x)}\cdot e_1;d\phi_{H(x)}\cdot(dH_x\cdot e_1))e_1^*(x)+(dH_{\tilde\phi(x)}\cdot e_1;d\phi_{H(x)}\cdot(dH_x\cdot e_2))e_2^*(x),\\
&\tilde\phi^*e_2^*(x)=(dH_{\tilde\phi(x)}\cdot e_2;d\phi_{H(x)}\cdot(dH_x\cdot e_2))e_2^*(x)+(dH_{\tilde\phi(x)}\cdot e_2;d\phi_{H(x)}\cdot(dH_x\cdot e_1))e_1^*(x),\\
&\tilde\phi^*e_3^*(x)=\eta(x),
\end{aligned}$$

where (e_1,e_2,e_3) is the basis dual to (e_1^*,e_2^*,e_3^*), $(\cdot\,;\cdot)$ is the scalar product on S^2 and η is the 1-form on S^3 solving the following elliptic problem

$$\begin{cases} d\eta=\tfrac12 H^*\phi^*\omega_{S^2},\\ d^*\eta=0,\end{cases}$$

where ω_{S^2} is the volume form on S^2. We observe that the operator

$$\begin{aligned}
L\colon C^2(\Omega,\mathbf{R}^4\setminus B_{1/2})&\longrightarrow C^0(\Omega,\mathbf{R}^4),\\
u&\longmapsto (d^*(u^*e_1^*),d^*(u^*e_2^*),d^*(u^*e_3^*),d^*(u^*\partial/\partial r)),
\end{aligned}$$

where $\Omega=B_2^4\setminus B_{1/2}^4$ is elliptic. Using a classical interpolation result (see [BBH]), we get that for any subdomain Ω' of Ω, one has

$$\|\nabla u\|^2_{L^\infty(\Omega')}\leqslant C\|u\|_{L^\infty(\Omega)}\|Lu\|_{L^\infty(\Omega)}+C\|u\|^2_{L^\infty(\Omega)}.$$

Applying this result to $u=\tilde\phi_k\circ(x/|x|)$, we obtain that

$$\|\nabla\tilde\phi_k\|^2_\infty\leqslant C\|\tilde\phi_k\|_\infty(\|\nabla^2\phi_k\|_\infty+\|\nabla\tilde\phi_k\|_\infty\|\nabla\phi_k\|_\infty)+\|\tilde\phi_k\|^2_\infty\leqslant Cd+\sqrt{k}\,\|\nabla\tilde\phi_k\|_\infty.$$

Thus we have found a family of liftings of the ϕ_k that satisfies (2.29). Now we can extend $\tilde{\phi}_k$ to a map from E into E by homogeneity: we take the flat metric on each fiber of the disc bundle E over S^2, and we take $F_k(x)=|x|_1\tilde{\phi}_k(x/|x|_1)$, where $|x|_1$ is the distance of x to the zero section $\simeq S^2$ of the disk bundle, and where we are using the linearity on the D^2-fibers. It is clear that F_k so defined on N_1 satisfies $\|\nabla F\|_{L^\infty(N_1)}\leqslant C\sqrt{k}$. On $N_2\simeq B^4$ we define F_k from $N_2\simeq B^4$ into $N_2\simeq B^4$ by $F_k(x)=|x|_2\tilde{\phi}_k(x/|x|_2)$, where this time $|x|_2$ is the distance to the center 0 of B^4, and where we are using the linearity on $B^4\subset\mathbf{R}^4$. By gluing the two pieces N_1 and N_2 together, we get a family of maps F_k from $\mathbf{CP}^2$ into $\mathbf{CP}^2$ satisfying (2.27) and (2.28). Thus, we have then F_k in the cases $(\mu,\nu,\xi)=(0,0,1)$, $(\mu,\nu,\xi)=(0,1,0)$ and $(\mu,\nu,\xi)=(1,0,0)$. We get F_k for general (μ,ν,ξ) by simply an iterative gluing of the previous ones.

Consider now a class z in $(\pi_p(N)\otimes\mathbf{R})^*$. Since $H^*(N)$ is generated by the classes ω_i, $\bar{\omega}_i$, α_i and β_i, each form at each node of every tree-graph K_l arising in the finite linear combination K representing z is a nonexact 2-form representing one of the classes above or a 4-form, wedge of two nonexact 2-forms of this family. We restrict to tree-graphs which are optimal in the sense of Definition 2.12. Let $u: S^p\to N$ be such that $z([u])\neq 0$. Consider now $\sum_l \lambda_l(F_k\circ u)^{K^l}$. It is a polynomial in $\sqrt{k}$ of the form

$$z([u])=\sum_l k^{(n_l+p)/2}\lambda_l\int_{S^p}u^{K^l},$$

where we have used the identity (2.6). Since K is optimal, we have $n_l\leqslant n_z$. Assume that, for every u with $z([u])\neq 0$, the coefficient in front of $k^{(n_z+p)/2}$ is always 0. Then, in representing z, we could remove all trees that contain n_z nodes, and deduce that the minimal number of nodes for representing z is strictly less than n_z, a contradiction. Thus we may choose u so that the coefficient in front of $k^{(n_z+p)/2}$ is nonzero, and we have

$$A_k=z([F_k\circ u])=ad^{(n_z+p)/2}+P\big(\sqrt{d}\,\big), \tag{2.30}$$

where $a\neq 0$ and $\deg P<n_z+p$. Observe now that we have

$$E_k=\inf\left\{\int_{S^p}|\nabla u|^p\,d\mathcal{H}^p : z([u])=A_k\right\}\leqslant\int_{S^p}|\nabla F_k\circ u|^p\leqslant C_u\|\nabla F_k\|_\infty^p\leqslant C_u k^{p/2},$$

which, combined with (2.30), implies that

$$\limsup_{k\to\infty}\frac{\log E_k}{\log A_k}\leqslant\frac{p}{n_z+p}.$$

From this inequality and (2.26), we deduce that

$$\lim_{k\to\infty}\frac{\log E_k}{\log A_k}=\frac{p}{n_z+p},$$

so that $p(n_z+p)^{-1}$ is optimal. Proposition 2.15 is proved. □

2.6. Rigidity property of linear combinations of tree-graphs and interpretation of homotopy integrals as linking numbers

One question we did not address yet is the invariance of the isomorphism

$$V^p \simeq \mathrm{Hom}(\pi_p(N), \mathbf{R})$$

under deformation of the geometric realization Ψ_N. In the previous subsections we have constructed the formal linear combination of tree-graphs of forms $K=\sum_l \lambda_l K^l$ which have the *homotopy property*: for any u,

$$\int_{S^p} u^K \text{ remains unchanged under homotopic deformation of } u,$$

and which therefore correspond to an element in $\mathrm{Hom}(\pi_p(N), \mathbf{R})$. This is why these integrals are also called *homotopy integrals*. Now we address the question of finding the formal linear combination of tree-graphs of forms of $\Psi_N(\mathcal{M}_N)$ which have the *rigidity property*: for every smooth u from S^p into N, $\int_{S^p} u^K$ is unchanged under a deformation of Ψ_N by adding an exact form to every closed generator of $\Psi_N(\mathcal{M}_N)$ (see [Nov1], [Nov2] and [Nov3]). For instance, for $\pi_3(S^2)$, the linear form

$$\int_{S^3} u^*\omega \wedge d^*\Delta^{-1}(u^*\omega'),$$

where $\int_{S^2}\omega=1$ and $\int_{S^2}\omega'=1$, remains unchanged when one adds arbitrary exact 2-forms $d\alpha$ and $d\alpha'$ to ω and ω'. This means that this particular tree-graph (made of two vertices connected by one segment and where one generator of $H^2(S^2)$ is assigned to each vertex) has the rigidity property. This problem of the rigidity property under deformation of geometric realization of minimal models was first raised in [Nov1], and sufficient conditions for this rigidity property to hold are given in [Nov2], [Nov3], [Mi1] and [Mi2].

We say that an element in $H^*(N)$ is in *general position* if the corresponding Poincaré dual is an integer multiplicity combination of simplices (i.e. their integral on cycles in $H_*(N, \mathbf{Z})$ are in $\mathbf{Z}$). The goal of this subsection is to prove the following result.

Proposition 2.16. *Let N be a compact simply connected manifold and let Ψ_N be a geometric realization of the minimal model $\mathcal{M}_N$. Suppose that $z\in\mathrm{Hom}(\pi_p(N), \mathbf{R})$ admits a representation via Ψ_N by a finite linear combination in $\mathbf{Z}$ of tree-graphs K_l of closed forms that are in general position. Assume also that K has the rigidity property and finally that, in each K^l, every pair of closed form connected by a segment have Poincaré duals that can be represented by disjoint closed polyhedral chains. Then*

$$\int_{S^p} u^K \in \mathbf{Z} \quad \text{for any } u\in C^\infty(S^p, N).$$

Remark 2.17. We can always choose a basis of $H^*(N)$ which is in general position.

In fact, first take free generators of $H_*(N,\mathbf{Z})$. They form a basis in $H_*(N,\mathbf{R})$ (see for instance [GMS2, I, §5.4.1, Theorem 8]) and the Poincaré duals of these classes form a basis of $H^*(N)$ in general position. We can then proceed to the construction of a geometric realization Ψ_N of the minimal model of N starting from these classes. In order to apply Proposition 2.16, it remains to check both the rigidity property of K representing z and whether each pair of forms connected by a segment in the tree-graph can be realized by disjoint cycles. Observe, as an illustration of Proposition 2.16, that all these conditions are fulfilled by each example we gave above ($\pi_p(\mathbf{CP}^2)$, $\pi_p(\mathbf{CP}^1\times\mathbf{CP}^1)$ and $\pi_p(\mathbf{CP}^2\#(\mathbf{CP}^1\times\mathbf{CP}^1))$ for $p=2,3,4$ except for $K_{\gamma_{13}}$ and $K_{\gamma_{23}}$ arising while computing $\pi_4(\mathbf{CP}^2\#(\mathbf{CP}^1\times\mathbf{CP}^1))$, where our geometric interpretation of $\int_{S^p}u^K$ is no longer valid.

Proof of Proposition 2.16. Let K^l be a tree-graph of closed forms arising in the finite linear combination of tree-graphs of forms $K=\sum_l\lambda_l K^l$ in the representation of z. Let $\omega_1,...,\omega_{n_{K^1}}$ be the closed forms at the nodes of K^l. Let C_i be the closed smooth simplicial chains representing the Poincaré duals of the ω_i's such that $C_{i_1}\cap C_{i_2}=\varnothing$ whenever $(i_1,i_2)\in\mathcal{I}$ (the set of pairs of indices whose corresponding nodes are connected by a segment in the tree-graph). Assume, to simplify the presentation, that the C_i's are smooth submanifolds of N of dimension $p_i=n-\deg\omega_i$ (for general smooth simplicial chains, the approach below requires a more technical discussion that we skip). Let N_i be an open tubular neighborhood of C_i (diffeomorphic to the normal bundle of C_i in N) that we choose sufficiently small in order to guarantee that $N_{i_1}\cap N_{i_2}=\varnothing$ for all $(i_1,i_2)\in\mathcal{I}$. We denote by π_i the orthogonal projection from N_i into C_i. We replace ω_i by a cohomologically equivalent representative of the Thom form of N_i which is supported in N_i and whose integral along each $(n-p_i)$-plane perpendicular to C_i gives 1. We keep using ω_i to denote this new representative of $[\omega_i]$. Since K has the rigidity property, $\int_{S^p}u^K$ is not altered by this change of geometric realization. Let S_i be an $\mathbf{R}^{q_i}$-vector bundle over C_i whose sum with N_i gives a trivial bundle $N_i\oplus S_i\simeq C_i\times\mathbf{R}^{n-p_i+q_i}$ and let $\widetilde{\omega}_i$ be a representative of the Thom class of S_i. Let $\Omega_i=u^{-1}(N_i)$, and let E_i be the pull-back bundle of S_i by $\pi_i\circ u$ over Ω_i:

$$E_i=(\pi_i\circ u)^{-1}S_i.$$

Let Π_i the projection map from E_i into N_i and ϕ_i be the canonical bundle map from E_i into S_i lifting $\pi_i\circ u$ and realizing an isomorphism from any fiber of E_i into the image fiber by $\pi_i\circ u$. Finally let Φ_i be the following map:

$$\begin{aligned}\Phi_i\colon E_i&\longrightarrow S_i\oplus N_i\simeq C_i\times\mathbf{R}^{n-p_i+q_i},\\ x&\longmapsto\phi_i(x)+u(\Pi_i(x)).\end{aligned}$$

Following [BT], we denote by $(\Pi_i)_*$ the integration on E_i along the fibers which assign a $(\bigwedge^{k-q_i} N_i)$-form to any $(\bigwedge^k E_i)$-form. Using the projection formula (Proposition 6.15) in [BT], we have for any q-form $\alpha\in C^\infty(\bigwedge^q N_i)$,

$$(\Pi_i)_*(\Phi_i^*\omega_i\wedge\widetilde{\omega}_i\wedge\Pi^*\alpha)=u^*\omega_i\wedge\alpha.$$

This implies in particular, for $q=p-\deg\omega_i$, that

$$\int_{E_i}\Phi_i^*(\omega_i\wedge\widetilde{\omega}_i)\wedge\Pi_i^*\alpha=\int_{S^p}u^*\omega_i\wedge\alpha. \tag{2.31}$$

We assume that each N_i avoids the north pole and denote by L the *linking map* defined by (2.19). Furthermore let $\Omega_{S^{p-1}}$ be a $(p-1)$-form on S^{p-1} satisfying $\int_{S^{p-1}}\Omega_{S^{p-1}}=1$. Following (2.17), we have that

$$(-1)^{m_K}\int_{S^p}u^{K^l}=\int_{S^p}...\int_{S^p}\bigwedge_{i=1}^{n_{K^l}+1}u^*\omega_i(x_i)\wedge\bigwedge_{(i_1,i_2)\in\mathcal{I}}L^*\Omega_{S^{p-1}}(x_{i_1},x_{i_2}). \tag{2.32}$$

Combining (2.31) and (2.32), we then have

$$(-1)^{m_K}\int_{S^p}u^{K^l}=\int_{E_1}...\int_{E_{n_K+1}}\bigwedge_{i=1}^{n_{K^l}+1}\Phi_i^*(\omega_i\wedge\widetilde{\omega}_i)(z_i)\wedge\bigwedge_{I\in\mathcal{I}}\Pi_I^*L^*\Omega_{S^{p-1}}(z_{i_1},z_{i_2}),$$

where $\Pi_I(z_{i_1},z_{i_2})=(\Pi_{i_1}(z_{i_1}),\Pi_{i_2}(z_{i_2}))$. Since the N_i's are disjoint and also disjoint from the north pole of S^p, $L\circ\Pi_I\in C^\infty(E_{i_1}\times E_{i_2},S^{p-1})$ for all $I\in\mathcal{I}$. Thus we have, for all $I\in\mathcal{I}$,

$$d(L^*\Omega_{S^{p-1}}(z_{i_1},z_{i_2}))=0\quad\text{in }\ E_{i_1}\times E_{i_2}. \tag{2.33}$$

Let $\Xi_i: S_i\oplus N_i\to C_i\times\mathbf{R}^{n-p_i+q_i}$ be a bundle isomorphism and let P_i be the canonical projection from $C_i\times\mathbf{R}^{n-p_i+q_i}$ into $\mathbf{R}^{n-p_i+q_i}$ which assigns X to (x,X). Using (2.33), since $\omega_i\wedge\widetilde{\omega}_i$ is cohomologous to

$$A_i=\Xi_i^*P_i^*(f_i(X)dX_1\wedge...\wedge dX_{n-p_i+q_i}),$$

where f_i is the characteristic function of the unit ball $B_1^{n-p_i+q_i}$ divided by its volume, we have

$$(-1)^{m_K}\int_{S^p}u^{K^l}=\int_{E_1}...\int_{E_{n_K+1}}\bigwedge_{i=1}^{n_{K^l}+1}\Phi_i^*A_i(z_i)\wedge\bigwedge_{I\in\mathcal{I}}\Pi_I^*L^*\Omega_{S^{p-1}}(z_{i_1},z_{i_2}).$$

Using Federer's coarea formula, we have, letting $r_i=n-p_i-q_i$,

$$\begin{aligned}(-1)^{m_{K^l}} \prod_{i=1}^{n_{K^l}+1} |B_1^{r_i}| \int_{S^p} u^{K^l} \\ = \int_{B_1^{r_1}} \dots \int_{B_1^{r_{n_K+1}}} \bigwedge_{i=1}^{n_K+1} d\xi_i \int_{\prod_{i=1}^{n_K+1}(P_i\circ\Xi_i\circ\Phi_i)^{-1}(\xi_i)} \bigwedge_{I\in\mathcal{I}} \Pi_I^* L^* \Omega_{S^{p-1}}(z_{i_1}, z_{i_2}).\end{aligned} \tag{2.34}$$

For a regular value $(\xi_1, ..., \xi_{n_K+1})$ of $\prod_{i=1}^{n_K+1}(P_i\circ\Xi_i\circ\Phi_i)$, we introduce the map

$$\begin{aligned}V_{\xi_1,\dots,\xi_{n_K+1}}: \prod_{i=1}^{n_K+1} (P_i\circ\Xi_i\circ\Phi_i)^{-1}(\xi_i) &\longrightarrow (S^{p-1})^{n_K}, \\ (x_1, ..., x_{n_K+1}) &\longmapsto \prod_{I\in\mathcal{I}} (L\circ\Pi_I)(x_{i_1}, x_{i_2}).\end{aligned}$$

A short computation using (2.6) shows that

$$\dim \prod_{i=1}^{n_K+1} (P_i\circ\Xi_i\circ\Phi_i)^{-1}(\xi_i) = \dim(S^{p-1})^{n_K}.$$

A standard deformation argument shows also that the topological degree M of $V_{\xi_1,\dots,\xi_{n_K+1}}$ is independent of $(\xi_1, ..., \xi_{n_K+1})$. Combining this fact together with (2.34) and the integral expression of the topological degree, we have shown that, modulo a sign, $\int_{S^p} u^{K^l}$ equals this topological degree, which is an integer. Proposition 2.16 follows. □

Remark 2.18. An alternative proof of Proposition 2.16 may be obtained, following [BT, pp. 230–234], by interpreting each element of the integrand as a geometric operation.

Here one interprets $u^*\omega_i$ as the integration operation on $u^{-1}(C_i)$, where C_i is the Poincaré dual to ω_i in N, one interprets the d^{-1} operation as taking a chain bounding a boundary and the wedge $\wedge$ as the intersection operation—see [BT, (6.31), p. 69] and also [Ri, Proposition 2.2]), where such an approach is taken.

3. Bubbling for $W^{1,p}$-weakly convergent sequences in $C^\infty(S^p, N)$

Given a sequence u_n in $C^\infty(S^p, N)$ which is $W^{1,p}$-weakly converging to a map u in $W^{1,p}(S^p, N)$, the goal of this section is to show that one can extract a subsequence $u_{n'}$ from u_n such that, for every $z\in\mathrm{Hom}(\pi_p(N), \mathbf{R})$,

$$\lim_{n\to\infty} z([u_{n'}]) = z([u]) + \sum_{i=1}^{I} z([w_i]),$$

where the w_j's are disjoint "bubbles" from u_n (i.e. maps in $W^{1,p}(\mathbf{R}^p, N)$, weak limits of the dilated map $u_n(r_n^j x + a_j^n)$, where the points $a_j^n \to a_j \in S^p$ and the scalings $r_n^j \to 0$ fast enough so that, for distinct j and j', both r_n^j and $r_n^{j'}$ are noncomparable to $|a_n^j - a_n^{j'}|$). We will in fact formulate that convergence using a modification of the "graph" approach of Giaquinta, Modica and Souček [GMS2], and introducing a variation of their notion of a Cartesian current.

3.1. The Cartesian current associated with a map u in $W^{1,p}(S^p, N)$ and a p-dimensional tree-graph K of closed forms from N

Let N be a compact simply connected Riemannian manifold, K be a p-dimensional tree-graph of closed forms, and u be a map in $W^{1,p}(S^p, N)$. Recall that the number n_K of segments is one less than the number of nodes. We first introduce the following definition.

Definition 3.1. To any map u in $W^{1,p}(S^p, N)$ we assign a map $\mathcal{U}_K$ from $(S^p)^{n_K+1}$ into $N^{n_K+1} \times (S^{p-1})^{n_K}$ defined by

$$\mathcal{U}_K(x_1, ..., x_{n_K+1}) = \left(u(x_1), ..., u(x_{n_K+1}), ..., \frac{\pi(x_i) - \pi(x_j)}{|\pi(x_i) - \pi(x_j)|}, ... \right),$$

where π is the stereographic projection of $S^p \setminus \{*\}$ onto $\mathbf{R}^p$ and the pairs (i,j) run over the set $\mathcal{I}_K$ of pairs (i,j) for which the nodes i and j are connected, in the graph K, by a segment going from j to i. Such a map will be called the *tree-graph map* associated with u and K.

Observe that with this notation, we have, using Lemma 2.9,

$$\int_{S^p} u^K = \int_{x_1} ... \int_{x_{n_K+1}} U_K^*(\omega^K) \wedge \mathcal{G}^K = \int_{(S^p)^{n_K+1}} \mathcal{U}_K^* \Big(\omega^K \bigwedge_{J \in \mathcal{I}_K} \Omega(X_J) \Big),$$

where, for any $J = (j_1, j_2) \in \mathcal{I}_K$, X^J is a variable in S^{p-1} and $\Omega(X^J)$ denotes the standard unit-volume form on $S^{p-1} \subset \mathbf{R}^p$ given by

$$\Omega(X^J) = \frac{1}{|S^{p-1}|} \sum_{k=1}^{p} X_k^J (-1)^{k-1} \, dX_1^J \dots dX_{k-1}^J dX_{k+1}^J \dots dX_p^J.$$

In the product space $(S^p)^{n_K+1} \times N^{n_K+1} \times (S^{p-1})^{n_K}$ we have the $p(n_K+1)$ integer-multiplicity rectifiable current Graph $(\mathcal{U}_K)$ defined by integration on the graph of $\mathcal{U}_K$: for any smooth $p(n_K+1)$-form Ψ in $(S^p)^{n_K+1} \times N^{n_K+1} \times (S^{p-1})^{n_K}$,

$$\operatorname{Graph}(\mathcal{U}_K)(\Psi) = \int_{(S^p)^{n_K+1}} \mathcal{V}_K^* \Psi, \quad \text{where } \mathcal{V}_K(x) = (x, \mathcal{U}_K(x)). \tag{3.1}$$

The following proposition shows that Graph $(\mathcal{U}_K)$ is indeed an integer rectifiable current whose boundary also has finite mass (and hence is also rectifiable [Fe, Theorem 4.2.16].)

Proposition 3.2. *Under the notation above we have the existence of a constant C_K independent of $u\in W^{1,p}(S^p,N)$ such that*

$$\mathbf{M}(\operatorname{Graph}(\mathcal{U}_K))\leqslant C_K[1+\|\nabla u\|_p^{p\,(n_K+1)}]. \tag{3.2}$$

Moreover, there exists a constant C_K such that

$$\mathbf{M}(\partial\operatorname{Graph}(\mathcal{U}_K))\leqslant C_K[1+\|\nabla u\|_p^{pn_K}]. \tag{3.3}$$

Proof. Any smooth $p(n_K+1)$-form Ψ in the product space

$$(S^p)^{n_K+1}\times N^{n_K+1}\times (S^{p-1})^{n_K}$$

can be written as a finite sum of smooth simple forms. Corresponding to a choice of non-negative integers $(m_1,...,m_{n_K+1},p_1,...,p_{n_K+1},q_1,...,q_{n_K})$ such that

$$\sum_{i=1}^{n_K+1} m_i+\sum_{j=1}^{n_K+1} p_j+\sum_{k=1}^{n_K} q_k=p(n_K+1),$$

with $m_i\leqslant p$, $p_j\leqslant\dim N$ and $q_k\leqslant p-1$, we may let Ψ' be the canonical component of Ψ on

$$\Big(\bigwedge\nolimits^{m_1}S^p\wedge...\wedge\bigwedge\nolimits^{p_{m_K+1}}S^p\Big)\wedge\Big(\bigwedge\nolimits^{p_1}N\wedge...\wedge\bigwedge\nolimits^{p_{n_K+1}}N\Big)\wedge\Big(\bigwedge\nolimits^{q_1}S^{p-1}\wedge...\wedge\bigwedge\nolimits^{q_{n_K}}S^{p-1}\Big).$$

Then Ψ' is the product of a function $f\in C^\infty(S^p,\mathbf{R})$ and a simple form

$$\alpha_1(x_1)\wedge...\wedge\alpha_{n_K+1}(x_{n_K+1})\wedge\beta_1(y_1)\wedge...\wedge\beta_{n_K+1}(y_{n_K+1})\wedge\gamma_1(z_1)\wedge...\wedge\gamma_{n_K}(z_{n_K}),$$

where each of the α_i's, β_j's and γ_k's is a simple form. We have

$$\begin{aligned}&\operatorname{Graph}(\mathcal{U}_K)(\Psi)\\&\quad=\|f\|_\infty\left|\int_{(S^p)^{n_K+1}}\bigwedge_{i=1}^{n_K+1}\alpha_i(x_i)\wedge\bigwedge_{j=1}^{n_K+1}u^*\beta_j(y_j)\wedge\bigwedge_{J=(j_1,j_2)\in\mathcal{I}_K}\pi^*\left(\frac{x_{j_1}-x_{j_2}}{|x_{j_1}-x_{j_2}|}\right)^*\gamma_J(z_j)\right|\\&\quad\leqslant C_K\|\Psi'\|_\infty\int_{(S^p)^{n_K+1}}\prod_{j=1}^{n_K+1}|\nabla u|^{p_j}(y_j)\prod_{J=(j_1,j_2)\in\mathcal{I}_K}\frac{1}{|\pi(x_{j_1})-\pi(x_{j_2})|^{q_J}}.\end{aligned} \tag{3.4}$$

We prove now, by induction on the number of nodes n_K+1 in the tree-graph K, that the integral

$$\int_{(S^p)^{n_K+1}}\prod_{i=1}^{n_K+1}|f_i|(x_i)\prod_{J=(j_1,j_2)\in\mathcal{I}_K}\frac{1}{|\pi(x_{j_1})-\pi(x_{j_2})|^{q_J}} \tag{3.5}$$

is bounded by

$$C_K \prod_{i=1}^{n_K+1} \|f_i\|_{p/p_i}. \tag{3.6}$$

To facilitate calculation, consider the sphere measure $\mu=\pi_{\#}(\mathcal{H}^p|_{S^p})$ on $\mathbf{R}^p$ so that

$$\|f\circ\pi^{-1}\|_{L^p(\mu)} = \|f\|_{L^p}$$

for $f\in L^p(S^p)$. Let us take an *end node* i_2 connected to the node i_1 so that $J=(i_1,i_2)\in\mathcal{I}_K$. Classical estimates on Riesz potentials (see [St]) give

$$\left\|\int_{x_{i_2}\in\mathbf{R}^p} \frac{|f_{i_2}|(\pi^{-1}(x_{i_2}))}{|x_{i_1}-x_{i_2}|^{q_J}}\right\|_{L^{p/(p_{i_2}+q_J-n)}(\mu)} \leqslant C_K\|f_{i_2}\circ\pi^{-1}\|_{L^{p/p_{i_2}}(\mu)} = C_K\|f_{i_2}\|_{p/p_{i_2}}. \tag{3.7}$$

Therefore

$$\left\|f_{i_1}(\pi^{-1}(x_{i_1}))\int_{x_{i_2}\in\mathbf{R}^p} \frac{|f_{i_2}|(\pi^{-1}(x_{i_2}))}{|x_{i_1}-x_{i_2}|^{q_J}}\right\|_{L^{p/(p_{i_1}+p_{i_2}+q_J-p)}(\mu)} \leqslant C_K\|f_{i_1}\|_{p/p_{i_1}}\|f_{i_2}\|_{p/p_{i_2}}. \tag{3.8}$$

Replacing then $f_{i_1}\circ\pi^{-1}$ by

$$f'_{i_1}(\pi^{-1}(x_{i_1})) = f_{i_1}(\pi^{-1}(x_{i_1}))\int_{x_{i_2}} |f_{i_2}|(\pi^{-1}(x_{i_2}))|x_{i_1}-x_{i_2}|^{-q_J},$$

replacing p_{i_1} by $p'_{i_1}=p_{i_1}+p_{i_2}+q_J-p$ and removing the node i_2 from the graph K, we are in the position to apply our induction assumption to this new graph, and this permits us to bound (3.5) by (3.6). Applying this fact to the inequality (3.4), we obtain (3.2).

We establish now (3.3). Suppose Φ be a smooth compactly supported $(p(n_K+1)-1)$-form in $(S^p)^{n_K+1}\times N^{n_K+1}\times(S^{p-1})^{n_K}$. This time we may consider the projection Φ' in

$$\Big(\bigwedge\nolimits^{m_1}S^p\wedge\ldots\wedge\bigwedge\nolimits^{m_{n_K+1}}S^p\Big)\wedge\Big(\bigwedge\nolimits^{p_1}N\wedge\ldots\wedge\bigwedge\nolimits^{p_{n_K+1}}N\Big)\wedge\Big(\bigwedge\nolimits^{q_1}S^{p-1}\wedge\ldots\wedge\bigwedge\nolimits^{q_{n_K}}S^{p-1}\Big),$$

where $m_1,...,m_{n_k+1},p_1,...,p_{n_K+1},q_1,...,q_{n_K}$ are non-negative integers such that

$$\sum_{i=1}^{n_K} m_i + \sum_{j=1}^{n_K+1} p_j + \sum_{k=1}^{n_K} q_k = p(n_K+1)-1$$

with $m_i\leqslant p$, $p_j\leqslant\dim N$ and $q_k\leqslant p-1$. Again, assume that Φ' is simple.

We also observe that for any smooth form γ in $\bigwedge^q S^{p-1}$, the following identity holds, in the sense of distributions, if $q<p-1$:

$$d\pi^*\left(\left(\frac{x-y}{|x-y|}\right)^*\gamma\right) = \pi^*\left(\frac{x-y}{|x-y|}\right)^*(d\gamma) \quad \text{in } \mathcal{D}'(\textstyle\bigwedge^q(\mathbf{R}^p\times\mathbf{R}^p)), \tag{3.9}$$

whereas if $\gamma\in C^\infty(\bigwedge^{p-1} S^{p-1})$, a short computation shows that, for any form

$$\phi\in C_0^\infty(\textstyle\bigwedge^p(S^p\times S^p)),$$

one has

$$\int_{S^p\times S^p} d\phi\wedge\pi^*\left(\frac{x-y}{|x-y|}\right)^*\gamma=(-1)^p\int_{S^{p-1}}\delta\wedge\pi^*\int_{\mathbf{R}^p}\Delta^*\phi, \tag{3.10}$$

where Δ is the diagonal map assigning (x,x) to x. Thus, because of (3.9) and (3.10), the d commutes with U_K^* modulo the operation which corresponds to summing all the pull-backs of forms obtained by removing one segment $J_0=(j_1,j_2)$ in K for which $q_J=n-1$, and by fusing the nodes j_1 and j_2: precisely, the form

$$\beta_{j_1}(y_{j_1})\wedge\beta_{j_2}(y_{j_2})\wedge\pi^*\left(\frac{x_{j_2}-x_{j_1}}{|x_{j_2}-x_{j_1}|}\right)^*\gamma_{J_0}\wedge\bigwedge_{\{j:J=(j,j_2)\in\mathcal{I}_K\}}\pi^*\left(\frac{x_{j_2}-x_j}{|x_{j_2}-x_j|}\right)^*\gamma_J$$

is changed into

$$\beta_{j_1}(y_{j_1})\wedge\beta_{j_2}(y_{j_1})\wedge\int_{S^{p-1}}\gamma_{J_0}\wedge\bigwedge_{\{j:J=(j,j_2)\in\mathcal{I}_K\}}\pi^*\left(\frac{x_{j_1}-x_j}{|x_{j_1}-x_j|}\right)^*\gamma_J.$$

We then obtain a finite linear combination of new tree-graphs and, applying (3.2), we obtain (3.3) and Proposition 3.2 is proved. □

3.2. The effect of bubbling for $z\in\mathrm{Hom}(\pi_p(N),\mathbf{R})$

In this section we will consider the behavior of a sequence of p-dimensional tree-graph forms corresponding to a $W^{1,p}$-weakly convergent sequence of maps in $C^\infty(S^p,N)$ and a fixed element of $\mathrm{Hom}(\pi_p(N),\mathbf{R})$. We first note that a strict sub-tree-graph L will have dimension $q<p$, and consider the behavior of the corresponding q-forms.

LEMMA 3.3. *Let u_n be a sequence weakly converging to u in $W^{1,p}(S^p,N)$ and L be a q-dimensional tree-graph of forms with $q<p$. Then*

$$u_n^L\rightharpoonup u^L\quad in\ \ L^{p/q}(S^p,\textstyle\bigwedge^q S^p).$$

Proof. We prove this lemma by induction on the number n_L+1 of nodes in L. If there is only one node, this result can be found in [GMS2]. Let $\omega\in\mathcal{D}^r(N)$ be, as in §2.2, the form at the summit of L. Then, there exists a family of tree-graphs $L_1,...,L_m$, whose dimensions sum to $q-r+m$, such that

$$u_n^L(x)=u_n^*\omega(x)\wedge\bigwedge_{i=1}^m\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u_n^{L_i}(x_i).$$

From the proof of Proposition 3.2, we have that

$$\|u_n^L\|_{p/q} \leqslant C_L \|\nabla u\|_p^{q+n_L},$$

where n_L+1 is the number of nodes in L. Extracting a subsequence if necessary, we may always assume that u_n^L converges weakly in $L^{p/q}$ and the goal is to show that this limit is u^L. Assuming that N is isometrically embedded in $\mathbf{R}^k$, we can write ω as the pull-back under the inclusion map of N of a form in $\mathbf{R}^k$:

$$\omega = \sum_{J\in\mathcal{J}} a_J\, dy_J \in \mathcal{D}^r(\mathbf{R}^k).$$

(Here $\mathcal{J}$ denotes the collection of increasing r-tuples in $\{1, ..., k\}$ and $dy_J = dy_{j_1} \wedge ... \wedge dy_{j_r}$ for $J=(j_1, ..., j_r) \in \mathcal{J}$.)

First, ignoring the coefficient functions $a_J(u_n)$ occurring in $u_n^*\omega$, we study the convergence of the q-forms

$$du_n^{j_1} \wedge ... \wedge du_n^{j_r} \wedge \bigwedge_{i=1}^m \int_{x_i\in S^p} \pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i). \tag{3.11}$$

Note that we may rewrite (3.11) as the sum of two terms

$$\begin{aligned} &d\left[u_n^{j_1}\, du_n^{j_2} \wedge ... \wedge du_n^{j_r} \wedge \bigwedge_{i=1}^m \int_{S^p} \pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)\right] \\ &\qquad + (-1)^r u_n^{j_1}\, du_n^{j_2} \wedge ... \wedge du_n^{j_d} \wedge d\left[\bigwedge_{i=1}^m \int_{S^p} \pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i)\right]. \end{aligned} \tag{3.12}$$

For fixed $n_L \geqslant 1$, we will also use induction on r. In case $r=1$, the bracketed expression in the first term is simply

$$u_n^{j_1} \wedge \bigwedge_{i=1}^m \int_{S^p} \pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^* \Omega_{S^{p-1}} \wedge u_n^{L_i}(x_i).$$

Here the functions $u_n^{j_1}$, being uniformly bounded in $W^{1,p}(S^p)$, converge, by Sobolev embedding, strongly in L^m, for all $m<\infty$, to u^{j_1}. The remaining wedge product is, by (3.7) and (3.8), weakly convergent, in some $L^{m'}$-space with $m'<\infty$, to the corresponding wedge product with $u^{L_i}(x_i)$ replacing $u_n^{L_i}(x_i)$. Multiplying and applying d, we conclude that the first term in (3.12) with $r=1$ converges, in the sense of distributions, to

$$d\left[u^{j_1} \wedge \bigwedge_{i=1}^m \int_{S^p} \pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^* \Omega_{S^{p-1}} \wedge u^{L_i}(x_i)\right].$$

In case $r\in\{2,3,...\}$, the induction assumption on r implies that

$$u_n^{j_1}\,du_n^{j_2}\wedge...\wedge du_n^{j_d}\wedge\bigwedge_{i=1}^m\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u_n^{L_i}(x_i)$$

converges weakly, in $L^{p/(q-1)}$-sense, to

$$u^{j_1}\,du^{j_2}\wedge...\wedge du^{j_d}\wedge\bigwedge_{i=1}^m\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u^{L_i}(x_i).$$

So again, the first term in (3.12) converges, in the sense of distributions, to the corresponding term with u_n replaced by u.

The second term in (3.12) equals, modulo a multiplication by a constant,

$$\sum_{l=1}^m(-1)^{s_l}a_J(u_n)u_n^{j_1}\,d\wedge u_n^{j_2}\wedge...\wedge du_n^{j_r}\wedge u_n^{L_l}\wedge\bigwedge_{i\neq l}\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u^{L_i}(x_i),$$

where s_l is some integer depending on l. Each of the terms of this sum corresponds to a $u_n^{L'_l}$, where the number of nodes in L'_l has been decreased by one, and now equals $n_L+1-1=n_L$. We can thus apply our induction assumption on the number of nodes and deduce that each of these $u_n^{L'_l}$ converges weakly in $L^{p/q}$ to the corresponding $u^{L'_l}$.

Thus, we have showed, by induction on r, that

$$du_n^{j_1}\wedge du_n^{j_2}\wedge...\wedge du_n^{j_r}\wedge\bigwedge_{i=1}^m\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u_n^{L_i}(x_i)$$

converges weakly in $L^{p/q}$ to

$$du^{j_1}\wedge du^{j_2}\wedge...\wedge du^{j_r}\wedge\bigwedge_{i=1}^m\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u^{L_i}(x_i).$$

Finally, we consider the coefficients $a_J(u_n)$. Since a_J is smooth and bounded on the compact submanifold N, Sobolev embedding implies that the sequence $a_J(u_n)$ converges strongly in every L^m-space to $a_J(u)$, and therefore (3.11) converges, in the distribution sense, to

$$a_J(u)\,du^{j_1}\wedge...\wedge du^{j_d}\wedge\bigwedge_{i=1}^m\int_{S^p}\pi^*\left(\frac{x-x_i}{|x-x_i|}\right)^*\Omega_{S^{p-1}}\wedge u^{L_i}(x_i).$$

Since it is bounded in $L^{p/q}$, this convergence also holds weakly in $L^{p/q}$ and Lemma 3.3 is proved. □

We are now ready to study the behavior of the smooth p-forms u_n^K on S^p associated with a $W^{1,p}$-weakly convergent sequence of maps $u_n \in C^\infty(S^p, N)$ and a finite linear combination $K = \sum_l \lambda_l K_l$ of tree-graphs constructed in §2. The weak convergence of the mappings u_n implies, by (2.20), the boundedness in L^1 of the sequence of p-forms u_n^K, or equivalently of the dual functions $* u_n^K \in C^\infty(S^p)$. Thus the corresponding sequence of *signed measures* on S^p,

$$(* u_n^K)\mathcal{H}^p|_{S^p}, \tag{3.13}$$

has a subsequence $(* u_{n_k}^K)\mathcal{H}^p|_{S^p}$ convergent to a signed Radon measure ν on S^p. Here, for notational simplicity, we indicate this by simply saying

$$u_{n_k}^K \rightharpoonup \nu \quad \text{weakly as Radon measures.}$$

The goal of this subsection is to prove the following proposition.

PROPOSITION 3.4. *Suppose that z belongs to* $\mathrm{Hom}(\pi_p(N), \mathbf{R})$ *and $K = \sum_l \lambda_l K_l$ is a formal linear combination of tree-graphs of closed forms on N associated with z for a given choice of geometric realization Ψ_N (as described in* §2). *For any sequence $u_n \in C^\infty(S^p, N)$ which $W^{1,p}$-weakly converges to a map $u \in W^{1,p}(S^p, N)$, there exist a subsequence u_{n_k}, finitely many points $a_1, ..., a_I$ in S^p and finitely many maps $w_1, ..., w_I$ from S^p to N, with each $z([w_i]) \neq 0$, such that the following assertions hold*:

(i) *The differential p-forms $u_{n_k}^K$ converge weakly as Radon measures*:

$$u_{n_k}^K \rightharpoonup u^K + \sum_{i=1}^I z([w_i])\delta_{a_i}. \tag{3.14}$$

(ii) *For k sufficiently large, the following identity holds*:

$$z([u_{n_k}]) = z([u]) + \sum_{i=1}^I z([w_i]). \tag{3.15}$$

(iii) *The following inequality holds in the sense of measures*:

$$|\nabla u|^p \,\mathcal{H}^p|_{S^p} + C_K \sum_{i=1}^I |z([w_i])|^{p/(p+n_K)} \delta_{a_i} \leqslant \liminf_{k\to\infty} |\nabla u_{n_k}|^p \,\mathcal{H}^p|_{S^p}, \tag{3.16}$$

where C_K is a positive constant depending only on K, and n_K is the total number of segments among all K^l in K.

(iv) *Given any bilipschitz homeomorphism Ψ of S^p, we have*

$$(u_{n_k} \circ \Psi)^K \rightharpoonup (u \circ \Psi)^K + \sum_{i=1}^I z([w_i])\delta_{\Psi^{-1}(a_i)}. \tag{3.17}$$

(v) *There exists a constant* $\varepsilon_{p,N}>0$ *depending only on* p *and* N *such that*

$$\lim_{r\to 0}\liminf_{k\to\infty}\int_{S^p\cap B_r(a_i)}|\nabla u_{n_k}|^p\,d\mathcal{H}^p\geqslant\varepsilon_{p,N} \tag{3.18}$$

whenever $z([w_i])\neq 0$ *for some* $i=1,...,I$.

(vi) *Let* B *be an open subdomain of* S^p, *and let* v_n *be another weakly converging sequence in* $W^{1,p}(S^p,N)$ *such that* $v_n=u_n$ *on* B *for all* $n\in\mathbf{N}$. *Then, as Radon measures,*

$$(u^K_{n_k}-v^K_{n_k})|_B\rightharpoonup(u^K-v^K)|_B, \tag{3.19}$$

where v *denotes the weak limit of* v_n *and* $|_B$ *is the restriction operator to the subdomain* B.

Remark 3.5. Using the notation introduced in formula (3.1), assertion (i) says that, for any continuous function f on S^p, the sequence

$$(\mathrm{Graph}\,\mathcal{U}^{n_k}_{K_l})\big(f(x_1)\omega^{K_l}(y_1,...,y_{n_{K_l}+1})\wedge\textstyle\bigwedge_{J\in\mathcal{I}_{K_l}}\Omega_{S^{p-1}}(X_J)\big)$$

converges to

$$(\mathrm{Graph}\,\mathcal{U}_{K_l})\big(f(x_1)\omega^{K_l}(y_1,...,y_{n_{K_l}+1})\wedge\textstyle\bigwedge_{J\in\mathcal{I}_{K_l}}\Omega_{S^{p-1}}(X_J)\big)+\sum_{i=1}^I f(a_i)z([v_i]),$$

where x_l always denotes the variable assigned to the summit of the graph K_l.

Proof of Proposition 3.4. In this proof, we will, for simplicity, not change notation when we pass to subsequences. In particular, we first pass to a subsequence to assume that

$$|\nabla u_n|^p\,\mathcal{H}^p|_{S^p}\rightharpoonup\mu\quad\text{and}\quad u^K_n\rightharpoonup\nu$$

converge weakly as Radon measures on S^p. There exists, by White's result in [Wh], a positive number $\varepsilon_{p,N}$ such that any map $u\in W^{1,p}(S^p,N)$ satisfying

$$\int_{S^p}|\nabla u|^p\,d\mathcal{H}^p\leqslant\varepsilon_{p,N} \tag{3.20}$$

is homotopically trivial. Thus, there are only finitely many points $a_1,...,a_I$ in S^p satisfying

$$\mu(\{a_1\})>\varepsilon_{p,N},\quad ...,\quad \mu(\{a_I\})>\varepsilon_{p,N}.$$

We will first verify that

$$\nu|_{S^p\setminus\{a_1,...,a_I\}}=u^K. \tag{3.21}$$

To do this, it suffices, by the Besicovitch covering lemma, to find, for each point $x_0 \in S^p \setminus \{a_1, ..., a_I\}$ and each positive δ, a positive number $r < \delta$ such that

$$\lim_{n\to\infty} \int_{S^p \cap B_r(x_0)} u_n^K = \int_{S^p \cap B_r(x_0)} u^K.$$

We may assume, for simplicity of the presentation, that x_0 does not coincide with the pole sent to infinity by the stereographic projection. The idea is to choose r so that $\int_{S^p \cap \partial B_r(x_0)} |\nabla u_n|^p$ is small, and then use the equation

$$\int_{S^p \cap B_r(x_0)} u_n^K = \int_{S^p} \hat{u}_n^K - \int_{S^p \setminus B_r(x_0)} \hat{u}_n^K, \tag{3.22}$$

with a certain extension $\hat{u}_n \in W^{1,p}(S^p, N)$ of the restriction $u_n|_{S^p \cap B_r(x_0)}$. We will show below how to choose $\hat{u}_n$ and the corresponding limit $\hat{u}$ on $S^p \setminus B_r(x_0)$ to have strong $W^{1,p}$-convergence there. This will take care of the convergence of the second term in (3.22). The first term will involve the topological quantities, namely $z([\hat{u}_n])$ and $z([\hat{u}])$, that will all vanish provided the total p-energies of the $\hat{u}_n$'s are all less than $\varepsilon_{p,N}$.

To choose r and find a suitable extension $\hat{u}_n$, first let ε be a small positive number, to be determined later, and choose a positive $s = s(\delta, \varepsilon)$ small enough so that

$$s < \min\{\delta, |x_0 - a_1|, ..., |x_0 - a_I|\} \quad \text{and} \quad \int_{S^p \cap B_s(x_0)} \mu \leqslant \frac{\varepsilon}{2}.$$

Passing to a subsequence, we may, by Fatou's lemma and Fubini's theorem, choose $r \in \left(\frac{1}{2}t, t\right)$ so that

$$\int_{S^p \cap \partial B_r(x_0)} |\nabla u_n|^p \, d\mathcal{H}^{p-1} \leqslant r\varepsilon$$

for all large n. This implies that

$$\|u_n\|_{W^{1-1/p,p}(S^p \cap \partial B_r(x_0), \mathbf{R}^k)} \leqslant C\varepsilon$$

(where $\mathbf{R}^k$ is an ambient space in which N is isometrically embedded). Since

$$\dim[S^p \cap \partial B_r(x_0)] = p - 1 < p,$$

there exist elements ξ_n in N such that

$$\|u_n - \xi_n\|_{L^\infty(S^p \cap \partial B_r(x_0))} \leqslant C\varepsilon, \tag{3.23}$$

where C is independent of s, ε and u_n. By (3.23), we can form a Whitney C^1-extension $\tilde{u}_n$ of $u_n|_{S^p \cap \partial B_r(x_0)}$ to $S^p \setminus B_r(x_0)$. Letting π_N denote the nearest-point projection from a tubular neighborhood of N onto N, we see that the map $\hat{u}: S^p \to N$ defined by

$$\begin{cases} \hat{u}_n = u & \text{on } S^p \cap B_r(x_0), \\ \hat{u}_n = \pi_N \circ \tilde{u}_n & \text{on } S^p \setminus B_r(x_0), \end{cases}$$

satisfies the small energy bound

$$\int_{S^p} |\nabla \hat{u}_n|^p \, d\mathcal{H}^p \leqslant C\varepsilon.$$

(The fact that constants are independent of r comes from the scaling invariance of the p-energy in $\mathbf{R}^p$.)

It is clear, since u_n converges weakly to u, that $\hat{u}_n$ converges to $\hat{u}$ and

$$\int_{S^p} |\nabla \hat{u}|^p \, d\mathcal{H}^p \leqslant C\varepsilon.$$

Moreover, our choice of r so that $\int_{\partial B_r(x_0)} |\nabla u_n|^p$ is uniformly bounded (with respect to n), and our choice of extension guarantee that $\hat{u}_n|_{S^p \setminus B_r(x_0)}$ converges *strongly* in $W^{1,p}(S^p \setminus B_r(x_0), N)$. Let $\hat{u}_n^K$ be the form obtained by replacing u_n by $\hat{u}_n$ in each of the forms $U^*_{K_l}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}$. By also insisting that $C\varepsilon < \varepsilon_{p,N}$, we then have $z([\hat{u}_n])=0$, so that

$$0 = \int_{S^p} \hat{u}_n^K = \sum_l \lambda_l \int_{x_1 \in \mathbf{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbf{R}^p} \widehat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}, \tag{3.24}$$

and similarly

$$0 = \int_{S^p} \hat{u}^K = \sum_l \lambda_l \int_{x_1 \in \mathbf{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbf{R}^p} \widehat{U}^*_{K_l,\infty}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}. \tag{3.25}$$

Next, letting

$$\Omega_0 = \pi[S^p \cap B_r(x_0)] \quad \text{and} \quad \Omega_1 = \mathbf{R}^p \setminus \pi[S^p \cap B_r(x_0)],$$

we write, for every tree-graph K_l arising in K,

$$\begin{aligned} \int_{S^p} \hat{u}_n^{K_l} &= \int_{x_1 \in \mathbf{R}^p} \cdots \int_{x_{n_{K_l}+1} \in \mathbf{R}^p} \widehat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \\ &= \sum_{(i_1,\ldots,i_{n_{K_l}+1}) \in \mathfrak{I}_l} \int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} \widehat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}, \end{aligned} \tag{3.26}$$

where $\mathfrak{I}_l$ denotes the set of n_{K_l}-tuples of ordered numbers taken from the set $\{0,1\}$. Thus, the integral we are interested in, namely $\int_{S^p \cap B_r(x_0)} u_n^K$, equals the single term in the sum given by the n_{K_l}-tuple $(0, \ldots, 0)$. For each of the remaining terms, $i_k = 1$ for some $k \in \{1, \ldots, n_{K_l+1}\}$, and we claim that we have the convergence

$$\begin{aligned} \lim_{n \to \infty} \int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} & \widehat{U}^*_{K_l,n}(\omega^{K_l}) \wedge \mathcal{G}^{K_l} \\ &= \int_{x_1 \in \Omega_{i_1}} \cdots \int_{x_{n_{K_l}+1} \in \Omega_{i_{n_{K_l}+1}}} \widehat{U}^*_{K_l,\infty}(\omega^{K_l}) \wedge \mathcal{G}^{K_l}. \end{aligned} \tag{3.27}$$

To verify this, we may assume that $i_1=1$. We observe that the modification of the orientation of segments in the graph only leads to a possible change of sign in the integrand $\widehat{U}^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l}$. Therefore, we may always assume that the node for which the variable is integrated in Ω_1 is the summit of our graph. We have chosen indexation so that this is the variable x_1. Let ω_1 be the form at the summit of the tree-graph K_l. We now can write

$$\begin{aligned}\int_{x_1\in\Omega_{i_1}}\cdots\int_{x_{n_{K_l}+1}\in\Omega_{i_{n_{K_l}+1}}}&\widehat{U}^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l}\\ =\int_{x_1\in\pi(S^p\setminus B_r(x_0))}&(\hat{u}_n\circ\pi^{-1})^*\omega_1(x_1)\wedge\bigwedge_k\int_{x_{j_k}}\left(\frac{x_1-x_{j_k}}{|x_1-x_{j_k}|}\right)^*\Omega_{S^{p-1}}\wedge\hat{u}_n^{K_{l,k}}(\pi^{-1}(x_{j_k})),\end{aligned}$$

where the j_k's are all the nodes connected to the summit and the $K_{l,k}$'s are the tree-graphs issued from these nodes. Denote by $d_1>0$ the degree of the form ω_1. From Lemma 3.3, we know that for every k, $\hat{u}_n^{K_{l,k}}$ converges weakly in L^{p/s_k} for some $s_k>0$, and the s_k's satisfy $\sum_k(s_k-1)=p-d_1$ (where we are using Proposition 3.2). Therefore, in the distribution sense,

$$\int_{x_{j_k}}\left(\frac{x_1-x_{j_k}}{|x_1-x_{j_k}|}\right)^*\Omega_{S^{p-1}}\wedge(\hat{u}_n^{K_{l,k}}\circ\pi^{-1})\rightharpoonup\int_{x_{j_k}}\left(\frac{x_1-x_{j_k}}{|x_1-x_{j_k}|}\right)^*\Omega_{S^{p-1}}\wedge(\hat{u}^{K_{l,k}}\circ\pi^{-1}).$$

Using classical estimates on Riesz integrals (see [St]), we obtain that

$$\int_{x_{i_k}}\left(\frac{x_1-x_{i_k}}{|x_1-x_{i_k}|}\right)^*\Omega_{S^{p-1}}\wedge\hat{u}_n^{K_{l,k}}(\pi^{-1}(x_{i_k}))$$

is uniformly bounded in $L^{p/(s_k-1)}$. Therefore, it converges weakly in $L^{p/(s_k-1)}$ to the limit

$$\int_{x_{j_k}}\left(\frac{x_1-x_{j_k}}{|x_1-x_{j_k}|}\right)^*\Omega_{S^{p-1}}\wedge\hat{u}^{K_{l,k}}\circ\pi^{-1}.$$

Since $\hat{u}_n^*\omega_1$ converges strongly to $\hat{u}^*\omega_1$ on $\Omega_2=\mathbf{R}^p\setminus B_r(x_0)$, (3.27) is proved. Therefore, combining (3.24), (3.25), (3.26) and (3.27), we obtain that

$$\begin{aligned}\lim_{n\to\infty}\sum_l\lambda_l\int_{x_1\in B_r(x_0)}\cdots\int_{x_{n_{K_l}+1}\in B_r(x_0)}&U^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l}\\ =\sum_l\lambda_l\int_{x_1\in B_r(x_0)}\cdots\int_{x_{n_{K_l}+1}\in B_r(x_0)}&U^*_{K_l,\infty}(\omega^{K_l})\wedge\mathcal{G}^{K_l}.\end{aligned}\tag{3.28}$$

Considering now the integral of

$$u_n^K=\sum_l\lambda_l u_n^{K_l}\quad\text{on }\ B_r(x_0).$$

We make the following elementary remark: Let f_n be some sequence of functions weakly converging in $L^{p/s}(\mathbf{R}^p\setminus B_r(x_0))$, for $2\leqslant s\leqslant p-1$, to a limit f. Then

$$\int_{\mathbf{R}^p\setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} f_n(y)\,dy$$

converges strongly in $L^q_{\mathrm{loc}}(B_r(x_0))$ for any $q>0$ and converges weakly in $L^{p/(s-1)}(B_r(x_0))$ to

$$\int_{\mathbf{R}^p\setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} f(y)\,dy.$$

Moreover, it is not difficult to check that

$$\left\|\int_{\mathbf{R}^p\setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} f_n(y)\,dy\right\|_{L^q(B_r(x_0))}$$

is uniformly bounded for some $q>p/(s-1)$. Therefore, we deduce that there exists some $q>p/(s-1)$ such that

$$\int_{\mathbf{R}^p\setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} f_n(y)\,dy \to \int_{\mathbf{R}^p\setminus B_r(x_0)} \frac{1}{|x-y|^{p-1}} f(y)\,dy \quad \text{in } L^q(B_r(x_0)).$$

Considering a decomposition of the domain $\mathbf{R}^p$ into $B_r(x_0)$ and $\mathbf{R}^p\setminus B_r(x_0)$ in computing the integral $\int_{\mathbf{R}^p} \hat{u}_n^{K_l}=\int_{x_1\in\mathbf{R}^p} \dots \int_{x_{n_{K_l}+1}\in\mathbf{R}^p} \widehat{U}^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l}$, only the convergence of the term

$$\int_{x_1\in B_r(x_0)} \dots \int_{x_{n_{K_l}+1}\in B_r(x_0)} U^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l}$$

is a-priori problematic. But, using the result of our efforts above in (3.28), we can deduce that

$$\int_{B_r(x_0)} u_n^K \to \int_{B_r(x_0)} u^K.$$

This holds for any x_0 in $S^p\setminus\{a_1, \dots, a_I\}$ and any r small enough. Therefore, (3.21) holds true, and we have that

$$u_n^K \rightharpoonup u^K + \sum_{i=1}^{I} m_i\delta_{a_i} \quad \text{as Radon measures}$$

for some real numbers $m_1, \dots, m_I$. A careful and classical concentration compactness blow-up study at each a_i, using similar arguments as above, shows that each number m_i may be identified as a sum of $z([v_k])$, where v_k are maps from S^p into N. One readily finds a single map $w_i\colon S^p\to S^p$ with $[w_i]=\sum_k [v_k]$. This ends the proof of (3.14) and (3.15).

The proof above shows that, *in* $B_r(a_i)$, the part of $u_n^{K_l}(x_1)$ given by

$$\sum_{(i_2,\dots,i_{n_{K_l}+1})\in\mathfrak{I}_l}\int_{x_2\in\Omega_{i_2}}\dots\int_{x_{n_{K_l}+1}\in\Omega_{i_{n_{K_l}+1}}}U^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l},$$

where $\mathfrak{I}_l$ is the set of n_{K_l}-tuples of numbers in $\{1,2\}$ such that at least one i_k is equal to 2, converges to

$$\sum_{(i_2,\dots,i_{n_{K_l}+1})\in\mathfrak{I}_l}\int_{x_2\in\Omega_{i_2}}\dots\int_{x_{n_{K_l}+1}\in\Omega_{i_{n_{K_l}+1}}}U^*_{K_l,\infty}(\omega^{K_l})\wedge\mathcal{G}^{K_l}.$$

Only the local term

$$V_{r,a_i}=\int_{x_2\in B_r(a_i)}\dots\int_{x_{n_{K_l}+1}\in B_r(a_i)}U^*_{K_l,n}(\omega^{K_l})\wedge\mathcal{G}^{K_l}$$

is concentrating. Using (2.20) (i.e. Lemma 2.11), we have

$$\left|\int_{B_r(a_i)}V_{r,a_i}\right|^{n/(n+n_{K_l})}\leqslant C_{K_l}\int_{B_r}|\nabla u_n|^p\,d\mathcal{H}^p.$$

Combining these facts with the result of White quoted at the beginning of the proof of the proposition, which says that

$$\lim_{r\to 0}\liminf_{k\to\infty}\int_{B_r(a_i)}|\nabla u_n|^p\,d\mathcal{H}^p\geqslant\varepsilon_{p,N}$$

as long as $z[w_i]\neq 0$, we get (3.16). In order to establish (3.17) it suffices to combine the proof of (3.14) with the last part of Remark 2.10. Fact (3.19) is also a direct consequence of the previous observations. This completes the proof of Proposition 3.4. □

4. Scans of p-dimensional tree-graph forms for maps of $\mathbf{R}^{p+1}$

4.1. Notation and definitions

Throughout this section, we will be discussing $W^{1,p}$-mappings

$$u\colon\mathbf{R}^{p+1}\longrightarrow N$$

by considering their restrictions to p-spheres.

Letting $\mathbf{R}^{p+1}\times\mathbf{R}_+$ parametrize the space of all p-dimensional spheres in $\mathbf{R}^{p+1}$ with $(c,r)\mapsto\partial B_r(c)$, we observe that, for almost all $(c,r)\in\mathbf{R}^{p+1}\times\mathbf{R}_+$, the *restriction*

$$u|_{\partial B_r(c)}\in W^{1,p}(\partial B_r(c),S^p).$$

To apply all the work of the previous chapters, we will consider the corresponding $W^{1,p}$-map of the unit sphere,

$$u_{c,r}: S^p \longrightarrow S^p, \quad u_{c,r}(x) = u(c+rx) \text{ for } x \in \partial B_r(c).$$

A *scan* $\mathcal{S}$ in $\mathbf{R}^{p+1}$ is simply a measurable map from the space of p-spheres to the space $\mathcal{M}(S^p)$ of signed Radon measures on S^p,

$$\mathcal{S}: \mathbf{R}^{p+1} \times \mathbf{R}_+ \longrightarrow \mathcal{M}(S^p).$$

As described in (3.13), we may use, for any integrable p-form ω on S^p, the same symbol ω to denote the corresponding absolutely continuous signed measure $(*\,\omega)\mathcal{H}^p|_{S^p} \in \mathcal{M}(S^p)$. Thus, a linear combination K of p-dimensional tree-graphs on S^p and a mapping $u \in C^\infty(\mathbf{R}^{p+1}, N)$ induce the *K-scan of the mapping u*,

$$(c,r) \longmapsto u^K_{c,r},$$

where $u^K_{c,r}$ is the p-form constructed in §2.2 using the Gauss integrals of §2.4. Even for nonsmooth $u \in W^{1,p}(\mathbf{R}^{p+1}, N)$, where $u_{c,r} \in W^{1,p}(S^p, N)$ for almost all (c,r), similar constructions and integrations give a K-scan of u.

We next consider the behavior of the K-scans of smooth mappings in a $W^{1,p}$-weakly convergent sequence.

4.2. Convergence of scans

The goal of this subsection is to prove the following proposition.

PROPOSITION 4.1. *Let K be a finite linear combination of tree-graphs of closed forms associated with a class $z \in \mathrm{Hom}(\pi_p(N), \mathbf{R})$. For any sequence $u_n \in C^\infty(\mathbf{R}^{p+1}, S^p)$ converging $W^{1,p}$-weakly to some limit $u \in W^{1,p}(\mathbf{R}^{p+1}, S^p)$, there exist a subsequence $u_{n'}$ and a limit scan $\mathcal{S}$,*

$$\mathcal{S}(c,r) = u^K_{c,r} + \sum_{i=1}^{I(c,r)} m_i(c,r)\,\delta_{a_i(c,r)},$$

with convergence in the following sense: for almost every $(c,r) \in \mathbf{R}^m \times \mathbf{R}_+$,

$$\sup_{n''} \int_{\partial B_r(c)} |\nabla u_{n''}|^p \, d\mathcal{H}^p < \infty$$

for some subsequence n'' of n', and, for any such subsequence, the differential p-forms

$$(u_{n''})^K_{c,r} \rightharpoonup \mathcal{S}(c,r)$$

weakly as Radon measures. Here the quantities $I(c,r) \in \mathbf{N}$, $m_i(c,r) \in \mathbf{R}_+ \cap z(\Pi_p(n))$ and $a_i(c,r) \in S^p$ are all measurable in (c,r).

Proof. Since the weak $W^{1,p}$-convergence implies strong L^1_{loc}-convergence, we may first pass to a subsequence, without changing notation, to guarantee that

$$\lim_{n\to\infty} u_n(x)=u(x) \quad \text{for a.e. } x\in\mathbf{R}^{p+1}. \tag{4.1}$$

To find the desired subsequence $u_{n'}$ and uniquely determine the limiting scan, let

$$\{(c_1,s_1),(c_2,s_2),\dots\}$$

be a countable dense subset of $\mathbf{R}^{p+1}\times\mathbf{R}_+$. By Fatou's lemma and Fubini's theorem,

$$\int_{r\in[s_1-1,s_1+1]} \liminf_{n\to\infty} \int_{\partial B_r(c_1)} |\nabla u_n|^p \, d\mathcal{H}^p \, dr \leqslant \sup_n \int_{\mathbf{R}^{p+1}} |\nabla u_n|^p \, dx < \infty,$$

and we may choose a number $r_1\in[s_1-1,s_1+1]$ and a subsequence $\alpha_1(n)$ so that

$$\sup_n \int_{\partial B_{r_1}(c_1)} |\nabla u_{\alpha_1(n)}|^p \, d\mathcal{H}^p < \infty.$$

Similarly, we inductively find, for $k=2,3,\dots$, numbers $r_k\in[s_k-1/k,s_k+1/k]$ and a subsequence $\alpha_k(n)$ of $\alpha_{k-1}(n)$ so that

$$\sup_n \int_{\partial B_{r_k}(c_k)} |\nabla u_{\alpha_k(n)}|^p \, d\mathcal{H}^p < \infty.$$

Then $\{(c_1,r_1),(c_2,r_2),\dots\}$ is also dense in $\mathbf{R}^{p+1}\times\mathbf{R}_+$, and the diagonal sequence $\alpha_n(n)$ gives

$$\sup_n \int_{\partial B_{r_k}(c_k)} |\nabla u_{\alpha_n(n)}|^p \, d\mathcal{H}^p < \infty$$

for all $k=1,2,\dots$. By Proposition 3.4, we may use another diagonal procedure to find a subsequence $u_{n'}$ so that, for every $k=1,2,\dots$, one has on S^p the weak convergences of Radon measures

$$\lim_{n\to\infty} |\nabla (u_{n'})_{c_k,r_k}|^p \, \mathcal{H}^p|_{S^p} = \mu_k$$

and

$$\lim_{n\to\infty} (u_{n'})^K_{c_k,r_k} = u^K_{c_k,r_k} + \sum_{i=1}^{I(c_k,r_k)} m_i(c_k,r_k)\delta_{a_i(c_k,r_k)}, \tag{4.2}$$

where μ_k is a positive Radon measure on S^p, $I(c_k,r_k)\in\{0,1,\dots\}$ and $m_i(c_k,r_k)\in\mathbf{R}_+$ and $a_i(c_k,r_k)\in S^p$ for $i=1,\dots,I(c_k,r_k)$. We will now show how these countably many points $a_i(c_k,r_k)$ and multiplicities $m_i(c_k,r_k)$ uniquely determine the limiting scan.

For this purpose, consider now an arbitrary point $c \in \mathbf{R}^{p+1}$. Then, by (4.1), we see that the exceptional set

$$X_c = \left\{ r \in \mathbf{R} \colon \mathcal{H}^p \left(\left\{ x \in \partial B_r(c) \colon \lim_{n' \to \infty} u_{n'}(x) \neq u(x) \right\} \right) > 0 \right\} \tag{4.3}$$

has, by Fubini's theorem, measure zero. Using now Fatou's lemma, we have that

$$\int_{r \in \mathbf{R}} \liminf_{n' \to \infty} \int_{\partial B_r(c)} (|\nabla u| + |\nabla u_{n'}|^p) \, d\mathcal{H}^p \leqslant 2 \sup_{n'} \int_{\mathbf{R}^{p+1}} |\nabla u_{n'}|^p \, dx < \infty,$$

and the set

$$Y_c = \left\{ r \in \mathbf{R} \colon \int_{\partial B_r(c)} |\nabla u|^p \, d\mathcal{H}^p + \liminf_{n' \to \infty} \int_{\partial B_r(c)} |\nabla u_{n'}|^p \, d\mathcal{H}^p = \infty \right\} \tag{4.4}$$

also has measure zero. Finally,

$$Z_c = \{ r \in \mathbf{R} \colon \mu_k(\partial B_r(c)) > 0 \text{ for some } k = 1, 2, \dots \} \tag{4.5}$$

is countable. Avoiding these three sets by taking $r \in \mathbf{R} \setminus (X_c \cup Y_c \cup Z_c)$, we may now find and fix a subsequence n'' (depending on r) so that

$$\sup_{n''} \int_{\partial B_r(c)} |\nabla u_{n''}|^p \, d\mathcal{H}^p < \infty.$$

The sequence of maps $u_{n''}|_{\partial B_r(c)}$ is then weakly sequentially compact in $W^{1,p}(\partial B_r(c), N)$ while the sequence of measures $(u_{n''})^K_{c,r}$ is weakly sequentially compact in $\mathcal{M}(S^p)$.

To prove the convergence of these sequences, we note that, being subsets of duals of separable spaces, bounded sets in $W^{1,p}(\partial B_r(c), N)$ and in $\mathcal{M}(S^p)$, with their weak* topologies, are metrizable. Thus, we may use the following elementary observation.

Remark 4.2. A sequentially-compact sequence in a metric space converges if and only if any two subsequences contain subsequences convergent to the *same* limit.

Since $r \notin X_c \cup Y_c$, we first conclude that

$$u_{n''} \rightharpoonup u \quad \text{in } W^{1,p}(\partial B_r(c), N).$$

Then, we see from Proposition 3.4 that any subsequence of $u_{n''}$ contains a subsequence $u_{n'''}$ giving the weak convergence of measures in the form

$$(u_{n'''})^K_{c,r} \rightharpoonup u^K_{c,r} + \sum_{i=1}^{\bar{I}} \overline{m}_i \delta_{\bar{a}_i}, \tag{4.6}$$

where $\bar{I}\in\{0,1,\dots\}$, $\bar{m}_i\in\mathbf{R}\setminus\{0\}$ and $\bar{a}_i\in S^p$ for $i=1,\dots,\bar{I}$. Any other subsequence of $u_{n''}$ contains, again by Proposition 3.4, a subsequence $u_{n''''}$ giving a weak convergence

$$(u_{n''''})^K_{c,r} \rightharpoonup u^K_{c,r}+\sum_{j=1}^{\bar{J}}\bar{n}_j\delta_{\bar{b}_j}. \tag{4.7}$$

Thus, to complete the proof, we need to establish the uniqueness

$$\sum_{i=1}^{\bar{I}}\bar{m}_i\delta_{\bar{a}_i}=\sum_{j=1}^{\bar{J}}\bar{n}_j\delta_{\bar{b}_j}. \tag{4.8}$$

To verify (4.8), take any point $\bar{a}_i$ and fix a positive number

$$\varrho<\min\{|\bar{a}_i-b|:\bar{a}_i\neq b\in\{\bar{a}_1,\dots,\bar{a}_{\bar{I}},\bar{b}_1,\dots,\bar{b}_{\bar{J}}\}\}.$$

We may choose an element (c_k,r_k) of our countable family above so that $r_k<\frac{1}{2}\varrho$ and $\bar{a}_i\in B_{r_k}(c_k)$. Thus $B_{r_k}(c_k)$ either does not intersect $\{\bar{b}_1,\dots,\bar{b}_{\bar{J}}\}$ or contains exactly one $\bar{b}_j$, which then coincides with $\bar{a}_i$.

Consider now the Lipschitz sphere

$$\Sigma=\partial(B_{r_k}(c_k)\cap B_r(c))=\bar{\Sigma}_1\cup\Sigma_2,$$

where

$$\Sigma_1=\Sigma\cap B_r(c)=\partial B_{r_k}(c_k)\cap B_r(c),$$
$$\Sigma_2=\Sigma\cap B_{r_k}(c_k)=\partial B_r(c)\cap B_{r_k}(c_k).$$

Letting $\Psi(x)=r_k^{-1}(x-c_k)$ for $x\in\bar{\Sigma}_1$, we see that Ψ extends to a bilipschitz homeomorphism of $\Psi\colon\Sigma\to S^p$ whose restriction to Σ_2 is a smooth diffeomorphism.

Concerning the region $\bar{\Sigma}_1$, we have the convergence, as Radon measures on S^p, of the original subsequence $(u_{n''})^K_{c_k,r_k}$. This convergence then restricts, by (3.19), to the open subset $\Psi(\Sigma_1)$ and, by (4.5), to the closure $\Psi(\bar{\Sigma}_1)$. Using (3.14) and (3.17), we deduce that

$$(u_{n''})^K_{c_k,r_k}|_{\Psi(\bar{\Sigma}_1)} \rightharpoonup u^K_{c_k,r_k}|_{\Psi(\bar{\Sigma}_1)}+\mathcal{R}, \tag{4.9}$$

where

$$\mathcal{R}=\sum_{a_i(c_k,r_k)\in\Sigma_1} m_i(c_k,r_k)\delta_{a_i(c_k,r_k)}.$$

Concerning the complementary region Σ_2, we may restrict the convergences of the two subsequences $(u_{n'''})^K_{c,r}$ and $(u_{n''''})^K_{c,r}$ to the open subset $\Psi(\Sigma_2)$ of S^p. By our choice of ϱ and by (3.17) and (3.19),

$$(u_{n'''}\circ\Psi^{-1})^K|_{\Psi(\Sigma_2)} \rightharpoonup (u\circ\Psi^{-1})^K|_{\Psi(\Sigma_2)}+\bar{m}_i\delta_{\Psi(c+ra_i)}, \tag{4.10}$$

and similarly, combining (4.2), (4.6), (3.17) and (3.19), we get that

$$(u_{n''''}\circ\Psi^{-1})^K|_{\Psi(\Sigma_2)} \rightharpoonup (u\circ\Psi^{-1})^K|_{\Psi(\Sigma_2)}+m\delta_{\Psi(c+ra_i)}, \tag{4.11}$$

where

$$\begin{aligned} &\text{either} \quad m=0, \quad \text{in case } B_{r_k}(c_k)\cap\{\bar b_1,...,\bar b_{\bar J}\}=\varnothing,\\ &\text{or} \quad m=\bar n_j \quad \text{and} \quad \bar b_j=\bar a_i. \end{aligned}$$

Finally we note that

$$\int_{S^p}(u_n\circ\Psi^{-1})^K=0 \tag{4.12}$$

for all n, because $u_n\circ\Psi$ is, by the continuity of u_n on $\overline{B_r(c)\cap B_{r_k}(c_k)}$, homotopic to zero in $[S^p,N]$.

Adding (4.9) (with n'' replaced by n''') and (4.10), integrating over S^p, using (4.12), and taking $\lim_{n'''\to\infty}$, we deduce that

$$0=\lim_{n'''\to\infty}\int_{S^p}(u_{n'''}\circ\Psi^{-1})^K=\left(\int_{S^p}(u\circ\Psi^{-1})^K\right)+\mathcal{R}(1)+\overline{m}_i.$$

Similarly, using (4.9) (with n'' replaced by n''''), (4.11) and (4.12), and taking $\lim_{n''''\to\infty}$, we see that

$$0=\lim_{n''''\to\infty}\int_{S^p}(u_{n''''}\circ\Psi^{-1})^K=\left(\int_{S^p}(u\circ\Psi^{-1})^K\right)+\mathcal{R}(1)+m.$$

Combining the last two equations we see that $m=\overline{m}_i\neq 0$ so that $\bar n_j=m=\overline{m}_i$ and $\bar b_j=\bar a_i$.

By repeating this argument, we deduce that the two sets $\{\bar b_1,...,\bar b_{\bar J}\}$ and $\{\bar a_1,...,\bar a_{\bar I}\}$ coincide and that the associated multiplicities are equal, which completes the proof of equation (4.8).

Finally, the measurability of the limiting scan

$$(c,r)\longmapsto u^K_{c,r}+\sum_{i=1}^{I(c,r)} m_i(c,r)\delta_{a_i(c,r)},$$

and hence of $I(c,r)$, $m_i(c,r)$ and $a_i(c,r)$, follows from the measurability of pointwise limits of sequences of measurable functions and the separability of $C^0(S^p)$. □

Remark 4.3. Concerning the convergence in Proposition 4.1, one may obtain convergence for a *single* subsequence, independent of (c,r), provided one does not insist on the $W^{1,p}$-weak (i.e. energy bounded) convergence of the restrictions $u_{n'}|_{\partial B(c,r)}$ for which one may nevertheless obtain L^p- and pointwise a.e.-convergence. For the space $\mathcal{M}(S^p)$, one must also use a suitable "flat distance". Here one may argue as in [HR1]. We do not pursue this in the present paper because our interest is primarily in the existence and topological and analytic properties of the limit scan $\mathcal{S}$.

4.3. Connecting $\pi_p(N)\otimes\mathbf{R}$ singularities

In this section we prove our main result: Theorem 1.1. This is an immediate consequence of Proposition 3.4 (i) (ii) and the following structure theorem on the rectifiability of the "bubbled scan" in Proposition 4.1.

THEOREM 4.4. *Suppose that* $z\in\mathrm{Hom}(\pi_p(N),\mathbf{R})$, K *is a corresponding finite linear combination of tree-graphs of forms*,

$$u_n\in C^\infty(\mathbf{R}^{p+1},N)\rightharpoonup u\in W^{1,p}(S^p,N) \quad \textit{weakly in } W^{1,p},$$

and $u_{n'}$ *is the subsequence with bubbled scan*

$$\mathcal{S}(c,r)=u^K_{c,r}+\sum_{i=1}^{I(c,r)} m_i(c,r)\,\delta_{a_i(c,r)},$$

as in Proposition 4.1. *Then there exist a countable union* Γ *of* C^1*-curves with measurable orientation* $\vec{\Gamma}$ *and a non-negative* $\mathcal{H}^1$ *measurable function* θ *from* Γ *into* $z(\pi_p(N))$ *such that*

$$\int_\Gamma \theta^{p/(p+n_z)}\,d\mathcal{H}^1\leqslant C_z\liminf_{n\to\infty}\int_{R^{p+1}}|\nabla u_n|^p\,dx \tag{4.13}$$

(*with* C_z *depending only on* z *and* n_z *as in Definition* 2.12), $\partial B_r(c)$ *is transverse to* Γ *for almost all* $(c,r)\in\mathbf{R}^{p+1}\times\mathbf{R}_+$, *and*

$$\mathcal{S}(c,r)=u^K_{c,r}+\sum_{a\in\Gamma\cap\partial B_r(c)}\mathrm{sgn}[\vec{\Gamma}(a)\cdot(a-c)]\theta(a)\delta_{(a-c)/r}. \tag{4.14}$$

Thus,

$$\{a_1(c,r),...,a_{I(c,r)}(c,r)\}=\Big\{\frac{a-c}{r}:a\in\Gamma\cap\partial B_r(c) \textit{ and } \theta(a)\neq 0\Big\},$$

and

$$m_i(c,r)=\mathrm{sgn}[\vec{\Gamma}(c+ra_i(c,r))\cdot a_i(c,r)]\theta(c+ra_i(c,r)).$$

We will need the following elementary lemma, a proof of which can be found in [HR1, Lemma 7.1].

LEMMA 4.5. *For* $\varepsilon>0$ *and for any sequence* $u_n\in W^{1,p}(\mathbf{R}^{p+1},N)$ *with*

$$L=\sup_n\int_{\mathbf{R}^{p+1}}|\nabla u_n|^p\,dx$$

being finite, *the* ε *energy concentration set*

$$E_\varepsilon=\Big\{c\in\mathbf{R}^{p+1}:\limsup_{r\to0}\liminf_{n\to\infty}\frac{1}{r}\int_{B_r(c)}|\nabla u_n|^p\,dx>\varepsilon\Big\}$$

has $\mathcal{H}^1(E_\varepsilon)\leqslant C_p\varepsilon^{-1}L$, *where* C_p *is a constant depending only on* p.

Proof. Suppose that $(c,r)\in\mathbf{R}^{p+1}\times\mathbf{R}_+$, $a\in\partial B_r(c)$ and $0<\varrho<r$. Then the boundary

$$\Sigma=\partial(B_\varrho(a)\cap B_r(c))$$

is uniformly bilipschitz homeomorphic to $\partial B_\varrho(a)$. In fact, one may define $\Psi:\Sigma\to\partial B_\varrho(a)$ to be the identity on $B_r(c)\cap\partial B_\varrho(a)$ and, on $B_\varrho(a)\cap\partial B_r(c)$, to be the radial projection away from the point

$$c+\frac{r-\varrho}{r}(a-c)$$

onto $\partial B_\varrho(a)\setminus B_r(c)$. One checks that $\operatorname{Lip}\Psi\leqslant 4$ and $\operatorname{Lip}\Psi^{-1}\leqslant 1$. Thus

$$4^{-p}\int_\Sigma|\nabla u|^p\,d\mathcal{H}^p\leqslant\int_{\partial B_\varrho(a)}|\nabla(u\circ\Psi^{-1})|^p\,d\mathcal{H}^p\leqslant 4^p\int_\Sigma|\nabla u|^p\,d\mathcal{H}^p.$$

For the corresponding map $(u\circ\Psi^{-1})_{a,\varrho}: S^p\to N$, we have the conformal invariance

$$\int_{S^p}|\nabla((u\circ\Psi^{-1})_{a,\varrho})|^p\,d\mathcal{H}^p=\int_{\partial B_\varrho(a)}|\nabla(u\circ\Psi^{-1})|^p\,d\mathcal{H}^p.$$

With $\varepsilon_1=10^{-1}4^{-p}\varepsilon_{p,N}$, where $\varepsilon_{p,N}$ is the constant introduced in (3.20), it follows that

$$\int_\Sigma|\nabla u|^p\leqslant 10\varepsilon_1\quad\Longrightarrow\quad[u\,|\,\Sigma]\sim[(u\circ\Psi^{-1})_{a,\varrho}]=0\ \text{ in }\ \pi_p(N).\tag{4.15}$$

Suppose that $c\in\mathbf{R}^{p+1}$ and $r\in\mathbf{R}_+\setminus(X_c\cup Y_c\cup Z_c)$, as in (4.3), (4.4) and (4.5). Consider the "bubbling points" in $\partial B_r(c)$:

$$A(c,r)=\{c+ra_1(c,r),c+ra_2(c,r),...,c+ra_{I(c,r)}(c,r)\}.$$

Also recall the classical fact (see, e.g., [Gi, Theorem 2.2]) that the set of energy density points of the $W^{1,p}$-map u,

$$Z_u=\left\{c\in\mathbf{R}^{p+1}:\limsup_{r\to0}\frac{1}{r}\int_{B_r(c)}|\nabla u|^p\,dx>0\right\},\tag{4.16}$$

has $\mathcal{H}^1(Z_u)=0$.

We will complete the proof in three steps.

Step 1. $A(c,r)\setminus Z_u\subset E_{\varepsilon_1}$.

Suppose, for contradiction, that there is a point

$$a=c+ra_i(c,r)\in A(c,r)\setminus(Z_u\cup E_{\varepsilon_1}).$$

By (4.4) and the fact that $a\notin Z_u\cup E_{\varepsilon_1}$, we can also choose a positive σ small enough so that $B_\sigma(a)\cap A(c,r)=\{a\}$,

$$\int_{\partial B_r(c)\cap B_\sigma(a)} |\nabla u|^p \, d\mathcal{H}^p \leqslant \varepsilon_1 \tag{4.17}$$

and

$$\frac{1}{\sigma}\int_{B_\sigma(a)} |\nabla u|^p \, dx + \liminf_{n'\to\infty} \frac{1}{\sigma}\int_{B_\sigma(a)} |\nabla u_{n'}|^p \, dx \leqslant \varepsilon_1.$$

Using now Fubini's theorem and Proposition 4.1, we can find a radius $\varrho\in\left[\frac{1}{2}\sigma,\sigma\right]$ and a subsequence $u_{n''}$ of u'_n such that

$$(u_{n''})^K_{a,\varrho} \rightharpoonup \mathcal{S}(a,\varrho), \tag{4.18}$$

and

$$\lim_{n''\to\infty} \int_{\partial B_\varrho(a)} |\nabla u_{n''}|^p \, d\mathcal{H}^p \leqslant 6\varepsilon_1. \tag{4.19}$$

Combining these facts with Proposition 3.4 and (4.15), we deduce that $(u_{n''})^K_{a,\varrho}$ cannot concentrate to produce any nonzero $m_i(a,\varrho)\delta_{a_i(a,\varrho)}$ so that

$$\mathcal{S}(a,\varrho) = u^K_{a,\varrho}. \tag{4.20}$$

Let $\Sigma=\partial(B_\varrho(a)\cap B_r(c))$ and $\Psi\colon\Sigma\to\partial B_\varrho(a)$ be as above. Using (4.2), (4.18), (4.20), (3.17) and (3.19) as in the proof of (4.11), we deduce that

$$(u_{n''}\circ\Psi^{-1})^K_{a,\varrho} \rightharpoonup (u\circ\Psi^{-1})^K_{a,\varrho} + m_i(c,r)\delta_{\varrho^{-1}(\Psi(a)-a)}. \tag{4.21}$$

Combining now (4.17), (4.19) and (4.15), we also have that the p-energy of $(u\circ\Psi^{-1})_{a,\varrho}$ on Σ is below the required energy for having a nonzero $\pi_p(N)$ homotopy class. Thus

$$\int_{S^p} (u\circ\Psi^{-1})^K_{a,\varrho} = 0. \tag{4.22}$$

Since the restriction of $u_{n''}$ to Σ is also null homotopic (because it extends as a smooth map on $B_r(c)\cap B_\varrho(a)$), we also have

$$\int_{S^p} (u_{n''}\circ\Psi^{-1})^K_{a,\varrho} = 0. \tag{4.23}$$

Combining (4.21), (4.22) and (4.23), we obtain that $m_i(r,c)=0$, a contradiction which completes the proof of Step 1.

Step 2. Choice of Γ.

Since, by Lemma 4.5, $\mathcal{H}^1(E_{\varepsilon_1})<\infty$, we see from the Besicovitch structure theorem [Fe, Theorem 3.3.13] that $\Lambda=E_{\varepsilon_1}\setminus\Gamma$ is *purely unrectifiable* for some countable union Γ of C^1-curves. This implies that, for any $c\in\mathbf{R}^{p+1}$,

$$\{r : \text{either } \partial B_r(c)\cap\Lambda\neq\varnothing \text{ or } \partial B_r(c) \text{ is not transverse to } \Gamma\}$$

has 1-dimensional Lebesgue measure zero. By Fubini's theorem, the set

$$\mathcal{B}_\Lambda=\{c\in\mathbf{R}^{p+1}:\mathcal{H}^1(\{r\in\mathbf{R}_+:\partial B_r(c)\cap\Lambda\neq\varnothing\})>0\} \tag{4.24}$$

has $(p+1)$-dimensional Lebesgue measure zero.

By Lemma 6.1 below, the set

$$\mathcal{A}_\Gamma=\{c\in\mathbf{R}^{p+1}:\mathcal{H}^1(T_{\Gamma,c})>0\}, \tag{4.25}$$

where

$$T_{\Gamma,c}=\{x\in\Gamma:\partial B_{|x-c|}(c) \text{ is not transverse to } \Gamma \text{ at } x\} \tag{4.26}$$

also has $(p+1)$-dimensional Lebesgue measure zero.

For a point $c\in\mathbf{R}^{p+1}\setminus(\mathcal{A}_\Gamma\cup\mathcal{B}_\Lambda)$, we deduce from (4.16), (4.25), (4.26) and (4.24) that the set

$$W_c=\{r\in\mathbf{R}_+:\partial B_r(c)\cap(Z_u\cup T_{\Gamma,c}\cup\Lambda\cup[E_{\varepsilon_1}\setminus(\Gamma\cup\Lambda)])\neq\varnothing\} \tag{4.27}$$

has $\mathcal{H}^1(W_c)=0$.

Step 3. Choice of multiplicity θ and orientation $\vec{\Gamma}$.

We fix one point $c_0\in\mathbf{R}^{p+1}\setminus(\mathcal{A}_\Gamma\cup\mathcal{B}_\Lambda)$ and, recalling (4.3), (4.4), (4.5) and (4.27), let

$$R_0=\bigcup\{A(c_0,r):r\in\mathbf{R}_+\setminus(W_{c_0}\cup X_{c_0}\cup Y_{c_0}\cup Z_{c_0})\}.$$

By Step 1 and (4.27),

$$R_0\subset\Gamma.$$

On the subset $\Gamma\setminus R_0$, we simply define $\theta\equiv 0$, and take any choice of measurable orientation $\vec{\Gamma}$ there.

For any point $a\in R_0$, one has $a\in A(c_0,|a-c_0|)$ and

$$a=c_0+|a-c_0|a_i(c_0,|a-c_0|)$$

for some $i\in\{1,...,I(c_0,r)\}$, and we may define

$$\theta(a)=|m_i(c_0,|a-c_0|)|$$

and choose the unique unit tangent vector $\vec{\Gamma}(a)$ so that

$$\operatorname{sgn}[\vec{\Gamma}(a)\cdot(a-c_0)] = \operatorname{sgn} m_i(c_0, |a-c_0|)$$

(because Γ is transverse to $\partial B_{|a-c_0|}(c_0)$ at a). This implies the $\mathcal{H}^1$-measurability of θ and $\vec{\Gamma}$, and inequality (4.13) follows from (3.14) and (3.16). The functions θ and $\vec{\Gamma}$ were chosen so that the formula (4.14) holds in case the center $c=c_0$ and the radius $r\in\mathbf{R}_+\setminus(W_{c_0}\cup X_{c_0}\cup Y_{c_0}\cup Z_{c_0})$.

It only remains to verify that, with this choice of θ and $\vec{\Gamma}$, the formula (4.14) is still true for an arbitrary c in $\mathbf{R}^{p+1}\setminus(\mathcal{A}_\Gamma\cup\mathcal{B}_\Lambda)$ and almost every $r>0$. Since $c_0\notin\mathcal{A}_\Gamma$, transversality a.e. implies that the set

$$\Upsilon c_0 = \{x\in\Gamma : |x-c_0| \in W_{c_0}\cup X_{c_0}\cup Y_{c_0}\cup Z_{c_0}\}$$

has measure $\mathcal{H}^1(\Upsilon_{c_0})=0$. It follows that

$$V_c = \{r\in\mathbf{R}_+ : \partial B_r(c)\cap\Upsilon_{c_0} \neq 0\}$$

also has $\mathcal{H}^1(V_c)=0$. Now, if we choose any radius

$$r\in\mathbf{R}_+\setminus(V_c\cup W_c\cup X_c\cup Y_c\cup Z_c),$$

then, for any point $a\in\Gamma\cap\partial B_r(c)$, the distance $r_0=|a-c_0|$ is also a good radius for c_0, that is, $r_0\notin(W_{c_0}\cup X_{c_0}\cup Y_{c_0}\cup Z_{c_0})$. In particular, we have the transversality of Γ at a with respect to both spheres $\partial B_r(c)$ and $\partial B_{r_0}(c_0)$.

For the orientations, there are four possibilities according to whether either of the two numbers

$$\operatorname{sgn}[\vec{\Gamma}(a)\cdot(a-c)] \quad \text{and} \quad \operatorname{sgn}[\vec{\Gamma}(a)\cdot(a-c_0)]$$

is positive or negative. To simplify the presentation, we assume that

$$\operatorname{sgn}[\vec{\Gamma}(a)\cdot(a-c)] = \operatorname{sgn}[\vec{\Gamma}(a)\cdot(a-c_0)] = +1 \tag{4.28}$$

(the other cases can be treated in a similar way). As above, we have, with $r=|a-c|$ and $r_0=|a-c_0|$, that

$$a = c_0 + r_0 a_i(c_0, r_0) = c + r a_j(c, r)$$

for some $i\in\{1, ..., I(c_0, r_0)\}$ and $j\in\{1, ..., I(c, r)\}$. To establish (4.14), it now suffices to show that

$$m_j(c, r) = m_i(c_0, r_0).$$

First note that for positive ϱ less than

$$\delta=\min\{\operatorname{dist}(a,A(c,r)\setminus\{a\}),\operatorname{dist}(a,A(c_0,r_0)\setminus\{a\})\},$$

one has

$$B_\varrho(a)\cap A(c,r)=\{a\}=B_\varrho(a)\cap A(c_0,r_0). \tag{4.29}$$

Let $\Omega=[B_r(c)\setminus B_{r_0}(c_0)]\cup[B_{r_0}(c)\setminus B_r(c)]$. By (4.28), the unit tangent vectors $\pm\vec{\Gamma}(a)$ lie outside the (closed) tangent cone $\operatorname{Tan}(\Omega,a)$. It follows that

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\mathcal{H}^1(\{\varrho\in[0,\sigma]:\Gamma\cap\partial B_\varrho(a)\cap\widetilde{\Omega}\neq\varnothing\})=0 \tag{4.30}$$

for some (slightly larger) open cone $\widetilde{\Omega}$ about a containing $a+\operatorname{Tan}(\Omega,a)$.

Also, since $x\notin Z_u$,

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\int_\sigma^{2\sigma}\int_{\partial B_\varrho(x)}|\nabla u|^p\,d\mathcal{H}^p\,d\varrho\leqslant\lim_{\sigma\to 0}\frac{1}{\sigma}\int_{B_\sigma(a)}|\nabla u|^p\,dx=0, \tag{4.31}$$

and clearly

$$\int_{\partial B_r(c)\cap B_\varrho(a)}|\nabla u|^p\,d\mathcal{H}^p+\int_{\partial B_{r_0}(c_0)\cap B_\varrho(a)}|\nabla u|^p\,d\mathcal{H}^p\to 0, \tag{4.32}$$

as $\varrho\to 0$.

We will now argue as in Step 1 using the two Lipschitz spheres

$$\Sigma=\partial(B_\varrho(a)\cap B_r(c))\quad\text{and}\quad\Sigma_0=\partial(B_\varrho(a)\cap B_{r_0}(c_0)).$$

From (4.30), (4.31), (4.32) and (4.15), we see that we may choose a sufficiently small positive $\varrho<\delta$ so that

$$\Gamma\cap\partial B_\varrho(a)\cap(\Omega\cup\widetilde{\Omega})=\varnothing, \tag{4.33}$$

such that the restrictions $u|_\Sigma$ and $u|_{\Sigma_0}$ have sufficiently small p-energies so that their $\Pi_p(N)$-homotopy classes both vanish, and so that there is, as in Proposition 4.1, weak convergence of subsequences of the p-forms $(u_{n'})^K_{a,\varrho}$ to $\mathcal{S}(a,\varrho)$.

Let

$$\Psi:\Sigma\longrightarrow\partial B_\varrho(a)\quad\text{and}\quad\Psi_0:\Sigma_0\longrightarrow\partial B_\varrho(a)$$

be bilipschitz homeomorphisms as in Step 1. Now, after translation and rescaling, we are considering the five sequences of maps from S^p to N,

$$(u_n)_{c,r},\quad(u_n)_{c_0,r_0},\quad(u_n)_{a,\varrho},\quad(u_n\circ\Psi^{-1})_{a,\varrho}\quad\text{and}\quad(u_n\circ\Psi_0^{-1})_{a,\varrho}.$$

Since $u_n|_\Sigma$ and $u_n|_{\Sigma_0}$ have smooth extensions to $\mathbf{R}^{p+1}$, they are null-homotopic and the integrals of the corresponding tree-graph forms vanish:

$$\int_{S^p}(u_n\circ\Psi^{-1})^K_{a,\varrho}=0=\int_{S^p}(u_n\circ\Psi_0^{-1})^K_{a,\varrho}.$$

Similarly,

$$\int_{S^p}(u\circ\Psi^{-1})^K_{a,\varrho}=0=\int_{S^p}(u\circ\Psi_0^{-1})^K_{a,\varrho},$$

because, as we saw above, the p-homotopy classes of $u|_\Sigma$ and $u|_{\Sigma_0}$ also vanish.

Our Proposition 4.1 allows us to pass to consecutive subsequences. Concerning the bubbling, we may, as before, consider the partition

$$\Sigma=(\Sigma\cap B_r(c))\cup(\Sigma\cap\overline{B_\varrho(a)}),$$

and similarly for Σ_0. On $\Sigma\cap\overline{B_\varrho(a)}$ (resp. $\Sigma_0\cap\overline{B_\varrho(a)}$), we have, by (4.9) and (4.29), the single term $m_j(c,r)\delta_{a_j(c,r)}$ (resp. $m_i(c_0,r_0)\delta_{a_i(c_0,r_0)}$), while, on $\Sigma\cap B_r(c)$ or $\Sigma_0\cap B_{r_0}(c_0)$, all the bubbling occurs, by (4.33), on the intersection

$$\Sigma\cap B_r(c)\cap\Sigma_0\cap B_{r_0}(c)=(\partial B_\varrho(a))\cap B_r(c)\cap B_{r_0}(c_0).$$

Putting this all together, we have for some subsequence $u_{n''}$,

$$\begin{aligned}0&=\lim_{n\to\infty}\int_{S^p}(u_{n''}\circ\Psi^{-1})^K_{a,\varrho}\\&=\int_{S^p}(u\circ\Psi^{-1})^K_{a,\varrho}+m_j(c,r)+\sum_{a+\varrho a_k(a,\varrho)\in B_r(c)}m_k(a,\varrho)\\&=0+m_j(c,r)+\sum_{a+\varrho a_k(a,\varrho)\in B_r(c)\cap B_{r_0}(c_0)}m_k(a,\varrho).\end{aligned}$$

Similarly,

$$0=0+m_i(c_0,r_0)+\sum_{a+\varrho a_k(a,\varrho)\in B_{r_0}(c_0)\cap B_r(c)}m_k(a,\varrho).$$

The last two equations now give the desired equality $m_j(c,r)=m_i(c_0,r_0)$. □

Remark 4.6. One may study such bubbling phenomena by considering restrictions to oriented *affine hyperplanes* A rather than p-spheres. The limiting scan, which is now a p-form-valued function defined on almost every oriented hyperplane A, will then be given by precisely the same data as before, namely the p-form u^K_A, the 1-rectifiable set Γ, the orientation $\vec{\Gamma}$, and the density θ. In particular, for $\mathcal{H}^1$-almost every $a\in\Gamma$ and the p-sphere $\partial B_r(c)$ transverse to Γ at a, the integer multiplicity of $\mathcal{S}(c,r)$ at a coincides with that induced by using the limit scan of the correspondingly oriented affine p-plane tangent to $\partial B_r(c)$ at a.

5. Scans of p-dimensional tree-graph forms for maps of $\mathbf{R}^m$

Here, for integers $m>p+1$, we describe how to generalize the results of the previous section to maps in $W^{1,p}(\mathbf{R}^m, N)$, where the bubbled scan, corresponding to a fixed $z\in \mathrm{Hom}(\pi_p(N), \mathbf{R})$, is $(m-p)$-dimensional. In the new version below of Proposition 4.1, we again obtain subconvergence of a sequence of scans associated with smooth maps to a limit scan, which is now defined on almost every p-dimensional Euclidean subsphere of $\mathbf{R}^m$. The action of z on the p-homotopy of the limit map u is again dual to a rectifiable bubbled scan, now of dimension $m-p$. Theorem 5.2 below gives a precise sense of how *the "boundary" of the bubbled scan equals the z-type topological singularity of the limit map u.*

For more discussion about the boundary of higher-dimensional scans and other general properties of scans, see [DH].

To parametrize the p-dimensional oriented Euclidean subspheres of $\mathbf{R}^m$, we first define, as in [Fe], the translation

$$\tau_c(x) = x+c \quad \text{for } c, x\in \mathbf{R}^m,$$

as well as the $(m-1)(m-2)\ldots(m-p-1)$-dimensional manifold

$$\mathcal{O}_{p+1} = \{\text{linear orthogonal embeddings } \phi: \mathbf{R}^{p+1} \to \mathbf{R}^m\}.$$

For each triple $(\phi, c, r)\in\mathcal{O}_{p+1}\times\mathbf{R}^m\times\mathbf{R}_+$, we have the oriented $(p+1)$-disk $\tau_{c\#}\phi_{\#}[B_r(0)]$ whose boundary is the *oriented p-dimensional Euclidean sphere*

$$S_{\phi,c,r} = \tau_{c\#}\phi_{\#}[\partial B_r(0)]$$

supported in the affine $(p+1)$-plane $c+\phi(\mathbf{R}^{p+1})$. For $u\in W^{1,p}(\mathbf{R}^m, N)$ and a.e. (ϕ, c, r), we also have the restriction map

$$u_{\phi,c,r}\in W^{1,p}(S^p, N), \quad u_{\phi,c,r}(x) = u(c+\phi(rx)).$$

Proposition 5.1. *Let K be a linear combination of tree-graphs of closed forms associated with a class $z\in\mathrm{Hom}(\pi_p(N), \mathbf{R})$. For any sequence $u_n\in C^\infty(\mathbf{R}^m, N)$ converging $W^{1,p}$-weakly to some limit $u\in W^{1,p}(\mathbf{R}^m, N)$, there exist a subsequence $u_{n'}$ and a limit scan $\mathcal{S}$,*

$$\mathcal{S}(\phi, c, r) = u^K_{\phi,c,r} + \sum_{i=1}^{I(\phi,c,r)} m_i(\phi, c, r)\delta_{a_i(\phi,c,r)},$$

so that, for almost every $(\phi, c, r)\in\mathcal{O}_{p+1}\times\mathbf{R}^m\times\mathbf{R}_+$ and for a subsequence n'' of n' with

$$\sup_{n''}\int_{S_{\phi,c,r}} |\nabla u_{n''}|^p \, d\mathcal{H}^p < \infty,$$

the differential p-forms

$$(u_{n''})^K_{\phi,c,r} \rightharpoonup \mathcal{S}(\phi,c,r)$$

weakly as Radon measures. Here the quantities $I(\phi,c,r)\in\mathbf{N}$, $m_i(\phi,c,r)\in\mathbf{R}_+\cap z(\Pi_p(n))$ *and* $a_i(\phi,c,r)\in S^p$ *are all measurable in* (ϕ,c,r).

Proof. Replacing $\mathbf{R}^{p+1}\times\mathbf{R}_+$ by $\mathcal{O}_{p+1}\times\mathbf{R}^m\times\mathbf{R}_+$, we may repeat many of the arguments from the proof of Proposition 4.1. There the proof involved obtaining the convergent subsequences, first, on a fixed countable dense family $\{\partial B_{r_k}(c_k)\}_{k=1}^{\infty}$ of spheres, and, second, on almost any other sphere, by determining the multiplicity at a point using a local homological connection to a region on one of the spheres $\partial B_{r_k}(c_k)$.

Now we will find it convenient to consider affine subspaces of $\mathbf{R}^m$ and use boundaries of coordinate rectangles in place of spheres for such homological connections. In particular, one may, as noted before, replace S^p by $\mathbf{R}^p$ in Proposition 3.4. We need some notation for affine subspaces. For integers $1\leqslant k\leqslant l\leqslant m$ and an l-dimensional affine subspace A of $\mathbf{R}^m$, let

$$\mathcal{A}_k(A)=\{k\text{-dimensional affine subspaces of } A\}.$$

Using the standard basis $e_1=(1,0,...,0),...,e_m=(0,...,1)$, we say that a subspace $D\in\mathcal{A}_k(\mathbf{R}^m)$ is a *coordinate affine subspace* if

$$D=\{a+r_1e_{\lambda_1}+...+r_ke_{\lambda_k}: r_1,...,r_k\in\mathbf{R}\} \tag{5.1}$$

for some point $a\in\mathbf{R}^m$ and choice of integers $1\leqslant\lambda_1<...<\lambda_k\leqslant m$. In case A is a coordinate affine subspace, we let

$$\mathcal{C}_k(A)=\{k\text{-dimensional coordinate affine subspaces of } A\}.$$

Starting with $\mathcal{D}_m=\{\mathbf{R}^m\}$, we may choose first a countable dense subset $\mathcal{D}_{m-1}$ of $\mathcal{C}_{m-1}(\mathbf{R}^m)$, and then, by downward induction, countable dense subsets $\mathcal{D}_i(A)$ of $\mathcal{C}_i(A)$ for all $A\in\mathcal{D}_{i+1}$ and let $\mathcal{D}_i=\bigcup_{A\in\mathcal{D}_{i+1}}\mathcal{D}_i(A)$. Letting $\mathcal{D}=\bigcup_{i=0}^{m-1}\mathcal{D}_i$, we may also insist that any nonempty intersection $D\cap E$, with $D,E\in\mathcal{D}$, also belongs to $\mathcal{D}$. As in the beginning of the proof of Proposition 4.1, we may find a fixed subsequence $u_{n'}$ and insist that

$$\sup_{n'}\int_D|\nabla u_{n'}|^p\,d\mathcal{H}^i<\infty$$

for all $D\in\mathcal{D}_i$, $i=p,...,m$, and that, at the level p, one has the pointwise a.e. convergence of the maps $u_{n'}|_D$ for each $D\in\mathcal{D}_p$ as well as the weak measure convergence of the corresponding differential p-forms $(u_{n'}|_D)^K$.

We now verify that *this convergence on every* $D \in \mathcal{D}_p$ *is sufficient to determine the limit scan* $\mathcal{S}$ *and the corresponding convergence to* $\mathcal{S}$ (*in the sense involving energy bounded subsequences* $u_{n''}|_A$ *as described above*) *on almost all affine* p*-planes* $A \in \mathcal{A}_p(\mathbf{R}^m)$ *parallel to any given fixed vector subspace*. This will follow by showing, by induction on $i \in \{p+1, ..., m\}$, that we have, for each $D \in \mathcal{D}_i$, such convergence on almost every $A \in \mathcal{A}_p(D)$ parallel to any fixed subspace. The first step $i=p+1$ follows very much as in the proof of Proposition 4.1 and Remark 4.6. Whereas before the key uniqueness was established by reference to the convergence on a region of a suitably chosen small p-sphere $\partial B_{r_k}(c_k)$, we may now refer to a piece of the (p-dimensional) boundary of a suitable small coordinate rectangle with faces in $\mathcal{D}_p(\mathbf{R}^{p+1})$.

We now assume inductively that $i \in \{p+2, ..., m\}$ and that we have the desired generic convergence for $A \in \mathcal{A}_p(E)$ with $E \in \mathcal{D}_{i-1}$. Fix $D \in \mathcal{D}_i$ and consider the family $\mathcal{A}$ of all affine p-dimensional subspaces of D parallel to a fixed p-dimensional subspace V. Suppose D has the form in equation (5.1). Consider now any i-dimensional coordinate rectangle R in D whose faces generate affine subspaces in $\mathcal{D}_{i-1}$. Thus

$$R = \left\{ a + \textstyle\sum_{j=1}^{i} c_j e_{\lambda_j} : b_j^- \leqslant c_j \leqslant b_j^+ \right\},$$

where each $D_j^\pm \equiv \{x \in D : x_{\lambda_j} = b_j^\pm\} \in \mathcal{D}_{i-1}$. Let π be the orthogonal projection of D onto D_1^-. By induction, one has suitable convergence on almost all $B \in \mathcal{A}_p(D_1^-)$ parallel to $\pi(V)$. Similarly for the p-planes parallel to V in $\pi^{-1}(B) \cap D_j^\pm$ for $j=2, ..., i$. For such a p-plane B which is thus good for all of (the countably many) such R, $\pi^{-1}(B)$ is a $(p+1)$-plane in D, and we may verify a suitable convergence on almost all p-planes $A \subset \pi^{-1}(B)$ parallel to V. In fact, to determine, as in the proof of Proposition 4.1, convergence near a point of such a p-plane A in $\pi^{-1}(B)$, one chooses a small rectangle R as above, notes that P splits the $(p+1)$-dimensional rectangle $R \cap \pi^{-1}(B)$ into two pieces. Using the boundary of one of these, we see that the total multiplicity of the delta mass limit along $P \cap R$ is determined by the remaining $2p+1$ sides which are contained in either the coordinate p-plane B or $\pi^{-1}(B) \cap D_j^\pm$, along either of which we already have the desired convergence. Taking almost all such A corresponding to almost all such B gives, by Fubini's theorem, almost all $A \in \mathcal{A}$. This completes the induction and gives, when $i=m$, the desired convergence along almost all p-planes $A \in \mathcal{A}_p(\mathbf{R}^m)$.

Finally, concerning convergence on the p-dimensional spheres, one already has, for almost all affine $(p+1)$-planes $Q \in \mathcal{A}_{p+1}(\mathbf{R}^m)$, convergence along almost all p-planes $A \in \mathcal{A}_p(Q)$. But then, for any fixed $c \in Q$, one also gets convergence on almost every Euclidean p-sphere in Q centered at c. To see this, one argues once again, as in Proposition 4.1, this time using regions of boundaries of $(p+1)$-dimensional rectangles in Q with faces lying in such A. An application of Fubini's theorem completes the proof. □

We can now describe the higher-dimensional generalization of our main result, Theorem 4.4, concerning the structure of bubbled scans.

THEOREM 5.2. *Suppose* K, z, $u_{n'}$, u *and* $\mathcal{S}$ *are as in Proposition* 5.1. *Then there exist a countable union* Γ *of* $(m-p)$*-dimensional* C^1*-submanifolds with measurable orientation* $\vec{\Gamma}$ *and a non-negative* $\mathcal{H}^{m-p}$*-measurable function* θ *from* Γ *into* $z(\pi_p(N))$ *such that*

$$\int_\Gamma \theta^{p/(p+n_z)}\, d\mathcal{H}^{m-p} \leqslant C_z \liminf_{n\to\infty} \int_{R^{p+1}} |\nabla u_n|^p\, dx, \tag{5.2}$$

(*with* C_z *depending only on* z *and* n_z, *as in Definition* 2.12) *and*, *for almost all* $(\phi,c,r)\in \mathcal{O}\times\mathbf{R}^m\times\mathbf{R}_+$, *the sphere* $S_{(\phi,c,r)}$ *is transverse to* Γ, *and the value of the limiting scan* $\mathcal{S}$ *on this sphere is given by*

$$\mathcal{S}(\phi,c,r)=u^K_{\phi,c,r}+\sum_{a\in\Gamma\cap S_{\phi,c,r}} \operatorname{sgn}[\vec{\Gamma}(a)\wedge\vec{S}_{\phi,c,r}(a)]\theta(a)\delta_{(\phi^{-1}(a-c))/r}.$$

Thus,

$$\{a_1(\phi,c,r),...,a_{I(\phi,c,r)}(\phi,c,r)\}=\left\{\frac{\phi^{-1}(a-c)}{r} : a\in\Gamma\cap S_{\phi,c,r}, \theta(a)\neq 0\right\},$$

and

$$m_i(\phi,c,r)=\operatorname{sgn}(\vec{\Gamma}\wedge\vec{S}_{\phi,c,r})(c+\phi[ra_i(c,r)])\theta(c+\phi[ra_i(c,r)]).$$

Here, $\operatorname{sgn}(\alpha e_1\wedge...\wedge e_m)=\operatorname{sgn}(\alpha)$.

Proof. Referring to the proof of Theorem 4.4, we readily verify that the density set

$$Z_u=\left\{c\in\mathbf{R}^{p+1} : \limsup_{r\to 0} r^{p-m}\int_{B_r(c)} |\nabla u|^p\, dx>0\right\}$$

has $\mathcal{H}^{m-p}$-measure zero and that the ε energy concentration set

$$E_\varepsilon=\left\{c\in\mathbf{R}^{p+1} : \limsup_{r\to 0}\liminf_{n\to\infty} r^{p-m}\int_{B_r(c)} |\nabla u_n|^p\, dx>\varepsilon\right\}$$

has $\mathcal{H}^{m-p}(E_\varepsilon)\leqslant C\varepsilon^{-1}\sup_n \int_{\mathbf{R}^{p+1}} |\nabla u_n|^p\, dx$. Also, we define as before the bubbling sets $A(\phi,c,r)=\{a_i(\phi,c,r)\}$. For Step 1, we will first verify that, for all $c\in\mathbf{R}^m$ and almost all ϕ and r,

$$A(\phi,c,r)\setminus Z_u\subset E_0. \tag{5.3}$$

For this, it will be sufficient, by Remark 4.6 and Fubini's theorem, to show that for $a\in\mathbf{R}^m\setminus E_0$ and almost every affine p-plane P through a, the sequence $u_{n'}|_P$ has *no bubbling* at a. To work with the family of all affine p-planes passing through a fixed point a, we use the following elementary integral formula.

Lemma 5.3. *There is a positive constant $c_{m,p}$ such that for any $f\in W^{1,p}(B_\sigma(a))$,*

$$\sigma^{p-m}\int_{B_\sigma(a)}|\nabla f|^p\,dx=c_{m,p}\int_{\phi\in\mathcal{O}_p}\int_{a+\phi(B_\sigma(0))}|\nabla f|^p\,d\mathcal{H}^p\,d\Theta(\phi). \tag{5.4}$$

Here Θ denotes the rotation invariant probability measure on $\mathcal{O}_p$.

Proof. Rescaling, we may assume that $\sigma=1$. The formula

$$\mu(E)=\int_{\phi\in\mathcal{O}_p}\mathcal{H}^{p-1}(E\cap(a+\phi(\mathbf{R}^p))\,d\Theta(\phi),$$

for every $\mathcal{H}^{m-1}$-measurable subset E of S^{m-1}, defines a finite positive rotation invariant measure on S^{m-1}. By the uniqueness of the Haar measure [Fe, §2.9] on S^{m-1}, μ must be some positive multiple $c_{m,p}$ of $\mathcal{H}^{m-1}|_{S^{m-1}}$. It then follows from the coarea formula and Fubini's theorem that

$$\begin{aligned}\int_{B_1(a)}|\nabla f|^p\,dx&=\int_0^1\int_{\partial B_s(a)}|\nabla f|^p\,d\mathcal{H}^{m-1}\,ds\\&=c_{m,p}\int_0^1\int_{\phi\in\mathcal{O}}\int_{a+\phi(\partial B_s(0))}|\nabla f|^p\,d\mathcal{H}^{p-1}\,d\Theta(\phi)\\&=c_{m,p}\int_{\phi\in\mathcal{O}}\int_0^1\int_{a+\phi(\partial B_s(0))}|\nabla f|^p\,d\mathcal{H}^{p-1}\,ds\,d\Theta(\phi)\\&=c_{m,p}\int_{\phi\in\mathcal{O}}\int_{a+\phi(B_1(0))}|\nabla f|^p\,d\mathcal{H}^{p}\,d\Theta(\phi).\end{aligned}$$

□

As before, we may, for $\varepsilon>0$, choose a positive σ small enough so that

$$\sigma^{p-m}\int_{B_\sigma(a)}|\nabla u|^p\,dx+\liminf_{n'\to\infty}\sigma^{p-m}\int_{B_\sigma(a)}|\nabla u_{n'}|^p\,dx\leqslant\varepsilon.$$

For $\phi\in\mathcal{O}_p$ consider the affine p-plane $a+\phi(\mathbf{R}^p)$. For a.e. such ϕ we may choose a positive $\sigma_\phi<\sigma$ so that

$$\int_{a+\phi(B_{\sigma_\phi}(0))}|\nabla u|^p\,d\mathcal{H}^p\leqslant\varepsilon,$$

and $u_{n'}|_{a+\phi(\mathbf{R}^p)}$ has no bubbling in $B_{\sigma_\phi}(a)\setminus\{a\}$. For each positive integer j, we consider the set

$$\Phi_j=\{\phi\in\mathcal{O}_p:\sigma_\phi>j^{-1}\text{ and }u_{n'}|_{a+\phi(\mathbf{R}^p)}\text{ has a bubble at }a\}.$$

From the lower bound inequality (3.20) and Remark 4.6, we conclude that

$$\liminf_{n'\to\infty}\int_{a+\phi(B_{1/j}(0))}|\nabla u_{n'}|^2\,d\mathcal{H}^p>C\varepsilon_{p,N}$$

for $\phi\in\Phi_j$. By Fatou's lemma and the above formula (5.4), we see that

$$C\varepsilon_{p,N}\Theta(\Phi_j) < \int_{\Phi_j} \liminf_{n'\to\infty} \int_{a+\phi(B_{1/j}(0))} |\nabla u_{n'}|^2 \, d\mathcal{H}^p \, d\Theta(\phi)$$
$$\leqslant \liminf_{n'\to\infty} j^{m-p} \int_{B_{1/j}(a)} |\nabla u_{n'}|^p \, dx < \varepsilon.$$

Taking ε sufficiently small, we conclude that $\Theta(\Phi_j)=0$ for each j which implies the inclusion (5.3).

Now, as in Step 2 of Theorem 4.4, we may work with the $(m-p)$-rectifiable part Γ of the $\mathcal{H}^{m-p}$ σ-finite set E_0. Following Remark 4.6, we see how to obtain, for $\mathcal{H}^{m-p}$-almost any $a\in\Gamma$ and any fixed oriented affine p-plane P through a transverse to the approximate tangent space $T_a\Gamma$, candidates for the multiplicity $\theta(a)$ and orientation $\vec{\Gamma}(a)$ suitable for an oriented p-sphere through a tangent to this particular P. Again, this implies the $\mathcal{H}^1$-measurability of θ and $\vec{\Gamma}$, and inequality (5.2) follows from (3.14), (3.16) and Lemma 5.4. Note that simply reversing the orientation of P will, by our conventions, lead to the same $\theta(a)$ and $\vec{\Gamma}(a)$.

Next, as before in Step 3 of Theorem 4.4, we must now establish the uniqueness that for almost every pair P,Q of oriented affine p-planes through a transverse to $T_a\Gamma$, each determines the *same* pair $\theta(a),\vec{\Gamma}(a)$. For this, we may henceforth assume that

$$\operatorname{dist}_{\mathrm{Haus}}(P\cap B_1(a), Q\cap B_1(a)) < \delta(P), \tag{5.5}$$

where $\delta(P)$ is an (easy to estimate) constant which depends only on the distance between $P\cap\partial B_1(a)$ and $T_a\Gamma$ and is to be determined below. For any such pair P,Q, we may again choose ϱ small enough so that there is no bubbling for $u_n'|_P$ in $P\cap B_\varrho(a)\setminus\{a\}$ or for $u_n'|_Q$ in $Q\cap B_\varrho(a)\setminus\{a\}$.

In general, it will, as before, suffice to obtain a bubble-free p-dimensional connection V between the $(p-1)$-sphere $S=P\cap\partial B_\varrho(a)$ and either $T=Q\cap\partial B_\varrho(a)$ or $T=-Q\cap\partial B_\varrho(a)$. Before, in the proof of Theorem 4.4, the connection V was simply given by oriented regions on the p-sphere $\partial B_\varrho(a)$ in $\mathbf{R}^{p+1}$ which avoided the affine line $a+T_a\Gamma$. Now we will obtain V as a finite chain of one-parameter rotations of p-spheres connecting S and and T which will avoid the $(m-p)$-dimensional affine space $a+T_a\Gamma$.

With any choice of subspaces

$$P=P_p\subset P_{p+1}\subset\ldots\subset P_m=\mathbf{R}^m \quad\text{and}\quad Q=Q_p\subset Q_{p+1}\subset\ldots\subset Q_m=\mathbf{R}^m,$$

with $\dim P_j=\dim Q_j=j$, one may explicitly associate, for $j=p,\ldots,m$, geodesic circles of rotations $\gamma_j\colon[0,2\pi]\to\mathbf{O}(m)$ along with $t_j\in[0,\pi]$ and $(p-1)$-spheres R_j so that $\gamma_j(0)=\mathrm{id}$ and

$$R_p=S,\quad R_{p+1}=\gamma_{p+1}(t_{p+1})(R_p),\quad \ldots,\quad R_m=\gamma_m(t_m)(R_{m-1})=T$$

as follows: Since $\dim(P_{m-1}\cap Q_{m-1})\geqslant m-2$, we first choose a circle of rotations γ_m which fixes $P_{m-1}\cap Q_{m-1}$, and choose t_m so that $\gamma_m(t_m)(P_{m-1})=Q_{m-1}$. Then,

$$\dim(\gamma_m(t_m)(P_{m-2})\cap Q_{m-2})\geqslant m-3,$$

and we may similarly choose γ_{m-1} to fix both $\gamma_m(t_m)(P_{m-2})\cap Q_{m-2}$ and $Q_{m-1}^{\perp}$, and then find t_{m-1} so that

$$\gamma_{m-1}(t_{m-1})\circ\gamma_m(t_m)(P_{m-2})=Q_{m-2}.$$

Continuing downward, we obtain all the desired γ_j and t_j so that finally

$$\gamma_m(t_m)\circ\gamma_{m-1}(t_{m-1})\circ\ldots\circ\gamma_m(t_m)\circ\gamma_{p+1}(t_{p+1})(S)=T.$$

Using the notation $\gamma_j(t,x)=\gamma_j(t)(x)$, we see that $\gamma_j([0,t_j]\times R_j)$ connects R_{j-1} to R_j and that

$$V=\bigcup_{j=p+1}^{m}\gamma_j([0,t_j]\times R_j)$$

is the desired finite chain of one-parameter rotations of p-spheres connecting S and T. Finally, defining the conical region

$$W=\{x\in\mathbf{R}^m:\operatorname{dist}(x,a+T_a\Gamma)>m^{-1}\operatorname{dist}((P\cup Q)\cap\partial B_{|x-a|}(a),a+T_a\Gamma)\},$$

we readily find the constant $\delta(P)$ so that $t_{p+1},...,t_m$ can be chosen small enough to insure that each

$$\gamma_j([0,t_j]\times R_j)\subset W. \tag{5.6}$$

To guarantee the nonbubbling along V, we need to discuss the energy estimates for this construction. First, we observe, as in (4.15), that conformal invariance implies that *for any ball Σ in a Euclidean p-sphere in $\mathbf{R}^m$ with*

$$\liminf_{n'\to\infty}\int_\Sigma|\nabla u_{n'}|^p<10\varepsilon_1, \tag{5.7}$$

the sequence $u_{n'}|_\Sigma$ will exhibit no bubbling.

In particular, since the inclusions (5.6) imply that $V\subset\bigcup_{p+1}^m\Sigma_j$, where each

$$\Sigma_j=W\cap\gamma_j([0,2\pi]\times R_j)$$

is a ball in the p-sphere $\gamma_j([0,2\pi]\times R_j)$, we now need only guarantee that (5.7) holds for each Σ_j. To obtain this, we argue as before that, for almost all such $a\in\Gamma$, there is no

p-energy concentration at a occurring in the cone W, which is uniformly transverse to $T_a\Gamma$. Thus one can, for any $\varepsilon>0$, choose an arbitrarily small σ so that

$$\liminf_{n'\to\infty} \sigma^{p-m+1} \int_{W\cap\partial B_\sigma(a)} |\nabla u_{n'}|^p\, dx < \varepsilon.$$

Arguing as in the proof of Lemma 5.4, this integral will dominate a double integral of $|\nabla u_{n'}|^p$ over $\widetilde{P}\cap W\cap\partial B_\sigma(a)$ as $\widetilde{P}$ varies over affine $(p+1)$-planes containing the fixed P. Moreover, we may even integrate over all possible choices of subspaces

$$P = P_p \subset P_{p+1} \subset \ldots \subset P_m = \mathbf{R}^m \quad \text{and} \quad Q = Q_p \subset Q_{p+1} \subset \ldots \subset Q_m = \mathbf{R}^m.$$

These lead to an estimate of integrals over the resulting Σ_j. Here the constant in the bound may depend on P and Q. Nevertheless, since there is an open set of corresponding R_j still satisfying (5.6), we may finally choose suitable Σ_j satisfying (5.7), from Fubini's theorem, by taking ε and then σ sufficiently small (depending on P and Q). □

6. Appendix

Let $\gamma: [0,L]\to\mathbf{R}^n$ be a C^1-curve, $t\in[0,L]$ and $a\in\mathbf{R}^n$. Then γ is, at $\gamma(t)$, *transverse* to the intersecting sphere $\partial B_{|\gamma(t)-a|}(a)$ if and only if

$$\dot\gamma(t)\cdot(\gamma(t)-a) \neq 0.$$

For most centers a, one has such transversality at most points of the curve. More precisely, we have the following result.

LEMMA 6.1. *For any C^1-embedded curve $\gamma: [0,L]\to\mathbf{R}^n$ with $\dot\gamma$ nonvanishing, the set*

$$\mathcal{A}_\gamma = \{a\in\mathbf{R}^n : \mathcal{H}^1(\{t\in[0,L] : \dot\gamma(t)\cdot(\gamma(t)-a)=0\})>0\}$$

is covered by countably many $(n-2)$-dimensional affine planes, and, in particular, has n-dimensional Lebesgue measure zero.

Proof. For $i=0,\ldots,n$, let G_i denote the Grassmannian of i-dimensional vector subspaces of $\mathbf{R}^n$, and consider the projection

$$\pi_i: \mathbf{R}^n\times G_i \longrightarrow \mathbf{R}^n, \quad \pi_i(a,P)=a,$$

for $(a,P)\in\mathbf{R}^n\times G_i$. For each such (a,P), also let

$$T_{(a,P)} = \{t\in[0,L] : \dot\gamma(t)\cdot(\gamma(t)-a)=0 \text{ and } \dot\gamma(t)\in P\}.$$

Letting

$$S_i=\{(a,P)\in\mathbf{R}^n\times G_i:\mathcal{H}^1(T_{(a,P)})>0\},$$

we readily infer that $S_0=\varnothing$. We also claim that $S_1=\varnothing$. In fact, otherwise, there is a convergent sequence $t_j\to t_0$ in some $T_{(a,P)}$ with $\dim P=1$. Assuming that γ is parameterized by arc-length, $\dot\gamma(t_j)$ is, for $j=0$ and i sufficiently large, the same unit vector v whose span is the line P. But then

$$\begin{aligned}1=v\cdot\dot\gamma(t_0)&=\lim_{j\to\infty} v\cdot\frac{\gamma(t_j)-\gamma(t_0)}{t_j-t_0}=\lim_{j\to\infty}\frac{\dot\gamma(t_j)\cdot\gamma(t_j)-\dot\gamma(t_0)\cdot\gamma(t_0)}{t_j-t_0}\\&=\lim_{j\to\infty}\frac{\dot\gamma(t_j)\cdot a-\dot\gamma(t_0)\cdot a}{t_j-t_0}=0,\end{aligned}$$

a contradiction. Thus, $S_1=\varnothing$.

Next, we note that

$$\mathcal{A}_\gamma=\pi_n(S_n)\supset\pi_{n-1}(S_{n-1})\supset\ldots\supset\pi_1(S_1)\ =\varnothing,$$

so that it suffices to show that each set

$$\mathcal{A}_i=\pi_i(S_i)\setminus\pi_{i-1}(S_{i-1})$$

is covered by countably many $(n-2)$-dimensional affine planes, for $i=2,3,...,n$.

Consider two points (a,P) and (a',P') in S_i, with $a,a'\in\mathcal{A}_i$ and $P\neq P'$. Then

$$\mathcal{H}^1(T_{(a,P)}\cap T_{(a',P')})=0, \tag{6.1}$$

because otherwise the inclusion

$$T_{(a,P)}\cap T_{(a',P')}\subset T_{(a,P\cap P')}$$

would imply that $\mathcal{H}^1(T_{(a,P\cap P')})>0$ and $a\in\pi_j(S_j)$ with $j=\dim(P\cap P')<i$, contradicting the fact that $a\notin\pi_{i-1}(S_{i-1})$. It now follows from (6.1) that distinct P give $\mathcal{H}^1$-essentially disjoint positive measure subsets of $[0,L]$. Thus

$$\mathcal{P}_i=\{P:(a,P)\in S_i \text{ for some } a\in\mathbf{R}^n\}$$

is countable.

Fixing $P\in\mathcal{P}_i$, we now claim that for any two distinct points a and a' satisfying

$$\mathcal{H}^1(T_{(a,P)}\cap T_{(a',P)})>0,$$

the vector $a-a'$ is orthogonal to P. In fact, otherwise, the $(n-1)$-dimensional vector space H orthogonal to $a-a'$ would not contain P, $\dim(P\cap H)=i-1$, and, as before, the inclusion

$$T_{(a,P)}\cap T_{(a',P)}\subset T_{(a,P\cap H)}$$

would imply that $\mathcal{H}^1(T_{(a,P\cap H)})>0$ and $a\in\pi_{i-1}(S_{i-1})$, contradicting the fact that $a\in\mathcal{A}_i$. It follows that the unique affine $(n-i)$-plane Q_a which is orthogonal to P and passes through a must coincide with the corresponding $(n-i)$-plane $Q_{a'}$. In other words, if $Q_a\neq Q_{a''}$ for two points a and a'' for which both $\mathcal{H}^1(T_{(a,P)})$ and $\mathcal{H}^1(T_{(a'',P)})$ are positive, then necessarily

$$\mathcal{H}^1(T_{(a,P)}\cap T_{(a'',P)})=0.$$

It follows, as before, that the family $\mathcal{Q}_P$ of all such orthogonal affine $(n-i)$-planes Q_a with $\mathcal{H}^1(T_{(a,P)})>0$ is countable because distinct ones give $\mathcal{H}^1$-essentially disjoint positive measure subsets of $[0,L]$.

We now conclude that

$$\mathcal{A}_\gamma\subset\bigcup_{i=2}^{n}\bigcup_{P\in\mathcal{P}_i}\bigcup_{Q\in\mathcal{Q}_P}Q$$

is covered by countably many $(n-2)$-dimensional affine planes. □

References

[ABO] Alberti, G., Baldo, S. & Orlandi, G., Functions with prescribed singularities. *J. Eur. Math. Soc.* (*JEMS*), 5 (2003), 275–311.

[ABL] Almgren, F., Browder, W. & Lieb, E. H., Co-area, liquid crystals, and minimal surfaces, in *Partial Differential Equations* (Tianjin, 1986), Lecture Notes in Math., 1306, pp. 1–22. Springer, Berlin–Heidelberg, 1988.

[Be] Bethuel, F., The approximation problem for Sobolev maps between two manifolds. *Acta Math.*, 167 (1991), 153–206.

[BBC] Bethuel, F., Brezis, H. & Coron, J. M., Relaxed energies for harmonic maps, in *Variational Methods* (Paris, 1988), Progr. Nonlinear Differential Equations Appl., 4, pp. 37–52. Birkhäuser, Boston, MA, 1990.

[BBH] Bethuel, F., Brezis, H. & Hélein, F., Asymptotics for the minimization of a Ginzburg–Landau functional. *Calc. Var. Partial Differential Equations*, 1 (1993), 123–148.

[BT] Bott, R. & Tu, L. W., *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics, 82. Springer, New York, 1982.

[BCL] Brezis, H., Coron, J. M. & Lieb, E. H., Harmonic maps with defects. *Comm. Math. Phys.*, 107 (1986), 649–705.

[DH] De Pauw, T. & Hardt, R., Rectifiable and flat scans in a metric space. In preparation.

[Fe] Federer, H., *Geometric Measure Theory*. Die Grundlehren der mathematischen Wissenschaften, 153. Springer, New York, 1969.

[Gi] Giaquinta, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*. Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983.

[GMS1] Giaquinta, M., Modica, G. & Souček, J., The Dirichlet energy of mappings with values into the sphere. *Manuscripta Math.*, 65 (1989), 489–507.

[GMS2] — *Cartesian Currents in the Calculus of Variations*. I, II. Ergebnisse der Mathematik und ihrer Grenzgebiete. A Series of Modern Surveys in Mathematics, 37, 38. Springer, Berlin–Heidelberg, 1998.

[GM] Griffiths, P. A. & Morgan, J. W., *Rational Homotopy Theory and Differential Forms*. Progress in Mathematics, 16. Birkhäuser, Boston, MA, 1981.

[Gr1] Gromov, M., *Metric Structures for Riemannian and Non-Riemannian Spaces*. Progress in Mathematics, 152. Birkhäuser, Boston, MA, 1999.

[Gr2] — Quantitative homotopy theory, in *Prospects in Mathematics* (Princeton, NJ, 1996), pp. 45–49. Amer. Math. Soc., Providence, RI, 1999.

[H] Hang, F., On the weak limits of smooth maps for the Dirichlet energy between manifolds. *Comm. Anal. Geom.*, 13 (2005), 929–938.

[HL1] Hang, F. & Lin, F., Topology of Sobolev mappings. *Math. Res. Lett.*, 8 (2001), 321–330.

[HL2] — Topology of Sobolev mappings. II. *Acta Math.*, 191 (2003), 55–107.

[HL3] — Topology of Sobolev mappings. III. *Comm. Pure Appl. Math.*, 56 (2003), 1383–1415.

[HR1] Hardt, R. & Rivière, T., Connecting topological Hopf singularities. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 2 (2003), 287–344.

[HR2] — Bubbling phenomena and weak convergence for maps in $W^{2,2}(B^5, S^3)$. In preparation.

[Mi1] Millionshchikov, D. V., Imbeddings of a minimal model of k-homotopy type into the algebras of smooth forms $\Lambda^*(M)$. *Uspekhi Mat. Nauk*, 43:2 (1988), 147–148 (Russian); English translation in *Russian Math. Surveys*, 43:2 (1988), 179–180.

[Mi2] — Some spectral sequences in analytic homotopy theory. *Mat. Zametki*, 47:5 (1990), 52–61, 159 (Russian); English translation in *Math. Notes*, 47 (1990), 458–464.

[Nov1] Novikov, S. P., The analytic generalized Hopf invariant. Multivalued functionals. *Uspekhi Mat. Nauk*, 39:5 (1984), 97–106 (Russian); English translation in *Russian Math. Surveys*, 39:5 (1984), 113–124.

[Nov2] — Analytical homotopy theory. Rigidity of homotopic integrals. *Dokl. Akad. Nauk SSSR*, 283 (1985), 1088–1091 (Russian); English translation in *Soviet Math. Dokl.*, 32 (1985), 285–288.

[Nov3] — Analytical theory of homotopy groups, in *Topology and Geometry—Rohlin Seminar*, Lecture Notes in Math., 1346, pp. 99–112. Springer, Berlin–Heidelberg, 1988.

[PR] Pakzad, M. R. & Rivière, T., Weak density of smooth maps for the Dirichlet energy between manifolds. *Geom. Funct. Anal.*, 13 (2003), 223–257.

[Ri] Rivière, T., Minimizing fibrations and p-harmonic maps in homotopy classes from S^3 into S^2. *Comm. Anal. Geom.*, 6 (1998), 427–483.

[SU] Schoen, R. & Uhlenbeck, K., Approximation theorems for Sobolev mappings. Preprint, 1984.

[St] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, 30. Princeton University Press, Princeton, NJ, 1970.

[Sul] Sullivan, D., Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (1977), 269–331.

[Wh] White, B., Homotopy classes in Sobolev spaces and the existence of energy minimizing maps. *Acta Math.*, 160 (1988), 1–17.

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