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Amalgamated free products of weakly rigid factors and calculation of their symmetry groups

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0. Introduction

We prove in this paper a series of rigidity results for amalgamated free product (hereafter abbreviated AFP) II_1 factors $M=M_1*_B M_2$, which can be viewed as von Neumann algebra versions of the "subgroup theorems" and "isomorphism theorems" for AFP groups in Bass–Serre theory. Our main "subalgebra theorem" shows that, under rather general conditions, any von Neumann subalgebra $Q\subset M$ with the *relative property* (T) in the sense of [P5] (also called a *rigid inclusion*), can be conjugated by an inner automorphism of M into either M_1 or M_2. We derive several "isomorphism theorems" in the case the amalgamation is over the scalars, $B=\mathbf{C}$, over a common Cartan subalgebra, $B=A$, or over a regular hyperfinite subfactor, $B=R$. The typical such statement shows that if $\theta: M\simeq N^t$ is an isomorphism from an AFP factor $M=M_1*_B M_2*_B\ldots*_B M_m$ onto the amplification by some $t>0$ of an AFP factor $N=N_1*_C N_2*_C\ldots*_C N_n$, $1\leqslant m,n\leqslant\infty$, with each M_i and each N_j containing a "large" subalgebra with the relative property (T), then $m=n$ and $\theta(B\subset M_i)$ is unitarily conjugate to $(C\subset N_i)^t$, for all i, after some permutation of indices.

When applied to the case $B=R$, these results allow us to obtain the first explicit calculations of outer automorphism groups of II_1 factors, and answer in the affirmative a problem posed by A. Connes in 1973, on whether there exist II_1 factors M with no outer automorphism, i.e. with $\mathrm{Out}(M)\stackrel{\text{def}}{=}\mathrm{Aut}(M)/\mathrm{Int}(M)=\{1\}$. More precisely, we show that if a group Γ is the free product of two infinite property (T) groups [K] with no non-trivial characters, for example $\Gamma=\mathrm{SL}(n_0,\mathbf{Z})*\mathrm{SL}(n_1,\mathbf{Z})$, $n_0,n_1\geqslant 3$, then there exist actions of Γ on the hyperfinite II_1 factor R such that the corresponding crossed product factors $M=R\rtimes\Gamma$ have both trivial fundamental group, $\mathfrak{F}(M)=\{1\}$, and trivial outer automorphism group, $\mathrm{Out}(M)=\{1\}$. In fact, the general result shows that for any separable compact abelian group K there exist factors M with $\mathfrak{F}(M)=\{1\}$ and $\mathrm{Out}(M)=K$.

In turn, when applied to the case of amalgamated free products over a common Cartan subalgebra, our "isomorphism theorem" provides a Bass–Serre type result for *orbit equivalence* (OE) of actions of free product groups

$$\Gamma=\Gamma_1*\ldots*\Gamma_n \quad\text{and}\quad \Lambda=\Lambda_1*\ldots*\Lambda_m$$

on the probability space. Thus, we show that if each Γ_i and each Λ_j has an infinite normal subgroup with the relative property (T) of Kazhdan–Margulis (for instance, if Γ_i and Λ_j are Kazhdan groups for all i and j), and if (σ,Γ) and (θ,Λ) are free, probability measure preserving (m.p.) actions with $\sigma|_{\Gamma_i}$ and $\theta|_{\Lambda_j}$ ergodic for all i and j, then $\sigma\sim_{\mathrm{OE}}\theta$ implies that $m=n$ and $\sigma|_{\Gamma_i}\sim_{\mathrm{OE}}\theta|_{\Lambda_i}$, for all i, after a permutation of the indices i. Note that the opposite implication holds true for arbitrary groups Γ_i and Λ_j, as shown by D. Gaboriau in [G2]. In fact, we derive the componentwise OE of actions under the weaker assumption that the group measure space factors associated with (σ,Γ) and (θ,Λ) are stably isomorphic, i.e. when σ and θ are *von Neumann equivalent* (vNE). We use this vNE Bass–Serre rigidity and [Fu1], [Fu2], [Ge1], [Ge2], [MoS], [P6] and [P8] to give examples of group measure space factors M from free ergodic m.p. actions σ of free product groups $\Gamma=\Gamma_1*\Gamma_2*...$ such that $\mathfrak{F}(M)=\{1\}$ and $\mathrm{Out}(M)=\mathrm{H}^1(\sigma,\Gamma)$, with explicit calculation of the abelian group $\mathrm{H}^1(\sigma,\Gamma)$.

Finally, when applied to the case $B=\mathbf{C}$, our results become von Neumann algebra analogues of Kurosh's classical theorems for free products of groups, similar to Ozawa's recent results of this type in [O], but covering a different class of factors than [O] and allowing amplifications. For instance, we show that if N_i, $2\leqslant i\leqslant n$, and M_j, $2\leqslant j\leqslant m$, are property (T) II_1 factors in the sense of Connes–Jones (e.g. if N_i and M_j are group factors associated with Kazhdan groups, [CJ]) then

$$M_1*M_2*...*M_m \overset{\theta}{\simeq} (N_1*N_2*...*N_n)^t$$

implies that $m=n$ and that $\theta(M_i)$ is inner conjugate to N_i^t for all i, after some permutation of indices. In fact, in its most general form our result only requires M_i and N_j to be *weakly rigid* (*w-rigid*), i.e. to have diffuse-regular subalgebras with the relative property (T). Taking $M=N$ and $M_j=P^{s_j}$, with $\{s_j\}_j=S$ being a multiplicative subgroup of $\mathbf{R}_+^*$ and P being a w-rigid II_1 factor with trivial fundamental group (for instance, the group factor $L(G)$ associated with $G=\mathbf{Z}^2\rtimes\mathrm{SL}(2,\mathbf{Z})$, cf. [P5]) and using a result of Dykema–Radulescu [DyR], we get $\mathfrak{F}(M)=S$ for $M=*_{s\in S}P^s$. This provides a completely new class of factors with arbitrary given $S\subset\mathbf{R}_+^*$ as fundamental group from the ones in [P8]. Indeed, the examples constructed in [P8] are group measure space factors, while the free group factors $*_{s\in S}P^s$ have no Cartan subalgebras, by results of Voiculescu [V2] (see [Sh] and Remark 6.6 in this paper).

The key technical result behind all these applications is the above mentioned "subalgebra theorem", of Bass–Serre type. We state it in details below, together with other main results in the paper, and also explain some of the ideas behind the proofs. An inclusion of finite von Neumann algebras $B\subset P$ will be called *homogeneous* if there exists $\{y_j\}_j\subset P$ with $E_B(y_i^*y_j)=\delta_{ij}$, for all i and j, and $\sum_i y_iB$ dense in P. This technical assumption is satisfied by all inclusions coming from (cocycle) crossed products and (generalized) group measure constructions, or Cartan inclusions. It is also satisfied when P is an arbitrary finite von Neumann algebra and $B=\mathbf{C}$. Following [P5], a von Neumann subalgebra $Q\subset P$ has the *relative property* (T) (or $Q\subset P$ is a *rigid inclusion*) if any "deformation" of id_P by completely positive subunital subtracial maps, $\phi_n\to\mathrm{id}_P$, is uniform on the unit ball of Q (see also [PeP]).

THEOREM 0.1. *Let* (M_i, τ_i), $i=1,2$, *be finite factors with a common von Neumann subalgebra* $B \subset M_i$, *such that* $\tau_1|_B = \tau_2|_B$ *and such that* $B \subset M_i$ *are homogeneous*, $i=1,2$. *Let* $Q \subset M = M_1 *_B M_2$ *be a diffuse von Neumann subalgebra with the relative property* (T) *such that no corner* qQq *of* Q *can be embedded into* B. *Then there exists a unique partition of* 1 *with projections* q_1' *and* q_2' *in the commutant of* Q *in* M *such that* $u_i(Qq_i')u_i^* \subset M_i$, $i=1,2$, *for some unitary elements* u_1 *and* u_2 *in* M. *Moreover*, *if the normalizer of* Q *in* M *generates a factor* N, *then there exists a unique* $i \in \{1,2\}$ *such that* $uQu^* \subset M_i$ *for some* $u \in \mathcal{U}(M)$, *which also satisfies* $uNu^* \subset M_i$.

The proof of this result takes §§2–5 of the paper. It uses "deformation/rigidity" and "intertwining" techniques from [P5], [P7] and [P8]. Thus, we embed $M = M_1 *_B M_2$ into the larger algebra $\widetilde{M} = M *_B (B \overline{\otimes} L(\mathbf{F}_2))$, whose aboundance of deformations is used to show that "rigid parts" of M have to concentrate on certain subspaces with "bounded word-length". This initial information is then used as a starting point in a word-reduction argument to obtain a Hilbert bimodule intertwining Q into one of the M_i's. The homogeneity condition is needed in order to measure the "size" of letters in the M_i's. To get a unitary element conjugating Q into M_i from this, we prove in §1 a series of results on the relative commutants and normalizers of subalgebras in AFP factors, using [P8, I, Theorem 2.1 and Corollary 2.3].

If we take $B = \mathbf{C}$ in Theorem 0.1 and use the fact that finite von Neumann algebras with the Haagerup property ([H], [Ch]) have no diffuse subalgebras with the relative property (T), then we get an analogue of Kurosh's isomorphism theorem for free products of groups.

THEOREM 0.2. *Let* (M_0, τ_{M_0}) *and* (N_0, τ_{N_0}) *be finite von Neumann algebras with Haagerup's compact approximation property. Let* M_i, $1 \leqslant i \leqslant m$, *and* N_j, $1 \leqslant j \leqslant n$, *be w-rigid* II_1 *factors*, *where* $m, n \geqslant 1$ *are some cardinals* (*finite or infinite*). *If* θ *is an isomorphism of* $M = *_{i=0}^m M_i$ *onto* N^t, *where* $N = *_{j=0}^n N_j$ *and* $t > 0$, *then* $m = n$ *and*, *after some permutation of indices*, $\theta(M_i)$ *and* N_i^t *are unitarily conjugate in* N^t, *for all* $i \geqslant 1$.

Ozawa's pioneering result of this type in [O] concerns free products of group factors $M_i = L(\Gamma_i)$ and $N_i = L(\Lambda_i)$ with each Γ_i and Λ_i being a product of two or more infinite conjugacy class (ICC) groups, either word hyperbolic (at least one of them) or amenable, typical examples being the groups $\mathbf{F}_{n_i} \times S_\infty$, not covered by Theorem 0.2 above. In turn, our typical M_i and N_i are factors from property (T) (more generally w-rigid) groups.

Letting $M_i = N_i$ for all i and $m < \infty$ in Theorem 0.2, it follows that if $\mathfrak{F}(M_i) = \{1\}$ for some $1 \leqslant i \leqslant m$ (for example if $M_i = L(\mathbf{Z}^2 \rtimes \mathbf{F}_k)$, with $2 \leqslant k < \infty$, cf. [P5]), then $\mathfrak{F}(M) = \{1\}$. Moreover, taking $m = \infty$ in Theorem 0.2 and using the "compression formula" for free products of infinitely many II_1 factors $(*_i M_i)^t \simeq *_i M_i^t$ in [DyR], we can include specific numbers into the fundamental group. Thus we get the following result.

COROLLARY 0.3. (1) *Let* $m \in \mathbf{N}$ *and let* $M_1, ..., M_m$ *be w-rigid* II_1 *factors. Let* (M_0, τ_{M_0}) *be a finite von Neuman algebra with Haagerup's compact approximation property. If one of the factors* M_i, $1 \leqslant i \leqslant m$, *has trivial fundamental group then so does*

$$M = M_0 * M_1 * ... * M_m.$$

(2) *If* $S \subset \mathbf{R}_+^*$ *is an arbitrary infinite (possibly uncountable) subgroup and* P *is a w-rigid* II_1 *factor with trivial fundamental group (e.g.* $P = L(\mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z}))$*), then the* II_1 *factor* $*_{s \in S} P^s$ *has fundamental group equal to* S.

Since a group measure space factor $M = L^\infty(X, \mu) \rtimes_\sigma (\Gamma_1 * \Gamma_2)$ associated with a free ergodic m.p. action $(\sigma, \Gamma_1 * \Gamma_2)$ on a probability space (X, μ) can alternatively be viewed as an AFP factor $M = M_1 *_A M_2$, where $A = L^\infty(X, \mu)$ and $M_i = A \rtimes_{\sigma|_{\Gamma_i}} \Gamma_i$, Theorem 0.1 allows us to obtain Bass–Serre type vNE and OE rigidity results for actions of free products of groups, as follows.

THEOREM 0.4. (vNE Bass–Serre rigidity) *Let* Γ_0 *and* Λ_0 *be groups with the Haagerup property and let* Γ_i, $1 \leqslant i \leqslant n \leqslant \infty$, *and* Λ_j, $1 \leqslant j \leqslant m \leqslant \infty$, *be* ICC *groups having normal non-virtually abelian subgroups with the relative property* (T). *Assume that either* Γ_0 *is infinite or* $n \geqslant 2$. *Let* σ *(resp.* θ*) be a free ergodic m.p. action of* $\Gamma = \Gamma_0 * \Gamma_1 * ...$ *(resp.* $\Lambda = \Lambda_0 * \Lambda_1 * ...$*) on the probability space* (X, μ) *(resp.* (Y, ν)*) such that* $\sigma_i = \sigma|_{\Gamma_i}$ *(resp.* $\theta_i = \theta|_{\Lambda_i}$*) is ergodic for all* $i \geqslant 1$. *Denote by* $M = L^\infty(X, \mu) \rtimes_\sigma \Gamma$, $N = L^\infty(Y, \nu) \rtimes_\theta \Lambda$, $M_i = L^\infty(X, \mu) \rtimes_{\sigma_i} \Gamma_i \subset M$ *and* $N_j = L^\infty(Y, \nu) \rtimes_{\theta_j} \Lambda_j \subset N$ *the corresponding group measure space factors. If* $\alpha: M \simeq N^t$ *is an isomorphism, for some* $t > 0$, *then* $m = n$ *and there is a permutation* π *of indices* $i \geqslant 1$ *and unitary elements* $u_i \in N^t$ *such that, for all* $i \geqslant 1$,

$$\mathrm{Ad}(u_i)(\alpha(M_i)) = N_{\pi(i)}^t \quad \textit{and} \quad \mathrm{Ad}(u_i)(\alpha(L^\infty(X, \mu))) = (L^\infty(Y, \nu))^t.$$

In particular, $\mathcal{R}_\sigma \simeq \mathcal{R}_\theta^t$ *and* $\mathcal{R}_{\sigma_i} \simeq \mathcal{R}_{\theta_{\pi(i)}}^t$ *for all* $i \geqslant 1$.

In particular, taking the isomorphism α between the group measure space factors in Theorem 0.4 to come from an orbit equivalence of the actions, one gets the following result.

COROLLARY 0.5. (OE Bass–Serre rigidity) *Let* Γ_i, $1 \leqslant i \leqslant n \leqslant \infty$, Λ_j, $1 \leqslant j \leqslant m \leqslant \infty$, σ *and* θ *be as in Theorem* 0.4. *If* $\mathcal{R}_{\sigma, \Gamma} \simeq \mathcal{R}_{\theta, \Lambda}^t$, *then* $n = m$ *and there exists a permutation* π *of the set of indices* $i \geqslant 1$ *such that* $\mathcal{R}_{\sigma_i, \Gamma_i} \simeq \mathcal{R}_{\theta_{\pi(i)}, \Lambda_{\pi(i)}}^t$ *for all* $i \geqslant 1$.

Like in [P8, II], the terminology "vNE rigidity" is used here in a broad sense, in the same spirit the terminology "OE rigidity" is being used in orbit equivalence ergodic theory ([Fu1], [MoS], [S], [Z]). It can designate results which from an isomorphism of group measure space factors derives orbit equivalence of the actions involved ("vNE/OE rigidity", like [P5, Theorem 6.2]), or even conjugacy of the actions ("vNE strong rigidity", e.g. [P8, II, Theorem 7.1]). Theorem 0.4 brings out a new type of vNE rigidity, which we have labeled "Bass–Serre" because of its analogy to group theory results. It is a "vNE/OE"-type result but stronger, as it derives not only the orbit equivalence of the "main actions" (σ,Γ) and (θ,Λ), but also the componentwise orbit equivalence of their restrictions (σ_i,Γ_i) and (θ_i,Λ_i).

The "vNE Bass–Serre rigidity" can be used in combination with OE rigidity results in orbit equivalence ergodic theory to get more insight on the group measure space factors involved. Thus, taking $\Gamma_0=\Lambda_0=\{1\}$ and $2\leqslant n,m<\infty$ in Theorem 0.4, by Gaboriau's results in [G1] it follows that the ℓ^2-Betti numbers of Γ_i and Λ_i must satisfy

$$\beta_k^{(2)}(\Gamma_i)=\frac{\beta_k^{(2)}(\Lambda_i)}{t},$$

for all $1\leqslant i\leqslant n=m$, and

$$\sum_{i=1}^{n}\beta_1^{(2)}(\Gamma_i)+(n-1)=\frac{1}{t}\biggl(\sum_{i=1}^{n}\beta_1^{(2)}(\Lambda_i)+(n-1)\biggr),$$

forcing $t=1$. Also, if we take $\Gamma=\Gamma_0*\Gamma_1*...$ and σ as in Theorem 0.4, and add the conditions $\mathrm{Out}(\mathcal{R}_{\sigma_1})=\{1\}$ and (σ_1,Γ_1) not OE to (σ_i,Γ_i), for all $i\neq1$, then $\mathrm{Out}(\mathcal{R}_\sigma)=\{1\}$ and $\mathrm{Out}(M)=\mathrm{H}^1(\sigma,\Gamma)$. Examples of actions (σ_1,Γ_1) with the associated orbit equivalence relation $\mathcal{R}_{\sigma_1,\Gamma_1}$ having trivial outer automorphism group are constructed in [Ge2], [Fu2] and [MoS], and we construct some more, using the Monod–Shalom rigidity theorem [MoS]. The group $\mathrm{H}^1(\sigma,\Gamma)$ can in turn be calculated by using [P6], thus getting explicit computations of $\mathrm{Out}(M)$ for the group measure space factors M. The fact that one can choose the action (σ,Γ) to be free yet have restrictions $\sigma|_{\Gamma_i}$ isomorphic to specific Γ_i-actions, for all i, is a consequence of [Tö], but we include a proof for the reader's convenience (see §7.3 and §A.1).

Theorem 0.1 is in fact used to obtain another (genuine) "vNE/OE rigidity" result in this paper, for free ergodic m.p. actions (σ,Γ) with Γ being a free product of infinite groups, $\Gamma=\Gamma_0*\Gamma_1$, and σ satisfying the relative property (T) of [P5, Definition 5.10], i.e. such that $L^\infty(X,\mu)\subset L^\infty(X,\mu)\rtimes_\sigma\Gamma$ is a rigid inclusion. This way we recover the uniqueness of the HT Cartan subalgebra (as defined in [P5, §6.1]) in the group-factors $L(\mathbf{Z}^2\rtimes\mathbf{F}_n)$ and their amplifications, one of the main results in [P5].

Similarly, we obtain rigidity results for crossed product factors $M=R\rtimes_\sigma(\Gamma_0*\Gamma_1)$ corresponding to actions $(\sigma,\Gamma_0*\Gamma_1)$ on the hyperfinite II_1 factor R, by regarding M as an AFP factor $M=(R\rtimes\Gamma_0)*_R(R\rtimes\Gamma_1)$. In fact, in this case we can control even better the groups of symmetries $\mathfrak{F}(M)$ and $\mathrm{Out}(M)$, with complete calculations. To state this result, let $f\mathcal{T}_R$ denote the class of actions $(\sigma,\Gamma_0*\Gamma_1)$ of free product groups $\Gamma_0*\Gamma_1$ on the hyperfinite factor R, satisfying the properties: (a) Γ_0 is free and indecomposable; (b) Γ_1 is w-rigid (which is true, e.g., if Γ_1 is an infinite Kazhdan group); (c) $R\subset R\rtimes_\sigma\Gamma_0$ is a rigid inclusion; (d) $\sigma|_{\Gamma_1}$ is a non-commutative Bernoulli Γ_1-action, i.e. R can be represented in the form $R=\overline{\bigotimes}_{g\in\Gamma_1}(M_{n\times n}(\mathbf{C}),\mathrm{tr})_g$, $n\geqslant 2$, with $\sigma|_{\Gamma_1}$ acting on it by left Bernoulli shifts; (e) $\sigma|_{\Gamma_1}$ is freely independent with respect to the normalizer $\mathcal{N}_0$ of $\sigma(\Gamma_0)$ in $\mathrm{Out}(R)$.

To show that such actions exist, we first prove that for any two countable sets of automorphisms S_1 and S_2 of R, there exist $\theta\in\mathrm{Aut}(R)$ such that S_1 and $\theta S_2\theta^{-1}$ are "freely independent" (see §8.2 and §A.2). Combining this with results from [Bu], [Ch], [Fe], [NPSa], [P5] and [Va], we deduce that for many arithmetic groups Γ_0 (in particular for $\Gamma_0=\mathrm{SL}(n,\mathbf{Z})$, for all $n\geqslant 2$) and any w-rigid group Γ_1, there exist actions $(\sigma,\Gamma_0*\Gamma_1)$ on R in the class $f\mathcal{T}_R$. Using Theorem 0.1, properties (a)–(e) above and [P7], we get the following results.

THEOREM 0.6. *For any* $\Gamma_0=\mathrm{SL}(n_0,\mathbf{Z})$, $n_0\geqslant 2$, *and any w-rigid group* Γ_1 *there exist actions* σ *of* $\Gamma_0*\Gamma_1$ *on* R *in the class* $f\mathcal{T}_R$. *If* $(\sigma,\Gamma_0*\Gamma_1)$ *is an* $f\mathcal{T}_R$ *action and we let* $M=R\rtimes_\sigma(\Gamma_0*\Gamma_1)$, *then* $\mathfrak{F}(M)=\{1\}$ *and* $\mathrm{Out}(M)=\mathrm{Char}(\Gamma_0)\times\mathrm{Char}(\Gamma_1)$.

THEOREM 0.7. *Given any compact abelian group* K, *there exist separable* II_1 *factors* M *with* $\mathfrak{F}(M)=\{1\}$ *and* $\mathrm{Out}(M)=K$. *For instance, if* $(\sigma,\Gamma_0*\Gamma_1)$ *is an* $f\mathcal{T}_R$ *action and* $M=R\rtimes_\sigma(\Gamma_0*\Gamma_1)$ *is the associated crossed product factor, with* $\Gamma_0=\mathrm{SL}(n,\mathbf{Z})$ *and* $\Gamma_1=\mathrm{SL}(m,\mathbf{Z})\times\widehat{K}$ *for some* $n,m\geqslant 3$, *then* $\mathfrak{F}(M)=\{1\}$ *and* $\mathrm{Out}(M)=K$. *Moreover, denoting by* $M^\infty=M\overline{\otimes}\mathcal{B}(\ell^2\mathbf{N})$ *the associated* II_∞ *factor, we have* $\mathrm{Out}(M^\infty)=K$.

The study of outer automorphisms of type-II von Neumann factors was at the core of Connes decomposition theory for factors of type III and his classification of amenable factors, in the early 70s [C1], [C3]. Two subsequent seminal papers [C2], [C4] gave the first indications that the outer symmetry groups $\mathrm{Out}(M)$ and $\mathfrak{F}(M)$ can reflect rigidity properties of non-amenable factors. In particular, it was shown in [C4] that $\mathfrak{F}(M)$ and $\mathrm{Out}(M)$ are countable for group factors associated with ICC groups with the property (T). The recent rigidity results in [P5], [P8] and [P9] provide explicit calculations of $\mathfrak{F}(M)$ for large families of group measure space factors M, and reduce the calculation of $\mathrm{Out}(M)$ to the computation of the commutants of the corresponding group actions. However, such commutants are difficult to compute, being left as an open problem even in the case of Bernoulli actions (see [P8, II]). The calculation of $\mathrm{Out}(M)$ that we obtain in this paper for crossed product factors arising from actions of free products of w-rigid groups on the probability space and on the hyperfinite factor thus give the first such explicit computations.

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1. Conjugating subalgebras in AFP factors

1.1. AFP algebras

Let (M_1, τ_1) and (M_2, τ_2) be finite von Neumann algebras with a common von Neumann subalgebra $B \subset M_i$, $i=1,2$, such that $\tau_1|_B = \tau_2|_B$. We denote by $(M_1 *_B M_2, \tau_1 * \tau_2)$ the finite von Neumann algebra *free product with amalgamation* (AFP) of $(M_1, \tau_1; B)$ and $(M_2, \tau_2; B)$, as defined in [V1] and [P2, pp. 384–385]. Thus, $M_1 *_B M_2$ has a dense $*$-subalgebra

$$B \oplus \bigoplus_{n \geqslant 1} \bigoplus_{\substack{i_j \in \{1,2\} \\ i_1 \neq i_2 \neq i_3 \neq \ldots \neq i_n}} \operatorname{sp}(M_{i_1} \ominus B)(M_{i_2} \ominus B) \ldots (M_{i_n} \ominus B) \tag{1.1}$$

with the trace $\tau = \tau_1 * \tau_2$ defined on *reduced words* by $\tau(x) = \tau_1(x) = \tau_2(x)$ for $x \in B$, and $\tau(x) = 0$ for $x = x_{i_1} x_{i_2} \ldots x_{i_n}$, with $x_{i_k} \in M_{i_k} \ominus B$, $i_k \in \{1,2\}$, $i_1 \neq i_2 \neq i_3 \neq \ldots \neq i_n$. Thus, the vector subspaces B and $\operatorname{sp}(M_{i_1} \ominus B)(M_{i_2} \ominus B) \ldots (M_{i_n} \ominus B) \subset M$ in the above sum are all mutually orthogonal with respect to the scalar product given by the trace τ. Also, their closure in $L^2(M, \tau)$ gives mutually orthogonal Hilbert B-bimodules,

$$L^2((M_{i_1} \ominus B)(M_{i_2} \ominus B) \ldots (M_{i_n} \ominus B)) \simeq \mathcal{H}^0_{i_1} \otimes_B \mathcal{H}^0_{i_2} \otimes_B \ldots \otimes_B \mathcal{H}^0_{i_n},$$

summing up to $L^2(M, \tau)$, where $\mathcal{H}^0_i = L^2(M_i) \ominus L^2(B)$.

1.2. Controlling intertwiners and relative commutants

In this subsection we prove a very useful "dichotomy-type" result for subalgebras Q of AFP factors $M = M_1 *_B M_2$. It shows that if Q sits in one of the factors, say M_1, then it can either be conjugated into the "core" B of the AFP algebra M, or else all its normalizers lie in M_1, and even all "intertwining" Hilbert Q-M_1 bimodules $\mathcal{H} \subset L^2(M)$ with $\dim(\mathcal{H}_{M_1}) < \infty$ must be entirely contained in $L^2(M_1)$! Also, in this second case, any bimodule intertwining Q into the other factor, M_2, vanishes.

The first results of this type was obtained in [P1], in the case of "plain" free product factors, $M=M_1*M_2$. The next theorem provides a sharp generalization of these results. The proof uses the basic "intertwining criteria" [P8, I, Theorem 2.1 and Corollary 2.3], following arguments similar to [P8, I, Theorem 3.1].

THEOREM 1.1. *Let (M_1,τ_1) and (M_2,τ_2) be finite von Neumann algebras and B be a common von Neumann subalgebra such that $\tau_1|_B=\tau_2|_B$. Let $M=M_1*_B M_2$, $0\neq q\in\mathcal{P}(M_1)$ and let $Q\subset qM_1q$ be a von Neumann subalgebra. Assume that no corner of Q can be embedded into B inside M_1, i.e. $Q'\cap q\langle M_1,B\rangle q$ contains no non-zero finite projections. If $0\neq\xi\in L^2(qM)$ satisfies*

$$Q\xi\subset L^2\biggl(\sum_{i=1}^{n}\xi_i M_k\biggr)$$

for some $k\in\{1,2\}$ and some $\xi_1,...,\xi_n\in L^2(M)$, then $k=1$ and $\xi\in L^2(M_1)$. In particular, $Q'\cap qMq\subset M_1$, the normalizer $\mathcal{N}_{qMq}(Q)$ of Q in qMq is contained in qM_1q, and if $x\in M$ satisfies $Qx\subset xM_2$ then $x=0$.

Proof. Let p denote the orthogonal projection of $L^2(M)$ onto the Hilbert subspace

$$\overline{Q\xi M_k}^{\|\cdot\|_2}\subset L^2(M).$$

Note that $p\in Q'\cap q\langle M,e_{M_k}\rangle q$ and $0\neq\mathrm{Tr}(p)<\infty$, where $\mathrm{Tr}=\mathrm{Tr}_{\langle M,e_{M_k}\rangle}$ denotes the canonical trace on $\langle M,e_{M_k}\rangle$. To prove that $k=1$ and $\xi\in M_k$ it is sufficient to show that $p\leqslant e_{M_k}$, or equivalently that $(1-e_{M_k})p(1-e_{M_k})=0$. Indeed, because then $Q\xi M_k\subset L^2(M_k)$, so in particular $\xi\in L^2(M_k)$ and $u\xi\in L^2(M_k)$ for all $u\in\mathcal{U}(Q)$. But since no corner of Q can be embedded into B inside M_1, by [P8, I, Corollary 2.3], it follows that for any $\varepsilon>0$ there exists $u\in\mathcal{U}(Q)$ such that $\|E_B(u)\|_2\leqslant\varepsilon$. Thus, if $k=2$, then $\xi,u\xi\in L^2(M_2)$ so that $u\xi=E_{M_2}(u\xi)=E_B(u)\xi$, and by the Cauchy–Schwarz inequality we have

$$\|\xi\|_1=\|u\xi\|_1=\|E_B(u)\xi\|_1\leqslant\|E_B(u)\|_2\|\xi\|_2\leqslant\varepsilon\|\xi\|_2.$$

Since $\varepsilon>0$ was arbitrary, this shows that $\xi=0$. Thus, the only possibility is that $k=1$, i.e. $\xi\in L^2(M_1)$.

By taking spectral projections, to show that $(1-e_{M_k})p(1-e_{M_k})=0$ it is in fact sufficient to show that if $f\in Q'\cap\langle M,e_{M_k}\rangle$ is a projection such that $0\neq\mathrm{Tr}(f)<\infty$ and $f\leqslant 1-e_{M_k}$, then $f=0$. To this end, we will show that $\|f\|_{2,\mathrm{Tr}}$ is arbitrarily small.

Thus, let $\eta_0=1,\eta_1,...,\eta_n,...\subset M$ be an orthonormal basis of M over M_k, i.e.

$$E_{M_k}(\eta_i^*\eta_j)=\delta_{ij}p_j\in\mathcal{P}(M_k)$$

and $\|\eta\|<\infty$, for all i and j. If we let $f_n=\sum_{i=1}^n \eta_i e_{M_k}\eta_i^*$ then, as f has finite trace and $f\leqslant 1-e_{M_k}=\sum_{i=1}^\infty \eta_i e_{M_k}\eta_i^*$, there exists $n\in\mathbf{N}$ such that $\|f_nf-f\|_{2,\mathrm{Tr}}<\varepsilon\|f\|_{2,\mathrm{Tr}}$. Thus, if $u\in\mathcal{U}(Q)$, then

$$\mathrm{Tr}(f_nuf_nu^*)\geqslant \mathrm{Tr}(ff_nfuf_nu^*)-|\mathrm{Tr}(ff_n(1-f)uf_nu^*)|-|\mathrm{Tr}((1-f)f_nuf_nu^*)|. \tag{1.2}$$

Using that f_nf is ε-close to f in the norm $\|\cdot\|_{2,\mathrm{Tr}}$ and that f commutes with $u\in Q$, we deduce that

$$\mathrm{Tr}(ff_nfuf_nu^*)=\mathrm{Tr}(f_nfuf_nfu^*)\geqslant(1-2\varepsilon-\varepsilon^2)\|f\|_{2,\mathrm{Tr}}^2. \tag{1.3}$$

Similarly, we have

$$|\mathrm{Tr}(ff_n(1-f)uf_nu^*)|+|\mathrm{Tr}((1-f)f_nfuf_nu^*)|\leqslant 2\varepsilon(1+\varepsilon)\|f\|_{2,\mathrm{Tr}}^2. \tag{1.4}$$

Combining (1.2)–(1.4), we get

$$\mathrm{Tr}(f_nuf_nu^*)\geqslant(1-4\varepsilon-3\varepsilon^2)\|f\|_{2,\mathrm{Tr}}^2 \quad \text{for all } u\in\mathcal{U}(Q). \tag{1.5}$$

On the other hand,

$$\mathrm{Tr}(f_nuf_nu^*)=\mathrm{Tr}\biggl(\sum_{i,j=1}^n \eta_ie_{M_k}\eta_i^*u\eta_je_{M_k}\eta_j^*u^*\biggr)=\sum_{i,j=1}^n\|E_{M_k}(\eta_iu\eta_j^*)\|_2^2. \tag{1.6}$$

Thus, in order to prove that $\|f\|_{2,\mathrm{Tr}}$ is small, it is sufficient to prove that for all $\eta_0,...,\eta_n\in M\ominus M_k$ and for all $\varepsilon>0$, there exists $u\in\mathcal{U}(Q)$ such that

$$\|E_{M_k}(\eta_iu\eta_j^*)\|_2\leqslant\varepsilon \quad \text{for all } 0\leqslant i,j\leqslant n.$$

Furthermore, by Theorem 1.1 and Kaplansky's density theorem, it is enough to prove this in the case where the η_i's are reduced words of the form $\eta_i=\delta_ix_i$ such that one of the following holds true: (a) δ_i is a reduced word that ends with a letter in $M_2\ominus B$ and x_i is either equal to 1 or contained in $M_1\ominus B$; (b) $k=2$, $\delta_i=1$ and $x_i\in M_1\ominus B$. Since $x_iux_j^*\in M_1$, if we set $y=x_iux_j^*-E_B(x_iux_j^*)\in M_1\ominus B$, then in both cases (a) and (b) the reduced word $\delta_iy\delta_j^*$ is perpendicular to M_k. Indeed, in case (a), $\delta_iy\delta_j^*$ lies in

$$...(M_{k'}\ominus B)(M_k\ominus B)(M_{k'}\ominus B)...,$$

where $\{k,k'\}=\{1,2\}$, and thus it has length at least 3, so $\delta_iy\delta_j^*\perp M_k$ by (1.1). In case (b), $\delta_iy\delta_j^*\in M_1\ominus B$, so it is perpendicular to $M_2=M_k$. Using $\delta_iy\delta_j^*\perp M_k$, we then get

$$E_{M_k}(\eta_iu\eta_j^*)=E_{M_k}(\eta_iy\eta_j^*)+E_{M_k}(\delta_iE_B(x_iux_j^*)\delta_j^*)=E_{M_k}(\delta_iE_B(x_iux_j^*)\delta_j^*),$$

implying that

$$\|E_{M_k}(\eta_i u\eta_j^*)\|_2 = \|E_{M_k}(\delta_i E_B(x_i u x_j^*)\delta_j^*)\|_2 \leqslant \|\delta_i\|\,\|\delta_j\|\,\|E_B(x_i u x_j^*)\|_2.$$

But by the hypothesis and [P8, I, Corollary 2.3], for any $\varepsilon>0$ we can find $u\in\mathcal{U}(Q)$ such that

$$\|E_B(x_i u x_j^*)\|_2 \leqslant \frac{\varepsilon}{\|\delta_i\|\,\|\delta_j\|}. \qquad \square$$

Note that, under the conditions of Theorem 1.1, not only the normalizer $\mathcal{N}_{qMq}(Q)$ of Q in qMq is contained in M_1, but also the normalizer of the von Neumann algebra generated by $\mathcal{N}_{qMq}(Q)$, and so on. In fact, even the unitary elements $u\in qMq$, with the property that $uQu^*\cap qM_1q$ is not embeddable into B, are contained in M_1. More generally, if $Q\subset qM_1q$ is a von Neumann subalgebra such that $Q'\cap q\langle M_1,B\rangle q$ contains no non-zero finite projections, and if we denote by $N_1=N(Q,M_1;B)$ the von Neumann subalgebra of qMq generated by unitary elements $u\in qMq$ such that $(uQu^*\cap M_1)'\cap q\langle M_1,B\rangle q$ contains no non-zero finite projections, then $N_1\subset M_1$. If we then repeat this operation, taking $N_2=N(N_1,M_1;B)$ to be the von Neumann algebra generated by all unitary elements $u\in qMq$ such that $uN_1u^*\cap q\langle M_1,B\rangle q$ contains no non-zero finite projections, then $N_2\subset M_1$. We can of course continue this procedure inductively until it "stops", i.e. until we reach an N_i such that $N(N_i,M_1;B)=N_i$. More, formally, consider the following definition.

Definition 1.2. Given $q\in\mathcal{P}(B)$ and $Q\subset qMq$, we consider by (transfinite) induction the strictly increasing family of von Neumann algebras $Q=N_0\subset N_1\subset...\subset N_j\subset...\subset N_i$, indexed by the first i ordinals, such that:

(a) for each $j<i$, $N_{j+1}=N(N_j,M_1;B)$ and $N_j\neq N_{j+1}$;

(b) $N(N_i,M_1;B)=N_i$;

(c) if $j\leqslant i$ has no "predecessor", then $N_j=\bigcup_{n<j}N_n$.

We then let $\widetilde{N}(Q,M_1;B)=N_i$ and call it the *weak quasi-normalizer* (*wq-normalizer*) of Q in qMq relative to $(M_1;B)$. Note that in fact both the definitions of $N(Q,M_1;B)$ and $\widetilde{N}(Q,M_1;B)$ make sense for any finite von Neumann algebra (M,τ) and von Neumann subalgebras $B,M_1\subset M$ and $Q\subset qMq$, with $q\in\mathcal{P}(M_1)$.

This definition is analogous to the definition of wq-normalizer of a subgroup $H\subset G$ used in [P5], [P6], [P8]. It is easy to see that $\widetilde{N}(Q,M_1;B)$ is the smallest von Neumann subalgebra P of qMq such that $(uPu^*\cap qM_1q)'\cap q\langle M,e_B\rangle q$ contains non-zero finite projections for all $u\in qMq\setminus P$. Theorem 1.1 thus implies the following result.

Corollary 1.3. *Let* (M_1,τ_1), (M_2,τ_2), $q\in B\subset M_i$, $i=1,2$, *and* $M=M_1*_B M_2$ *be as in Theorem* 1.1. *Let* $Q\subset qM_1q$ *be a von Neumann subalgebra such that no corner of* Q *can be embedded into* B *inside* M_1, *i.e.* $Q'\cap q\langle M_1,B\rangle q$ *contains no non-zero finite projections. Then* $\widetilde{N}(Q,M_1;B)\subset M_1$.

We will make repeated use of the following application of Theorem 1.1, which shows that if one of the algebras M_i involved in an amalgamated free product $M=M_1*_B M_2$ contains a regular subalgebra Q, then Q must necessarily be contained in B, modulo inner conjugacy.

Corollary 1.4. *Let* (M_1,τ_1), (M_2,τ_2) *and* $B\subset M_i$, $i=1,2$, *be as in Theorem* 1.1, *and let* $M=M_1*_B M_2$. *Let* $Q\subset qM_1q$ *be a von Neumann subalgebra, for some* $q\in\mathcal{P}(B)$ *with* $qBq\neq qM_2q$. *Assume that* Q *is regular in* qMq. *Then one can embed a corner of* Q *into* B *inside* M_1, *i.e.* $Q'\cap q\langle M_1,e_B\rangle q$ *contains non-zero finite-trace projections.*

Proof. If $Q'\cap q\langle M_1,e_B\rangle q$ contains no non-zero finite-trace projection then, by Theorem 1.1, the normalizer $\mathcal{N}_M(Q)$ of Q in qMq is contained in M_1. Since $\mathcal{N}_M(Q)''=qMq$, this implies that $qMq=qM_1q$, thus $qM_2q=qBq$, a contradiction. □

1.3. Locating subalgebras by means of normalizers

In this and the next subsections we prove that if a subalgebra of $M=M_1*_B M_2$ is normalized by "many" unitary elements in M_1, then it must necessarily be contained in M_1. This technical result will in fact not be needed until §7, where it plays a key role in the proof of the Bass–Serre type Theorem 7.7. The proof uses the intertwining criteria in [P8] and a careful asymptotic analysis of elements written in the AFP expansion (1.1). We first prove the result assuming that the subalgebra we want to "locate" is unitarily conjugate to a subalgebra of B. This assumption will be shown to be redundant in §1.4, in the case when $B=A$ is Cartan in M.

Proposition 1.5. *Let* Λ_1 *and* Λ_2 *be discrete groups and* $\sigma:\Lambda\to\mathrm{Aut}(B,\tau)$ *be an action of* $\Lambda=\Lambda_1*\Lambda_2$ *on the finite von Neumann algebra* (B,τ). *Let*

$$M=B\rtimes_\sigma\Lambda=M_1*_B M_2,$$

where $M_i=B\rtimes_{\sigma|_{\Lambda_i}}\Lambda_i$, $i=1,2$. *Let* $q\in\mathcal{P}(B)$, $B_0\subset qBq$ *be a von Neumann subalgebra,* $u\in\mathcal{U}(qMq)$ *and set* $\mathcal{N}=\{v\in\mathcal{U}(qM_1q):v(uqB_0qu^*)v^*=uqB_0qu^*\}$. *Assume that no corner of* $\mathcal{N}''$ *can be embedded into* B *inside* M_1. *Then* $uqB_0qu^*\subset M_1$.

Proof. Assume that there exists $b_0\in qB_0q$, $\|b_0\|\leqslant 1$, such that

$$d_0=ub_0u^*-E_{M_1}(ub_0u^*)$$

satisfies $c=\|d_0\|_2>0$. To get a contradiction, we first show that all of the unit ball of $uqBqu^*$ can be embedded into a set of the form $\sum_{g\in F}(B)_1u_g$ with $F\subset\Lambda$ finite and $(B)_1$ denoting the unit ball of B. We need the following lemma.

LEMMA 1.6. *Let (B,τ) be a finite von Neumann algebra, $\sigma:\Lambda\to\mathrm{Aut}(B,\tau)$ be an action, $M=B\rtimes_\sigma G$ be the corresponding crossed product finite von Neumann algebra and $\{u_g\}_g\subset M$ be the canonical unitary elements. For any finite set in the unit ball of M, $S_0\subset(M)_1$ and any $\varepsilon>0$, there exists $F\subset\Lambda$ finite such that $x(B)_1y^*\subset_\varepsilon\sum_{g\in F}(B)_1u_g$ for all $x,y\in S_0$.*

Proof. By Kaplansky's density theorem, there exists a finite set $F_0\subset\Lambda$ and elements $\{b_g^x\in B:x\in S_0 \text{ and } g\in F_0\}$, such that $x_0=\sum_{g\in F_0}b_g^xu_g$ satisfies $\|x_0\|\leqslant 1$ and $\|x-x_0\|_2\leqslant\frac{1}{2}\varepsilon$ for all $x\in S_0$. If we put $F=F_0F_0^{-1}$, then we clearly have $x_0By_0^*\subset\sum_{g\in F}Bu_g$ for all $x,y\in S_0$. On the other hand, if $b\in B$ satisfies $\|b\|\leqslant 1$ and we let $x_0by_0^*=\sum_{g\in F}b_gu_g$ then $b_g=E_B((x_0by_0^*)u_g^*)$ and thus $\|b_g\|\leqslant\|(x_0by_0^*)u_g^*\|\leqslant 1$. This implies that $\|xby^*-x_0by_0^*\|_2\leqslant\varepsilon$ and thus $xby^*\in_\varepsilon\sum_{g\in F}(B)_1u_g$. □

By Lemma 1.6, it follows that there exists $F\subset\Lambda$ finite such that

$$uq(B)_1qu^*\subset_{\varepsilon/2}\sum_{g\in F}(B)_1u_g,$$

where $\varepsilon=\frac{1}{4}c^2$. Let $N=\mathcal{N}''\subset qM_1q$.

For any $v\in\mathcal{N}\subset qM_1q$ we then have

$$v(ub_0u^*)v^*\in_{\varepsilon/2}\sum_{g\in F}(B)_1u_g$$

as well. Since

$$v(ub_0u^*)v^*=vd_0v^*+v(E_{M_1}(ub_0u^*))v^*,$$

with $vd_0v^*\perp M_1$ and $v(E_{M_1}(ub_0u^*))v^*\in M_1$, by Pythagoras' theorem it follows that we have $vd_0v^*\in_{\varepsilon/2}\sum_{g\in F_0}(B)_1u_g$, where $F_0=F\setminus\Lambda_1$. Now let $d_1\in\sum_{g\in F_0}(B)_1u_g$ be such that $\|d_0-d_1\|_2\leqslant\frac{1}{2}\varepsilon$. We have thus shown that

> there exists $F_0\subset\Lambda\setminus\Lambda_1$ finite and $d_1\in\sum_{g\in F_0}(B)_1u_g$ with $\|d_1\|\leqslant|F_0|$ and $\|d_1\|_2\geqslant c-\frac{1}{8}c^2>0$, such that $vd_1v^*\in_\varepsilon\sum_{g\in F_0}(B)_1u_g$ for all $v\in\mathcal{N}$, where $\varepsilon=\frac{1}{4}c^2$. (1.7)

Now note that by the condition satisfied by the algebra $\mathcal{N}''=N$, from [P8, I, Corollary 2.3] it follows that

> for all $K\subset\Lambda_1$ finite and $\delta>0$, there exists $v\in\mathcal{N}$ such that if ξ denotes the projection of v onto the Hilbert space $\bigoplus_{h\in\Lambda_1\setminus K}L^2(B)u_h$, then $\|v-\xi\|_2\leqslant\delta$. (1.8)

At this point, we need the following lemma.

Lemma 1.7. *Let* (B,τ), (σ,Λ), M *and* $\{u_g\}_g$ *be as in Lemma* 1.6, *and* $\Lambda=\Lambda_1*\Lambda_2$. *Let* $F_0\subset\Lambda\setminus\Lambda_1$ *be a finite set. Then, there exists* $K=K(F_0)\subset\Lambda_1$ *finite such that any* $\xi\in L^2(B\rtimes\Lambda_1)$ *supported by* $\Lambda_1\setminus K$ *satisfies* $\xi\big(\sum_{g\in F_0}Bu_g\big)M_1\perp\sum_{g\in F_0}Bu_g$.

Proof. Note that each irreducible alternating word $g\in F_0$ has at least one letter from Λ_2. Let K_0 denote the set of elements in Λ_1 that can appear as first letter in a word g in F_0 (including the trivial letter e) and set $K=K_0K_0^{-1}$. Then, $(\Lambda_1\setminus K)K_0\cap K_0=\varnothing$. Now note that if $\xi\in L^2(B\rtimes\Lambda_1)$ is supported by $\Lambda_1\setminus K$ then any element η in $\xi\big(\sum_{g\in F}Bu_g\big)$ is supported on elements $g\in G$ that begin with a letter in $(\Lambda_1\setminus K)K_0\subset\Lambda_1\setminus K_0$. Moreover, this is still the case for elements of the form ηx, for $x\in M_1$. In turn, any g in the support of an element in $\sum_{g\in F_0}Bu_g$ begins with a letter in K_0. Thus, the two vector spaces $\xi\big(\sum_{g\in F_0}Bu_g\big)M_1$ and $\sum_{g\in F_0}Bu_g$ are supported on disjoint subsets of $\Lambda_1*\Lambda_2$ and are thus perpendicular. □

We now continue the proof of Proposition 1.5. Let $K=K(F_0)$ be given by Lemma 1.7, for the finite set $F_0\subset\Lambda\setminus\Lambda_1$ from (1.7). Let $\delta=\varepsilon/|F_0|$ and choose $\xi\in L^2(B\rtimes\Lambda_1)$ supported on $\Lambda_1\setminus K$, as given by (1.8). Then

$$\|\xi d_1v^*-vd_1v^*\|_2\leqslant\|\xi-v\|_2\|d_1\|\leqslant\delta|F_0|\leqslant\varepsilon,$$

which, together with (1.7), implies that $\xi d_1v^*\in_{2\varepsilon}\sum_{g\in F_0}Bu_g$. But, by Lemma 1.7, we have $\xi d_1v^*\perp\sum_{g\in F_0}Bu_g$. Thus, $\|\xi d_1v^*\|_2\leqslant 2\varepsilon$. On the other hand,

$$\|\xi d_1v^*\|_2=\|\xi d_1\|_2\geqslant\|d_1\|_2-\|\xi-v\|_2\|d_1\|\geqslant c-\tfrac{1}{8}c^2-\varepsilon>2\varepsilon,$$

a contradiction which ends the proof of Proposition 1.5. □

1.4. A Cartan conjugacy result

We now prove that if the "core" B of an AFP algebra $M=M_1*_BM_2$ is maximal abelian and regular (and thus Cartan) in M, then any other Cartan subalgebra $A_0\subset M$ which is normalized by "many" unitary elements in M_1 is unitarily conjugate to $B=A$. Note that it strenghtens both Corollary 1.4 and Proposition 1.5, in the case where the core $B=A$ is abelian and Cartan in M.

Theorem 1.8. *Let* Λ_1 *and* Λ_2 *be infinite discrete groups and* $\sigma:\Lambda\to\mathrm{Aut}(A,\tau)$ *be a free ergodic action of* $\Lambda=\Lambda_1*\Lambda_2$ *on a diffuse abelian von Neumann algebra* (A,τ). *Let* $M=A\rtimes_\sigma\Lambda=M_1*_AM_2$, *where* $M_i=A\rtimes_{\sigma|_{\Lambda_i}}\Lambda_i$, $i=1,2$. *Let* $q\in\mathcal{P}(A)$ *and* $A_0\subset qMq$ *be a Cartan subalgebra such that no corner of* $(\mathcal{N}_{qMq}(A_0)\cap qM_1q)''$ *can be embedded into* A *inside* M_1. *Then* $A_0\subset qM_1q$ *and there exists* $u\in\mathcal{U}(qM_1q)$ *such that* $uA_0u^*=Aq$.

LEMMA 1.9. *Let Λ_1 and Λ_2 be discrete groups and let $\sigma:\Lambda=\Lambda_1*\Lambda_2\to\mathrm{Aut}(B,\tau)$ be a trace-preserving action on a finite von Neumann algebra (B,τ). Let $M=B\rtimes_\sigma\Lambda=M_1*_B M_2$, where $M_1=B\rtimes_{\sigma|_{\Lambda_1}}\Lambda_1\subset M$. Let $q\in\mathcal{P}(B)$ and assume that $A_0\subset qMq$ is a diffuse abelian von Neumann subalgebra such that no corner of A_0 can be embedded into qM_1q inside M. Then for any $\varepsilon>0$ there exist $F\subset\Lambda\setminus\Lambda_1$ finite and $x_1,x_2\in\sum_{g\in F}Bu_g$ such that any $u\in\mathcal{N}_{qMq}(A_0)$ satisfies*

$$\|ux_1u^*x_2-x_2ux_1u^*\|_2\leqslant\varepsilon\quad\textit{and}\quad\sqrt{\tau(q)}-\varepsilon\leqslant\|ux_1u^*x_2\|_2\leqslant\sqrt{\tau(q)}+\varepsilon.$$

Proof. By the assumption on A_0, it follows from [P8, I, Corollary 2.3] that there exists $a_1\in\mathcal{U}(A_0)$ such that $\|E_{M_1}(a_1)\|_2<\frac{1}{4}\varepsilon$. Thus, we can find $F_1\subset\Lambda\setminus\Lambda_1$ finite and $x_1\in\sum_{g\in F_1}Bu_g$ such that $\|a_1-x_1\|_2\leqslant\frac{1}{4}\varepsilon$. Repeating the above argument, we can now find $a_2\in\mathcal{U}(A_0)$, a finite set F with $F_1\subset F\subset\Lambda\setminus\Lambda_1$ and $x_2\in\sum_{g\in F}Bu_g$ such that

$$\|a_2-x_2\|_2\leqslant\frac{\varepsilon}{4\|x_1\|}.$$

Using these inequalities, we get for $u\in\mathcal{N}_{qMq}(A_0)$ (in fact for all $u\in M$ with $\|u\|\leqslant 1$),

$$\begin{aligned}\|ux_1u^*x_2-ua_1u^*a_2\|_2&\leqslant\|ux_1u^*(x_2-a_2)\|_2+\|u(x_1-a_1)u^*a_2\|_2\\&\leqslant\|x_1\|\|x_2-a_2\|_2+\|x_1-a_1\|_2\leqslant\frac{\|x_1\|\varepsilon}{4\|x_1\|}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.\end{aligned}$$

Similarly, it follows that $\|x_2ux_1u^*-a_2ua_1u^*\|_2\leqslant\frac{1}{2}\varepsilon$ for all $u\in(M)_1$. Finally, if $u\in\mathcal{N}_{qMq}(A_0)$ then $ua_1u^*a_2=a_2ua_1u^*$ (because A_0 is abelian) and $\|ua_1u^*a_2\|_2=\sqrt{\tau(q)}$. Thus, by combining this with the above inequalities, we get the desired estimates. □

LEMMA 1.10. *With the same notation and assumptions as in Lemma* 1.9 *above, let $F\subset\Lambda\setminus\Lambda_1$ be a finite set and let $x_1,x_2\in\sum_{g\in F}Bu_g$. Then, for all $\varepsilon>0$, there exists $K=K(F,\varepsilon)\subset\Lambda_1$ finite and $\delta=\delta(F,\varepsilon)>0$ such that if $u\in(M_1)_1$ satisfies $\|E_B(uu_g^*)\|_2\leqslant\delta$ for all $g\in K$, then it must also satisfy $ux_1u^*x_2\perp_\varepsilon x_2ux_1u^*$.*

Proof. Since $F\subset\Lambda\setminus\Lambda_1$ is finite, by the free decomposition $\Lambda=\Lambda_1*\Lambda_2$, one readily deduces that there exists $K=K^{-1}\subset\Lambda_1$ finite such that $(\Lambda_1\setminus K)F(\Lambda_1\setminus K)F$ has empty intersection with $F(\Lambda_1\setminus K)F(\Lambda_1\setminus K)$. Next, let $u\in(M_1)_1$ and set $u'=\sum_{g\in K}E_B(uu_g^*)u_g$ and $u''=u-u'$. Then u'' is supported on $\Lambda_1\setminus K$ and we have the decomposition

$$ux_1u^*x_2=u''x_1(u'')^*x_2+u'x_1u^*x_2+ux_1(u')^*x_2-u'x_1(u')^*x_2.$$

Let $x_{1,2}=u''x_1(u'')^*x_2$. Then, $x_{1,2}$ is supported on $(\Lambda_1\setminus K)F(\Lambda_1\setminus K)F$ and we have the following estimate:

$$\|ux_1u^*x_2-x_{1,2}\|_2\leqslant(2\|u\|+\|u'\|)\|x_1\|\,\|x_2\|\,\|u'\|_2\leqslant(2+|K|)\|x_1\|\,\|x_2\|\,\|u'\|_2.$$

Similarly, if we set $x_{2,1}=x_2u''x_1(u'')^*$, then $x_{2,1}$ is supported on $F(\Lambda_1\setminus K)F(\Lambda_1\setminus K)$ and $\|x_2ux_1u^*-x_{2,1}\|_2\leqslant(2+|K|)\|x_1\|\,\|x_2\|\,\|u'\|_2$.

Next, we show that K and $\delta=\varepsilon(12|K|(\|x_1\|\,\|x_2\|+1))^{-3/2}$ satisfy the conclusion. To this end, let $u\in(M_1)_1$ be such that $\|E_B(uu_g^*)\|_2\leqslant\delta$ for all $g\in K$. Then

$$\|u'\|_2=\bigg(\sum_{g\in K}\|E_B(uu_g^*)\|_2^2\bigg)^{1/2}\leqslant\frac{\varepsilon}{|K|(12(\|x_1\|\,\|x_2\|+1))^{3/2}},$$

hence

$$\|ux_1u^*x_2-x_{1,2}\|_2\leqslant\frac{\varepsilon}{4(\|x_1\|\,\|x_2\|+1)}\quad\text{and}\quad\|x_2ux_1u^*-x_{2,1}\|_2\leqslant\frac{\varepsilon}{4(\|x_1\|\,\|x_2\|+1)}.$$

Also, we have

$$\|x_{1,2}\|_2\leqslant\|x_{1,2}-ux_1u^*x_2\|_2+\|ux_1u^*x_2\|_2\leqslant\frac{\varepsilon}{4(\|x_1\|\,\|x_2\|+1)}+\|x_1\|\,\|x_2\|\leqslant\|x_1\|\,\|x_2\|+1.$$

But, by the way we have chosen K, $x_{1,2}$ and $x_{2,1}$ have disjoint supports. Hence $x_{1,2}\perp x_{2,1}$. Thus

$$\begin{aligned}|\langle ux_1u^*x_2,x_2ux_1u^*\rangle|&\leqslant|\langle ux_1u^*x_2-x_{1,2},x_2ux_1u^*\rangle|+|\langle x_{1,2},x_2ux_1u^*-x_{2,1}\rangle|\\&\leqslant\|ux_1u^*x_2-x_{1,2}\|_2\|x_2ux_1u^*\|_2+\|x_{1,2}\|_2\|x_2ux_1u^*-x_{2,1}\|_2\\&\leqslant\frac{\varepsilon\|x_1\|\,\|x_2\|}{4(\|x_1\|\,\|x_2\|+1)}+\frac{\varepsilon(\|x_1\|\,\|x_2\|+1)}{4(\|x_1\|\,\|x_2\|+1)}\\&<\varepsilon.\end{aligned}$$

□

PROPOSITION 1.11. *With the same notation and assumptions as in Lemmas* 1.9 *and* 1.10, *let* $q\in\mathcal{P}(B)$ *and let* $A_0\subset qMq$ *be a diffuse abelian von Neumann subalgebra. Assume that no corner of* $(\mathcal{N}_{qMq}(A_0)\cap qM_1q)''\subset qMq$ *can be embedded into* B *inside* M_1. *Then a corner of* A_0 *can be embedded into* qM_1q *inside* qMq.

Proof. Assume that no corner of A_0 embeds into M_1. Apply first Lemma 1.9 for $\varepsilon=\frac{1}{4}\tau(q)^{1/2}$ to deduce that there exists $F\subset\Lambda\setminus\Lambda_1$ finite and $x_1,x_2\in M$ supported on F such that if $u\in\mathcal{N}_{qMq}(A_0)$ then

$$\|ux_1u^*x_2-x_2ux_1u^*\|_2\leqslant\tfrac{1}{4}\tau(q)^{1/2}\quad\text{and}\quad\tfrac{3}{4}\tau(q)^{1/2}\leqslant\|ux_1u^*x_2\|_2\leqslant\tfrac{5}{4}\tau(q)^{1/2}.$$

It then follows that $|\langle ux_1u^*x_2,x_2ux_1u^*\rangle|\geqslant\frac{1}{4}\tau(q)$ for all $u\in\mathcal{N}_{qMq}(A)$.

By Lemma 1.10, there exist $K\subset\Lambda_1$ finite and $\delta>0$ such that if $u\in(M_1)_1$ satisfies $\|E_B(uu_g^*)\|_2\leqslant\delta$ for all $g\in K$, then $ux_1u^*x_2\perp_{\tau(q)/4}x_2ux_1u^*$. But this implies that we cannot find $u\in\mathcal{N}=\mathcal{N}_{qMq}(A_0)\cap qM_1q$ such that $\|E_B(uu_g^*)\|_2\leqslant\delta$ for all $g\in K$. By [P8, I, Corollary 2.3], this contradicts the fact that no corner of $\mathcal{N}''$ embeds into B inside M_1. □

The next result provides a useful "transitivity" property for the subordination relation considered in Corollary 1.4.

LEMMA 1.12. *Let M be a finite von Neumann algebra, B_0 and M_1 be von Neumann subalgebras of M and Q be a von Neumann subalgebra of M_1. Assume that there exist projections $q_0 \in B_0$ and $q_1 \in M_1$, a unital isomorphism of $q_0 B_0 q_0$ into $q_1 M_1 q_1$ and a partial isometry $v \in M$ such that $v^* v \in (q_0 B_0 q_0)' \cap q_0 M q_0$, $vv^* \in \psi(q_0 B_0 q_0)' \cap q_1 M q_1$ and $vb = \psi(b) v$ for all $b \in q_0 B_0 q_0$. Denote by q' the support projection of $E_{M_1}(vv^*) \in \psi(q_0 B_0 q_0)' \cap q_1 M_1 q_1$ and let $B_1 = \psi(q_0 B_0 q_0) q'$. If a corner of $B_1 = \psi(q_0 B_0 q_0) q'$ can be embedded into Q inside M_1, then a corner of B_0 can be embedded into Q inside M.*

Proof. Indeed, if $p_1 \in \mathcal{P}(B_1)$, $v_1 \in M_1 p_1$ is a non-zero partial isometry and

$$\psi_1 : p_1 B_1 p_1 \longrightarrow Q$$

is a (not necessarily unital) isomorphism such that $v_1 b = \psi_1(b) v_1$ for all $b \in p_1 B_1 p_1$, then $v_1 v \neq 0$ and $v_1 v b = \psi_1(\psi(b)) v_1 v$ for all $b \in q_0 B_0 q_0$. □

Proof of Theorem 1.8. By Proposition 1.11 and [P8, I, Theorem 2.1], there exist projections $q_0 \in A_0 \subset qMq$ and $q_1 \in qM_1 q$, a unital isomorphism of $A_0 q_0$ into $q_1 M_1 q_1$ and a partial isometry $v \in M$ such that $v^* v = q_0, vv^* \in \psi(A_0 q_0)' \cap q_1 M q_1$ and $va = \psi(a) v$ for all $a \in A_0 q_0$. Let q' be the support projection of $E_{M_1}(vv^*)$ and note that if we set $A_1 = \psi(A_0 q_0) \subset q_1 M_1 q_1$, then $q' \in A_1' \cap q_1 M_1 q_1$. By replacing, if necessary, ψ by $q' \psi(\cdot) q'$ and shrinking $q_0 \in A_0$ accordingly, we may assume that $q_1 = q'$.

Now, if a corner of A_1 can be embedded into A inside M_1, then, by Lemma 1.12, a corner of A_0 can embedded into A inside M, so, by [P5, §A.1], the two Cartan subalgebras $A_0, Aq \subset qMq$ are unitarily conjugate. If in turn no corner of A_1 can be embedded into A inside M_1 then, by Theorem 1.1, we have $vv^* \in A_1' \cap q_1 M q_1 \subset q_1 M_1 q_1$, implying that $v A_0 v^* \subset q_1 M_1 q_1$. By spatiality, $v A_0 v^*$ is Cartan in $q_1 M q_1$, which, by Corollary 1.4, implies that a corner of $v A_0 v^*$ can be embedded into A inside M_1. By [P5, §A.1], this implies that A_0 and Aq are unitarily conjugate in qMq. On the other hand, by Proposition 1.5, we have $A_0 \subset qM_1 q$. But then Theorem 1.1 applies to show that A_0 and Aq are conjugate in $qM_1 q$ as well. □

2. Deformation of AFP factors

Throughout this section, (M_1, τ_1) and (M_2, τ_2) will be finite von Neumann algebras with $B \subset M_i$ being a common von Neumann subalgebra such that $\tau_1|_B = \tau_2|_B$, as in §1. We describe, in this section, several useful ways to deform the identity map on the AFP algebra $M = M_1 *_B M_2$ by subunital subtracial completely positive (c.p.) maps which arise naturally from the amalgamated free product structure of M. By a *deformation* of id_M we mean here a sequence ϕ_n of subunital subtracial c.p. maps on M such that

$$\lim_{n\to\infty} \|\phi_n(x)-x\|_2 = 0 \quad \text{for all } x \in M.$$

2.1. Amalgamated free product of c.p. maps

We first recall the definition of amalgamated product of c.p. maps from [Bo], and establish some basic properties.

LEMMA 2.1. *Let $\phi_i: M_i \to M_i$ be subunital subtracial c.p. maps with $\phi_1(b)=\phi_2(b)$, for all $b\in B$, and $\tau\circ\phi_i=\lambda\tau$, $i=1,2$, with $0<\lambda\leqslant 1$. Then, $\phi_1 *_B \phi_2: M_1 *_B M_2 \to M_1 *_B M_2$ is a well-defined subunital subtracial c.p. map. Moreover, the map $(\phi_1,\phi_2)\mapsto \phi_1 *_B \phi_2$ is continuous with respect to the topologies given by pointwise $\|\cdot\|_2$-convergence.*

Proof. Since $\tau\circ\phi_i=\lambda\tau$, $i=1,2$, we have that $\tau\circ(\phi_1 *_B^{\mathrm{alg}} \phi_2)=\lambda\tau$, and hence, by [Bo], it extends uniquely to a c.p. map $\phi_1 *_B \phi_2$ on $M_1 *_B M_2$, which is then subtracial.

To show that this correspondence is continuous, suppose that $\varepsilon>0$ and $x_1, ..., x_n \in M_1 *_B M_2$. As $M_1 *_B^{\mathrm{alg}} M_2$ is dense in $M_1 *_B M_2$, let $x_1', ..., x_n' \in M_1 *_B^{\mathrm{alg}} M_2$ be such that

$$\|x_j - x_j'\|_2 \leqslant \tfrac{1}{3}\varepsilon \quad \text{for all } j \leqslant n.$$

Hence there exist $m, l\in \mathbf{N}$ and $1\in F_i \subset M_i$ finite such that each x_j' is the sum of at most m products of lenght at most l from $F_1\cup F_2$. Let $N=\max_{x\in F_1\cup F_2}\|x\|$. Then, if $\phi_i': M_i \to M_i$ are subunital subtracial c.p. maps with $\|\phi_i'(x)-\phi_i(x)\|_2<\varepsilon/(3mlN^l)$ for all $x\in F_i$, $i=1,2$, then repeated use of the triangle inequality together with the fact that subunital c.p. maps are contractions in the uniform norm shows that

$$\|\phi_1' *_B \phi_2'(x_j) - \phi_1 *_B \phi_2(x_j)\|_2 < \varepsilon \quad \text{for all } j \leqslant n. \qquad \square$$

Remark 2.2. In general, the free product of two subunital subtracial c.p. maps need not be subtracial. In fact, given any c.p. map ϕ_1 which is unital and subtracial, but not tracial, the c.p. map $\phi=\phi_1 * \mathrm{id}$ is not subtracial. Even more so, the Radon–Nikodym derivative $d\tau\circ\phi/d\tau$ of any such free product c.p. map ϕ is unbounded. To see this, let $x\in M_1$ be such that $\tau(x)=0$, but $\tau\circ\phi_1(x)\neq 0$, let $v\in M_2$ be a partial isometry with $\tau(v)=0$ and set $p=vv^*$. Then, we have

$$\begin{aligned}\tau\circ\phi((vxv^*)^*(vxv^*)) &= |\tau\circ\phi_1(x)|^2\tau(p) + \|v(\phi_1(x)-\tau\circ\phi_1(x))v^*\|_2^2 \\ &= |\tau\circ\phi_1(x)|^2\tau(p) + \tau(p)^2\|\phi_1(x)-\tau\circ\phi_1(x)\|_2^2.\end{aligned}$$

Thus, if we choose v such that $\tau(p)\to 0$, then

$$\frac{\tau\circ\phi((vxv^*)^*(vxv^*))}{\tau((vxv^*)^*(vxv^*))}\to\infty.$$

We note that in [Bo] it was assumed that a free product of subtracial c.p. maps is subtracial in order to show that the Haagerup property is preserved by free products. But although the above argument shows that this fact does not hold true unless the c.p. maps are actually tracial, the result on the Haagerup property is still valid, since, by [Jol, Proposition 2.2], one can take the c.p. maps given by the Haagerup property to be unital and tracial.

2.2. Deformation by automorphisms

Let $\alpha_i\in\mathrm{Aut}(M_i,\tau_{M_i})$, $i=1,2$, be such that $\alpha_1(b)=\alpha_2(b)$ for all $b\in B$. Then, since automorphisms are unital c.p. maps, we have that $\alpha=\alpha_1*_B\alpha_2$ is a unital tracial c.p. map. Moreover, α restricted to the dense subalgebra $M_1*_B^{\mathrm{alg}}M_2$ is an automorphism, and so, by continuity, we have that α is an automorphism.

Hence, if $\alpha_i^t\in\mathrm{Aut}(M_i)$, $t\in\mathbf{R}$, is a one-parameter group of automorphisms of M_i which is pointwise $\|\cdot\|_2$-continuous and satisfies $\alpha_i^t|_B=\mathrm{id}_B$, for $i=1,2$, then α^t gives a deformation of the identity of M by automorphisms. In particular, we have the following result.

LEMMA 2.3. *Let $v_j\in\mathcal{U}(B'\cap M_j)$, $j=1,2$. Then, there is a pointwise $\|\cdot\|_2$-continuous one-parameter group of automorphisms $\{\alpha^t\}_{t\in\mathbf{R}}\subset\mathrm{Aut}(M)$ such that*

$$\alpha_1=\mathrm{Ad}(v_1)*_B\mathrm{Ad}(v_2).$$

Proof. Let $h_j=h_j^*\in B'\cap M_j$ be such that $\exp(\pi i h_j)=v_j$. Here and in the proof of Lemma 2.4 below, but not anywhere else in the paper, i stands for $\sqrt{-1}$. Define

$$\alpha_j^t=\mathrm{Ad}(\exp(\pi t i h_j)),\quad t\in\mathbf{R},\ \ j=1,2,$$

and the above observation applies. □

For the next lemma, we let $\widetilde{M}=M*_B(B\overline{\otimes}L(\mathbf{F}_2))$. Note that, if we let $L(\mathbf{F}_2)=L(\mathbf{Z}*\mathbf{Z})=L(\mathbf{Z})*L(\mathbf{Z})$ and $\widetilde{M}_j=M_j*_B(B\overline{\otimes}L(\mathbf{Z}))$, $j=1,2$, then $\widetilde{M}=\widetilde{M}_1*_B\widetilde{M}_2$. Also, if $u_1\in L(\mathbf{Z}*1)\subset L(\mathbf{F}_2)$ and $u_2\in L(1*\mathbf{Z})\subset L(\mathbf{F}_2)$ are the canonical generating unitary elements, then $u_j\in B'\cap\widetilde{M}_j$, $j=1,2$. We will use the algebra $\widetilde{M}$ as framework for the main deformation of M, Lemma 2.4 below. The action of M on the (1.1)-decomposition of the AFP algebra $\widetilde{M}=\widetilde{M}_1*_B\widetilde{M}_2$ can be viewed as the analogue of the action of an amalgamated free product group $\Lambda_1*_H\Lambda_2$ on the Bass–Serre tree with vertices $\Lambda_1/H\cup\Lambda_2/H$.

In turn, the "graded deformation" below is inspired by the graded deformations of crossed product algebras involving Bernoulli actions in [P7] and [P8].

LEMMA 2.4. *There exists a pointwise* $\|\cdot\|_2$*-continuous one-parameter group of automorphisms* $\{\theta_t\}_{t\in\mathbf{R}}$ *and a period-*2 *automorphism* β *of* $\widetilde{M}$ *such that*

(a) $\theta_0=\mathrm{id}$ *and* $\theta_1=\mathrm{Ad}(u_1)*_B\mathrm{Ad}(u_2)$;

(b) $\beta\theta_t\beta=\theta_{-t}$ *for all* $t\in\mathbf{R}$;

(c) $M\subset\widetilde{M}^\beta$.

Proof. Let A_j be the von Neumann subalgebra generated by u_j, and let $h_j\in A_j$ be self-adjoint elements with spectrum in $[-\pi,\pi]$ such that $u_j=\exp(\pi i h_j)$. Set

$$u_j^t=\exp(\pi i t h_j),\quad j=1,2,\;\; t\in\mathbf{R}.$$

Then, $\theta_t=\mathrm{Ad}(u_1^t)*_B\mathrm{Ad}(u_2^t)\in\mathrm{Aut}(\widetilde{M})$, for all $t\in\mathbf{R}$, defines a pointwise $\|\cdot\|_2$-continuous one-parameter group of automorphisms which satisfies (a).

Let β be the unique automorphism of $\widetilde{M}$ satisfying $\beta|_M=\mathrm{id}_M$ and $\beta(u_j)=u_j^*$, $j=1,2$. Then β is clearly a period-2 automorphism and it satisfies (c) by definition. Also, for $x\in M=M_1*_B M_2$, we have

$$\begin{aligned}\beta\theta_t\beta(x)&=\beta\theta_t(x)=\beta(\mathrm{Ad}(\exp(\pi i t h_1))*\mathrm{Ad}(\exp(\pi i t h_2)))(x)\\&=(\mathrm{Ad}(\exp(-\pi i t h_1))*\mathrm{Ad}(\exp(-\pi i t h_2)))(x)=\theta_{-t}(x)\end{aligned}$$

for all $x\in M$. Similarly, for u_1 and u_2 we have

$$\beta\theta_t\beta(u_j)=\beta\theta_t(u_j^*)=\beta(u_j^*)=u_j=\theta_{-t}(u_j).$$

Since u_1, u_2 and M generate $\widetilde{M}$ as a von Neumann algebra, it follows that

$$\beta\theta_t\beta=\theta_{-t}\quad\text{for all } t. \qquad\square$$

2.3. Deformation by free products of multiples of the identity

Recall that if $\mathcal{H}_i^0=L^2(M_i)\ominus L^2(B)$ then we may decompose $L^2(M_1*_B M_2)$ in the usual way as

$$L^2(M_1*_B M_2)=L^2(B)\oplus\bigoplus_{n\geqslant 1}\bigoplus_{\substack{i_j\in\{1,2\}\\ i_1\neq i_2\neq i_3\neq\ldots\neq i_n}}\mathcal{H}_{i_1}^0\otimes_B\mathcal{H}_{i_2}^0\otimes_B\ldots\otimes_B\mathcal{H}_{i_n}^0.$$

For each $L\in\mathbf{N}$ we let $\widehat{E}_L$ be the projection onto the subspace

$$\bigoplus_{n\geqslant L}\bigoplus_{\substack{i_j\in\{1,2\}\\ i_1\neq i_2\neq i_3\neq\ldots\neq i_n}}\mathcal{H}_{i_1}^0\otimes_B\mathcal{H}_{i_2}^0\otimes_B\ldots\otimes_B\mathcal{H}_{i_n}^0.$$

Let $\{c_n\}_{n\geqslant 1}\subset[0,1)$ be such that $c_n\nearrow 1$. Then, by Lemma 2.1, we have the following result (see also [Pe]).

LEMMA 2.5. *The c.p. maps* $\phi_n = (c_n \mathrm{id}) *_B (c_n \mathrm{id})$ *give a deformation of the identity of* M. *Moreover*, ϕ_n *commutes with* $\widehat{E}_L$ *as operators on* $L^2(M)$ *and*

$$\|\phi_n \circ \widehat{E}_L(x)\|_2 \leqslant c_n^L \|x\|_2 \quad \textit{for all } n, L \in \mathbf{N} \textit{ and } x \in M.$$

Proof. This is trivial by the definitions. □

2.4. Deformation by subalgebras

For each $i \in \{1,2\}$, let $N_i^j \subset M_i$ be an increasing sequence of von Neumann subalgebras such that $B \subset N_i^1$ and

$$\overline{\bigcup_{j\geqslant 1} N_i^j} = M_i.$$

Let $E_i^j: M_i \to M_i$ be the conditional expectation onto N_i^j. Then $E^j = E_1^j *_B E_2^j$ gives a sequence of conditional expectations of M onto $N_1^j *_B N_2^j$, which, by Lemma 2.1, converges to the identity pointwise, i.e.

$$\overline{\bigcup_{j\geqslant 1} N_1^j *_B N_2^j} = M.$$

A particular case of such a deformation, which works whenever $B' \cap M_i$ is diffuse, is given by $N_i^j = p_i^j M_i p_i^j \oplus B(1-p_i^j)$, with $p_i^j \in \mathcal{P}(B' \cap M_i)$ satisfying $p_i^j \nearrow 1_{M_i}$, $i=1,2$. Indeed, this clearly implies that

$$\overline{\bigcup_{j\geqslant 1} (p_1^j M_1 p_1^j \oplus B(1-p_1^j)) *_B (p_2^j M_2 p_2^j \oplus B(1-p_2^j))} = M.$$

3. Deformation/rigidity arguments

In this section we investigate the effect that the deformations considered in §2 have on the relatively rigid subalgebras of $M_1 *_B M_2$. To this end, first recall from [P5, §4] that if $Q \subset M$ is a von Neumann subalgebra of the finite von Neumann algebra (M, τ), then $Q \subset M$ is called a *rigid inclusion* (or Q is a *relatively rigid subalgebra* of M) if any deformation of id_M by subunital subtracial c.p. maps $\{\phi_n\}_{n\geqslant 1}$ tends uniformly to id_Q on the unit ball of Q, i.e. $\lim_{n\to\infty} \sup\{\|\phi_n(y) - y\|_2 : y \in Q \text{ and } \|y\| \leqslant 1\} = 0$. This property does not in fact depend on the choice of the trace τ on M and can be given several other equivalent characterizations (see [P5] and [PeP]). The following result provides yet another characterization of relative rigidity, by showing that it is enough to consider deformations by unital tracial c.p. maps.

THEOREM 3.1. *Let N be a finite von Neumann algebra with countable decomposable center, and Q be a von Neumann subalgebra. Then, the following are equivalent*:

(i) *the inclusion $Q\subset N$ is rigid*;

(ii) *there exists a normal faithful tracial state τ on N such that for all $\varepsilon>0$ there exist $F=F(\varepsilon)\subset N$ finite and $\delta=\delta(\varepsilon)>0$ such that if $\phi\colon N\to N$ is a normal c.p. map with $\tau\circ\phi=\tau$, $\phi(1)=1$ and $\|\phi(x)-x\|_2\leqslant\delta$ for all $x\in F$, then*

$$\|\phi(b)-b\|_2\leqslant\varepsilon \quad \textit{for all } b\in Q \textit{ with } \|b\|\leqslant 1;$$

(iii) *condition* (ii) *is satisfied for any normal faithful tracial state τ on N.*

Proof. (i)⇒(iii) and (iii)⇒(ii) are both trivial and so it is enough to show (ii)⇒(i). That is, assuming that (ii) holds, we must show that the following condition holds:

$$\begin{aligned}&\text{for all } \varepsilon>0, \text{ there exist } F'=F'(\varepsilon)\subset N \text{ finite and } \delta'=\delta'(\varepsilon)>0 \text{ such that if}\\ &\phi\colon N\to N \text{ is a normal c.p. map with } \tau\circ\phi\leqslant\tau,\ \phi(1)\leqslant 1 \text{ and } \|\phi(x)-x\|_2\leqslant\delta'\\ &\text{for all } x\in F', \text{ then } \|\phi(b)-b\|_2\leqslant\varepsilon \text{ for all } b\in Q \text{ with } \|b\|\leqslant 1.\end{aligned} \tag{3.1}$$

By [PeP, Lemma 3], we may also assume that ϕ is symmetric in the above condition, i.e. $\tau(\phi(x)y)=\tau(x\phi(y))$ for all $x,y\in N$. Let $F=F\big(\frac{1}{2}\varepsilon\big)$, and $\delta=\delta\big(\frac{1}{2}\varepsilon\big)$ be given from (ii). Let $F'=F\cup\{1\}$ and $\delta'=\min_{x\in F}\big\{\frac{1}{2}\delta, \delta/(8\|x\|^2+1), \frac{1}{8}\varepsilon^2\big\}$, suppose that $\phi\colon N\to N$ is a normal symmetric c.p. map with $\tau\circ\phi\leqslant\tau$, $\phi(1)\leqslant1$ and $\|\phi(x)-x\|_2\leqslant\delta'$ for all $x\in F'$. Let $a=\phi(1)=d\tau\circ\phi/d\tau$ and define ϕ' by $\phi'(x)=\phi(x)+(1-a)^{1/2}x(1-a)^{1/2}$. Then, ϕ' is a normal c.p. map with $\phi'(1)=1$. Moreover, as ϕ is symmetric, so is ϕ' and hence $\tau\circ\phi'=\tau$.

Also, it follows that for each $x\in F$ we have

$$\begin{aligned}\|\phi'(x)-x\|_2 &\leqslant \|\phi(x)-x\|_2+\|(1-a)^{1/2}x(1-a)^{1/2}\|_2\\ &\leqslant \|\phi(x)-x\|_2+\|(1-a)x(1-a)\|_2^{1/2}\|x\|^{1/2}\\ &\leqslant \|\phi(x)-x\|_2+\|1-a\|_2\|x\|\\ &\leqslant \delta.\end{aligned}$$

Hence, by (ii), we have $\|\phi'(b)-b\|_2\leqslant\frac{1}{2}\varepsilon$ for all $b\in Q$ with $\|b\|\leqslant1$. Thus

$$\begin{aligned}\|\phi(b)-b\|_2 &\leqslant \|\phi'(b)-b\|_2+\|(1-a)^{1/2}x(1-a)^{1/2}\|_2\\ &\leqslant \|\phi'(b)-b\|_2+\|1-a\|_2\leqslant\varepsilon\end{aligned}$$

for all $b\in Q$ with $\|b\|\leqslant1$. □

COROLLARY 3.2. *Let (M_1,τ_1) and (M_2,τ_2) be finite von Neumann algebras and let $Q\subset M_1$ be a von Neumann subalgebra such that the inclusion $Q\subset M_1*M_2$ is rigid. Then, the inclusion $Q\subset M_1$ is rigid.*

Proof. Let $\varepsilon>0$ be given. Since $Q\subset M_1*M_2$ is rigid, we can find $F=F(\varepsilon)\subset M$ finite and $\delta=\delta(\varepsilon)>0$ satisfying condition (ii) of Theorem 3.1. By Lemma 2.1, there exists $F'\subset M_1$ finite and $\delta'>0$ such that if $\phi: M_1\to M_1$ is a normal unital tracial c.p. map such that $\|\phi(x)-x\|_2\leqslant\delta'$ for all $x\in F'$, then $\|\phi*\mathrm{id}_{M_2}(x)-x\|_2\leqslant\delta$ for all $x\in F$. Hence, by our choice of F and δ, we have that $\|\phi(b)-b\|_2=\|\phi*\mathrm{id}_{M_2}(b)-b\|_2\leqslant\varepsilon$ for all $b\in Q$ with $\|b\|\leqslant1$. Thus, by Theorem 3.1, it follows that $Q\subset M_1$ is a rigid inclusion. □

We will now use the deformation in Lemma 2.4 to exploit the relative rigidity of subalgebras $Q\subset M_1*_B M_2\subset \widetilde{M}_1*_B\widetilde{M}_2$. This "deformation/rigidity" argument is inspired by [P7, Lemmas 4.3–4.8] and [P8, I, §4].

PROPOSITION 3.3. *Let (M_1,τ_1) and (M_2,τ_2) be finite von Neumann algebras with a common von Neumann subalgebra $B\subset M_i$, $i=1,2$, such that $\tau_1|_B=\tau_2|_B$. Let $M=M_1*_B M_2$, $\widetilde{M}_i=M_i*_B(B\overline{\otimes}L(\mathbf{Z}))$, $i=1,2$, and $\widetilde{M}=\widetilde{M}_1*_B\widetilde{M}_2=M*_B(B\overline{\otimes}L(\mathbf{F}_2))$, as in Lemma 2.4. Let $\theta=\mathrm{Ad}(u_1)*\mathrm{Ad}(u_2)\in\mathrm{Aut}(\widetilde{M})$, where $u_1,u_2\in L(\mathbf{F}_2)$ are the canonical generators of $L(\mathbf{F}_2)$, as in Lemma 2.4. Let $Q\subset M$ be a von Neumann subalgebra such that $Q\subset\widetilde{M}$ is a rigid inclusion and assume that no corner of Q can be embedded into B inside M, i.e. $Q'\cap\langle M,B\rangle$ contains no non-zero finite projections. Then, there exists a non-zero partial isometry $v\in\widetilde{M}$ such that $vy=\theta(y)v$ for all $y\in Q$.*

Proof. Since $Q\subset\widetilde{M}$ is rigid, there exist $F\subset\widetilde{M}$ finite and $\delta>0$ such that if $\phi:\widetilde{M}\to\widetilde{M}$ is a subunital subtracial c.p. map with $\|\phi(x)-x\|_2\leqslant\delta$ for all $x\in F$, then $\|\phi(u)-u\|_2\leqslant\frac12$ for all $u\in\mathcal{U}(Q)$. Using the continuity of $t\mapsto\theta_t$, we can find $n\geqslant1$ such that $\|\theta_{1/2^n}(x)-x\|\leqslant\delta$ for all $x\in F$, which implies $\|\theta_{1/2^n}(u)-u\|\leqslant\frac12$ for all $u\in\mathcal{U}(Q)$.

Now, let a be the unique element of minimal $\|\cdot\|_2$-norm in

$$K=\overline{\mathrm{co}}^{\,w}\{\theta_{1/2^n}(u)u^*:u\in\mathcal{U}(Q)\}.$$

From $\|\theta_{1/2^n}(u)u^*-1\|_2\leqslant\frac12$ for all $u\in\mathcal{U}(Q)$, we get that $\|a-1\|_2\leqslant\frac12$, so $a\neq0$. Since $\theta_{1/2^n}(u)Ku^*=K$ and $\|\theta_{1/2^n}(u)au^*\|_2=\|a\|_2$ for all $u\in\mathcal{U}(Q)$, we deduce, using the uniqueness of a, that $au=\theta_{1/2^n}(u)a$ for all $u\in\mathcal{U}(Q)$. Using standard arguments, we can replace a by the partial isometry in its polar decomposition, thus getting a non-zero partial isometry $v\in M$ such that $vu=\theta_{1/2^n}(u)v$ for all $u\in\mathcal{U}(Q)$.

In what follows we show by induction that for any $k\geqslant0$ there exists a non-zero partial isometry $v_k\in\widetilde{M}$ such that

$$v_k u=\theta_{1/2^{n-k}}(u)v_k \quad \text{for all } u\in\mathcal{U}(Q), \tag{3.2}$$

which for $k=n$ gives the conclusion. Since we have already constructed v_0, we only need to construct v_{k+1}, given v_k. Note first that if v_k satisfies (3.2) then $v_k^*v_k\in Q'\cap\widetilde{M}$ and $v_kv_k^*\in\theta_{1/2^{n-k}}(Q)'\cap\widetilde{M}$.

By applying the automorphism β to the equality (3.2) and using the properties of β, we get

$$\begin{aligned}\beta(v_k)u &= \beta(v_k)\beta(u) = \beta(\theta_{1/2^{n-k}}(u))\beta(v_k) \\ &= \theta_{-1/2^{n-k}}(\beta(u))\beta(v_k) = \theta_{-1/2^{n-k}}(u)\beta(v_k) \quad \text{for all } u \in \mathcal{U}(Q).\end{aligned} \tag{3.3}$$

By replacing u by u^* and taking conjugates in (3.3), we obtain

$$u\beta(v_k^*) = \beta(v_k^*)\theta_{-1/2^{n-k}}(u) \quad \text{for all } u \in \mathcal{U}(Q),$$

which combined with (3.1) gives

$$v_k\beta(v_k^*)\theta_{-1/2^{n-k}}(u) = v_k u\beta(v_k^*) = \theta_{1/2^{n-k}}(u)v_k\beta(v_k^*) \quad \text{for all } u \in \mathcal{U}(Q). \tag{3.4}$$

By applying $\theta_{1/2^{n-k}}$ to (3.4), we further get, for $u\in\mathcal{U}(Q)$, the identity

$$\theta_{1/2^{n-k}}(v_k\beta(v_k^*))u = \theta_{1/2^{n-k-1}}(u)\theta_{1/2^{n-k}}(v_k\beta(v_k^*)). \tag{3.5}$$

Since no corner of Q can be embedded into B inside M, we can apply Theorem 1.1 to conclude that $Q'\cap\widetilde{M}\subset M$. Thus, since $v_k^*v_k\in Q'\cap\widetilde{M}$ and $M\subset\widetilde{M}^{\beta}$, we get $\beta(v_k^*v_k)=v_k^*v_k$, implying that $v_k\beta(v_k^*)$ is a partial isometry with the same left support as v_k. Thus, by (3.5), $w=\theta_{1/2^{n-k}}(v_k\beta(v_k^*))$ is a non-zero partial isometry satisfying $wu=\theta_{1/2^{n-k-1}}(u)w$ for all $u\in\mathcal{U}(Q)$, and the inductive step follows. □

Proposition 3.4. *As in* §2.3, *denote by* $\widehat{E}_L$ *the orthogonal projection of* $L^2(M)$ *onto the Hilbert space spanned by reduced words of length* $\geqslant L$. *If* $Q\subset M_1 *_B M_2$ *is a rigid inclusion, then for any* $\varepsilon>0$, *there exists* $L\in\mathbf{N}$ *such that* $\|\widehat{E}_L(x)\|_2<\varepsilon$ *for all* $x\in Q$ *with* $\|x\|\leqslant 1$.

Proof. Let $\phi_n: M\to M$ be as in Lemma 2.5, for some $c_n\nearrow 1$, then, by Lemma 2.1, we have $\lim_{n\to\infty}\|\phi_n(x)-x\|_2=0$ for all $x\in M$. Thus, since $Q\subset M$ is rigid, there exists $l\in\mathbf{N}$ such that $\|\phi_l(x)-x\|_2<\frac{1}{2}\varepsilon$ for all $x\in Q$ with $\|x\|\leqslant 1$. Let $L\in\mathbf{N}$ be such that $c_l^L<\frac{1}{2}\varepsilon$. Then

$$\|\widehat{E}_L(x)\|_2 \leqslant \|\widehat{E}_L(x-\phi_l(x))\|_2 + \|\phi_l\circ\widehat{E}_L(x)\|_2 \leqslant \|x-\phi_l(x)\|_2 + c_l^L\|x\|_2 < \varepsilon$$

for all $x\in Q$ with $\|x\|\leqslant 1$. □

4. Existence of intertwining bimodules

In the previous section we saw that a relatively rigid subalgebra Q of a finite AFP von Neumann algebra $M=M_1*_B M_2$ can be "located" by certain c.p. deformations of id_M. In this section we will use this information to prove that $L^2(M)$ must contain non-trivial Hilbert bimodules intertwining Q into either M_1 or M_2. The rather long and technical proof will proceed by contradiction, assuming that Q cannot be intertwined in neither M_1 nor M_2, inside M. We first show that this implies that Q cannot be intertwined in neither M_1 nor M_2 inside $\widetilde{M}=(M_1*_B M_2)*_B(B\overline{\otimes}L(\mathbf{F}_2))$ either. By Proposition 3.4 and [P8, I, Corollary 2.3], this shows that Q must contain "at infinity" elements with uniformly bounded free length and at least two "very large letters" in M_1, M_2 or $L(\mathbf{F}_2)$. This will be shown to contradict Proposition 3.3.

To "measure" the letters in M_i, we will need these algebras to have nice orthonormal bases over B, in the following sense.

Definition 4.1. Let (M,τ) be a separable finite von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. A sequence of elements $\{\eta_n\}_{n\geqslant 0}\subset M$ satisfying the conditions $\eta_0=1$, $E_B(\eta_i^*\eta_j)=\delta_{ij}$ for all i and j, and $\sum_{n=0}^{\infty}\eta_n B$ dense in $L^2(M,\tau)$, is called a *bounded homogeneous orthonormal basis* (BHOB) of M over B. An inclusion $B\subset M$ having a BHOB is said to be *homogeneous*.

LEMMA 4.2. *Let (M,τ) be a separable finite von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. Assume that one of the following conditions holds true*:

(a) $B=\mathbf{C}\,1$;

(b) $B=A\subset M$ *is Cartan* (*i.e.* B *is maximal abelian and regular in* M) *and* M *is of type* II_1;

(c) $B=N\subset M$ *is an irreducible inclusion of* II_1 *factors and* B *is regular in* M.

Then $B\subset M$ is homogeneous, *moreover in both cases* (b) *and* (c), *M has a* BHOB *made of unitary elements in $\mathcal{N}_M(B)$.*

Proof. Case (a) is clear by the Gram–Schmidt algorithm. Case (c) is trivial once we notice that such $N\subset M$ is a crossed product inclusion $N\subset M=N\rtimes_{\sigma,v}G$ for some cocycle action σ of a discrete countable group G on N, with 2-cocycle v. Indeed, in this case the canonical unitary elements $\{u_g\}_g\subset M$ implementing the action σ provide a BHOB of M over N.

To prove case (b), we first show that given any $n\geqslant 1$ and any $v_0=1, v_1, ..., v_{n-1}\in\mathcal{N}_M(A)$, with $E_A(v_i^*v_j)=0$ for $0\leqslant i,j<n$, $i\neq j$, and $v_n'\in\mathcal{GN}_M(A)$, with $(v_n')^*v_n'\neq 1$, there exists a non-zero $v\in\mathcal{GN}_M(A)$ such that $v^*v_n'=0$, $v_n'v^*=0$ and $E_A(v^*v_j)=0$ for all $1\leqslant j\leqslant n-1$. Indeed, first note that, by [D], there exists $u\in\mathcal{N}_M(A)$ such that $v_n'=u(1-p)$, where $p=1-(v_n')^*v_n'\in\mathcal{P}(A)$. Since $Ap\subset pMp$ is Cartan with pMp of type II_1, it follows that

there exists $w\in\mathcal{N}_{pMp}(Ap)$ such that $w\notin\sum_{i=0}^{n-1} pu^*v_iAp$ (or else, it would follow that pMp would be finite-dimensional over Ap, and thus of type I, implying that M has a type-I direct summand, a contradiction). Then $w_i=pu^*v_ip\in\mathcal{GN}(Ap)$, $0\leqslant i\leqslant n-1$, satisfy

$$0\neq w-\sum_{i=0}^{n-1} w_iE_A(w_i^*w)\in\mathcal{GN}(A)$$

by [D]. But then $v=u\big(w-\sum_{i=0}^{n-1} w_iE_A(w_i^*w)\big)$ clearly satisfies all required conditions.

Now, to finish the proof of (b), let $\{u_n\}_{n\geqslant 0}\subset\mathcal{N}_M(A)$ be a sequence of unitary elements normalizing A, dense in $\mathcal{N}_M(A)$ in the $\|\cdot\|_2$-norm, with $u_0=1$ and with each u_n appearing with infinite multiplicity. It is sufficient to construct a sequence $\{v_n\}_{n\geqslant 0}\subset\mathcal{N}_M(A)$ such that for all $m\geqslant 0$ we have

$$E_A(v_i^*v_j)=\delta_{ij} \text{ for all } 0\leqslant i,j\leqslant m \quad\text{and}\quad u_m\in\sum_{i=0}^{m} v_iA. \tag{4.1}$$

Assume that we have constructed $v_0=1, v_1, ..., v_{n-1}$ satisfying (4.1) for $m=n-1$, for some $n\geqslant 1$. Let $v_n''=u_n-\sum_{i=0}^{n-1} v_iE_A(v_i^*u_n)\in\mathcal{GN}_M(A)$. Let $v_n'\in\mathcal{GN}_M(A)$ be maximal with the properties $v_n'(v_n'')^*v_n''=v_n''$ and $v_n'\perp\sum_{i=0}^{n-1} v_iA$. By applying the first part of the proof to $v_0, v_1, ..., v_{n-1}, v_n'$ and using the maximality, it follows that v_n' is a unitary element. But then $v_n=v_n'$ clearly satisfies (4.1). □

THEOREM 4.3. *Let (M_1,τ_1) and (M_2,τ_2) be finite von Neumann algebras and $B\subset M_i$, $i=1,2$, be a common von Neumann subalgebra such that $\tau_1|_B=\tau_2|_B$. Suppose that the inclusions $B\subset M_1$ and $B\subset M_2$ are homogeneous. Let Q be a von Neumann subalgebra of $M=M_1*_BM_2$ such that the inclusion $Q\subset M$ is rigid. Then, for either $i=1$ or $i=2$, there exists a non-zero projection f in $Q'\cap\langle M,e_{M_i}\rangle$ of finite trace* $\mathrm{Tr}=\mathrm{Tr}_{\langle M,e_{M_i}\rangle}$.

Proof. By taking spectral projections, it is sufficient to show that there exists $j\in\{1,2\}$ and $a\in Q'\cap\langle M,e_{M_i}\rangle$, with $0\leqslant a\leqslant 1$ and $0\neq\mathrm{Tr}(a)<\infty$. Assume, by contradiction, that there are no such elements in $Q'\cap\langle M,e_{M_i}\rangle$, $i=1,2$. If we identify Q with the diagonal subalgebra $\{x\oplus x:x\in Q\}$ in $M\oplus M$, then this is equivalent to saying that Q cannot be intertwined into $M_1\oplus M_2$ inside $M\oplus M$. By [P8, I, Corollary 2.3], this implies the following.

Fact 1. For all $\varepsilon>0$ and for all $y_1, ..., y_n\in M$, there exists $w\in\mathcal{U}(Q)$ such that

$$\|E_{M_i}(y_jwy_k^*)\|_2<\varepsilon \quad\text{for } i\in\{1,2\} \text{ and } j,k\in\{1,...\,n\}.$$

For the next part of the proof, we need to introduce some notation. Thus, for $i\in\{1,2\}$, let $\{\xi_i^j\}_{j\geqslant 0}\subset M_i$ be a BHOB of M_i over B. Also take $\{\xi_3^j\}_{j\geqslant 0}=\{u_g\}_{g\in\mathbf{F}_2}\subset L(\mathbf{F}_2)$ such that $\xi_3^0=u_e=1$. Then using the notation $\widetilde{M}=M*_B(B\otimes L(\mathbf{F}_2))$ as in Proposition 3.3 a simple exercise shows that

$$\beta=\{1\}\cup\{\xi_{i_1}^{j_1}\dots\xi_{i_n}^{j_n}:n\in\mathbf{N},\ i_k\in\{1,2,3\},\ j_k\geqslant 1 \text{ and } i_1\neq i_2\neq i_3\neq\dots\neq i_n\}$$

is a BHOB of $\widetilde{M}$ over B.

For each $n_0\in\mathbf{N}$, let

$$S_{n_0}=\{1\}\cup\{\xi_{i_1}^{j_1}\dots\xi_{i_n}^{j_n}:n\leqslant n_0,\ i_k\in\{1,2\},\ 1\leqslant j_k\leqslant n_0 \text{ and } i_1\neq i_2\neq i_3\neq\dots\neq i_n\},$$
$$\widetilde{S}_{n_0}=\{1\}\cup\{\xi_{i_1}^{j_1}\dots\xi_{i_n}^{j_n}:n\leqslant n_0,\ i_k\in\{1,2,3\},\ 1\leqslant j_k\leqslant n_0 \text{ and } i_1\neq i_2\neq i_3\neq\dots\neq i_n\}.$$

Also, for $i\in\{1,2\}$, let $\widetilde{S}_{n_0}^{R,i}$ (resp. $\widetilde{S}_{n_0}^{L,i}$) be the subset of $\widetilde{S}_{n_0}$ consisting of 1 and the vectors in $\widetilde{S}_{n_0}$ such that $i_n\neq i$ (resp. $i_1\neq i$). Note that, if $\zeta,\zeta'\in\widetilde{S}_{n_0}^{R,i}$ and $x\in M_i$ is such that $E_B(x)=0$, then $\|\zeta' x\zeta^*\|_2=\|x\|_2$. Also if $\zeta\in\widetilde{S}_{n_0}$ and $b\in B$ then $\|\zeta b\|_2=\|b\|_2$.

We now strengthen Fact 1 so that the elements $y_1,\dots,y_n$ may be taken in $\widetilde{M}$.

Fact 2. For all $\varepsilon>0$ and all $y_1,\dots,y_n\in\widetilde{M}$, there exists $w\in\mathcal{U}(Q)$ such that

$$\|E_{M_i}(y_j^*wy_k)\|_2<\varepsilon \quad \text{for } i\in\{1,2\} \text{ and } j,k\in\{1,\dots n\}.$$

As our basis for $\widetilde{M}$ is made up of bounded vectors, by first approximating the y_k's on the right of w and then approximating the y_j's on the left of w, we may assume that all of the y_j's are basis elements, and then use the triangle inequality to deduce the general case. Also, as E_{M_i} is B-bimodular, it is enough to suppose that the y_j's all lie in β. Thus, we only need to show that for all $\varepsilon>0$ and for all $n_0\in\mathbf{N}$, there exists $w\in\mathcal{U}(Q)$ such that $\|E_{M_i}(\zeta^*w\zeta')\|_2<\varepsilon$ for all $\zeta,\zeta'\in\widetilde{S}_{n_0}$.

To prove this, we first use Fact 1 to deduce that there exists $w\in\mathcal{U}(Q)$ such that $\|E_{M_i}(\zeta_0^*w\zeta_0')\|_2<\varepsilon$ for all $\zeta_0,\zeta_0'\in S_{n_0}$. Then, if $\zeta,\zeta'\in\widetilde{S}_{n_0}$, we may find $\zeta_1,\zeta_1'\in\widetilde{S}_{n_0}^{L,1}\cap\widetilde{S}_{n_0}^{L,2}$ and $\zeta_0,\zeta_0'\in S_{n_0}$ such that $\zeta=\zeta_0\zeta_1$ and $\zeta'=\zeta_0'\zeta_1'$. If $\zeta_1=\zeta_1'=1$, then from the above we have that $\|E_{M_i}(\zeta^*w\zeta')\|_2=\|E_{M_i}(\zeta_0^*w\zeta_0')\|_2<\varepsilon$. Otherwise, we have

$$\|E_{M_i}(\zeta^*w\zeta')\|_2\leqslant\|E_M(\zeta^*w\zeta')\|_2\leqslant\|\zeta_1^*E_B(\zeta_0^*w\zeta_0')\zeta_1'\|_2\leqslant\|E_{M_1}(\zeta_0^*w\zeta_0')\|_2<\varepsilon.$$

This proves Fact 2.

We continue by showing that there are elements of $\mathcal{U}(Q)$ ("at infinity") which are almost orthogonal to the subspaces having at most one "large letter" from $M_1\cup M_2$. Specifically, let $\mathcal{H}_{n_0}=\overline{\mathrm{sp}}(\widetilde{S}_{n_0}M_1\widetilde{S}_{n_0}^*\cup\widetilde{S}_{n_0}M_2\widetilde{S}_{n_0}^*)\subset L^2(\widetilde{M})$.

Let $\zeta_1,\zeta_1'\in\widetilde{S}_{n_0}^{R,i}$ and $\zeta_2,\zeta_2'\in\widetilde{S}_{n_0}^{R,j}$. Then, for all $b_1,b_2\in B$ and for all $K,L>n_0$, we have that $E_B((\zeta_1'\xi_i^K b_1\zeta_1^*)^*(\zeta_2'\xi_j^L b_2\zeta_2^*))=\delta_{\zeta_1'\zeta_2'}E_B((\xi_i^K b_1\zeta_1^*)^*(\xi_j^L b_2\zeta_2^*))$, and we also have that $E_B((\xi_j^L b_2\zeta_2^*)(\xi_i^K b_1\zeta_1^*)^*)=\delta_{\zeta_1\zeta_2}E_B((\xi_j^L b_2)(\xi_i^K b_1)^*)$. Hence,

$$\begin{aligned}\langle\zeta_1'\xi_i^K b_1\zeta_1^*,\zeta_2'\xi_j^L b_2\zeta_2^*\rangle &= \tau\circ E_B((\zeta_1'\xi_i^K b_1\zeta_1^*)^*(\zeta_2'\xi_j^L b_2\zeta_2^*))\\ &= \delta_{\zeta_1'\zeta_2'}\tau\circ E_B((\xi_i^K b_1\zeta_1^*)^*(\xi_j^L b_2\zeta_2^*))\\ &= \delta_{\zeta_1'\zeta_2'}\delta_{\zeta_1\zeta_2}\delta_{ij}\langle\xi_i^K b_1,\xi_j^L b_2\rangle.\end{aligned}$$

Also, if $\zeta\in\widetilde{S}_{n_0}$ and $b\in B$, then $\langle\zeta_1'\xi_i^K b_1\zeta_1^*,\zeta b\rangle=\tau\circ E_B((\zeta b)^*\zeta_1'\xi_i^K b_1\zeta_1^*)=0$.

Hence, we have the following direct sum of orthogonal subspaces of $\mathcal{H}_{n_0}$:

$$\mathcal{H}_{n_0}'=\bigoplus_{\zeta\in\widetilde{S}_{n_0}}\mathcal{H}_\zeta\oplus\bigoplus_{\substack{i\in\{1,2\}\\ \zeta,\zeta'\in\widetilde{S}_{n_0}^{R,i}}}\mathcal{H}_{i,\zeta',\zeta},$$

where $\mathcal{H}_\zeta=\overline{\zeta B}$ and $\mathcal{H}_{i,\zeta',\zeta}=\overline{\mathrm{sp}}\;\zeta'\{\xi_i^K\}_{K>n_0}B\zeta^*$.

Since $\{\xi_3^j\}_{j\geqslant 0}=\{u_g\}_{g\in\mathbf{F}_2}$, we may find $m_0>2n_0$ such that

$$\{\xi_3^j\}_{j=1}^{n_0}\{\xi_3^j\}_{1\leqslant j\leqslant n_0}^*\subset\{\xi_3^j\}_{j=1}^{m_0}.$$

We will show that $\mathcal{H}_{n_0}\subset\mathcal{H}_{m_0}'$.

Let $\mathcal{K}_0$ be the Hilbert space generated by all vectors of the form $\eta' b\eta^*$, where $b\in B$, $\eta=\xi_{i_1}^{j_1}\dots\xi_{i_n}^{j_n}$, $\eta'=\xi_{k_1}^{l_1}\dots\xi_{k_m}^{l_m}$, $i_n\neq k_m$, $n+m\leqslant m_0$, and $j_p,l_p\leqslant m_0$ for all p. If $\zeta,\zeta'\in\widetilde{S}_{n_0}$, and $x\in M_i\ominus B$, then we may find $\zeta_1,\zeta_1'\in\widetilde{S}_{n_0}^{R,i}$ and $\zeta_0,\zeta_0'\in\widetilde{S}_{n_0}\cap M_i$ such that $\zeta=\zeta_1\zeta_0$, and $\zeta'=\zeta_1'\zeta_0'$. We have that if P is the projection onto the subspace $\overline{\mathrm{sp}}\{\xi_i^j\}_{j=1}^{m_0}B$ then $\zeta_1'(\zeta_0'x\zeta_0^*-P(\zeta_0'x\zeta_0^*)-E_B(\zeta_0'x\zeta_0^*))\zeta_1^*\in\mathcal{H}_{m_0}'$, $\zeta_1'P(\zeta_0'x\zeta_0^*)\zeta_1^*\in\mathcal{K}_0$, and

$$\zeta_1'E_B(\zeta_0'x\zeta_0^*)\zeta_0^*\in\widetilde{S}_{n_0}B\widetilde{S}_{n_0}^*.$$

If $\zeta,\zeta'\in\widetilde{S}_{n_0}$ and $b\in B$ then, if ζ and ζ' do not end with a letter in the same algebra, we have that $\zeta' b\zeta^*\in\mathcal{K}_0$. Also, if both ζ and ζ' end with something in $\{\xi_3^j\}_{j\geqslant 0}$, then, as $L(\mathbf{F}_2)$ commutes with B and since $\{\xi_3^j\}_{j=1}^{n_0}\{\xi_3^j\}_{1\leqslant j\leqslant n_0}^*\subset\{\xi_3^j\}_{j=1}^{m_0}$, we may rewrite $\zeta' b\zeta^*$ to see that it is in $\mathcal{K}_0$, otherwise as above we may find $\zeta_1,\zeta_1'\in\widetilde{S}_{n_0}^{R,i}$ and $\zeta_0,\zeta_0'\in\widetilde{S}_{n_0}\cap M_i$ such that $\zeta=\zeta_1\zeta_0$ and $\zeta'=\zeta_1'\zeta_0'$, and then decompose $\zeta' b\zeta^*$ into parts in $\mathcal{H}_{m_0}'$, $\mathcal{K}_0$ and something in $\widetilde{S}_{n_0}B\widetilde{S}_{n_0}$ with shorter words. Hence, by induction, to show that $\mathcal{H}_{n_0}\subset\mathcal{H}_{m_0}'$ it is enough to show that $\mathcal{K}_0\subset\mathcal{H}_{m_0}'$.

Let η and η' be as above, and take $b\in B$. If $n=0$, i.e. $\eta=1$, then $\eta' b\eta^*=\eta' b\in\mathcal{H}_{m_0}$. Also, if $i_n=3$, then, since $L(\mathbf{F}_2)$ commutes with B and $(\xi_{i_n}^{j_n})^*\in\{\xi_3^j\}_{j=1}^{m_0}$, we can rewrite $\eta' b\eta^*$ so that η and η' are still in $\widetilde{S}_{m_0}$ but such that the length of η is shorter. If

$i_n = i \in \{1,2\}$ then, since $E_B((b\xi_{i_n}^{j_n})^*) = 0$, as above we may replace $(b\xi_{i_n}^{j_n})^*$ by $P((b\xi_{i_n}^{j_n})^*)$ and $(b\xi_{i_n}^{j_n})^* - P((b\xi_{i_n}^{j_n})^*)$, and in so doing rewrite $\eta' b \eta^*$ as a sum of things where the word on the right has shorter length plus something in $\mathcal{H}'_{m_0}$. Thus, by induction, we have shown that $\mathcal{K}_0 \subset \mathcal{H}'_{m_0}$ and so $\mathcal{H}_{n_0} \subset \mathcal{H}'_{m_0}$.

Let P_{n_0} be the orthogonal projection of $L^2(\widetilde{M})$ onto $\mathcal{H}_{n_0}$.

Fact 3. For all $\varepsilon > 0$ and for all $y_1, ..., y_n \in \widetilde{M}$, there exists $w \in \mathcal{U}(Q)$ such that $\|P_{n_0}(y_j^* w y_k)\|_2^2 < \varepsilon$ for all j and k.

Let m_0 be as above. Then, by Fact 2, there exists $w \in \mathcal{U}(Q)$ such that

$$\|E_{M_j}(\zeta^* y_k^* w^* y_j \zeta')\|_2^2 < \tfrac{1}{3}\varepsilon |\widetilde{S}_{m_0}|^2$$

for all $j, k \leqslant n$ and for all $\zeta, \zeta' \in \widetilde{S}_{m_0}$.

Thus, for all $\zeta \in \widetilde{S}_{m_0}$ and all $b \in B$, we have

$$|\langle \zeta b, y_j^* w y_k \rangle|^2 = |\tau(E_B(y_k^* w^* y_j \zeta) b)|^2 \leqslant \|E_{M_1}(y_k^* w^* y_j \zeta)\|_2^2 \, \|b\|_2^2 < \tfrac{1}{3}\varepsilon |\widetilde{S}_{m_0}|^2 \|\zeta b\|_2^2,$$

and so

$$\|P_{\mathcal{H}_\zeta}(y_j^* w y_k)\|_2^2 < \tfrac{1}{3}\varepsilon |\widetilde{S}_{m_0}|^2.$$

Also, for $i \in \{1,2\}$, all $\zeta, \zeta' \in \widetilde{S}_{m_0}^{R,i}$ and all $\xi \in \overline{\mathrm{sp}}\{\xi_i^K\}_{K > m_0} B$, we have

$$\begin{aligned} |\langle \zeta' \xi \zeta^*, y_j^* w y_k \rangle|^2 = |\tau(E_{M_i}(\zeta^* y_k^* w^* y_j \zeta')\xi)|^2 &\leqslant \|E_{M_i}(\zeta^* y_k^* w^* y_j \zeta')\|_2^2 \, \|\xi\|_2^2 \\ &< \tfrac{1}{3}\varepsilon |\widetilde{S}_{m_0}|^2 \|\zeta' \xi \zeta^*\|_2^2, \end{aligned}$$

and so

$$\|P_{\mathcal{H}_{i,\zeta',\zeta}}(y_j^* w y_k)\|_2^2 < \tfrac{1}{3}\varepsilon |\widetilde{S}_{m_0}|^2.$$

Therefore

$$\begin{aligned} \|P_{n_0}(y_j^* w y_k)\|_2^2 &\leqslant \|P_{\mathcal{H}'_{m_0}}(y_j^* w y_k)\|_2^2 \\ &= \sum_{\zeta \in \widetilde{S}_{m_0}} \|P_{\mathcal{H}_\zeta}(y_j^* w y_k)\|_2^2 + \sum_{i=1}^{2} \sum_{\zeta' \in \widetilde{S}_{m_0}^{R,i}} \sum_{\zeta \in \widetilde{S}_{m_0}^{L,i}} \|P_{\mathcal{H}_{i,\zeta',\zeta}}(y_j^* w y_k)\|_2^2 < \varepsilon \end{aligned}$$

for each $j, k \leqslant n$. Thus, we have proved Fact 3.

Next we note that if $Q' \cap \langle M, e_B \rangle$ contains a non-zero finite-trace projection, then so does $Q' \cap \langle M, e_{M_1} \rangle$ and so, by our assumption, we are in the position of applying Proposition 3.3.

Hence, there exists a non-zero partial isometry $v\in\widetilde{M}$ such that $vy=\theta(y)v$ for all $y\in Q$. Let $\varepsilon>0$ and take n_1 large enough so that there exists $v_0^*\in\operatorname{sp}\widetilde{S}_{n_1}B$ satisfying $\|v-v_0\|_2<\frac{1}{6}\varepsilon$. Using the notation in Proposition 3.4, let $L\in\mathbf{N}$ be such that

$$\|\widehat{E}_L(x)\|_2 < \tfrac{1}{12}\varepsilon\|v_0\|^2 \quad \text{for all } x\in Q \text{ with } \|x\|\leqslant 1,$$

also let $n_0=n_1+3L$ and let m_0 be as above so that $\mathcal{H}_{n_0}\subset\mathcal{H}'_{m_0}$. Then, as our basis is bounded, we have that v_0 is bounded so, by Fact 3, there exists $w\in\mathcal{U}(Q)$ such that $\|P_{m_0}(v_0w)\|_2<\frac{1}{4}\varepsilon\|v_0\|$. Take $w_0\in M$ such that $\widehat{E}_L(w_0)=0$ and $\|w-w_0\|_2<\frac{1}{6}\varepsilon\|v_0\|^2$.

Let $\mathcal{K}\subset L^2(\widetilde{M})$ be the right Hilbert B-module generated by 1 and all the vectors $\xi_{i_1}^{j_1}\dots\xi_{i_n}^{j_n}\in\beta$ such that if $i_{k_0}=3$ for some $k_0\leqslant n$, then $j_k\leqslant n_1$ for all $k<k_0$. Note that $\mathcal{K}M\subset\mathcal{K}$ and so, since $w_0\in M$ and $v_0\in\mathcal{K}$, we have that $v_0w_0\in\mathcal{K}$.

Let $\zeta,\zeta'\in\widetilde{S}_{m_0}^{R,i}$, $K>m_0$, and $b\in B$, then since $K>n_1$ we have that

$$P_{\mathcal{K}}(\zeta'\xi_i^K b\zeta) = \begin{cases} \zeta'\xi_i^K b\zeta, & \text{if } \zeta\in M \text{ and } \zeta'\in\mathcal{K},\\ 0, & \text{otherwise.}\end{cases}$$

Hence, $P_{\mathcal{K}}(\mathcal{H}_{n_0})\subset P_{\mathcal{K}}(\mathcal{H}'_{m_0})\subset\mathcal{H}'_{m_0}\subset\mathcal{H}_{m_0}$.

Let us write v_0^* and w_0 in β as

$$v_0^* = \sum_{\xi_v\in\beta}\xi_v b_{\xi_v} \quad \text{and} \quad w_0 = \sum_{\xi_w\in\beta}\xi_w b_{\xi_w}.$$

Take $\xi_v,\xi_w\in\beta$ such that $b_{\xi_v}\neq 0$ and $b_{\xi_w}\neq 0$, where $\xi_v=\xi_{i_1}^{j_1}\dots\xi_{i_n}^{j_n}$ and $\xi_w=\xi_{k_1}^{l_1}\dots\xi_{k_m}^{l_m}$. Thus,

$$\theta(\xi_w b_{\xi_w})(\xi_v b_{\xi_v})^* = (u_{k_1}\xi_{k_1}^{l_1}u_{k_1}^*\dots u_{k_m}\xi_{k_m}^{l_m}u_{k_m}^* b_{\xi_w})(b_{\xi_v}^*(\xi_{i_n}^{j_n})^*\dots(\xi_{i_1}^{j_1})^*).$$

Let us assume that $n\geqslant 3m$ by adding on 1's at the end of this word, if necessary. If $k_s\leqslant n_0$, for all $1\leqslant s<m$, then, since $v_0^*\in\operatorname{sp}\widetilde{S}_{n_1}B$ and $m<L$, by decomposing $u_{k_m}^* b_{\xi_w} b_{\xi_v}^*(\xi_{i_n}^{j_n})^*$ as its expectation onto B plus something with terms in $B\otimes L(\mathbf{F}_2)$ and zero expectation onto B, we write $\theta(\xi_w b_{\xi_w})(\xi_v b_{\xi_v})^*$ as something in $\mathcal{H}_{n_0}$ plus something in $\mathcal{K}^\perp$. Otherwise, if $k_s>n_0$ for some $s<m$, then, by decomposing

$$u_{k_m}(\xi_{k_m}^{l_m}(u_{k_m}^* b_{\xi_w} b_{\xi_v}^*(\xi_{i_n}^{j_n})^*)(\xi_{i_{n-1}}^{j_{n-1}})^*)(\xi_{i_{n-2}}^{j_{n-2}})^*$$

just as above into its expectation onto B plus something with terms in $B\otimes L(\mathbf{F}_2)$ and zero expectation onto B, we write $\theta(\xi_w b_{\xi_w})(\xi_v b_{\xi_v})^*$ as something with shorter words plus something in $\mathcal{K}^\perp$. Hence, by induction, we have shown that $\theta(\xi_w b_{\xi_w})(\xi_v b_{\xi_v})^*\in\mathcal{H}_{n_0}+\mathcal{K}^\perp$, and hence also $\theta(w_0)v_0\in\mathcal{H}_{n_0}+\mathcal{K}^\perp$.

As $P_{\mathcal{K}}(\mathcal{H}_{n_0})\subset\mathcal{H}_{m_0}$ we have that $P_{\mathcal{K}}(\theta(w_0)v_0)\subset\mathcal{H}_{m_0}$. Thus

$$\begin{aligned}|\langle v_0w_0,\theta(w_0)v_0\rangle| &= |\langle v_0w_0,P_{\mathcal{K}}(\theta(w_0)v_0)\rangle| \\ &= |\langle P_{m_0}(v_0w_0),P_{\mathcal{K}}(\theta(w_0)v_0)\rangle| \\ &\leqslant \|P_{m_0}(v_0w_0)\|_2(\|w_0-w\|_2\|v_0\|+\|v_0\|_2) \\ &\leqslant (\|P_{m_0}(v_0w)\|_2+\|v_0\|\,\|w_0-w\|_2)(\|w_0-w\|_2\|v_0\|+\|v_0\|_2) \\ &< \left(\tfrac14\varepsilon\|v_0\|+\tfrac16\varepsilon\|v_0\|\right)\left(\tfrac16\varepsilon\|v_0\|+\|v_0\|\right) \\ &< \tfrac14\varepsilon.\end{aligned}$$

Hence we have shown that

$$\begin{aligned}\|v\|_2^2=\|vw\|_2^2=\langle vw,\theta(w)v\rangle &\leqslant 2\|v-v_0\|_2+|\langle v_0w,\theta(w)v_0\rangle| \\ &\leqslant 2\|v-v_0\|_2+2\|v_0\|^2\|w-w_0\|_2+|\langle v_0w_0,\theta(w_0)v_0\rangle|<\tfrac13\varepsilon+\tfrac13\varepsilon+\tfrac14\varepsilon<\varepsilon,\end{aligned}$$

which contradicts the assumption that v is non-zero. □

5. Rigid subalgebras in AFP factors: general Bass–Serre type results

We have shown in Theorem 4.3 that if Q is a relatively rigid von Neumann subalgebra of an AFP algebra $M=M_1*_BM_2$, then there exists a non-trivial Hilbert bimodule $\mathcal{H}\subset L^2(M)$ intertwining Q into one of the M_i's. We now deduce that a corner of Q can be conjugated by a unitary element into that same M_i. When M_1 and M_2 are factors, one can in fact uniquely partition 1 with projections $q_1,q_2\in Q'\cap M$ such that Qq_i is unitarily conjugate into M_i, $i=1,2$. This general Bass–Serre type result will be used in the next sections to derive more specific statements in the cases $B=\mathbf{C}$, $B=A$, abelian Cartan, and $B=R$, the hyperfinite II_1 factor.

THEOREM 5.1. *Let (M_i,τ_i), $i=0,1,2$, be finite von Neumann algebras with a common von Neumann subalgebra $B\subset M_i$, $i=0,1,2$, such that $\tau_0|_B=\tau_1|_B=\tau_2|_B$, and such that the inclusions $B\subset M_j$ are homogeneous. Let $M=M_0*_BM_1*_BM_2$. Let $Q\subset M$ be a relatively rigid diffuse von Neumann subalgebra. Assume that no corner of Q can be embedded into M_0 inside M.*

(1) *There exist $i\in\{1,2\}$, projections $q\in Q$ and $q''\in Q'\cap M$ with $qq''\neq0$, and a unitary element $u\in\mathcal{U}(M)$ such that $uqQqq''u^*\subset M_i$.*

(2) *If M_1 and M_2 are factors, then there exists a unique pair of projections $q_1,q_2\in Q'\cap M$ such that $q_1+q_2=1$ and $u_i(Qq_i)u_i^*\subset M_i$ for some unitary elements $u_i\in\mathcal{U}(M)$, $i=1,2$. Moreover, these projections lie in the center of $Q'\cap M$.*

Proof. (1) By Theorem 4.3 and [P8, I, Theorem 2.1], there exist $i\in\{0,1,2\}$, non-zero projections $q\in\mathcal{P}(Q)$ and $p\in\mathcal{P}(M_i)$, an isomorphism ψ of qQq into pM_ip, and a non-zero partial isometry $v\in M$ such that $vv^*\in(qQq)'\cap qMq$, $v^*v\in\psi(qQq)'\cap pMp$ and $xv=v\psi(x)$ for all $x\in qQq$. By hypothesis, i cannot be equal to 0, and thus $i\in\{1,2\}$. Note that, by shrinking q if necessary, we may assume that $xv=0$ for $x\in qQq$ implies $x=0$. Also, if we denote by q' the support projection of $E_{M_1}(v^*v)$, then, by replacing if necessary ψ by $q'\psi(\cdot)q'$, it follows that we may assume that $q'=p$.

Now note that if a corner of $\psi(qQq)$ can be embedded into pBp inside pM_ip, then, by Lemma 1.12, a corner of Q can be embedded into B (and thus into $M_0\supset B$ as well) inside M, contradicting the hypothesis. Thus, no corner of $\psi(qQq)\subset pM_ip$ can be embedded into B inside M_i, so we can apply Theorem 1.1 to conclude that $Q_0'\cap pMp\subset pM_ip$. Hence, $v^*v\in Q_0'\cap pMp\subset M_i$. Taking $q''=vv^*$ and a unitary element $u\in M$ such that $uqq''=v$, the statement follows.

(2) Let $z=z(q)$ denote the central support of q in Q and note that zq'' is then the central support of qq'' in Qq''. By the factoriality of M_i, $i=1,2$, it follows that there exists a unitary element $u\in\mathcal{U}(M)$ such that $Qq''z\subset uM_iu^*$. (Indeed, this is because whenever $Q_0\subset N$ is an inclusion of finite von Neumann algebras, $q_0\in\mathcal{P}(Q_0)$ and $N_0\subset N$ is a subfactor with $qQ_0q\subset N_0$, then there exists $u\in\mathcal{U}(N)$ such that $Q_0z(q)\subset uN_0u^*$.) Thus, the projection $p_0'=q''z\in Q'\cap M$ together with the unitary element u satisfy the condition $uQp_0'u^*\subset M_i$.

Let $\mathcal{F}$ be the set of all families of mutually orthogonal projections

$$\{p_i'\}_{i\in I}\subset\mathcal{P}(Q'\cap M),$$

with the property that for all $i\in I$ there exists $j(i)\in\{1,2\}$ (unique by Theorem 1.1) and $v_i\in\mathcal{U}(M)$ such that $v_iQp_i'v_i^*\subset M_{j(i)}$. The set $\mathcal{F}$ is clearly inductively ordered with respect to the order given by inclusion. Let $\{p_i'\}_{i\in I}$ be a maximal element. Let

$$q_1'=\sum_{j(i)=1}p_i',\quad q_2'=\sum_{j(i)=2}p_i'$$

and set $q'=1-q_1'-q_2'$.

Assume that $q'\neq0$. Since Q is diffuse, there exists $q\in\mathcal{P}(Q)$ such that $\tau(q'q)=1/n$ for some integer $n\geqslant1$. Let $\widetilde{Q}\subset M$ be a von Neumann algebra isomorphic to $M_{n\times n}(qQqq')$ with $qQqq'$ equal to the upper-left corner $qq'\widetilde{Q}qq'$ and qq' having central trace $1/n$ in $\widetilde{Q}$. By [P5, §4], $\widetilde{Q}\subset M$ is a rigid inclusion. Thus, we can apply the first part of the proof to get $i\in\{1,2\}$, $0\neq\tilde{q}'\in\widetilde{Q}'\cap M$ and a unitary element $w\in M$ such that $w\widetilde{Q}\tilde{q}'w^*\subset M_i$. Since qq' has scalar central trace in $\widetilde{Q}$, it follows that the projection

$$p=qq'\tilde{q}'\in(qQq)'\cap qMq=q(Q'\cap M)q$$

is non-zero. Thus $p=qq''$ for some projection $q''\in Q'\cap M$ with $q''\leqslant q'$. Since $p\leqslant\tilde{q}'$, we also have $w(qQqq'')w^*\subset M_i$, implying that if we let $p'=z(q)q''$ (where $z(q)$ is the central support of q in Q) then $p'\in Q'\cap M$, $p'\leqslant q'$ and there exists a unitary element u in M such that $u(Qp')u^*\subset M_i$. But then $\{p_i'\}_{i\in I}\cup\{p'\}$ lies in $\mathcal{F}$, thus contradicting the maximality of $\{p_i'\}_{i\in I}$.

We have thus shown that $q_1'+q_2'=1$. On the other hand, by the factoriality of the M_k's, $k=1,2$, for each fixed k we can choose the unitary elements $\{v_i:j(i)=k\}$ which satisfy $v_i(Qp_i')v_i^*\subset M_k$ so that $v_ip_i'v_i^*$ be mutually orthogonal projections in M_k. Taking $u_k\in\mathcal{U}(M)$ to be a unitary element extending $\sum_{j(i)=k} v_i$, it follows that $u_k(Qq_k')u_k^*\subset M_k$, $k=1,2$.

This proves the existence part of fact (2). But the uniqueness part is then clear, since if p_1', p_2' is another pair of projections in $Q'\cap M$ satisfying $p_1'+p_2'=1$, $v_i(Qp_i')v_i^*\subset M_i$ for some $v_i\in\mathcal{U}(M)$, $i=1,2$, and we assume $x=p_1'q_2'\neq 0$, then the partial isometry w in the polar decomposition of x lies in $Q'\cap M$, and if we denote $p=ww^*$ then $v_1(Qp)v_1^*\subset M_1$ while $u_2w^*(Qp)wu_2^*\subset M_2$, contradicting Theorem 1.1.

To finish the proof of (2), we need to show that q_1' and q_2' are in the center of $Q'\cap M$. Since $q_1'+q_2'=1$, this amounts to showing that their central supports in $Q'\cap M$ are disjoint.

Assume by contradiction that there exist non-zero projections $q_i''\leqslant q_i', q_i''\in Q'\cap M$ with $u'q_1''(u')^*=q_2''$ for some $u'\in\mathcal{U}(Q'\cap M)$. But then $u_k(Qq_k'')u_k^*\subset M_k$, $k=1,2$, are diffuse and are conjugate by the unitary element $u_2u'u_1^*$, contradicting Theorem 1.1 again. □

THEOREM 5.2. *Let I be a set of indices with $0\in I$ and (M_i,τ_i), $i\in I$, be a family of finite von Neumann algebras with a common von Neumann subalgebra $B\subset M_i$, such that $\tau_0|_B=\tau_i|_B$ for all i. Assume that M_i are factors for $i\neq 0$, and that the inclusions $B\subset M_i$ are homogeneous for all $i\in I$. Denote by $M=*_{B,i\in I}M_i$ the free product with amalgamation over B of the algebras M_i, $i\in I$. Let $t>0$ and $Q\subset M^t$ be a relatively rigid diffuse von Neumann subalgebra such that no corner of Q can be embedded into M_0 inside M and such that the normalizer of Q in M^t generates a factor N. Then there exists a unique $i\in I\setminus\{0\}$ and a unitary element $u\in M^t$ such that $uQu^*\subset M_i^t$. Moreover, such u satisfies $uNu^*\subset M_i^t$, and in fact $u\widetilde{N}u^*\subset M_i^t$, where $\widetilde{N}=\widetilde{N}(N,M_1^t;B)$ is as in Definition* 1.2.

Proof. Note first that the fact that $Q\subset M^t$ is rigid implies that Q is countably generated (see, e.g., [PeP]). Thus, there exists a countable subset $S\ni 0$ of indices $i\in I$ such that $Q\subset(*_{B,i\in S}M_i)^t$. By Corollary 3.2, $Q\subset(*_{B,i\in S}M_i)^t$ is rigid and, by Theorem 1.1, all of N is contained in $(*_{B,i\in S}M_i)^t$. This shows that it is sufficient to prove the statement in the case when M_i, $i\geqslant 0$, is a sequence of algebras.

Moreover, since $Q\subset M^t$ is rigid and since the factors $\widetilde{M}(K,t)=M_0*_B(*_{B,k\in K}M_k)^t$, with K being a finite subset of $\{1,2,...\}$, tend to M^t, it follows by [P5] that there exists a non-zero projection $q'\in Q'\cap M^t$, a unitary element $v\in M^t$ and a finite set $K\subset\{1,2,...\}$ such that $v(Qq')v^*\subset\widetilde{M}(K,t)$. But $q'\in N$ and, by [P8, I, Lemma 3.5], Qq' is quasi-regular in $q'Nq'$, so, by Theorem 1.1, we have $v(q'Nq')v^*\subset\widetilde{M}(K,t)$. Since N is a factor, v can be modified so that $vNv^*\subset\widetilde{M}(K,t)$. In particular $vQv^*\subset\widetilde{M}(K,t)$.

Since Q is diffuse, there exists $q\in\mathcal{P}(Q)$ such that $\tau(q)\leqslant t^{-1}$. Thus, we may assume that $v(qQq)v^*\subset\widetilde{M}_K\stackrel{\text{def}}{=}M_0*_B(*_{B,k\in K}M_k)$, and notice that, by [P5, Proposition 4.7], the unital inclusion $v(qQq)v^*\subset p\widetilde{M}_Kp$ is rigid, where $p=vqv^*$. Since K is finite, Theorem 5.1 applies to get $i\in K$, $0\neq q_i'\in(vqQqv^*)'\cap p\widetilde{M}_Kp$ and a unitary element $w\in\widetilde{M}_K$ such that $w(vqQqv^*q_i')w^*\subset M_i$. Moreover, since $\tau(q)\leqslant t^{-1}$, we can view $w(vqQqv^*q_i')w^*$ as a (possibly non-unital) subalgebra of M_i^t. Since $q_i'\in(vqQqv^*)'\cap pMp$, it follows that $q_i'\in vqNqv^*$ (recall that N is generated by the normalizer of Q in M). By [P8, I, Lemma 3.5] and Theorem 1.1 again, it follows that $w(q_i'vqNqvq_i')w^*\subset M_i$, implying that $wq_i'vq$ can be extended to a unitary element $u\in M^t$ such that $uNu^*\subset M_i^t$. Thus $uQu^*\subset M_i^t$. Also, by Theorem 1.1, i is unique with this property, while, by Corollary 1.3, it follows that $u\widetilde{N}u^*\subset M_i^t$. □

6. Amalgamation over C: free product factors with prescribed $\mathfrak{F}(M)$

We first apply Theorem 5.1 to plain free product factors, where the result becomes an analogue of the classical Kurosh theorem for groups. The first Kurosh-type results in operator algebra framework were obtained by N. Ozawa in [O]. He proved that if N is a non-prime non-hyperfinite II_1 subfactor of a free product $M=M_1*M_2$ of semiexact finite factors M_1 and M_2, then N can be unitarily conjugated into either M_1 or M_2 (this is an analogue of the "Kurosh subgroup theorem"). As a consequence, he showed that if two free products $*_iM_i$ and $*_jN_j$ of non-hyperfinite non-prime semiexact factors N_i and M_j are isomorphic, then the "length" of the two free products must be the same and each N_i is unitarily conjugate to M_i, after some permutation of indices (this is an analogue of the "Kurosh isomorphism theorem").

In turn, our results cover different classes of algebras. Thus, our analogue of the "Kurosh subgroup theorem" allows M_1 and M_2 to be arbitrary finite von Neumann algebras, but only gives information about relatively rigid subalgebras Q of M_1*M_2. Our corresponding "isomorphism theorem", which in fact we obtain for amplifications of free products, will require the factors N_i and M_j to be either w-rigid, i.e. to have diffuse regular relatively rigid subalgebras, or to be group measure space factors associated with actions of w-rigid ICC groups. In particular, it holds for II_1 factors N_i and M_j with the property (T) (in the sense of [CJ]), and more generally for tensor products of property (T) II_1 factors with arbitrary finite factors. Moreover, since the factors $N=L(\mathbf{Z}^2\rtimes\mathbf{F}_n)$ in [P5] are w-rigid and have trivial fundamental group (see Corollary 7.18 for a different proof), this will allow us to obtain large classes of factors with trivial fundamental group, different from the ones in [P5] and [P8]. More generally, using also [DyR], we construct a completely new class of factors with prescribed fundamental group which, unlike the ones in [P5] and [P8], have no Cartan subalgebras (by [V2], cf. Remark 6.6 below).

THEOREM 6.1. *Let (M_i,τ_i), $i=0,1,2$, be finite von Neumann algebras and let $M=M_0*M_1*M_2$. Assume that no direct summand of (M_0,τ_0) has relatively rigid diffuse von Neumann subalgebras (which, e.g., holds if $M_0=\mathbf{C}$, or more generally if M_0 has the Haagerup property). Let $Q\subset M$ be a relatively rigid diffuse von Neumann subalgebra of M.*

(1) *There exist $i\in\{1,2\}$, $q\in\mathcal{P}(Q)$, $q'\in\mathcal{P}(Q'\cap M)$ and $u\in\mathcal{U}(M)$ such that $qq'\neq 0$ and $uqQqq'u^*\subset M_i$.*

(2) *If, in addition, M_1 and M_2 are factors, then there exists a unique pair of projections $q_1',q_2'\in Q'\cap M$ such that $q_1'+q_2'=1$ and $u_i(Qq_i')u_i^*\subset M_i$ for some unitary elements $u_i\in\mathcal{U}(M)$, $i=1,2$. Moreover, $q_1',q_2'\in\mathcal{Z}(Q'\cap M)$.*

(3) *If instead of M_1 and M_2 we consider a whole family of finite factors M_i, $i\geqslant 1$, we take a rigid inclusion $Q\subset M^t=(M_0*M_1*M_2*...)^t$, for some $t>0$, and we assume that the normalizer of Q in M^t generates a factor N, then there exists a unique $i\geqslant 1$ and a unitary element $u\in M^t$ such that $uQu^*\subset M_i^t$. Moreover, such a u satisfies $uNu^*\subset M_i^t$, and in fact $u\widetilde{N}u^*\subset M_i^t$, where $\widetilde{N}=\widetilde{N}(N,M_i^t;\mathbf{C})$ is as in Definition 1.2.*

Proof. As in the proof of Theorem 5.2, note that Q is relatively rigid implies that Q is countably generated. Thus, there exist countably generated von Neumann subalgebras $M_i^0\subset M_i$, $i=0,1,2$, such that $Q\subset M_0^0*M_1^0*M_2^0$. Hence, to prove (1), it is clearly sufficient to prove it in the case where the M_i's are countably generated, $i=0,1,2$. But then each $\mathbf{C}\subset M_i$ is homogeneous by Lemma 4.2. Let us show that no corner of Q can be embedded into M_0 inside M. Assume that this is not true. By [P8, I, Theorem 2.1], it follows that there exist non-zero projections $q\in Q$ and $p\in M_0$, a unital isomorphism ψ of qQq into pM_0p and a non-zero partial isometry $v\in M$ such that $vv^*\in(qQq)'\cap qMq$, $v^*v\in\psi(qQq)'\cap pMp$ and $xv=v\psi(x)$ for all $x\in qQq$. Let $q''=vv^*\in Q'\cap M$.

Since $\psi(qQq)\subset pM_0p$ is a diffuse von Neumann subalgebra, by Theorem 1.1, it follows that $\psi(qQq)'\cap pMp\subset pM_0p$. Thus $v^*v\in pM_0p$. This shows that $v^*qQqv\subset pM_0p$, which in turn implies that $qQqq''\subset wM_0w^*$ for some unitary element $w\in M$ extending v. Since $Q\subset M$ is rigid, by [P5, Proposition 4.7], $qQqq''\subset qq''Mqq''$ is also rigid, which trivially implies that $qQqq''\oplus(1-qq'')\mathbf{C}\subset M$ is rigid. But then it follows that

$$w^*(qQqq''\oplus(1-qq'')\mathbf{C})w\subset M_0$$

is rigid by Corollary 3.2. By taking a suitable amplification of $w^*(qQqq'')w$ in M_0 and using again [P5, Proposition 4.7], this implies that a direct summand of M_0 contains a relatively rigid diffuse von Neumann subalgebra, a contradiction.

Altogether, this shows that the conditions required in Theorem 5.1 are satisfied, so part (1) of the statement follows as a particular case of that theorem.

For part (2), simply notice that if M_1 and M_2 are factors, then the countably generated von Neumann subalgebras M_0^0, M_1^0, M_2^0 with the property $Q\subset M_0^0*M_1^0*M_2^0$ can be chosen so that M_1^0 and M_2^0 are factors as well, so Theorem 5.1 (2) applies.

Part (3) follows then from part (2) and Theorem 5.2. □

Definition 6.2. A finite von Neumann algebra (M,τ) is *weakly rigid* (w-rigid) if it contains a regular relatively rigid diffuse von Neumann subalgebra, i.e. a subalgebra $Q\subset M$ such that $\mathcal{N}_M(Q)''=M$ and $Q\subset M$ is a rigid inclusion (or Q is a relatively rigid subalgebra of M [P5]). Note that if G is a w-rigid group as defined in [P3], [P5], [P6] and [P8], i.e. G contains an infinite normal subgroup with the relative property (T) of Kazhdan–Margulis ([M]; see also [dHV]), then $L(G)$ is w-rigid. Also, if M is w-rigid and P is an arbitrary finite von Neumann algebra, then $M\overline{\otimes}P$ is w-rigid.

Theorem 6.3. *Let (M_0,τ_{M_0}) and (N_0,τ_{N_0}) be finite von Neumann algebras which have no relatively rigid diffuse subalgebras (which, e.g., is true if M_0 and N_0 have Haagerup's compact approximation property). Let $M_1,...,M_m$ and $N_1,...,N_n$ be* II_1 *factors, where $n,m\geqslant 1$ are some cardinals (finite or infinite) and assume that each M_i and each N_j is w-rigid. If θ is an isomorphism of $M=*_{i=0}^{m}M_i$ onto N^t, where $N=*_{j=0}^{n}N_j$ and $t>0$, then $m=n$ and, after some permutation of indices, $\theta(M_i)$ and N_i^t are unitarily conjugate in N^t for all $i\geqslant 1$.*

Proof. For each $1\leqslant i\leqslant m$ let $Q_i\subset M_i$ be a regular relatively rigid diffuse von Neumann subalgebra. Since M_i are factors and $Q_i\subset M_i$ are regular inclusions, it follows that for any $s>0$ the factor M_i^s contains a regular diffuse relatively rigid subalgebra. To see this, note that $\mathcal{N}_{M_i}(Q_i)$ acts ergodically on the center of Q_i, so $\mathcal{Z}(Q_i)$ is either diffuse or atomic. In both cases we can find projections $q'\in\mathcal{Z}(Q_i)$ and $q\in Q_iq'$ such that the central trace of q in Q_i is a scalar mutiple of q' and $\tau(q)=s/k$, for some $k\geqslant t$. By [P8, I, Lemma 3.5], it follows that the inclusion $qQ_iq\subset qM_iq$ is regular and, by [P5, Proposition 4.7], it is rigid as well. But then $M_{k\times k}(qQ_iq)\subset M_{k\times k}(qM_iq)=M_i^s$ is regular and rigid ([P5]).

Moreover, note that any diffuse von Neumann subalgebra $B_i\subset M_i^s$ satisfies

$$B_i'\cap M^s\subset M_i^s.$$

To see this, note first that by taking direct sums of k copies of B_i embedded diagonally into $M_{k\times k}(M_i^s)=M_i^{ks}$, with k sufficiently large, we may assume that $s\geqslant 1$. Then take $p\in\mathcal{P}(B_i)$ with $\tau(p)=1/s$ and note that if we assume by contradiction that

$$B_i'\cap M^s\neq B_i'\cap M_i^s,$$

then $(pB_ip)'\cap pM^sp\neq(pB_ip)'\cap pM_i^sp$. But $(pM_i^sp\subset pMp)\simeq(M_i\subset M)$ and M splits off M_i as a free product. Thus, by Theorem 1.1, the relative commutant in $pM^sp=M$ of the diffuse subalgebra $pB_ip\subset pM_i^sp=M_i$ must be contained in pM_i^sp, contradicting the assumption.

Taking now $s=1/t$, it follows that $M_i^{1/t}$ has a diffuse regular relatively rigid subalgebra, implying that $P_i=\theta^{1/t}(M_i^{1/t})$ has such a subalgebra B_i as well. In particular, the inclusion $B_i\subset N$ is rigid. In addition, $B_i'\cap N\subset P_i$. Since the inclusion $B_i\subset N$ is rigid and regular, by Theorem 6.1 (3), there exists a unique $j(i)\in\{1,2,...,n\}$ and a unitary $u_i\in N$ such that $B_i\subset u_iN_{j(i)}u_i^*$ and $P_i\subset u_iN_{j(i)}u_i^*$. Thus, there exists a unique $j(i)$ such that for some unitary element $v_i\in N^t$ we have $\theta(M_i)=P_i^t\subset v_iN_{j(i)}^tv_i^*$.

Similarly, by applying the above to θ^{-1}, we get for each $1\leqslant j\leqslant n$ a unique $1\leqslant k(j)\leqslant m$ and a unitary element in $w_j\in M$ such that $\theta^{-1}(N_j^t)\subset w_jM_{k(j)}w_j^*$. Altogether, for each $1\leqslant i\leqslant m$ we get

$$M_i=\theta^{-1}(\theta(M_i))\subset\theta^{-1}(v_iN_{j(i)}^tv_i^*)\subset u_iM_{k(j(i))}u_i^*, \tag{6.1}$$

where $u_i=w_{j(i)}\theta^{-1}(v_i)$. By Theorem 1.1, it follows that $k(j(i))=i$, i.e. $k\circ j=\mathrm{id}$. Similarly, $j\circ k=\mathrm{id}$. Thus $m=n$, j and k are onto isomorphisms and the inclusions (6.1) are in fact equalities. $\square$

In the next statement, for a finite permutation $\pi\in S_m$ and $1\leqslant i\leqslant m$ we denote by $m(\pi,i)$ the cardinality of the set $\{\pi^k(i):k\geqslant 1\}$.

Corollary 6.4. *Let $m\in\mathbf{N}$ and let $M_1,...,M_m$ be w-rigid II_1 factors. Let (M_0,τ_{M_0}) be a finite von Neuman algebra which contains no rigid diffuse von Neumann subalgebras. Let $M=*_{i=0}^mM_i$. Then*

$$\mathfrak{F}(M)\subset\bigcup_{\pi\in S_m}\bigcap_{i=1}^m\mathfrak{F}(M_i)^{m(\pi,i)^{-1}}\subset\bigcap_{i=1}^m\mathcal{F}(M_i)^{1/m!}.$$

In particular, if one of the factors M_i, $1\leqslant i\leqslant m$, has trivial fundamental group, then so does M.

Proof. For $t\in\mathfrak{F}(M)$, let $\theta: M\to M^t$ be an isomorphism. Applying the previous theorem, we get that there exists $\pi\in S_m$ such that $\theta(M_i)$ and $M_{\pi(i)}^t$ are unitarily conjugate in

M^t. In particular, we have that $M_i \cong M^t_{\pi(i)}$ for all $1 \leqslant i \leqslant m$ which, by induction, implies that $M_i \cong M^{t^k}_{\pi^k(i)}$ for all $1 \leqslant i \leqslant m$, $k \in \mathbf{N}$.

Fixing i and letting $k = m(\pi, i)$, we obtain that $t^{m(\pi,i)} \in \mathfrak{F}(M_i)$, or equivalently $t \in \mathfrak{F}(M_i)^{m(\pi,i)^{-1}}$. By intersecting over all values of i, taking the union over all possible permutations and noticing that $m(\pi, i)$ divides m, the result follows. □

For the next corollary, we denote by $\mathcal{X}_{\text{fin}}$ the set of all finite tuples of positive numbers $\{t_i\}_{i=1}^n \subset \mathbf{R}_+^*$, $n \geqslant 2$, and by $\mathcal{X}_\infty$ the set of all infinite sequences $\{t_i\}_{i \geqslant 1} \subset \mathbf{R}_+^*$. Also, we let $\mathcal{X} = \mathcal{X}_{\text{fin}} \cup \mathcal{X}_\infty$. If $X = \{t_i\}_i$ and $Y = \{s_j\}_j$ are in $\mathcal{X}$, then we write $X \sim Y$ if both have the same "length" and there exists a permutation (bijection) π of the (common) set of indices $\{1, 2, ...\}$ such that $s_{\pi(i)} = t_i$ for all i.

Given a II_1 factor M and $X = \{t_i\}_i \in \mathcal{X}$, we let $M^X = *_i M^{t_i}$. Note that if $X, Y \in \mathcal{X}$ and $X \sim_\pi Y$, then π induces a natural isomorphism $\theta_\pi \colon M^X \simeq M^Y$, in the obvious way. For $t > 0$ and $X = \{t_i\}_i \in \mathcal{X}$ we let $tX = \{tt_i\}_i \in \mathcal{X}$.

COROLLARY 6.5. *Let M be a w-rigid II_1 factor with trivial fundamental group, e.g. $M = L(\mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z}))$ (cf. [P5]).*

(1) *If $X, Y \in \mathcal{X}$, then $M^X \simeq M^Y$ if and only if $X \sim Y$, which holds if and only if $M^X * L(\mathbf{F}_k) \simeq M^Y * L(\mathbf{F}_k)$ for some $1 \leqslant k \leqslant \infty$.*

(2) *$\mathfrak{F}(M^X) = \{1\}$ for all $X \in \mathcal{X}_{\text{fin}}$. Moreover, if we denote by $\mathcal{X}_0$ the set of elements X in $\mathcal{X}_{\text{fin}}$ with $\min X = 1$, then $\{M^X : X \in \mathcal{X}_0\}$ is a continuous family of mutually non-stably isomorphic II_1 factors.*

(3) *For each $X \in \mathcal{X}_\infty$ let $\mathcal{S}_X = \{t \in \mathbf{R}_+^* : tX \sim X\}$. Then $\mathfrak{F}(M^X) = \mathcal{S}_X$. In particular, if $S \subset \mathbf{R}_+^*$ is an infinite countable subgroup and $X \in \mathcal{X}_\infty$ has the elements of S as entries, each one repeated with the same (possibly infinite) multiplicity, then $\mathfrak{F}(M^{tX}) = S$ for all $t > 0$. Moreover, $M^{t_1 X}$ and $M^{t_2 X}$ are stably isomorphic if and only if t_1 and t_2 are in the same class in $\mathbf{R}_+^*/S$. Thus, $\{M^{tX} : t \in \mathbf{R}_+^*/S\}$ is a continuous family of mutually non-stably isomorphic II_1 factors all with fundamental group equal to S.*

(4) *If $S \subset \mathbf{R}_+^*$ is an arbitrary infinite (possibly uncountable) subgroup then the II_1 factor $M^S = *_{s \in S} M^s$ has fundamental group equal to S.*

Proof. (1) If $M^X \cong M^Y$ then, by Theorem 6.3, X and Y have the same length and there exists a bijection π of the corresponding (finite or infinite) set of indices $\{1, 2, ...\}$ such that $M^{t_i} = M^{s_{\pi(i)}}$ for all i. Since M has trivial fundamental group, this implies that $t_i = s_{\pi(i)}$ for all i, and thus $X \sim_\pi Y$. The second equivalence has exactly the same proof, using Theorem 6.3 with $M_0 = L(\mathbf{F}_k)$.

(2) If $X = \{t_i\}_{i=1}^n$ and $Y = \{s_j\}_{j=1}^m$ are in $\mathcal{X}_{\text{fin}}$ and $M^X \simeq (M^Y)^t$, then

$$M^X * L(\mathbf{F}_\infty) \simeq (M^Y)^t * L(\mathbf{F}_\infty).$$

But, by [DyR], the last factor is isomorphic to $M^{tY} * L(\mathbf{F}_\infty)$. Thus,

$$M^X * L(\mathbf{F}_\infty) \simeq M^{tY} * L(\mathbf{F}_\infty),$$

which, by part (1), implies that $X \sim tY$. Thus, if $X = Y$ (resp. $X, Y \in \mathcal{X}_0$) then $t=1$ and we get $\mathfrak{F}(M^X) = \{1\}$ (resp. $X \sim Y$). This implies both statements.

(3) By [DyR], if $X \in \mathcal{X}_\infty$ then $(M^X)^t \cong M^{tX}$. Thus $M^X \cong (M^X)^t$ if and only if $X \sim tX$, which readily implies all statements.

(4) It is easy to see that, in fact, the proof of the amplification formula $(M^S)^t \simeq M^{tS}$ in [DyR] does not depend on the fact that the infinite set S is countable. Thus, since for $t \in S$ we have $tS = S$ as sets, it follows that $(M^S)^t = M^{tS} = M^S$. Hence $S \subset \mathfrak{F}(M^S)$. Conversely, if $s \in \mathbf{R}_+^*$ satisfies $(M^S)^s \simeq M^S$, then $M^{sS} \simeq M^S$, which, by Theorem 6.3, implies that $tS = S$, so that $s \in S$. □

Remarks 6.6. Dima Shlyakhtenko pointed out to us that, by combining Voiculescu's initial argument for showing that $L(\mathbf{F}_n)$ has no Cartan subalgebras, with Kenley Jung's "monotonicity" [Ju], it follows that any free product of type-II_1 factors which embed into R^ω is "Cartan-less" (see [Sh] for a detailed argument). Thus, unlike the examples of factors with prescribed fundametal group in [P8], which are group measure space factors associated with equivalence relations coming from Connes–Størmer Bernoulli actions, the examples of factors M that we produce here (in Corollary 6.5) have no Cartan subalgebras, and altogether no diffuse hyperfinite "core". In particular, they cannot be written as crossed products of the form $M = R \rtimes \Gamma$ with R being the hyperfinite factor.

It is interesting to note that Theorem 5.1 can be used to give a completely new proof of the by now classical result of Connes and Jones showing that property (T) factors cannot be embedded into the free group factor [CJ]. Thus, rather than using Haagerup's property (i.e. "deformation by compact c.p. maps"), as the original proof does, this new proof uses a "deformation by automorphisms" of the free group factors.

COROLLARY 6.7. ([CJ]) *For every n, $2 \leqslant n \leqslant \infty$, the free group von Neumann algebra $L(\mathbf{F}_n)$ contains no relatively rigid diffuse subalgebra.*

Proof. If we write $L(\mathbf{F}_n)$ as $L(\mathbf{Z}) * L(\mathbf{Z}) * \ldots * L(\mathbf{Z})$ and then apply recursively the first part of Theorem 5.1 and Corollary 3.2, it follows that a corner of $L(\mathbf{Z})$ contains a rigid diffuse von Neumann subalgebra, a contradiction. □

7. Amalgamation over Cartan subalgebras: vNE/OE rigidity results

In this section we apply Theorem 5.1 to study group measure space factors of the form $A \rtimes_\sigma \Gamma$, where Γ is a free product of groups $\Gamma = \Gamma_0 * \Gamma_1 * \ldots$ and σ is a free ergodic m.p. action of Γ on $A = L^\infty(X, \mu)$, for a probability space (X, μ). Such a factor can alternatively be viewed as a free product with amalgamation $M=M_0*_A M_1*_A\dots$, where $M_i=A\rtimes_{\sigma_i}\Gamma_i$, $\sigma_i=\sigma|_{\Gamma_i}$, with $A\subset M$, with the algebra of coefficients A of the crossed product $A\rtimes\Gamma$ now becoming the "core" of the amalgamated free product. It is this form that will allow us to use Theorem 5.1.

Following [P8], we regard an isomorphism of such group measure space factors as a *von Neumann equivalence* (vNE) of the corresponding actions $(\sigma, *_i\Gamma_i)$. Thus, the main result we prove in this section is a rigidity result showing that vNE of actions of free products of groups Γ_i satisfying some weak rigidity conditions (of property (T) type) entails the orbit equivalence (OE) of the actions σ, with componentwise OE of the actions (σ_i, Γ_i). Due to its analogy to similar statements on (amalgamated) free products of groups in Bass–Serre theory, we refer to this as vNE *Bass–Serre rigidity*. We note that when applied to isomorphisms of group measure space factors that come from OE of the actions, they give OE *Bass–Serre rigidity* results.

Since we study the group measure space factors $M=A\rtimes(\Gamma_0*\Gamma_1*\dots)$ as AFP factors $(A\rtimes\Gamma_0)*_A(A\rtimes\Gamma_1)*_A\dots$, it is worth noticing that if $M=M_0*_A M_1*_A\dots$ is an AFP factor coming from Cartan subalgebra inclusions $A=L^\infty(X,\mu)\subset M_i$, $i\geqslant 0$, then it follows that the AFP "core" A is regular in M, but in general it may not be maximal abelian (and thus not Cartan) in M. For instance, if A is a Cartan subalgebra of a II_1 factor N, then A is not maximal abelian in $M=N*_A N$, because for any $u\in\mathcal{N}_N(A)$ with $E_A(u)=0$ the element $u*u^{-1}$ is still perpendicular to A yet acts trivially on it. For more on general properties of AFP factors arising from Cartan inclusions, we refer the reader to [Ko], [U1] and [U2].

In case $M_i=A\rtimes_{\sigma_i}\Gamma_i$, with each σ_i being a free m.p. action, then there exists a unique m.p. action σ of $\Gamma=\Gamma_0*\Gamma_1*\dots$ on A such that $\sigma|_{\Gamma_i}=\sigma_i$ for all i, and we still have the natural identification $M=A\rtimes_\sigma\Gamma=M_0*_A M_1*_A\dots$, as in the case when σ is a free action mentioned above. Then A is Cartan in M if and only if σ is a free action, i.e. if and only if σ_i are "freely independent" actions (in the obvious sense). For general equivalence relations (or Cartan subalgebras), the definition of "free independence" was formulated by Gaboriau [G1] and is recalled below. Recall that if B is a finite von Neumann algebra, p and q are non-zero projections in B and $\theta: pBp\to qBq$ is a $*$-morphism, then θ is called *properly outer* if $b\in B$ and the condition $\theta(x)b=bx$ for all $x\in B$ implies that $b=0$.

Definition 7.1. ([G1]) Let $\{\mathcal{R}_i\}_{i\in I}$ be a family of countable measurable measure-preserving equivalence relations on the same standard non-atomic probability space (X,μ) (see e.g. [FM]). We alternatively view each $\mathcal{R}_i$ as a pseudogroup of *local m.p. isomorphisms* $\phi: Y_1\simeq Y_2$ with $Y_1, Y_2\subset X$ measurable and the graph of ϕ contained in $\mathcal{R}_i$ [D], [FM]. We say that $\{\mathcal{R}_i\}_i$ are *freely independent* if for any n and any properly outer local isomorphisms $\phi_j\in\mathcal{R}_{i_j}$, $1\leqslant j\leqslant n$, $i_j\in I$, with $i_j\neq i_{j+1}, 1\leqslant j\leqslant n-1$, the product $\phi_1\phi_2\ldots\phi_n$ is properly outer.

In the case when each of the equivalence relations $\mathcal{R}_i$ is generated by properly outer automorphisms, Definition 7.1 can be viewed as a particular case of the following.

Definition 7.1$'$. Let (B,τ) be a finite von Neumann algebra and $S_i\subset\mathrm{Aut}(B,\tau)$, $i\in I$, be a family of sets of τ-preserving automorphisms, with each $\theta\in S_i$ either properly outer or equal to id_B. We say that $\{S_i\}_i$ are *freely independent* if for any n and any $\theta_j\in S_{i_j}\setminus\{\mathrm{id}_B\}$, $1\leqslant j\leqslant n$, $i_j\in I$, with $i_1\neq i_2\neq\ldots\neq i_n$, the product $\theta_1\theta_2\ldots\theta_n$ is properly outer.

The next lemma translates the freeness conditions in Definition 7.1 into the framework of operator algebras (see [U2]).

Lemma 7.2. *Let (M_n,τ_n), $n\geqslant 1$, be finite von Neumann algebras with a common Cartan subalgebra $A\subset M_n$ such that $\tau_n|_A=\tau_m|_A$ for all n and m. Then*

$$A\subset M_1 *_A M_2 *_A \ldots$$

is a Cartan subalgebra if and only if the equivalence relations $\mathcal{R}_n=\mathcal{R}_{A\subset M_n}$, $n\geqslant 1$, are freely independent.

Proof. Since A is clearly regular in $M=M_1 *_A M_2 *_A\ldots$, all we need to prove is that $A\subset M$ is maximal abelian if and only if $\{\mathcal{R}_n\}_{n\geqslant 1}$ are freely independent. But this is trivial by the definitions of freeness and of the amalgamated free product over A. □

The next result, essentially due to Törnquist [Tö], shows that a sequence of actions of countable groups (or merely countable m.p. equivalence relations) can be made "freely independent" by conjugating each one of them with a suitable m.p. automorphism. We include a proof, based on Lemma A.1 in the appendix, for the reader's convenience.

Proposition 7.3. (1) *Let (X,μ) be a standard non-atomic probability space and $\sigma_n: G_n\to\mathrm{Aut}(X,\mu)$ be free m.p. actions of discrete countable groups G_n, $n\geqslant 1$. Then there exists a free m.p. action σ of $G=*_{n=1}^{\infty}G_n$ on (X,μ) such that $\sigma|_{G_n}$ is conjugate to σ_n, for all $n\geqslant 1$. More generally, if $\{\mathcal{R}_n\}_{n\geqslant 1}$ are standard equivalence relations on (X,μ) then there exists an equivalence relation $\mathcal{R}$ on (X,μ) generated by a family of freely independent subequivalence relations $\mathcal{R}_n'\subset\mathcal{R}$, $n\geqslant 1$, such that $\mathcal{R}_n\simeq\mathcal{R}_n'$ for all n.*

(2) *Let (M_n,τ_n) be countably generated finite von Neumann algebras with a common diffuse Cartan subalgebra $A\subset M_n$, $n\geqslant 0$, such that $\tau_n|_A=\tau_m|_A$ for all n and m. Then there exist Cartan subalgebra inclusions $\{A\subset N_n\}_{n\geqslant 0}$ such that $(A\subset N_n)\simeq(A\subset M_n)$ for all n, and such that A is a Cartan subalgebra in $N_0 *_A N_1 *_A N_2\ldots$.*

Proof. (1) This is an immediate application by induction of Lemma A.1, once we notice that any $\mathcal{R}_n$ can be extended to a countable m.p. equivalence relation $\mathcal{S}_n$ on (X,μ) which is generated by countably many properly outer m.p. automorphisms.

(2) By Lemma 7.2, we only need to make the equivalence relations $\mathcal{R}_{A\subset M_n}$ freely independent, so the first part applies. □

We also notice the following general "compression formula" for restrictions of "free products of equivalence relations" $\mathcal{R}=*_{i=1}^n\mathcal{R}_i$, i.e. for relations $\mathcal{R}$ that are generated by freely independent subequivalence relations $\mathcal{R}_i\subset\mathcal{R}, 1\leqslant i\leqslant n$.

PROPOSITION 7.4. (1) *Let M_i, $1\leqslant i\leqslant n$, be II_1 factors, for some $2\leqslant n\leqslant\infty$, with a common Cartan subalgebra A and assume that $A\subset M=M_1*_AM_2*_A...*_AM_n$ is Cartan. If $p\in A$ is a projection of trace $1/m$ for some integer $m\geqslant 1$ then the Cartan subalgebra inclusion $Ap\subset pMp$ is naturally isomorphic to $Ap\subset M_0*_{Ap}pM_1p*_{Ap}...*_{Ap}pM_np$, where $(Ap\subset M_0)=(Ap\subset Ap\rtimes\mathbf{F}_{(n-1)(m-1)})$ for some free action of the free group $\mathbf{F}_{(n-1)(m-1)}$ on Ap.*

(2) *Let $\mathcal{R}_1,...,\mathcal{R}_n$ be freely independent countable ergodic m.p. equivalence relations on the same standard probability space (X,μ) and denote the equivalence relation they generate by $\mathcal{R}$. If $Y\subset X$ is a subset of measure $1/m$ then the restriction $\mathcal{R}^Y$ of $\mathcal{R}$ to Y is generated by the freely independent ergodic subequivalence relations $\mathcal{R}_i^Y$, $1\leqslant i\leqslant n$, and $\mathcal{R}_0$, where $\mathcal{R}_i^Y$ is the restriction of $\mathcal{R}_i$ to Y and $\mathcal{R}_0$ is generated by a free m.p. action of a free group with $(n-1)(m-1)$ generators $\mathbf{F}_{(n-1)(m-1)}$ on Y.*

Proof. It is clearly sufficient to prove part (1). By the representation of AFP algebras (1.1), the von Neumann algebra generated by pM_ip, $1\leqslant i\leqslant n$, in pMp is isomorphic to the AFP algebra $pM_1p*_{Ap}...*_{Ap}pM_np$. On the other hand, since $\tau(p)=1/m$ and each M_i is a factor, by Dye's Theorem [D], there exist matrix units $\{e_i^{jk}\}_{1\leqslant j,k\leqslant m}$ in the normalizing groupoid $\mathcal{GN}_{M_i}(A)$ of $A\in M_i$ such that $e_i^{1,1}=p$ and $e_i^{jj}=e_{i'}^{jj}$, for all $1\leqslant i,i'\leqslant n$ and for all $1\leqslant j\leqslant m$. Let $u_i^j=e_1^{1,j}e_i^{j,1}\in pMp$, $2\leqslant i\leqslant n$, $2\leqslant j\leqslant m$, and notice that there are $(n-1)(m-1)$ such unitary elements. The expansion (1.1) of $M_1*_A...*_AM_n$ implies that $\{u_i^j\}_{i,j}$ are the generators of a free group $\mathbf{F}_{(n-1)(m-1)}$, all of whose elements $\neq 1$ are perpendicular to Ap. Since $A\subset M$ is Cartan, this implies that the action induced by $\mathbf{F}_{(n-1)(m-1)}$ on Ap is free. Moreover, if we let $M_0=Ap\vee\{u_i^j\}_{i,j}''\simeq A_p\rtimes\mathbf{F}_{(n-1)(m-1)}$, then it is immediate to check that if $v=v_{i_1}v_{i_2}...v_{i_j}\in pMp$ is an "alteranting word", with v_{i_l} in the normalizing groupoid of Ap in $pM_{i_l}p$ for all l, and $i_1\neq i_2\neq...\neq i_k$ in $\{0,1,...,n\}$, $E_{Ap}(v_{i_l})=0$ for all l, then v has expectation 0 on Ap as well, $E_{Ap}(v)=0$. Thus, if we denote by $N\subset pMp$ the von Neumann algebra generated by $pM_ip, 0\leqslant i\leqslant n$, then $(Ap\subset N)=(Ap\subset M_0*_{Ap}pM_1p*_{Ap}...*_{Ap}pM_np)$.

Finally, since $e_1^{j,1}u_i^j=e_i^{j,1}$, we have that M is generated by $N=pNp$ and the matrix unit $\{e_1^{jk}\}_{j,k}$. This also implies that pMp is generated by $\bigvee_{i=0}^n pM_ip$. Altogether, this shows that $(Ap\subset pMp)=(Ap\subset N)=(Ap\subset M_0*_{Ap}pM_1p*_{Ap}...*_{Ap}pM_np)$. □

The above result shows in particular that if $\mathbf{F}_n\curvearrowright X$ is a free m.p. action on the probability space with restrictions to each of the generators of $\mathbf{F}_n$ acting ergodically on X, then the amplification of the corresponding orbit equivalence relation $\mathcal{R}_{\mathbf{F}_n}$ by $1/m$ is an equivalence relation that can be induced by a free ergodic m.p. action of a free group with $m(n-1)+1$ generators, thus recovering the "compression formula" $\mathcal{R}_{\mathbf{F}_n}^{1/m}=\mathcal{R}_{\mathbf{F}_{m(n-1)+1}}$ in [Hj]. On the other hand, Proposition 7.4 can be viewed as an AFP version of the "compression formula" for plain free product factors in [DyR].

Theorem 7.7 below will require the following notation.

Notation 7.5. Let $\{\Gamma_{ij}\}_{j=0}^{n_i}$ be discrete countable groups, $1\leqslant n_i\leqslant\infty$, $i=1,2$. Let $G_i=\Gamma_{i,0}*\Gamma_{i,1}*...*\Gamma_{i,n_i}$, $i=1,2$. Let $\sigma_i: G_i\to\operatorname{Aut}(X_i,\mu_i)$ be a free m.p. action on a standard probability space (X_i,μ_i), $i=1,2$. Let $A_i=L^\infty(X_i,\mu_i)$, $M_i=A_i\rtimes_{\sigma_i}G_i$ and $M_{ij}=A_i\rtimes_{\sigma_{ij}}\Gamma_{ij}$, where $\sigma_{ij}=\sigma_i|_{\Gamma_{ij}}$ for all $0\leqslant j\leqslant n_i$, $i=1,2$.

The general result that we prove shows that, under suitable weak rigidity conditions on the groups Γ_{ij}, an isomorphism between the factors M_1 and M_2 must take each of the "component inclusions" $(A_i\subset M_{ij})$ onto each other, modulo some permutation of indices and unitary conjugacy. Since the weak rigidity assumption on the Γ_{ij}'s is somewhat technical, we display the conditions separately and give right away a list of examples when they are satisfied.

Assumption 7.6. $\Gamma_{1,0}$ and $\Gamma_{2,0}$ have the Haagerup property, and if both $\Gamma_{1,0}$ and $\Gamma_{2,0}$ are finite then we must have $n_i\geqslant 2$ for at least one $i\in\{1,2\}$. For each $j\geqslant 1$, $i=1,2$, Γ_{ij} contains a subgroup H_{ij} with the following properties:

(a) The subgroup H_{ij} is non-virtually abelian and the pair (Γ_{ij},H_{ij}) has the relative property (T) ([M]; see also [dHV]).

(b) The normalizer N_{ij} of H_{ij} in Γ_{ij} is ICC in Γ_{ij} (i.e. $|\{hgh^{-1}:h\in N_{ij}\}|=\infty$ for all $g\in\Gamma_{ij}\setminus\{e\}$) and σ_{ij} is ergodic on N_{ij}.

(c) For any proper intermediate subgroup $N_{ij}\subset N'_{ij}\subset\Gamma_{ij}$ there exists $g\in\Gamma_{ij}\setminus N'_{ij}$ such that $g(N'_{ij})g^{-1}\cap N'_{ij}$ is non-virtually abelian.

Note that condition (c) above on the inclusion $N_{ij}\subset\Gamma_{ij}$ is similar to the *wq-normal* condition in [P6] and [P8]. It is equivalent to the existence of a well-ordered strictly increasing family of intermediate subgroups $\{G_l:0\leqslant l\leqslant L\}$ such that $G_0=N_{ij}$, $G_L=\Gamma_{ij}$ and $G_{k+1}=\{g\in\Gamma_{ij}:gG_kg^{-1}\cap G_k$ is non-virtually abelian$\}$ for all k.

If Γ_{ij}, $j\geqslant 1$, is ICC and has a normal non-virtually abelian subgroup with the relative property (T), then conditions (a), (b) and (c) are trivially satisfied.

Related to conditions (a) and (b), note that if a non-virtually abelian group H is normal in an ICC group G, then $L(H)$ has no type-I summand, i.e. it is of type II_1. Indeed, this is because G acts ergodically on the center of $L(H)$, so if $L(H)$ has non-zero type-I part then it is homogeneous of type I_n for some $2\leqslant n<\infty$, contradicting [T], [Ka].

Theorem 7.7. (vNE Bass–Serre rigidity) *With Notation* 7.5 *and Assumption* 7.6, *if* $\theta: M_1 \simeq M_2^t$ *for some* $t>0$, *then* $n_1=n_2$ *and there exist a permutation* π *of indices* $j \geqslant 1$ *and unitary elements* $u_j \in M_2^t$ *such that*, *for all* $j \geqslant 1$, $\mathrm{Ad}(u_j)(\theta(M_{1,j}))=M_{2,\pi(j)}^t$ *and* $\mathrm{Ad}(u_j)(\theta(A_1))=A_2^t$. *In particular*, $\mathcal{R}_{\sigma_1} \simeq \mathcal{R}_{\sigma_2}^t$ *and* $\mathcal{R}_{\sigma_{1,j}} \simeq \mathcal{R}_{\sigma_{2,\pi(j)}}^t$ *for all* $j \geqslant 1$. *Also*, *if* $\Gamma_{1,0}=\Gamma_{2,0}=1$, $2 \leqslant n_1 < \infty$, *then the existence of such an isomorphism forces* t *to be* 1.

Proof. We denote by $Q_{ij} \subset M_{ij}$ the "rigid part" of M_{ij}, i.e. $Q_{ij}=L(H_{ij})$, where $H_{ij} \subset \Gamma_{ij}$ is a subgroup satisfying properties (a), (b) and (c) in Assumption 7.6. Also, we denote by P_{ij} the von Neumann algebra generated by the normalizer of Q_{ij} in M_{ij}. Thus, $P_{ij} \supset L(N_{ij})$, where N_{ij} is the normalizer of H_{ij} in Γ_{ij} in (b). Notice that Assumption 7.6 implies that $P_{ij}' \cap M_{ij}=\mathbf{C}$ so that by Theorem 1.1 we also have $P_{ij}' \cap M_i=\mathbf{C}$, for all $j \geqslant 1$, $i=1,2$. Also, note that Q_{ij} is relatively rigid in M_{ij}, and thus in M_i.

Assume first that $t \leqslant 1$. For simplicity, let $A_2=A$, $M_2=M$, $G_2=G$, $\Gamma_{2,j}=\Gamma_j$ and let $\{u_g\}_{g\in G} \subset M$ denote the canonical unitary elements. Let $q \in A$ be so that $\tau(q)=t$ and $\theta(1)=q$. Fix $i \geqslant 1$. Since $Q_{1,i} \subset M_1$ is a rigid inclusion, $Q=\theta(Q_{1,i}) \subset qMq$ is a rigid inclusion [P5].

Let us show that no corner of Q can be embedded into $M_{2,0}$ inside M. Assume by contradiction that there exist non-zero $q_0 \in \mathcal{P}(Q)$ and $p_0 \in \mathcal{P}(M_{2,0})$, a unital isomorphism ψ of q_0Qq_0 into $p_0M_{2,0}p_0$, and a non-zero partial isometry $v \in M_{2,0}$ such that $v^*v \in (q_0Qq_0)' \cap q_0Mq_0$, $vv^* \in \psi(q_0Qq_0)' \cap p_0Mp_0$ and $vy=\psi(y)v$ for all $y \in q_0Qq_0$. Since Q is of type II, $\psi(q_0Qq_0) \subset M_{2,0}$ is of type II, so no corner of $\psi(q_0Qq_0)$ can be embedded into A. By Theorem 1.1, this implies that $\psi(q_0Qq_0)' \cap M \subset M_{2,0}$. Thus

$$vQv^* = v(q_0Qq_0)v^* \subset M_{2,0}.$$

But $\Gamma_{2,0}$ has the Haagerup property, so, by [P5], there exist unital trace-preserving A-bimodular c.p. maps ϕ_n on $M_{2,0}$ such that $\phi_n \to \mathrm{id}_{M_{2,0}}$ and ϕ_n is compact relative to A. Then $\phi_n * \mathrm{id} \to \mathrm{id}_M$. But, by [P5], $vQv^* \subset vv^*Mvv^*$ is a rigid inclusion. By [P5] again, this implies that

$$\lim_{n\to\infty} \|(\phi_n * \mathrm{id})(x)-x\|_2=0$$

uniformly for $x \in (vQv^*)_1$. Since $(\phi_n * \mathrm{id})(x)=\phi_n(x)$ for $x \in M_{2,0} \supset vQv^*$, this implies that the maps ϕ_n, which are A-bimodular and compact relative to A, tend uniformly to the identity on the unit ball of the type-II_1 algebra vQv^*. By [P5], this implies that a corner of vQv^* can be embedded into A inside $M_{2,0}$, a contradiction.

Since no corner of Q can be embedded into $M_{2,0}$ inside M, we can apply Theorem 5.1. Thus, there exist $j=j(i) \geqslant 1$, a non-zero projection $q' \in Q' \cap qMq$ and a unitary element $u \in M$, such that $uQq'u^* \subset qM_{2,j}q=q(A \rtimes \Gamma_{2,j})q$. Since $P=\theta(P_{1,j})$ is generated by the normalizer of Q, we have $q' \in P$ and, by [P8, I, Lemma 3.5 (1)], $Qq' \subset q'Pq'$ is quasi-regular.

By Theorem 1.1, it follows that $uq'Pq'u^* \subset qM_{2,j}q$. Since P is a factor and a corner of it is contained in $qM_{2,j}q$, it follows that u can be suitably modified in order to satisfy $uPu^* \subset qM_{2,j}q$. If $v \in qMq$ is such that $P_v = vPv^* \cap P$ is of type II_1, then $P_v \subset qM_{2,j}q$ and no corner of P_v can be embedded into the abelian algebra A. Since $(P_v)v = vP \subset vM_{2,j}$, by Theorem 1.1 it follows that $v \in M_{2,j}$. By applying this recursively, by Assumption 7.6 (c) it follows that $\theta(L(\Gamma_{1,i})) \subset qM_{2,j}q$.

Thus, if $\{u_{1,h} : h \in G_1\}$ denotes the canonical unitary elements in $M_1 = A_1 \rtimes G_1$, then $v_h = \theta(u_{1,h}) \in qMq$, $h \in G_1$, are in the normalizer of $\theta(A_1)$ in qMq. Thus, the unitary elements $\{uv_hu^* : h \in \Gamma_{1,j}\} \subset qM_{2,j}q$ normalize $u\theta(A_1)u^*$ and they generate a II_1 von Neumann algebra $N \subset qM_{2,j}q$. By Theorem 1.8, this implies that $u\theta(A_1)u^* \subset M_{2,j}$ and $w(u\theta(A_1)u^*)w^* = Aq$ for some $w \in \mathcal{U}(qM_{ij}q)$. Taking $v = wu$, it follows that

$$v(\theta(L(\Gamma_{1,i})))v^* \subset M_{2,j} \quad \text{and} \quad v\theta(A_1)v^* \subset M_{2,j}.$$

Thus $v\theta(M_{1,i})v^* \subset M_{2,j}$.

This shows that if $t \leqslant 1$, then for all $i \geqslant 1$ there exists $j = j(i) \geqslant 1$ (unique by Theorem 1.1) and a unitary element $v \in M_2^t$ such that

$$\mathrm{Ad}(v)\theta(M_{1,i}) = M_{2,j}^t \quad \text{and} \quad \mathrm{Ad}(v)(\theta(A_1)) = A_2^t.$$

Let us now consider the case $t > 1$. Let $n \geqslant t$ be an integer and note that if we let $B = M_{n\times n}(A_2) \subset M_{n\times n}(M_2) = M$, $G = G_2$ and extend σ_2 to the action σ which acts trivially on $M_{n\times n}(\mathbf{C}) \subset M_{n\times n}(M_2) = M$, then $M = B \rtimes_\sigma G_2$. Let $q \in A = D_n \otimes A_2$ be a projection of trace t/n.

The hypothesis then states that $\theta : M_1 \simeq qMq$ is an onto isomorphism. Fix $i \geqslant 1$ and let $Q = \theta(Q_{1,i}) \subset qMq$ and $P = \theta(P_{1,i}) \subset qMq$. As in the case $t \leqslant 1$, it follows that there exist $j = j(i) \geqslant 1$ and a unitary element u in $qMq = M_2^t$ such that

$$uPu^* \subset q(B \rtimes \Gamma_{2,j})q = M_{2,j}^t.$$

In particular, $\{\theta(u_{1,h}) : h \in \Gamma_{1,i}\} \subset qM_{2,j}q$ and they normalize $u\theta(A_1)u^*$. By applying Theorem 1.8 to $A_0 = u\theta(A_1)u^* \subset qBq$, it follows that $u(\theta(A_1))u^* \subset q(B \rtimes \Gamma_{2,j})q$ as well, so that $\mathrm{Ad}(u)(\theta(M_{1,i})) \subset M_{2,j}^t$. Since $\mathrm{Ad}(u)(\theta(A_1))$ is regular in $M_2^t = qMq$, by Corollary 1.4 and [P5, §A.1] it follows that there exists a unitary element in $M_{2,j}$ that conjugates $\theta(A_1)$ onto A_2^t.

Since we have dealt with both cases $t \geqslant 1$ and $t \leqslant 1$, we can apply the above equally well to θ and θ^{-1}, to obtain the following: for all $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant n_2$, there exist unitary elements $u_i \in M_2^t$ and $v_j \in M_1$, and indices $j(i) \in \{1, 2, ..., n_2\}$ and $i(j) \in \{1, 2, ..., n_1\}$

such that

$$\theta(A_1\subset M_{1,i})\subset u_i(A_2\subset M_{2,j(i)})^t u_i^*,\quad 1\leqslant i\leqslant n_1,$$
$$\theta^{-1}((A_2\subset M_{2,j})^t)\subset v_j(A_1\subset M_{1,i(j)})v_j^*,\quad 1\leqslant j\leqslant n_2,$$

so, altogether,

$$\theta(M_{1,k})\subset \mathrm{Ad}(u_k\theta(v_{j(k)}))(M_{1,i(j(k))}),\quad 1\leqslant k\leqslant n_1,$$

which, by Theorem 1.1, implies $i(j(k))=k$ for all k. Similarly, $j(i(k))=k$ for all k. Thus, $n_1=n_2=n$ and j defines a permutation π of the set of indices $1\leqslant i\leqslant n$.

To prove the last part, note that the equivalence $\mathcal{R}_{\sigma_1}\simeq\mathcal{R}_{\sigma_2}^t$ and [G1] imply that

$$\sum_{j=1}^n \beta_1(\Gamma_{1,j})+(n-1)=\beta_1(\mathcal{R}_{\sigma_1})=\frac{\beta_1(\mathcal{R}_{\sigma_2})}{t}=\frac{1}{t}\biggl(\sum_{j=1}^n \beta_1(\Gamma_{2,j})+(n-1)\biggr).$$

On the other hand, by the equivalence $\mathcal{R}_{\sigma_{1,j}}\simeq\mathcal{R}^t_{\sigma_{2,\pi(j)}}$, we get $\beta_1(\Gamma_{1,j})=\beta_1(\Gamma_{2,\pi(j)})/t$ for all $j\geqslant 1$, while, by [BeVa] and Assumption 7.6, all $\Gamma_{1,j}$'s have 0 as first ℓ^2-Betti number, $\beta_1(\Gamma_{1,j})=0$. (Indeed, this follows easily from [BeVa, Corollary 4] and the argument in the proof of [P6, Lemma 2.4].) Altogether we get $n-1=(n-1)/t$, implying that $t=1$. □

Before stating specific OE applications, recall from [Fu1] that two groups Γ and Λ are said to be *measure equivalent* (ME) with *dilation constant* $t>0$ if there exists free m.p. actions (σ,Γ) and (θ,Λ) such that $\mathcal{R}_\sigma\simeq\mathcal{R}_\theta^t$. We will use the notation $\Gamma\sim_{\mathrm{OE}_t}\Lambda$ to denote this property. It was recently proved in [G2] that if $\Gamma_i\sim_{\mathrm{OE}_1}\Lambda_i$ for all $i\geqslant 0$, then $*_j\Gamma_j\sim_{\mathrm{OE}_1} *_j\Lambda_j$. Note that an alternative proof of this fact follows from Proposition 7.3 (1) (see also [MoS, Comment 2.27]).

The OE rigidity result below, of Bass–Serre type, can alternatively be viewed as a converse to Gaboriau's ME result above, for free products of w-rigid ICC groups. Note however that we need the actions involved to be "separately ergodic" (which is not assumed in [G2]).

Corollary 7.8. (OE Bass–Serre rigidity) *Let Γ_0 and Λ_0 be Haagerup groups, and Γ_i and Λ_j, $1\leqslant i\leqslant n\leqslant\infty$, $1\leqslant j\leqslant m\leqslant\infty$, be* ICC *groups having normal non-virtually abelian subgroups with the relative property* (T). *Let σ (resp. θ) be a free ergodic m.p. action of $\Gamma=\Gamma_0*\Gamma_1*...$ (resp. $\Lambda=\Lambda_0*\Lambda_1*...$) on the probability space such that $\sigma_j=\sigma|_{\Gamma_j}$ (resp. $\theta_j=\theta|_{\Lambda_j}$) is ergodic for all $j\geqslant 1$. If $\mathcal{R}_{\sigma,\Gamma}\simeq\mathcal{R}^t_{\theta,\Lambda}$, then $n=m$ and there exists a permutation π of the set of indices $\geqslant 1$ such that $\mathcal{R}_{\sigma_i,\Gamma_i}\simeq\mathcal{R}^t_{\theta_{\pi(i)},\Lambda_{\pi(i)}}$ for all $i\geqslant 1$.*

The condition on the groups Γ_i and Λ_j, $i,j\geqslant 1$, in Corollary 7.8 can be weakened by using the full generality of Theorem 7.7.

COROLLARY 7.8′. *The same statement as in Corollary* 7.8 *holds true if we assume that each* $(\sigma|_{\Gamma_i}, \Gamma_i)$ *(resp.* $(\theta|_{\Lambda_i}, \Lambda_i)$*),* $i \geqslant 1$*, satisfies Assumption* 7.6.

COROLLARY 7.9. *Let* Γ_i*,* $0 \leqslant i \leqslant n$*,* $\Gamma = \Gamma_0 * \Gamma_1 * ...$ *and* σ *be as in Corollaries* 7.8 *or* 7.8′*. Assume that* $\mathrm{Out}(\mathcal{R}_{\sigma_1,\Gamma_1}) = \{1\}$ *and* (σ_1, Γ_1) *is not orbit-equivalent to* (σ_i, Γ_i) *for any* $i \neq 1$*. Then* $\mathrm{Out}(\mathcal{R}_{\sigma,\Gamma}) = \{1\}$ *and* $\mathrm{Out}(A \rtimes_\sigma \Gamma) = \mathrm{H}^1(\sigma, \Gamma)$*. Also, if* n *is finite and either all* Γ_i *are finitely generated or there exists* $i \geqslant 1$ *with* $\beta_n(\Gamma_i) \neq 0, \infty$*, then*

$$\mathfrak{F}(A \rtimes_\sigma \Gamma) = \{1\}.$$

Proof. The first part is trivial by Theorem 7.7. The last part follows from [G1]. □

Outer automorphism groups of equivalence relations are usually hard to calculate and there are only a few special families of group-actions (σ_1, Γ_1) for which one knows that $\mathrm{Out}(\mathcal{R}_{\sigma_1}) = \{1\}$ (cf. [Ge2], [Fu2] and [MoS]). Similarly for the 1-cohomology group H^1, where the only known calculations are in [Ge1], [PSa] and [P6]. Below we recall some examples from [Ge2], [Fu2] and [MoS], where both calculations can be made. We add a new construction of examples, in Example 7.12 below, which uses the Monod–Shalom OE rigidity theorem to calculate Out and [P6] to calculate H^1.

Example 7.10. (Gefter [Ge2], Furman [Fu2]) Take Γ_1 to be a lattice in $\mathrm{SO}(p, q)$, with $p > q \geqslant 2$ and notice that Γ_1 has $\mathrm{rk}_{\mathbf{R}} \geqslant 2$ (thus has property (T)) and admits a dense embedding into the compact Lie group $\mathrm{SO}(n)$, where $n = p + q$. Let σ_1 be the action by left translation of Γ_1 on the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$. Then $\mathrm{Out}(\mathcal{R}_{\sigma_1}) = \{1\}$.

Example 7.11. (Monod–Shalom [MoS]) Let $\mathcal{G} = \mathrm{SO}(n)$, $n \geqslant 5$, and let Λ_0 be an ICC torsion-free Kazhdan group which admits a dense embedding into $\mathcal{G}$ and has no outer automorphisms (such groups exist for any $n \geqslant 5$, for instance lattices in $\mathrm{SO}(p, q)$ with $p > q \geqslant 2$ and $n = p + q$ as in Example 7.10). Let K be any torsion-free group embeddable into $\mathcal{G}$ and $K_0 \subset K$ be a non-trivial subgroup such that K_0 is not isomorphic to K (for instance, $K_0 = \mathbf{F}_r \subset \mathbf{F}_s = K$, for some $s > r \geqslant 1$). Let $\Gamma_1 = (\Lambda_0 * K) \times (\Lambda_0 * K_0)$ and note that any automorphism of the group Γ_1 is inner on $\Lambda_0 \times \Lambda_0$ (by Kurosh or Bass–Serre). Let σ_1 be the action of $\Gamma_1 = (\Lambda_0 * K) \times (\Lambda_0 * K_0)$ on $\mathcal{G}$ by left-right translation. Notice that this action is free ergodic on $\Lambda_0 \times \Lambda_0$ and that, by [MoS], one can choose the embedding $K \subset \mathcal{G}$ such that σ_1 is free on Γ_1 (by considering all embeddings gKg^{-1}, $g \in \mathcal{G}$, and using a Baire category argument). Then Γ_1 is in the class $\mathcal{C}_{\mathrm{geom}}$ of Monod–Shalom and $\mathrm{Out}(\mathcal{R}_{\sigma_1}) = \{1\}$.

Example 7.12. Let this time Λ_0 be any torsion-free ICC group with only inner automorphisms and which cannot be decomposed as $\mathbf{Z} * \Lambda_0'$ (note that if Λ_0 is w-rigid then it does have this latter "free indecomposability" property). Let K be any torsion-free group with $K_0 \subset K$ being a non-trivial subgroup such that K_0 is not isomorphic to K. Let $\Lambda=\Lambda_0*K$ and $\Gamma_1=(\Lambda_0*K)\times(\Lambda_0*K_0)\subset\Lambda\times\Lambda$. Denote by σ_1 the action of Γ_1 on the product probability space $(X,\mu)=\prod_{g\in\Lambda}(X_0,\mu_0)_g$ by left-right (double) Bernoulli shifts. Then $\mathrm{Out}(\mathcal{R}_{\sigma_1})=\{1\}$. Indeed, noticing that $\Lambda_0*K, \Lambda_0*K_0\in\mathcal{C}_{\mathrm{geom}}$ and that σ_1 is separately ergodic (on Λ_0*K and Λ_0*K_0), it follows from [MoS] that any automorphism $\theta\in\mathrm{Aut}(\mathcal{R}_{\sigma_1})$ is an inner perturbation of a conjugacy $\sigma_1\sim\sigma_1\circ\gamma$ with respect to some $\gamma\in\mathrm{Aut}(\Gamma_1)$. But Γ_1 has only inner automorphisms (again by Kurosh or Bass–Serre). Thus, any such θ is an inner perturbation of an automorphism of the probability space that commutes with $\sigma_1(\Gamma_1)$. But this commutant is trivial if for instance (X_0,μ_0) is atomic with non-equal weights, as shown by the following proposition.

PROPOSITION 7.13. *Let Λ be a countable discrete group with two subgroups Λ_1 and Λ_2 such that $\Lambda_0=\Lambda_1\cap\Lambda_2$ satisfies $|\{hgh^{-1}:h\in\Lambda_0\}|=\infty$ for all $g\in\Lambda$, $g\neq e$. Let (B_0,τ_0) be a finite von Neumann algebra and let $(B,\tau)=\prod_{g\in\Lambda}(B_0,\tau_0)_g$. Let σ be the action of $\Lambda_1\times\Lambda_2$ on (B,τ) given by $\sigma_{h_1,h_2}((x_g)_g)=(x'_g)_g$, where $x'_g=x_{h_1^{-1}gh_2}$ for $g\in\Lambda$, $h_1\in\Lambda_1$ and $h_2\in\Lambda_2$. Then σ is a free separately mixing action of $\Lambda_1\times\Lambda_2$ on (B,τ) and the following are true*:

(1) *If $\theta\in\mathrm{Aut}(B,\tau)$ commutes with $\sigma(\Lambda_1\times\Lambda_2)$ then there exists a unique*

$$\theta_0\in\mathrm{Aut}(B_0,\tau_0)$$

such that θ is the product action given by θ_0, i.e. $\theta=\prod_g(\theta_0)_g$.

(2) *Any $\delta\in\mathrm{Aut}(\Lambda)$ satisfying $\delta(\Lambda_i)=\Lambda_i$, $i=1,2$, induces an automorphism $\Delta=\Delta(\delta)\in\mathrm{Aut}(B,\tau)$, by $\Delta((b_g)_g)=(\delta(b_g))_g$, which satisfies $\Delta\sigma\Delta^{-1}=\sigma\circ\delta$.*

(3) *If $(B_0,\tau_0)=(L^\infty(X_0,\mu_0),\int\cdot\,d\nu_0)$ for some atomic probability space (X_0,μ_0), then the commutant of $\sigma(\Lambda_1\times\Lambda_2)$ in $\mathrm{Aut}(X,\mu)$ is equal to $\mathrm{Aut}(X_0,\mu_0)=\mathrm{Aut}(B_0,\tau_0)$. Moreover, if σ is conjugate to another double Bernoulli shift σ' with atomic base space (X'_0,μ'_0), then $(X_0,\mu_0)\simeq(X'_0,\mu'_0)$.*

Proof. Part (2) is trivial. To prove part (1), it is sufficient to show that any θ commuting with σ must take the subalgebra $B_0^e=\ldots 1\otimes(B_0)_e\otimes 1\ldots$ of B into itself. Indeed, because if we let $\theta_0=\theta|_{B_0^e}$ and regard it as an automorphism of B_0 then $\theta\circ(\prod_g(\theta_0)_g)^{-1}$ still commutes with $\sigma(\Lambda_1\times\Lambda_2)$ and it acts as the identity on B_0^e, and thus on $\sigma(g_1,g_2)(B_0^e)$ for all $g_1,g_2\in\Lambda_0$. Since $\Lambda_1\Lambda_2=\Lambda$, the latter generate all of B. Thus $\theta=\prod_g(\theta_0)_g$.

To show that θ leaves B_0^e globally invariant, it is sufficient to show that the fixed point algebra $\{b\in B:\sigma(g,g)(b)=b \text{ for all } g\in\Lambda_0\}$ coincides with B_0^e. This in turn follows trivially from the fact that for any finite subset $F\subset\Lambda\setminus\{e\}$ there exists $g\in\Lambda_0$ such that $gFg^{-1}\cap F=\varnothing$. To see that this latter property holds true, note that if some finite set $\varnothing\neq F\subset\Lambda\setminus\{e\}$ would satisfy $|gFg^{-1}\cap F|\geqslant 1$, for all $g\in\Lambda_0$, then the "left-right" representation $\pi(g)(f)=\lambda(g)\varrho(g)(f)$ on $\ell^2(\Lambda)$ would satisfy $\langle\pi(g)(\chi_F),\chi_F\rangle\geqslant|F|^{-1}$ for all $g\in\Lambda_0$.

Taking the element f of minimal Hilbert norm in $\overline{\mathrm{co}}^w\{\pi(g)(\chi_F):g\in\Lambda_0\}\subset\ell^2(\Lambda)$, it follows that $f\geqslant 0$, $f\neq 0$ (because $\langle f,\chi_F\rangle\geqslant|F|^{-1}$) and $\pi(g)(f)=f$ for all $g\in\Lambda$. But then any appropriate "level set" K for f will be finite non-empty, and will satisfy $\pi(g)(\chi_K)=\chi_K$ for all $g\in\Lambda_0$, i.e. $gKg^{-1}=K$ for all $g\in\Lambda_0$, implying that Λ has elements with finite conjugacy class, a contradiction.

The first part of (3) is trivial by (2). Then to see that conjugacy of double Bernoulli shifts entails isomorphism of the base spaces, note that σ conjugate to σ' implies that all "diagonal" actions $\sigma\otimes\sigma''$ and $\sigma'\otimes\sigma''$ must also be conjugate, for all σ'', thus having isomorphic commutants in Aut. Taking σ'' to be itself a double Bernoulli Γ_1-action of base (X_0'',μ_0''), it follows that $\mathrm{Aut}((X_0,\mu_0)\times(X_0'',\mu_0''))\simeq\mathrm{Aut}((X_0',\mu_0')\times(X_0'',\mu_0''))$ for all (X_0'',μ_0''), which easily implies the result. □

Notation 7.14. Denote by $w\mathcal{T}_2$ the class of groups G that have a non-virtually abelian subgroup $H_0\subset G$ such that: (G,H_0) is a property (T) pair; the normalizer H of H_0 in G satisfies $|\{hgh^{-1}:h\in H\}|=\infty$ for all $g\in G\setminus\{e\}$; the wq-normalizer of H in G generates G. Note that any group in $w\mathcal{T}_2$ is ICC and that if G is ICC and has a normal non-abelian relatively rigid subgroup then $G\in w\mathcal{T}_2$. Thus, any group of the form $G=H_0\times K$ with H_0 being ICC Kazhdan and K being either ICC or equal to 1, is w-rigid and thus in $w\mathcal{T}_2$. Also, if $G\in w\mathcal{T}_2$ then $(G*K_0)\times K\in w\mathcal{T}_2$ for any ICC group K and any arbitrary group K_0.

COROLLARY 7.15. *Let Γ_0 be a Haagerup group and $\Gamma_i\in w\mathcal{T}_2$ for $1\leqslant i\leqslant n$, where $1\leqslant n\leqslant\infty$. Assume that Γ_1 is as in Examples* 7.10–7.12. *Then $\Gamma=\Gamma_0*\Gamma_1*...$ has a free ergodic m.p. action σ with* $\mathrm{Out}(\mathcal{R}_\sigma)=\{1\}$. *Moreover, the following are true*:

(1) *If Γ_1 is as in Examples* 7.11 *or* 7.12, *then there exist uncountably many non-stably orbit-equivalent actions σ of Γ with* $\mathrm{Out}(\mathcal{R}_\sigma)=\{1\}$ *and* $\mathfrak{F}(\mathcal{R}_\sigma)=\{1\}$.

(2) *If in addition Γ_0 is a product of amenable groups, then given any discrete countable abelian group K, the uncountable family of actions σ in* (1) *can be taken to satisfy* $\mathrm{H}^1(\sigma,\Gamma)=\mathbf{G}_0^{n-1}\times\mathbf{G}\times\prod_{j\geqslant 1}\mathrm{Char}(\Gamma_j)\times K^{n-1}$, *where $\mathbf{G}$ is the Polish group $\mathcal{U}(L^\infty(\mathbf{T},\lambda))$ and* $\mathbf{G}_0=\mathbf{G}/\mathbf{T}$.

Proof. By Proposition 7.3, we can take the free m.p. action σ of Γ on $A=L^\infty(X,\mu)$ so that for each $i\neq 1$, $\sigma_i=\sigma|_{\Gamma_i}$ is a (left) Bernoulli Γ_i-action, or a quotient of it as in [P6], and of one of the forms in Examples 7.10–7.12 for $i=1$. Notice that in Example 7.10 the group Γ_1 has property (T) and is ICC, and thus $\Gamma_1\in w\mathcal{T}_2$. Then, in both Examples 7.11 or 7.12, Γ_1 has $\Lambda_0\times\Lambda_0$ as a relatively rigid subgroup, which is wq-normal in Γ_1, with $\Lambda\times\Lambda$ being ICC in Γ_1. Furthermore, since $\mathrm{Out}(\mathcal{R}_{\sigma_i})$ is huge for $i\geqslant 2$ and trivial for $i=1$, σ_1 cannot be stably orbit-equivalent to σ_i, $i\geqslant 2$. The existence of "many actions" σ in the cases of Examples 7.11 and 7.12 follows from the existence of uncountably many non-stably OE relations (σ_1,Γ_1) of the form in Example 7.11 (cf. [MoS]), and respectively of the form in Example 7.12 (by Proposition 7.13 (3) and [MoS]).

Finally, the calculation of the 1-cohomology groups follows from [P6, Corollary 2.12, Lemmas 3.1, 3.2] and the fact that in all the cases of Examples 7.10, 7.11 and 7.12 one has $\mathrm{H}^1(\sigma_1,\Gamma_1)=\mathrm{Char}(\Gamma_1)$. Indeed, in case σ_1 is as in Example 7.10 or 7.11, then this calculation follows from [Ge1] and [P6, Corollary 2.12], while in the case of Example 7.12 the calculation is in [P6]. □

Corollary 7.16. *Let* $\Gamma={*}_{i\geqslant 0}\Gamma_i$, K *and* σ *be as in Corollary* 7.15 (2), *and let* $A=L^\infty(X,\mu)$ *and* $M=A\rtimes_\sigma\Gamma$. *Then* $\mathfrak{F}(M)=\{1\}$ *and*

$$\mathrm{Out}(M)=\mathrm{H}^1(\sigma,\Gamma)=\mathbf{G}_0^{n-1}\times\mathbf{G}\times\prod_{j\geqslant 1}\mathrm{Char}(\Gamma_j)\times K^{n-1}.$$

Proof. This is trivial by Corollaries 7.9 and 7.15. □

Note that $\mathrm{Out}(M)$ is abelian and non-locally compact in all examples in Corollary 7.16 above, but if we denote by $\widetilde{\mathrm{Out}}(M)$ the quotient of $\mathrm{Out}(M)$ by the connected component of id_M (which is closed in $\mathrm{Out}(M)$, with the latter being a Polish group in all examples considered), then $\widetilde{\mathrm{Out}}(M)$ is the quotient of $\prod_{j\geqslant 1}\mathrm{Char}(\Gamma_j)\times K^{n-1}$ by the connected component of 1, which for $n<\infty$ is a totally disconnected separable locally compact group.

We end this section by mentioning another rigidity result, which from an isomorphism of group measure space factors corresponding to relatively rigid actions of free products of groups derives the orbit equivalence of the actions. This type of results were first obtained in [P5] for HT group actions, and in [P8] for Bernoulli shift actions of groups containing infinite subgroups (not necessarily normal) with the relative property (T).

Theorem 7.17. (vNE/OE rigidity) *Let* (M_j,τ_j) *be type*-II_1 *von Neumann algebras with a common Cartan subalgebra* $A\subset M_i$, $i=1,2$, *such that* $\tau_1|_A=\tau_2|_A$. *Assume that* $M=M_1*_A M_2$ *is a factor and* A *is Cartan in* M. (N.B. *By Lemma* 7.2, *this is the same as requiring that* $\mathcal{R}_{A\subset M_i}$, $i=1,2$, *are freely independent*). *If* $A_0\subset M^t$ *is a rigid Cartan subalgebra*, *for some* $t>0$, *then there exists a unitary element* $u\in M^t$ *such that* $uA_0u^*=A^t$.

Proof. It is clearly sufficient to prove this in the case $t=1$. If some corner of A_0 can be embedded into A inside M, then the statement follows by [P5, §A.1]. If we assume that this is not the case, then we can apply Theorem 7.7 to get a non-zero $p\in\mathcal{P}(A_0)$ such that $vA_0pv^*\subset M_i$ for some $i\in\{1,2\}$ and $v\in\mathcal{U}(M)$. Moreover, since M is a factor and A_0 is Cartan in M, we may assume that vpv^* is central in M_i, so in particular $p_1=vpv^*$ lies in A (the latter being maximal abelian in M_i). Then vA_0pv^* is Cartan in p_1Mp_1, so, by Corollary 1.4, we have $p_1M_2p_1=Ap_1$, contradicting the fact that M_2 is of type II. □

COROLLARY 7.18. *Let $\mathcal{F}$ be the class of groups that can be written as a free product of two (or more) infinite groups. Let $\sigma: G \to (X, \mu)$ be a free ergodic m.p. action of a group $G \in \mathcal{F}$ and denote by $M = L^\infty(X, \mu) \rtimes_\sigma G$ the corresponding group measure space II_1 factor, with $A = L^\infty(X, \mu) \subset M$ being the corresponding Cartan subalgebra. Let M_0 be a II_1 factor with a relatively rigid Cartan subalgebra $A_0 \subset M_0$ and let $\mathcal{R}_0 = \mathcal{R}_{A_0 \subset M_0}$.*

(1) *If $\theta: M_0 \simeq M^t$ for some $t > 0$, then θ can be perturbed by an inner automorphism so as to take A_0 onto A^t. In particular, $\mathcal{R}_0 \simeq \mathcal{R}_\sigma^t$ and thus $\beta_n^{(2)}(\mathcal{R}_0) = \beta_n^{(2)}(G)/t$ for all n.*

(2) *If $G \in \mathcal{F}$ satisfies $\beta_1^{(2)}(G) \neq 0, \infty$ for some n (for instance, if $G = \Gamma_1 * \Gamma_2$, with Γ_1 and Γ_2 being finitely generated infinite groups, in which case $\beta_1^{(2)}(G) \neq 0, \infty$) and the action σ is relatively rigid, then $\mathfrak{F}(M) = \{1\}$.*

Proof. All statements are trivial by Theorem 7.17 and [G1]. □

Note that Corollary 7.18 (2) above shows in particular that $\mathfrak{F}(L(\mathbf{Z}^2 \rtimes \mathbf{F}_n)) = \{1\}$ for any $\mathbf{F}_n \subset \mathrm{SL}(2, \mathbf{Z})$, with $2 \leqslant n < \infty$, thus giving a new proof (but still using Gaboriau's work [G1]) of one of the main results in [P5].

Definition 7.19. A countable measurable standard m.p. equivalence relation $\mathcal{R}$ is an FT *equivalence relation* if it is of the form $\mathcal{R} = \mathcal{R}_\sigma^t$, where $t > 0$ and (σ, Γ) are free ergodic m.p. actions on the probability space (X, μ) with the following properties: (a) The group Γ is a free product of two (or more) infinite groups; (b) σ is relatively rigid, in the sense of [P5, Definition 5.10.1], i.e. $L^\infty(X, \mu) \subset L^\infty(X, \mu) \rtimes_\sigma G$ is a rigid inclusion [P5, Definition 4.2.1]. The above Corollary 7.18 thus shows that all OE invariants for FT equivalence relations $\mathcal{R}$ are in fact vNE invariants for $\mathcal{R}$, i.e. are isomorphism invariants of the associated group measure space II_1 factors $M = L(\mathcal{R}, w)$, where $w \in \mathrm{H}^2(\mathcal{R})$. We denote by $\mathcal{FT}$ the class of all such II_1 factors M.

Note that if $\mathcal{R} = \mathcal{R}_{\sigma,\Gamma}^t$ for some free ergodic m.p. action (σ, Γ) with Γ being a free product of two infinite groups, then $\mathcal{R}$ is HT_s in the sense of [P5] if and only if it is FT and Γ has the Haagerup property. Thus, all equivalence relations coming from amplifications of actions σ of non-amenable subgroups $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ on $L^\infty(\mathbf{T}^2, \lambda)$ are FT actions. However, actions σ of groups such as $\mathrm{SL}(n, \mathbf{Z}) * H$, with H being an infinite group and $\sigma|_{\mathrm{SL}(n,\mathbf{Z})}$ being isomorphic to the canonical action of $\mathrm{SL}(n, \mathbf{Z})$ on $(\mathbf{T}^n, \lambda)$, give FT equivalence relations which are not HT_s. Thus, the class $\mathcal{FT}$ provides additional group measure space II_1 factors for which orbit equivalence invariants of the actions, such as Gaboriau's ℓ^2-Betti numbers, become isomorphism invariants of the factors.

We end by mentioning an application of Proposition 7.3 (1) which brings some light to [P5, Problems 5.10.2 and 6.12.1] and to the problem of existence of "many" non-OE actions for non-amenable groups, as a consequence of [GP, Corollary 7].

Corollary 7.20. (1) *The class of groups* Γ *which admit free ergodic relatively rigid m.p. actions on the probability space is closed to free products with arbitrary groups* Γ'. *Also*, *if* Γ *is an* HT_s *group* (*i.e.* Γ *has Haagerup's property and admits relatively rigid actions*, *see* [P5, 6.11]), *then* $\Gamma*\Gamma'$ *is* HT_s *for any* Γ' *with Haagerup property.*

(2) *If* Γ *admits a relatively rigid action* (*e.g. if* $\Gamma\subset\mathrm{SL}(2,\mathbf{Z})$ *is non-amenable*, *or* Γ *is an arithmetic lattice in an absolutely simple non-compact Lie group with trivial center, cf.* [P5] *and* [Va]) *and* Γ' *is an arbitrary infinite amenable group*, *then* $\Gamma*\Gamma'$ *has uncountably many non-stably* OE *free ergodic m.p. actions on the probability space. Also*, *if* Γ_0 *is an arbitrary group and* Γ_1 *and* Γ_2 *are non-trivial amenable groups*, *at least one having more than two elements*, *then* $\Gamma_0*\Gamma_1*\Gamma_2$ *has uncountably many non-stably* OE *free ergodic m.p. actions on the probability space.*

Proof. Part (1) is a trivial consequence of Proposition 7.3 (1) and of the (trivial) property of relatively rigid equivalence relations $\mathcal{R}$ that any $\mathcal{R}_0$ that contains $\mathcal{R}$ is also relatively rigid (see e.g. [P5, Proposition 4.6.2]).

Part (2) is just the combination of [GP, Corollary 7]) and Proposition 7.3. □

8. Amalgamation over *R*: factors with no outer automorphisms

In this section we prove another rigidity result for AFP factors, this time in the case $M=M_0*_R M_1*_R*\ldots$, where R is the hyperfinite II_1 factor. As an application, we obtain factors M with $\mathrm{Out}(M)=\{1\}$, thus answering a well-known problem posed by A. Connes in 1973.

Like in the group measure space case in §7, we only consider crossed product inclusions $(R\subset M_i)=(R\subset R\rtimes_{\sigma_i}\Gamma_i)$, with the σ_i being freely independent, i.e. inducing a free action σ of $\Gamma=\Gamma_0*\Gamma_1*\ldots$ on R. Thus, M will be viewed alternatively as a crossed product factor $M=R\rtimes_\sigma\Gamma$, with the algebra of coefficients R having trivial relative commutant in M.

The key assumption is that the action $(\sigma,*_i\Gamma_i)$ has the relative property (T), i.e. that $R\subset M$ is a rigid inclusion in the sense of [P5]. The rigidity result shows the uniqueness, modulo unitary conjugacy, of the "core" R of such factors. Since the normalizer of R in M completely encodes the group Γ, we can completely recover the isomorphism class of the groups Γ_i, by classical Bass–Serre theory. The result is similar to the vNE/OE rigidity Theorem 7.17 (where however only the orbit equivalence class of Γ could be recovered) and to the unique crossed product decomposition result in [P9]. But since we also get the componentwise unitary conjugacy of the factors M_i, it is again a Bass–Serre type rigidity result.

Through this theorem, the calculation of $\mathrm{Out}(M)$ and $\mathfrak{F}(M)$ reduces to the calculation of the commutant of $\sigma(\Gamma)$ in $\mathrm{Out}(R)$, like in [P9], where however no such commutant could be calculated! This time, due to Bass–Serre arguments and the possibility of choosing the actions (σ_i, Γ_i) with prescribed properties (cf. Proposition 8.2 below), we can control such commutants and calculate $\mathrm{Out}(M)$ completely for large classes of factors.

LEMMA 8.1. *Let M_n, $n \geqslant 0$, be II_1 factors with a common subfactor $N \subset M_n$. Then $N \subset M = M_0 *_N M_1 *_N M_2 *_N \ldots$ is irreducible and regular if and only if $N \subset M_n$ is regular, irreducible for all $n \geqslant 1$, and the groups of outer automorphisms*

$$\Gamma_n = \{\mathrm{Ad}(u) : u \in \mathcal{N}_{M_n}(N)\}/\mathcal{U}(N), \quad n \geqslant 0,$$

on N are freely independent.

Proof. This is trivial by the definitions of freeness and of amalgamated free product over N, respectively. □

The result below is the analogue for actions on the hyperfinite II_1 factor R of the result on the existence of freely independent actions on the probability space in Proposition 7.3. It shows the existence of free actions σ of groups $\Gamma = \Gamma_0 * \Gamma_1 * \Gamma_2 * \ldots$ on R such that the restriction of σ to each individual group Γ_j is conjugate to a prescribed free action of Γ_j on R. It will be frequently used in this section. The proof relies on Lemma A.2 in the appendix.

PROPOSITION 8.2. *Let $\sigma_n : \Gamma_n \to \mathrm{Aut}(R)$ be free actions of countable discrete groups G_n, $n \geqslant 0$. Then there exists a free action σ of the group $G = *_n \Gamma_n$ on R such that $\sigma|_{\Gamma_n}$ is conjugate to σ_n for all $n \geqslant 0$.*

Proof. For each $n \geqslant 0$ let $\widetilde{G}_n = G_0 * G_1 * \ldots * G_n$. Assume that we have constructed a map $\tilde{\sigma}_n$ of $\widetilde{G}_n$ into $\mathrm{Aut}(R)$ such that the quotient map $\tilde{\sigma}_n'$ of G into $\mathrm{Aut}(R)/\mathrm{Int}(R)$ is a faithful group morphism with $\tilde{\sigma}_n|_{G_j}$ conjugate to σ_j, for all $0 \leqslant j \leqslant n$. We then apply Lemma A.2 to $\{\tilde{\sigma}_n(g) : g \in \widetilde{G}_n\} \cup \{\sigma_{n+1}(h) : h \in G_{n+1}\}$ to get an automorphism θ_{n+1} of R such that $\tilde{\sigma}_n(\widetilde{G}_n)$ and $\theta_{n+1}\sigma_{n+1}(G_{n+1})\theta_{n+1}^{-1}$ are freely independent. Denoting by $\tilde{\sigma}_{n+1}$ the map of $\widetilde{G}_{n+1} = \widetilde{G}_n * G_{n+1}$ into $\mathrm{Aut}(R)$ which restricted to $\widetilde{G}_n$ equals $\tilde{\sigma}_n$ and restricted to G_{n+1} equals $\theta_{n+1}\sigma_{n+1}(G_{n+1})\theta_{n+1}^{-1}$, the statement follows by induction. □

THEOREM 8.3. *Let $G_i = \Gamma_{i,0} * \Gamma_{i,1} * \ldots * \Gamma_{i,n_i}$ with Γ_{ij}, $0 \leqslant j \leqslant n_i$, being non-trivial groups, for some $1 \leqslant n_i \leqslant \infty$, $i = 1, 2$. For each $i = 1, 2$ let $\sigma_i : G_i \to \mathrm{Aut}(N_i)$ be a free ergodic action on a II_1 factor N_i. Let $M_i = N_i \rtimes_{\sigma_i} G_i$, $i = 1, 2$, and assume that $N_i \subset M_i$ are rigid inclusions, $i = 1, 2$. Let $\theta : M_1 \simeq M_2^t$ for some $t > 0$. Then the following are true:*

(1) *There exists $u \in \mathcal{U}(M_2^t)$ such that $\mathrm{Ad}(u)(\theta(N_1)) = N_2^t$. Thus, $G_1 \simeq G_2$, and σ_1 and σ_2^t are cocycle conjugate actions with respect to the identification $G_1 \simeq G_2$.*

(2) *If in addition* $\Gamma_{i,0}$ *are free groups and* Γ_{ij} *are free, indecomposable and not equal to the infinite cyclic group, for all* $1\leqslant j\leqslant n_i$, $i=1,2$, *then* $\Gamma_{1,0}\simeq\Gamma_{2,0}$, $n_1=n_2$ *and there exists a permutation* π *of the indices* $j\geqslant 1$ *and unitary elements* $u_j\in M_2^t$ *such that*

$$\mathrm{Ad}(u_j)(\theta(M_{1,j}))=M_{2,\pi(j)}^t \text{ and } \mathrm{Ad}(u_j)(\theta(N_1))=N_2^t \quad \text{for all } j\geqslant 1.$$

In particular, $\Gamma_{1,j}\simeq\Gamma_{2,\pi(j)}$, *and* $\sigma_{1,j}$ *and* $\sigma_{2,\pi(j)}^t$ *are cocycle conjugate with respect to this identification of groups, for all* $j\geqslant 1$.

Proof. We first prove that a corner of $\theta^{-1}(N_2^t)$ can be embedded into N_1 inside M_1. Assume that this is not the case. By applying recursively Theorem 5.1 (2), it follows that there exist a unitary element $u\in\mathcal{U}(M_1)$ and some $1\leqslant j\leqslant n_1$ such that $u\theta^{-1}(N_2^t)u^*\subset M_{1,j}$. Since $u\theta^{-1}(N_2^t)u^*$ is regular in M_1, using again the assumption by contradiction, Corollary 1.4 implies that a corner of $u\theta^{-1}(N_2^t)u^*$ can be embedded into N_1 inside $M_{1,j}$ (and thus inside M_1 as well). Altogether, this shows that a corner of $\theta^{-1}(N_2^t)$ can be embedded into N_1 inside M_1.

Similarly, a corner of $\theta(N_1)^{1/t}=\theta^{1/t}(N_1^{1/t})$ can be embedded into N_2 inside M_2. Thus, a corner of N_1 can be embedded into $\theta^{-1}(N_2^t)$ inside M_1. Since both N_1 and $\theta^{-1}(N_2^t)$ are regular in M_1, with $\mathcal{N}_{M_1}(N_1)/\mathcal{U}(N_1)\simeq G_1$ and with the other similar quotient isomorphic to G_2, and since both G_1 and G_2 are ICC (being free products of non-trivial groups), the unitary conjugacy of N_1 and $\theta^{-1}(N_2^t)$ in M_1 (equivalently, of $\theta(N_1)$ and N_2^t in M_2^t) follows from the following general result.

Lemma 8.4. *Let* M *be a* II_1 *factor and* $P,Q\subset M$ *be irreducible regular subfactors. Assume that* $\Gamma=\mathcal{N}_M(P)/\mathcal{U}(P)$ *and* $\Lambda=\mathcal{N}_M(Q)/\mathcal{U}(Q)$ *are* ICC *groups. Also, assume that each one of the inclusions* $P\subset M$ *and* $Q\subset M$ *is an amplification of a genuine crossed product inclusion. If* $L^2(M)$ *contains non-zero* P-Q *Hilbert bimodules* $\mathcal{H},\mathcal{K}\subset L^2(M)$ *such that* $\dim({}_P\mathcal{H})<\infty$ *and* $\dim(\mathcal{K}_Q)<\infty$, *then* P *and* Q *are unitarily conjugate in* M.

Proof. We first prove that $L^2(M)$ is generated by irreducible P-Q Hilbert bimodules that are finite-dimensional both as left P modules and as right Q modules. We will actually prove this by only using the fact that P and Q are quasi-regular in M. Note that, by [P5, §1.4], $\mathcal{H}^0=\mathcal{H}\cap M$ is dense in $\mathcal{H}$ and contains an orthonormal basis over Q. Similarly, since Q is quasi-regular in M and it is a factor, $L^2(M)$ is generated by Hilbert Q-Q bimodules $\mathcal{H}_\beta$ such that $\mathcal{H}_\beta^0=\mathcal{H}_\beta\cap M$ is dense in $\mathcal{H}_\beta$ and contains both left and right orthonormal bases over Q. But then $\mathcal{H}^0\cdot\mathcal{H}_\beta^0$ span all of $L^2(M)$ and are finite-dimensional over Q. Equivalently, $P'\cap J_MQ'J_M$ is generated by projections that have finite trace in $J_MQ'J_M$. Similarly, $P'\cap JQ'J$ is generated by projections that have finite trace in P'. Thus, $\mathcal{A}=P'\cap JQ'J$ is generated by projections that are finite with respect to both traces, thus corresponding to Hilbert P-Q bimodules which are finite-dimensional both from right and left. Since P and Q are factors, by [J], each such bimodule is a direct sum of irreducible bimodules.

Let now $\mathcal{H}\subset L^2(M)$ be an irreducible P-Q bimodule. By [J], we have

$$\dim({}_P\mathcal{H})\dim(\mathcal{H}_Q)\geqslant 1$$

and the equality means that the orthogonal projection $p_{\mathcal{H}}$ of $L^2(M)$ onto $\mathcal{H}$ satisfies $p_{\mathcal{H}}\in P'\cap\langle M,e_Q\rangle$, $\mathrm{Tr}_{\langle M,e_Q\rangle}(p_{\mathcal{H}})=1$ and $p_{\mathcal{H}}\langle M,e_Q\rangle p_{\mathcal{H}}=Pp_{\mathcal{H}}$. Thus, by [P3, proof of Lemma 1], $up_{\mathcal{H}}u^*=e_Q$ for some $u\in\mathcal{U}(M)$, which also satisfies $uPu^*=Q$.

Assume now that $\mathrm{Tr}(p_{\mathcal{H}})>1$. By [P8, I, Theorem 2.1], there exist a projection $p\in P$, a unital isomorphism $\psi: pPp\to Q$ and a partial isometry $v\in M$ such that

$$vv^*=p,\quad q'=v^*v\in\psi(pPp)'\cap Q\quad\text{and}\quad xv=v\psi(x)\ \text{ for all }\ x\in pPp.$$

Moreover, the finite-dimensionality plus irreducibility of $\mathcal{H}$ as a P-Q bimodule, implies that $Q_1=\psi(pPp)$ has finite index in Q and trivial relative commutant in Q, and that q' is minimal in $Q_1'\cap M$.

By appropriately amplifying $Q\subset M$, we may assume that this inclusion is a genuine crossed product inclusion $Q\subset Q\rtimes_\sigma\Gamma$. Denote by $\{u_g\}_g\subset M=Q\rtimes_\sigma\Gamma$ the canonical unitary elements implementing σ on Q. Let $q'=\sum_g x_g u_g$, with $x_g\in Q$. By identification of Fourier series, it follows that $x_g u_g\in Q_1'\cap M$ for all g. Thus $x_g x_g^*=x_g u_g u_g^* x_g^*\in Q_1'\cap Q=\mathbf{C}$, so that all x_g are scalar multiples of unitary elements in Q. Let $K_0\subset\Gamma$ be the support of this Fourier expansion of q'. Let also $K\subset\Gamma$ be the set of all $k\in\Gamma$ such that u_k can be perturbed by a unitary element in Q so as to fix Q_1 pointwise. Since $[Q:Q_1]<\infty$ and $Q_1'\cap Q=\mathbf{C}$, K is a finite subgroup of Γ and $K_0\subset K$.

Since $Q_1'\cap Q=\mathbf{C}$, by Connes' vanishing 1-cocycle for finite groups, the unitary elements $w_k\in Q$ satisfying $\mathrm{Ad}(w_k u_k)|_{Q_1}=\mathrm{id}_{Q_1}$ can be chosen of the form $\sigma_k(w)w^*$, $k\in K$, for some unitary element $w\in Q_1$. Thus, by perturbing all $\{u_g\}_{g\in\Gamma}$ by a 1-cocycle, we may assume that $\mathrm{Ad}(u_k)$ act trivially on Q_1. Let $\Gamma_0\subset\Gamma$ be the subgroup of all $g\in\Gamma$ such that u_g can be perturbed by a unitary element in Q so as to normalize Q_1. Clearly, $K\subset\Gamma_0$ and K is normal in Γ_0. We will prove that Γ_0 has finite index in Γ, thus contradicting the hypothesis.

By the minimality of q' in $Q_1'\cap M$, it follows that q' is minimal in the group algebra $L(K)=\mathrm{sp}\{u_k:k\in K\}$. Identify $pPp\subset pMp$ with $Q_1q'\subset q'Mq'$ via $\mathrm{Ad}(v)$. Let $\{v_h:h\in\Lambda\}$ be a choice of canonical unitary elements in $M=P\rtimes\Lambda$, which we assume commute with $p\in P$ (we can do that for each h by perturbing if necessary with unitary elements in the factor P). For each $h\in\Lambda$, $h\neq e$, let $v_h=\sum_g x_g^h u_g\in Q\rtimes_\sigma\Gamma$ be the Fourier expansion of v_h, and denote by θ_h the action implemented by $\mathrm{Ad}(v_h)$ on $Q_1\simeq Q_1q'=pPp$. Thus,

$v_h y=\theta_h(y)v_h$ for all $y\in Q_1$. Identifying the Fourier series in $\{u_g\}_g$, this implies that $\theta_h(y)(x_g^h u_g)=(x_g^h u_g)y$ for all $y\in Q_1$. As before, this implies that each x_g^h is a scalar multiple of a unitary element in Q and that $\mathrm{Ad}(x_g^h u_g)$ normalizes Q_1. Thus, the support K_h of the Fourier series for v_h is contained in Γ_0. Since $Q_1q'\subset q'Mq'$ is the closure of the span of elements in Q_1v_hq', $h\in\Lambda$, and each v_h is supported on Γ_0, as a Fourier expansion in $\{u_g\}_g$ with coefficients in Q, it follows that $q'Mq'\subset\bigoplus_{g\in\Gamma_0}L^2(Q)u_g$. In particular, since $q'\in L(K)\subset L(\Gamma_0)$, we get $q'L(\Gamma)q'=q'L(\Gamma_0)q'$, which clearly implies that Γ_0 has finite index in Γ. But this implies that Γ_0 is also ICC, so in particular it cannot have a non-trivial normal subgroup K. This contradiction finishes the proof. □

End of proof of Theorem 8.3. By Lemma 8.4, $\theta(N_1)$ and N_2^t are conjugate by a unitary element, so we may assume that $\theta(N_1)=N_2^t$. Thus, θ induces an isomorphism between the groups $\Gamma_1=\mathcal{N}_{M_1}(N_1)/\mathcal{U}(N_1)$ and $\Gamma_2=\mathcal{N}_{M_2}(N_2)/\mathcal{U}(N_2)$. But then, by the classical Kurosh theorem (see e.g. [LS]) and the condition on "free indecomposability" of the groups Γ_{ij}, it follows that $n_1=n_2=n$ and that there exists a permutation π of the indices $1\leqslant j\leqslant n$ such that $g_j\Gamma_{1,j}g_j^{-1}=\Gamma_{2,\pi(j)}$, for some elements $g_i\in G$. Thus, $u_j=u_{g_j}$ normalizes N_2^t and $\mathrm{Ad}(u_j)$ takes $\theta(M_{1,j})$ onto $M_{2,\pi(j)}^t$. □

Notation 8.5. We denote by $f\mathcal{T}_R$ the class of free actions $\sigma:\Gamma_0*\Gamma_1\to\mathrm{Aut}(R)$ on the hyperfinite II_1 factor R, with the properties:

(1) Γ_0 is free indecomposable; Γ_1 is w-rigid (in particular free indecomposable);

(2) $\sigma_0=\sigma|_{\Gamma_0}$ has the relative property (T), i.e. $R\subset R\rtimes_{\sigma_0}\Gamma_0$ is a rigid inclusion; $\sigma_1=\sigma|_{\Gamma_1}$ is a non-commutative Bernoulli shift action of Γ_1 on $R=\overline{\otimes}_g(N_0,\tau_0)_g$, where $N_0=R$ or $N_0=M_{n\times n}(\mathbf{C})$ for some $n\geqslant 2$;

(3) $\sigma(\Gamma_1)$ and the normalizer of $\sigma(\Gamma_0)$ in $\mathrm{Out}(R)$ (which is countable by [P5]) are freely independent.

Lemma 8.6. *Let Γ_1 be an arbitrary w-rigid group and $\Gamma_0=\mathrm{SL}(n,\mathbf{Z})$, $n\geqslant 2$, or more generally Γ_0 be a free indecomposable arithmetic lattice in an absolutely simple non-compact Lie group with trivial center. Then $\Gamma_0*\Gamma_1$ has $f\mathcal{T}_R$ actions on R.*

Proof. By [M], [Bu], [Fe], [Va], any such Γ_0 has a free ergodic action on some $\mathbf{Z}^m$ such that the pair $(\mathbf{Z}^m\rtimes\Gamma_0,\mathbf{Z}^m)$ has the relative property (T) of Kazhdan–Margulis [M]. By [Ch] and [NPSa], it follows that Γ_0 admits a free action σ_0 on the hyperfinite II_1 factor R such that $R\subset R\rtimes_{\sigma_0}\Gamma_0$ has the relative property (T). By [P5], it follows that the normalizer $\mathcal{N}_0$ of $\sigma_0(\Gamma_0)$ in $\mathrm{Out}(R)$ is countable. Let σ' be a fixed copy Bernoulli shift action of Γ_1. By Proposition 8.2, it follows that there exists an automorphism θ of R such that $\theta(\sigma'(\Gamma_1))\theta^{-1}$ is freely independent from $\mathcal{N}_0$.

Thus, if we denote by σ the unique action of $\Gamma=\Gamma_0*\Gamma_1$ on R given by $\sigma|_{\Gamma_0}=\sigma_0$ and $\sigma|_{\Gamma_1}=\theta\sigma'\theta^{-1}$, then conditions (1)–(3) in Notation 8.5 are all satisfied. □

THEOREM 8.7. *Let* $\sigma_i\colon \Gamma_{i,0}*\Gamma_{i,1}\to \mathrm{Aut}(R_i)$, $i=1,2$, *be* $f\mathcal{T}_R$ *actions*, $G_i=\Gamma_{i,0}*\Gamma_{i,1}$ *and* $M_i=R_i\rtimes_{\sigma_i}G_i$, $i=1,2$. *If* $\theta\colon M_1\simeq M_2^t$ *is an isomorphism, for some projection* $t>0$, *then* $t=1$ *and there exist a unitary element* $u\in M_2$, *a character* γ *of* $\Gamma_{2,0}$ *and isomorphisms* $\delta\colon G_1\simeq G_2$ *and* $\Delta\colon R_1\simeq R_2$ *such that* $\theta_0=\mathrm{Ad}(u)\circ\theta^\gamma\circ\theta$ *satisfies* $\theta_0(xu_g^1)=\Delta(x)u_{\delta(g)}^2$ *for all* $x\in R_1$, $g\in G_1$, *where* $\{u_g^i\}_g\subset M_i$ *are the canonical unitary elements implementing* σ_i, $i=1,2$. *Moreover, any other isomorphism* $\theta'\colon M_1\simeq M_2$ *is a perturbation of* θ *by an automorphism of* M_2 *of the form* $\mathrm{Ad}(v)\circ\theta^{\gamma'}$ *for some* $v\in\mathcal{U}(M_2)$ *and* $\gamma'\in\mathrm{Char}(G_2)$.

Proof. Since $R_i\subset R_i\rtimes_{\sigma_i}\Gamma_{i,0}$ are rigid inclusions, $R_i\subset R_i\rtimes_{\sigma_i}G_i=M_i$ are rigid as well. We can thus apply Theorem 8.3 to get a unitary element $v\in M_2^t$ such that

$$v(\theta(R_1))v^*=R_2^t \quad\text{and}\quad v(\theta(M_{1,1}))v^*=(M_{2,j})^t \ \text{ for some } j\in\{0,1\},$$

where $M_{ij}=R_i\rtimes\Gamma_{ij}$, $i=1,2$, $j=0,1$. But by [P5] or [P8], $R_1\subset M_{1,1}$ is not rigid, while $(R_2\subset M_{2,0})^t$ is rigid, so the only possibility is that $j=1$. Thus,

$$(R_1\subset M_{1,1})\simeq(R_2\subset M_{2,1})^t$$

and both inclusions come from crossed products associated with non-commutative Bernoulli shift actions of w-rigid groups. By [P7], this implies that $t=1$.

On the other hand, $\mathrm{Ad}v\circ\theta$ induces an isomorphism $\delta\colon\Gamma_{1,0}*\Gamma_{1,1}\simeq\Gamma_{2,0}*\Gamma_{2,1}$, which takes $\Gamma_{1,1}$ onto $\Gamma_{2,1}$. By Kurosh's theorem, $\delta(\Gamma_{1,0})=g\Gamma_{2,0}g^{-1}$ for some $g\in G_2$. But by [GoS], the groups $g\Gamma_{2,0}g^{-1}$ and $\Gamma_{2,1}$ can generate $\Gamma_{2,0}*\Gamma_{2,1}$ only if $g=g_1g_2$ for some $g_i\in\Gamma_{2,i}$, $i=0,1$. By conjugating with g, we may thus assume that the unitary element v is such that $\theta_0=\mathrm{Ad}(v)\circ\theta$ induces an isomorphism $\delta\colon G_1\simeq G_2$ which takes $\Gamma_{1,j}$ onto $\Gamma_{2,j}$, $j=0,1$. Thus, after identifying R_1 with R_2 via $\Delta=\theta_0|_{R_1}$ and $G_1\simeq G_2$ via δ, we are left with finding all automorphisms α of M_2 that take R_2 onto itself and take the canonical unitary elements u_g into unitary elements w_gu_g, $g\in G_2$, for some $w\colon G_2\to\mathcal{U}(R_2)$ a 1-cocycle for σ_2.

By [P7], this implies that w is co-boundary modulo scalars when restricted to $\Gamma_{2,1}$, i.e. $w_g\in\mathbf{C}\sigma_g(w)w^*$ for all $g\in\Gamma_{2,1}$, for some unitary element $w\in R_2$. Thus, by replacing α by $\mathrm{Ad}(w^*)\circ\alpha$, we may assume that $\alpha(u_g)\in\mathbf{C}u_g$ for all $g\in\Gamma_{2,1}$. Thus $\alpha|_{R_2}\in\mathrm{Aut}(R_2)$ commutes with $\sigma_2(\Gamma_{2,1})$, while still normalizing $\sigma_2(\Gamma_{2,0})$. But, by condition (3) in Notation 8.5, the latter condition implies that $\alpha|_{R_2}$ is freely independent from $\sigma_2(\Gamma_{2,1})$. This contradicts the commutation condition with $\sigma_2(\Gamma_{2,1})$, unless $\alpha|_{R_2}$ is inner. By perturbing α by $\mathrm{Ad}(w_0)$ for an appropriate $w_0\in\mathcal{U}(R_2)$, we may thus assume that $\alpha_{R_2}=\mathrm{id}_{R_2}$. Thus, α is given by a character of G_2. □

The above theorem shows in particular that the fundamental group of any $f\mathcal{T}_R$ factor $M=R\rtimes_\sigma(\Gamma_0*\Gamma_1)$, corresponding to an $f\mathcal{T}_R$ action $(\sigma,\Gamma_0*\Gamma_1)$, is trivial, while its

Out-group is equal to $\mathrm{Char}(\Gamma_0)\times\mathrm{Char}(\Gamma_1)$. By Lemma 8.6, one can take $\Gamma_0=\mathrm{SL}(n,\mathbf{Z})$, which has only trivial characters, thus making $\mathrm{Aut}(M)=\mathrm{Char}(\Gamma_1)$, with Γ_1 being an arbitrary w-rigid group. For instance, one can take $\Gamma_1=\mathrm{SL}(3,\mathbf{Z})\times H$, where H is an arbitrary discrete abelian group, and hence getting $\mathrm{Aut}(M)=\widehat{H}$. We thus obtain the following result.

Corollary 8.8. *Let* $\sigma\colon\Gamma_0*\Gamma_1\to\mathrm{Aut}(R)$ *be an* $f\mathcal{T}_R$ *action and let*

$$M=R\rtimes_\sigma(\Gamma_0*\Gamma_1).$$

Then the following are true:

(1) $\mathfrak{F}(M)=\{1\}$ *and* $\mathrm{Out}(M)=\mathrm{Char}(\Gamma_0)\times\mathrm{Char}(\Gamma_1)=\mathrm{Out}(M^\infty)$.

(2) *Given any compact abelian group* K, *there exists* $(\sigma,\Gamma_0*\Gamma_1)$ *such that the corresponding* $f\mathcal{T}_R$ *factor* M *satisfies* $\mathrm{Out}(M)=K=\mathrm{Out}(M^\infty)$. *For instance*, *if* $\Gamma_0=\mathrm{SL}(n,\mathbf{Z})$ *and* $\Gamma_1=\mathrm{SL}(m,\mathbf{Z})\times\widehat{K}$ *for some* $n,m\geqslant 3$, *then* $\mathrm{Out}(M)=K$.

Remark 8.9. One can use Remark A.3 (2) in place of Lemma A.2 in all the above proofs, to construct more II_1 factors with small calculable symmetry groups. Thus, let $f\mathcal{T}'_R$ be the class of free actions $\sigma\colon\Gamma_0*\Gamma_1\to\mathrm{Aut}(R)$ on the hyperfinite II_1 factor R, satisfying the properties:

(a) Γ_0 and Γ_1 are free, indecomposable and not equal to $\mathbf{Z}$;

(b) $R\subset R\rtimes_{\sigma_0}\Gamma_0$ is a rigid inclusion and $\sigma_1=\sigma|_{\Gamma_1}$ is a non-cocycle conjugate to $\sigma_0=\sigma|_{\Gamma_0}$ (note that this is indeed the case if σ_1 is a non-commutative Bernoulli shift);

(c) $\sigma(\Gamma_1)$ is freely independent with respect to the set $\mathcal{N}(\sigma_0(\Gamma_0))\cup\mathcal{N}^{\mathrm{op}}(\sigma_0(\Gamma_0))$, consisting of all automorphisms and anti-automorphisms of R^∞ that normalize $\sigma_0(\Gamma_0)$.

By Remark A.3 (2) and Lemma 8.6, it follows that there exist such actions σ for any linear group Γ_0 as in Lemma 8.6 and for any free indecomposable Γ_1. It then follows as in Corollary 8.8 that the corresponding crossed product II_1 factors $M=R\rtimes_\sigma(\Gamma_0*\Gamma_1)$ satisfy $\mathrm{Out}(M)=\mathrm{Char}(\Gamma_0*\Gamma_1)$. Moreover, if $t\in\mathfrak{F}(M)$ and $\theta\colon M\simeq M^t$, then θ must normalize $\sigma(\Gamma_0)$, contradicting condition (c) above. Thus, $\mathfrak{F}(M)=\{1\}$. Notice that to get this calculation we no longer have to use the results in [P7] on the fundamental group of $R\subset R\rtimes_{\sigma_1}\Gamma_1$. Similarly, if α is an anti-automorphism of M, then the same argument shows that it must normalize $\sigma_0(\Gamma_0)$, in contradiction to the choice (c). Altogether, this shows that in addition to the properties $\mathfrak{F}(M)=\{1\}$ and $\mathrm{Out}(M)=\mathrm{Char}(\Gamma_0*\Gamma_1)$, the factors M in the class $f\mathcal{T}'_R$ have no anti-automorphisms either. This provides a fairly large new family of factors with this latter property, after Connes' first examples in [C2]. Thus, if we choose the groups Γ_0 and Γ_1 without characters, e.g. $\Gamma_i=\mathrm{SL}(n_i,\mathbf{Z})$, $n_i\geqslant 3$, then the resulting factors M have no outer symmetries at all. Moreover, noticing that $f\mathcal{T}'_R$ factors are w-rigid, it follows that any factor of the form $N*M$, with N being a property (T) II_1 factor (e.g. $N=L(\mathrm{PSL}(n,\mathbf{Z}))$, $n\geqslant 3$) and $M\in f\mathcal{T}'_R$, has all the above properties (as a consequence of Theorem 6.3 and of the properties of M), and in addition has no Cartan subalgebras (by [V2], cf. Remark 6.6)!

Appendix. Constructing freely independent actions

We now prove the technical results needed in the proofs of Propositions 7.3 and 8.2, which established the existence of free actions of groups $\Gamma=\Gamma_0*\Gamma_1*...$ on $A=L^\infty(X,\mu)$ and on R with restrictions to Γ_i isomorphic to prescribed actions (σ_i,Γ_i), for all i.

More precisely, we prove that given any countable set $\{\theta_n\}_{n\geqslant 1}$ of properly outer automorphisms of (X,μ) (resp. of R) the set $\mathcal{V}$ of $\theta\in\mathrm{Aut}(X,\mu)$ (resp. $\theta\in\mathrm{Aut}(R)$) with the property that all alternating words $\theta_{i_0}\theta\theta_{i_1}\theta^{-1}\theta_{i_2}\theta\theta_{i_3}\theta^{-1}...$ are properly outer, is G_δ-dense in $\mathrm{Aut}(X,\mu)$ (resp. $\mathrm{Aut}(R)$). Writing $\mathcal{V}$ as an intersection of open sets $\mathcal{V}_n$ is obvious, and the non-trivial part is to show that each $\mathcal{V}_n$ is dense. To prove the density, in the commutative case (Lemma A.1) we use a maximality argument inspired from [P3], while in the hyperfinite case (Lemma A.2) we use directly a result from [P3], not having to re-do such a maximality argument.

The idea of using Baire category, in both the proofs of Lemmas A.1 and A.2, was triggered by [MoS, Remark 2.27] and [Tö, category lemma]. In fact, the commutative case A.1 below is essentially contained in [Tö]. We have included the complete proof, with a different treatment of the "density", for the reader's convenience.

LEMMA A.1. *Let (X,μ) be a standard non-atomic probability space and*

$$\{\theta_n\}_{n\geqslant 1}\subset\mathrm{Aut}(X,\mu)$$

be a sequence of properly outer m.p. automorphisms of (X,μ). Denote by $\mathcal{V}\subset\mathrm{Aut}(X,\mu)$ the set of all $\theta\in\mathrm{Aut}(X,\mu)$ with the property that $\theta_{i_0}\prod_{j=1}^n\theta\theta_{i_{2j-1}}\theta^{-1}\theta_{i_{2j}}$ is properly outer, for all $n\geqslant 1$, $i_1,i_2,...,i_{2n-1}\in\{1,2,3,...\}$ and $i_0,i_{2n}\in\{0,1,2,...\}$, where $\theta_0=\mathrm{id}_X$. Then $\mathcal{V}$ is a G_δ-dense subset of $\mathrm{Aut}(X,\mu)$. In particular, $\mathcal{V}\neq\varnothing$.

Proof. Let $A=L^\infty(X,\mu)$ and $\tau=\int\cdot\,\mathrm{d}\mu$. Denote by $\mathcal{F}$ the set of all finite partitions of the identity $\{p_i\}_i\subset\mathcal{P}(A)$. If $\varrho\in\mathrm{Aut}(X,\mu)$, then we still denote by ϱ the automorphism that it induces on (A,τ). As usual, $\mathrm{Aut}(A,\tau)$ is endowed with the topology given by pointwise $\|\cdot\|_2$-convergence, with respect to which it is metrizable and complete.

For each $\varrho\in\mathrm{Aut}(A,\tau)$, let $k(\varrho)=\inf\big\{\big\|\sum_i\varrho(p_i)p_i\big\|_2:\{p_i\}_i\in\mathcal{F}\big\}$. Note that ϱ is properly outer if and only if $k(\varrho)=0$. Also, if $\mathcal{D}_n$ denotes the set of $\varrho\in\mathrm{Aut}(A,\tau)$ with $k(\varrho)<1/n$ then $\mathcal{D}_n$ is clearly open in $\mathrm{Aut}(A,\tau)$. Given an n-tuple $(\theta_1,...,\theta_n)\subset\mathrm{Aut}(A,\tau)$, we denote by $\mathcal{V}_n = \mathcal{V}(\theta_1, ..., \theta_n)$ the set of all $\varrho \in \mathrm{Aut}(A,\tau)$ with

$$\theta_{i_0} \prod_{j=1}^{l} \varrho\theta_{i_{2j-1}}\varrho^{-1}\theta_{i_{2j}} \in \mathcal{D}_n,$$

for all $1 \leqslant l \leqslant n$ and all choices $i_1, i_2, ..., i_{2l-1} \in \{1,2,...,n\}$ and $i_0, i_{2l} \in \{0,1,2,...,n\}$.

It is immediate to see that $\mathcal{V}_n$ is open in $\mathrm{Aut}(A,\tau)$ and that $\bigcap_{n\geqslant 1} \mathcal{V}_n = \mathcal{V}$. We have to prove that each $\mathcal{V}_n$ is also dense in $\mathrm{Aut}(A,\tau)$, i.e. that any fixed $\varrho' \in \mathrm{Aut}(A,\tau)$ can be approximated arbitrarily well (in the point $\|\cdot\|_2$-norm convergence on A) by some $\varrho \in \mathcal{V}_n$.

By replacing, if necessary, $\{\theta_j\}_j$ by the properly outer automorphisms

$$\{\theta_j\}_j \cup \{\varrho'\theta_k(\varrho')^{-1}\}_k,$$

it follows that in order to prove the density of $\mathcal{V}_n$ it is sufficient to prove that id_A is in the closure of $\mathcal{V}_n = \mathcal{V}(\theta_1, ..., \theta_n)$, for any n-tuple of properly outer automorphisms.

To this end we will use the ultrapower II_1 factor R^ω [McD] as a framework. Thus, we choose a free ultrafilter ω on $\mathbf{N}$ and let $R^\omega = \ell^\infty(\mathbf{N}, R)/\mathcal{I}_\omega$, where $\mathcal{I}_\omega$ is the ideal associated with the trace $\tau_\omega((x_n)_n) = \lim_{n\to\omega} \tau(x_n)$, i.e. $\mathcal{I}_\omega = \{x = (x_n)_n : \tau_\omega(x^*x) = 0\}$.

We regard the abelian von Neumann algebra (A,τ) as a Cartan subalgebra of R. By [D], given any $\varrho \in \mathrm{Aut}(A,\tau)$, any finite set $F \subset A$ and $\varepsilon > 0$, there exists $v \in \mathcal{N}_R(A)$ such that $\|\varrho(a) - \mathrm{Ad}(v)(a)\|_2 \leqslant \varepsilon$ for all $a \in F$. Thus, there exist $u_n \in \mathcal{N}_R(A)$ such that $u = (u_n)_n \in R^\omega$ satisfies $uau^* = \varrho(a)$ for all $a \in A \subset A^\omega \subset R^\omega$. Note that ϱ is properly outer if and only if $E_{A'\cap R^\omega}(u) = 0$.

Write $A = \overline{\bigcup_m D_m}^{w}$, for some increasing sequence of finite-dimensional subalgebras of A. Let $\mathcal{N}_m$ denote $D_m' \cap \mathcal{N}_R(A)$, i.e. the part of the normalizer of A in R that leaves D_m be pointwise fixed. To prove that id_A is in the closure of $\mathcal{V}_n$, it is sufficient to prove the following fact:

Let $U_0 = 1, U_1, U_2, ..., U_n \in \mathcal{N}_{R^\omega}(A)$ with $E_{A'\cap R^\omega}(U_i) = 0$ for all $i \neq 0$. For all $m \geqslant 1$, there exists $u \in \mathcal{N}_m$ such that $E_{A'\cap R^\omega}\big(u_{i_0} \prod_{j=1}^{l} uU_{i_{2j-1}}u^*U_{i_{2j}}\big) = 0$ for all $1 \leqslant l \leqslant n$ and all $i_1, i_2, ..., i_{2l-1} \in \{1,2,...,n\}$ and $i_0, i_{2l} \in \{0,1,2,...,n\}$. (A.1)

We construct u by a maximality argument, "patching together" partial isometries in $\mathcal{G}_m \stackrel{\mathrm{def}}{=} \{vp : v \in \mathcal{N}_m \text{ and } p \in \mathcal{P}(A)\} = D_m' \cap \mathcal{GN}_R(A)$.

Thus, for each $v \in \mathcal{G}_m$ with $vv^* = v^*v$ and each $1 \leqslant k \leqslant 2n$, we denote by $\mathcal{V}_k^v$ the set of all products of the form $\prod_{j=0}^{k-1}(U_{i_j}v^{\alpha_j})U_{i_k}$, where $i_j \in \{1,2,...,n\}$ for $1 \leqslant j \leqslant k-1$, $i_0, i_k \in \{0,1,...,n\}$, $\alpha_j \in \{\pm 1\}$ and $\alpha_1 \neq \alpha_2 \neq ... \neq \alpha_k$. We also put $\mathcal{V}_0^v = \{U_i : 1 \leqslant i \leqslant n\}$. Note that if $i_0, i_{2l} \in \{0,1,...,n\}$ and $i_1, i_2, ..., i_{2l-1} \in \{1,2,...,n\}$, then $U_{i_0}\prod_{j=1}^{l} vU_{2j-1}v^*U_{2j} \in \mathcal{V}_{2l}^v$. We let

$$\mathcal{W} = \{v \in \mathcal{G}_m : vv^* = v^*v \text{ and } E_{A'\cap R^\omega}(x) = 0 \text{ for all } x \in \mathcal{V}_k^v,\ 1 \leqslant k \leqslant 2n\}.$$

We endow $\mathcal{W}$ with the order given by: $v \leqslant v'$ if $v = v'v^*v$. $(\mathcal{W}, \leqslant)$ is clearly inductively ordered. Let $v_0 \in \mathcal{W}$ be a maximal element. Assume that $0 \neq p = 1 - v_0 v_0^* \in \mathcal{P}(A)$. Since $E_{A'}(x) = 0$ for all $x \in \mathcal{V}_k^{v_0}$, $1 \leqslant k \leqslant 2n$, it follows that $E_{A'}(pxp) = 0$ as well. Since all such elements $w = pxp$ satisfy $ww^*, w^*w \in A$ and $wAw^* = Aww^*$, by [D] it follows that

$$\text{there exists } 0 \neq q_1 \in \mathcal{P}(Ap) \text{ such that } q_1 D_m = \mathbf{C} q_1 \text{ and } q_1 w q_1 = E_{A'}(w) q_1 = 0 \text{ for all } w \in \bigcup_{k=0}^{2n} \mathcal{V}_k^{v_0}. \tag{A.2}$$

Let $v_1 \in \mathcal{N}_{q_1 R q_1}(Aq_1)$. Note that $v_1 \in \mathcal{G}_m$ and $v_1 v_1^* = v_1^* v_1 = q_1$. Set $u = v_0 + v_1$. We will show that if $v_1 \in \mathcal{N}_{q_1 R q_1}(Aq_1)$ is chosen appropriately, then $u \in \mathcal{W}$, thus contradicting the maximality of v_0. Write

$$x = \prod_{j=0}^{k-1} (U_{i_j} u^{\alpha_j}) U_{i_k} = \sum_\beta \prod_{j=0}^{k-1} (U_{i_j} v_{\beta_j}^{\alpha_j}) U_{i_k} = \sum_\beta y_\beta,$$

where the sum is taken over all choices $\beta = (\beta_j)_{j=0}^{k-1} \in \{0,1\}^k$. We will show that v_1 can be taken such that $E_{A'}(y_\beta) = 0$ for all β and all $x \in \bigcup_{k=0}^{2n} \mathcal{V}_k^u$. For $\beta = (0, 0, ..., 0)$ we have $y_\beta = \prod_{j=0}^{k-1} (U_{i_j} v_0^{\alpha_j}) U_{i_k} \in \mathcal{V}_k^{v_0}$, so that $E_{A'}(y_\beta) = 0$, by the fact that $v_0 \in \mathcal{W}$.

The k terms y_β corresponding to just one occurrence of v_1 (i.e. $\beta = (\beta_1, ..., \beta_k)$ with all $\beta_i = 0$ except one), are of the form $w_0 v_1 w_1$, with $w_0, w_1 \in \bigcup_{j=0}^{k-1} \mathcal{V}_j^{v_0}$. Thus, each w_i satisfies $w_i w_i^*, w_i^* w_i \in A$ and $w_i A w_i^* = A w_i w_i^*$, $i = 0, 1$. By shrinking q_1 recursively, we may assume that $(w_0^* w_0) q_1 (w_1 w_1^*)$ is either equal to 0 or to q_1, for all such y_β and all $0 \leqslant k \leqslant 2n$. For the y_β for which $(w_0^* w_0) q_1 (w_1 w_1^*) = 0$ we have $y_\beta = 0$ and there is nothing to prove. For the y_β with $(w_0^* w_0) q_1 (w_1 w_1^*) = q_1$, take $u_0, u_1 \in \mathcal{N}_{R^\omega}(A)$ such that $u_0 q_1 = w_0 q_1$ and $q_1 u_1 = q_1 w_1$. Then $E_{A'}(y_\beta) = E_{A'}(u_0 v_1 u_1) = u_0 E_{A'}(v_1 u_1 u_0) u_0^*$, so that

$$\|E_{A'}(y_\beta)\|_1 = \|E_{A'}(v_1 u_1 u_0)\|_1.$$

Shrinking q_1 recursively again, all conditions so far are still satisfied, while we can assume that $q_1 u_1 u_0 q_1$ is either 0 or an element in $A' \cap R^\omega$, for all u_0 and u_1 coming from all w_0 and w_1 arising as above. Thus, if we take $v_1 \in \mathcal{N}_{q_1 R q_1}(Aq_1)$ to be properly outer, then in both cases $E_{A'}(v_1 u_1 u_0) = 0$ for all u_0 and u_1. We have thus shown that q_1 and v_1 can be chosen so that for all $\beta = (\beta_1, ..., \beta_h)$ having just one occurrence of 1 we have $E_{A'}(y_\beta) = 0$ as well.

Finally, the y_β with at least two occurrences of v_1 can be written as

$$y_\beta = x_0 (v_1^\alpha w v_1^{\alpha'}) x_1,$$

with $w \in \mathcal{V}_l^{v_0}$ for some $0 \leqslant l \leqslant k-2$, $\alpha, \alpha' \in \{\pm 1\}$ and partial isometries x_i. Thus

$$\|E_{A'}(y_\beta)\|_1 \leqslant \|y_\beta\|_1 \leqslant \|v_1^\alpha w v^{\alpha'}\|_1 = \|q_1 w q_1\|_1 = \|E_{A'}(w) q_1\|_1 = 0,$$

the last equality by (A.2).

Altogether, $E_{A'}(x)=0$ for all $x\in\bigcup_{k=0}^{2n}\mathcal{V}_k^u$, showing that $u\in\mathcal{W}$. But this contradicts the maximality of v_0. Thus v_0 must be a unitary element, finishing the proof that $\mathcal{V}_n$ is dense in $\mathrm{Aut}(A,\tau)$ and thus the proof of the statement. □

LEMMA A.2. *Let $\{\theta_n\}_{n\geqslant 1}$ be a sequence of properly outer automorphisms of the hyperfinite* II_1 *factor R. Denote by $\mathcal{V}$ the set of all automorphisms θ of R such that any automorphism of R of the form $\theta_{i_0}\prod_{j=1}^{n}\theta\theta_{i_{2j-1}}\theta^{-1}\theta_{i_{2j}}$ is outer, for all $n\geqslant 1$, $i_1,i_2,...,i_{2n-1}\in\{1,2,3,...\}$ and $i_0,i_{2n}\in\{0,1,2,...\}$, where $\theta_0=\mathrm{id}_R$. Then $\mathcal{V}$ is a G_δ-dense subset of* $\mathrm{Aut}(R)$. *In particular*, $\mathcal{V}\neq\varnothing$.

Proof. Let $\{u_n\}_{n\geqslant 1}$ be a sequence of unitary elements in R, dense in $\mathcal{U}(R)$ in the $\|\cdot\|_2$-norm and with each element repeated infinitely many times. For each $x\in R$ and $\varrho\in\mathrm{Aut}(R)$, denote by $k(\varrho,x)$ the unique element of minimal $\|\cdot\|_2$-norm in

$$K(\varrho,x)=\overline{\mathrm{co}}^{\,w}\{\varrho(v)xv^*:v\in\mathcal{U}(R)\}.$$

Let $\mathcal{D}_n$ be the set of automorphisms ϱ of R with the property that $\|k(\varrho,u_i)\|_2<1/n$ for all $1\leqslant i\leqslant n$.

We claim that $\mathcal{D}_n$ is open in $\mathrm{Aut}(R)$. To see this, let $\varrho\in\mathcal{D}_n$ and for each $1\leqslant i\leqslant n$ choose $v_1^i,v_2^i,...,v_{m_i}^i\in\mathcal{U}(R)$ such that

$$\left\|\frac{1}{m_i}\sum_{j=1}^{m_i}\varrho(v_j^i)u_iv_j^{i*}\right\|_2<\frac{1}{n}.$$

If $\delta>0$ is sufficiently small, then any $\varrho'\in\mathrm{Aut}(R)$ satisfying $\|\varrho'(v_j^i)-\varrho(v_j^i)\|_2<\delta$, for all $1\leqslant i\leqslant n$ and for all $1\leqslant j\leqslant m_i$, will satisfy

$$\left\|\frac{1}{m_i}\sum_{j=1}^{m_i}\varrho'(v_j^i)u_iv_j^{i*}\right\|_2<\frac{1}{n},$$

implying that $\|k(\varrho',u_i)\|_2<1/n$ for all $1\leqslant i\leqslant n$. Thus $\varrho'\in\mathcal{D}_n$.

Denote by $\mathcal{V}_n=\mathcal{V}_n(\theta_1,...,\theta_n)$ the set of all $\varrho\in\mathrm{Aut}(R)$ with the property that

$$\sigma_{i_0}\prod_{j=1}^{l}\varrho\theta_{i_{2j-1}}\varrho^{-1}\theta_{i_{2j}}\in\mathcal{D}_n$$

for all $1\leqslant l\leqslant n$ and all choices $i_1,i_2,...,i_{2l-1}\in\{1,2,...,n\}$ and $i_0,i_{2l}\in\{0,1,2,...,n\}$. Since $\mathcal{D}_n$ is open and

$$\mathrm{Aut}(R)\ni\varrho\longmapsto\theta_{i_0}\prod_{j=1}^{l}\varrho\theta_{i_{2j-1}}\varrho^{-1}\theta_{i_{2j}}$$

is a continuous map, for each m and i_j as before, it follows that $\mathcal{V}_n$ is open in $\mathrm{Aut}(R)$.

It is immediate to see that $\bigcap_{n\geqslant 1}\mathcal{V}_n=\mathcal{V}$. Thus, in order to show that $\mathcal{V}$ is a G_δ-dense subset of $\mathrm{Aut}(R)$, we have to prove that each $\mathcal{V}_n$ is dense in $\mathrm{Aut}(R)$. Moreover, arguing as in the proof of Lemma A.1, we see that, by replacing if necessary $\{\theta_n\}_{n\geqslant 1}$ with the sequence $\{\theta'_n\}_{n\geqslant 1}=\{\theta_n\}_{n\geqslant 1}\cup\{\varrho\theta_m\varrho^{-1}\}_{m\geqslant 1}$, it is enough to show that id_R is in the closure of $\mathcal{V}_n$. We will prove that in fact id_R is in the closure of $\mathcal{V}_n\cap\mathrm{Int}(R)$, i.e., given any finite-dimensional subfactor $R_0\subset R$, there exists $u\in\mathcal{U}(R'_0\cap R)$ such that $\mathrm{Ad}(u)\in\mathcal{V}_n$. In turn, this will be an easy consequence of [P3, Theorem 2.1].

To make the ideas more transparent, let us consider first the case when $\{\theta_n\}_{n\geqslant 0}$ is an enumeration of the automorphisms of a free cocycle action of a countable group Γ on R, $\theta\colon\Gamma\to\mathrm{Aut}(R)$. Let $M=R\rtimes\Gamma$ and $U_0=1, U_1, U_2, ...\in\mathcal{U}(M)$ be the canonical unitary elements implementing $\theta_0,\theta_1,...$. Since the action is free (i.e. θ_g is non-inner for all $g\neq e$), we have $R'\cap M=\mathbf{C}$. Fix a free ultrafilter ω on $\mathbf{N}$. We view $R\subset M$ as subalgebras of constant sequences in the ultrapower II_1 factor M^ω.

Since $(R'_0\cap R)'\cap M=R_0$, by [P3, Theorem 2.1], there exists a unitary element $V\in(R'_0\cap R)^\omega\subset M^\omega$ such that

$$VRV^*\vee M\simeq R*_{R_0}M.$$

In particular, if $w\in\mathcal{U}(R'_0\cap R)$ is a Haar unitary element and we put $U=VwV^*\in VRV^*$, then for any choice of $1\leqslant l\leqslant n$, $i_1,i_2,...,i_{2l-1}\in\{1,2,...,n\}$, $i_0,i_{2l}\in\{0,1,2,...,n\}$ and $1\leqslant r\leqslant n$, the unitary elements

$$\mathrm{Ad}\biggl(U_{i_0}\prod_{j=1}^{l}(UU_{i_{2j-1}}U^{-1})U_{i_{2j}}\biggr)(U^k)u_rU^{-k},\quad k=1,2,... \tag{A.3}$$

are mutually orthogonal with respect to the scalar product given by the trace. To see this, we need to show that

$$\tau\biggl(\mathrm{Ad}\biggl(U_{i_0}\prod_{j=1}^{l}(UU_{i_{2j-1}}U^{-1})U_{i_{2j}}\biggr)(U^k)u_rU^{-k}\biggr)=0\quad\text{for all }\ k\neq 0.$$

This amounts to showing that

$$\tau\biggl(U^{-k}\biggl(\prod_{j=1}^{l}UU_{i_{2j-1}}U^{-1}U_{i_{2j}}\biggr)U^k\biggl(\prod_{j=1}^{l}U^{-1}_{i_{2l-2j}}UU^{-1}_{i_{2l-2j+1}}U^{-1}\biggr)u_r\biggr)=0,$$

which does indeed hold true, because after some appropriate word-reduction we are left with a word of alternating "letters" $U^k\in VRV^*\ominus R_0$, for all $k\neq 0$, and $U_{i_j},U^*_{i_j}\in M\ominus R\subset M\ominus R_0$, for all $i_j\neq 0$.

By the orthogonality of the elements in (A.3), it follows that for large enough N we have

$$\left\| \frac{1}{N} \sum_{k=1}^{N} \mathrm{Ad}\left(U_{i_0} \prod_{j=1}^{l} (UU_{i_{2j-1}}U^{-1}) U_{i_{2j}} \right) (U^k) u_r U^{-k} \right\|_2 < \frac{1}{n} \tag{A.4}$$

for all choices of $1 \leqslant l \leqslant n$, $i_1, i_2, ..., i_{2l-1} \in \{1, 2, ..., n\}$, $i_0, i_{2l} \in \{0, 1, 2, ..., n\}$ and $1 \leqslant r \leqslant n$. Writing U as a sequence of unitary elements in $R_0' \cap R$, $U = (v_m)$, it follows that for m large enough $u = v_m$ satisfies

$$\left\| \frac{1}{N} \sum_{k=1}^{N} \theta_{i_0} \left(\prod_{j=1}^{l} \mathrm{Ad}(u) \theta_{i_{2j-1}} \mathrm{Ad}(u^*) \theta_{i_{2j}} \right) (u^k) u_r u^{-k} \right\|_2 < \frac{1}{n} \tag{A.5}$$

for all $1 \leqslant l \leqslant n$, all $i_1, i_2, ..., i_{2l-1} \in \{1, 2, ..., n\}$, $i_0, i_{2l} \in \{0, 1, 2, ..., n\}$ and all $1 \leqslant r \leqslant n$. Thus $\mathrm{Ad}(u) \in \mathcal{V}_n$, finishing the proof of this particular case.

Now, in the general case we can take $\theta_0 = \mathrm{id}_R$, θ_1, θ_2, ... to be a lifting in $\mathrm{Aut}(R)$ of an injective group morphism $\Gamma \to \mathrm{Out}(R)$, with Γ generated by n elements. Notice that the automorphisms $\theta_j \otimes \theta_j^{\mathrm{op}}$ on $R \overline{\otimes} R$ induce a cocycle action $\tilde{\theta}: \Gamma \to \mathrm{Aut}(R \overline{\otimes} R^{\mathrm{op}})$ (see e.g. [P4, §3]), so we can consider the crossed product factor $\widetilde{M} = R \overline{\otimes} R^{\mathrm{op}} \rtimes \Gamma$. Denote by $U_n \in \widetilde{M}$ the canonical unitary elements implementing $\theta_n \otimes \theta_n^{\mathrm{op}}$. By [P4] again, we can view $R \overline{\otimes} R \subset \widetilde{M}$ as the symmetric enveloping inclusion associated with a "diagonal subfactor" $N \subset R \simeq \mathbf{M}_{n+1}(N)$, with the embedding of N given by $x \oplus \theta_1(x) \oplus ... \oplus \theta_n(x)$. Moreover, the associated Jones tower $N \subset R \subset N_1 \subset ... \nearrow N_\infty$ can be viewed as a sequence of subalgebras of $\widetilde{M}$, making a non-degenerate commuting square:

$$\begin{array}{ccc} R \overline{\otimes} R^{\mathrm{op}} & \subset & \widetilde{M} \\ \cup & & \cup \\ R \vee R' \cap N_\infty & \subset & N_\infty. \end{array}$$

As before, we view $\widetilde{M}$ as the algebra of constant sequences in $\widetilde{M}^\omega$. Since

$$(R_0' \cap R)' \cap N_\infty = R_0 \vee R' \cap N_\infty$$

and each $R' \cap N_k$ is finite-dimensional, we can apply [P3, Theorem 2.1] to get a unitary element $V \in (R_0' \cap R)^\omega \subset N_\infty^\omega$ such that $V N_\infty V^* \vee N_\infty \simeq N_\infty *_{R_0 \vee R' \cap N_\infty} N_\infty$. By the above commuting square, we then also have $V \widetilde{M} V^* \vee \widetilde{M} \simeq \widetilde{M} *_{R_0 \vee R^{\mathrm{op}}} \widetilde{M}$.

Like before, take a Haar unitary element $w \in \mathcal{U}(R_0' \cap R)$ and let $U = V w V^* \in V R V^*$. For any choice of $1 \leqslant l \leqslant n$, $i_1, i_2, ..., i_{2l-1} \in \{1, 2, ..., n\}$, $i_0, i_{2l} \in \{0, 1, 2, ..., n\}$ and $1 \leqslant r \leqslant n$, the unitary elements

$$\mathrm{Ad}\left(U_{i_0} \prod_{j=1}^{l} (UU_{i_{2j-1}}U^{-1}) U_{i_{2j}} \right) (U^k) u_r U^{-k}, \quad k = 1, 2, ..., \tag{A.3$'$}$$

are then mutually orthogonal with respect to the scalar product given by the trace. Indeed, as $U^k \in VRV^* \ominus R_0 \subset VN_\infty V^* \ominus R_0 \vee N' \cap N_\infty$ for $k \neq 0$, and $U_{i_j}, U_{i_j}^* \in \widetilde{M} \ominus R \vee R^{\rm op}$ for $i_j \neq 0$, it follows that

$$\tau\left(\mathrm{Ad}\left(U_{i_0}\prod_{j=1}^{l}(UU_{i_{2j-1}}U^{-1})U_{i_{2j}}\right)(U^k)u_rU^{-k}\right)=0 \quad \text{for } k\neq 0,$$

showing the orthogonality in (A.3′).

Now, by the orthogonality of the elements in (A.3′), it follows that for large enough N, we have

$$\left\|\frac{1}{N}\sum_{k=1}^{N}\mathrm{Ad}\left(U_{i_0}\prod_{j=1}^{l}(UU_{i_{2j-1}}U^{-1})U_{i_{2j}}\right)(U^k)u_rU^{-k}\right\|_2<\frac{1}{n} \tag{A.4′}$$

for all choices of $1\leqslant l\leqslant n$, $i_1, i_2, ..., i_{2l-1} \in \{1,2,...,n\}$, $i_0, i_{2l} \in \{0,1,2,...,n\}$ and $1\leqslant r\leqslant n$. Writing U as a sequence of unitary elements in $R_0'\cap R$, $U=(v_m)$, it follows again that for large enough m the unitary element $u=v_m$ satisfies

$$\left\|\frac{1}{N}\sum_{k=1}^{N}\theta_{i_0}\left(\prod_{j=1}^{l}\mathrm{Ad}(u)\theta_{i_{2j-1}}\mathrm{Ad}(u^*)\theta_{i_{2j}}\right)(u^k)u_ru^{-k}\right\|_2<\frac{1}{n} \tag{A.5′}$$

for all $1\leqslant l\leqslant n$, $i_1, i_2, ..., i_{2l-1} \in \{1,2,...,n\}$, $i_0, i_{2l} \in \{0,1,2,...,n\}$ and $1\leqslant r\leqslant n$. Thus

$$\mathrm{Ad}(u)\in\mathcal{V}_n. \qquad \square$$

Remarks A.3. The proofs of both Lemmas A.1 and A.2 can of course be carried out without using the hyperfinite II_1 factor R and its ultrapower R^ω as framework, working exclusively in the spaces $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(R)$, respectively. But while it is straightforward to re-write the proof of Lemma A.1 this way, the proof of Lemma A.2 then becomes much more tedious, as one can no longer use results from [P3]. Instead, one has to go through a similar maximality argument as in the proof of Lemma A.1, but with the non-commutativity requiring some complicated estimates, similar to [P3, pp. 189–192].

When written in the "$\mathrm{Aut}(X,\mu)$ framework", a suitable adaptation of the proof of Lemma A.1 shows the following result.

(1) Let $(X,\mathcal{X},\nu)$ be a standard probability space and denote by $\mathcal{A}$ the set of all measurable isomorphisms ϱ of X into X such that $\nu\circ\varrho$ is non-singular with respect to ν. Let μ be another measure (not necessarily finite) on $(X,\mathcal{X})$, equivalent to ν. Denote by $\mathrm{Aut}(X,\mu)$ the μ-preserving automorphisms in $\mathcal{A}$. If $\{\theta_n\}_{n\geqslant 1}\in\mathcal{A}$ are properly outer, then

the set $\mathcal{V}\subset\mathrm{Aut}(X,\mu)$ of all $\theta\in\mathrm{Aut}(X,\mu)$ with the property that $\theta_{i_0}\prod_{j=1}^{n}\theta\theta_{i_{2j-1}}\theta^{-1}\theta_{i_{2j}}$ is properly outer, for all $n\geqslant 1$, $i_1,i_2,...,i_{2n-1}\in\{1,2,3,...\}$ and $i_0,i_{2n}\in\{0,1,2,...\}$, where $\theta_0=\mathrm{id}_X$, is a G_δ-dense subset of $\mathrm{Aut}(X,\mu)$.

In turn, the proof of the non-commutative case in the "Aut(R) framework" can be adapted to show the following more general result.

(2) For each $n\geqslant 1$, let $\theta_n: R^\infty\to R^\infty$ be either an endomorphism or an anti-endomorphism of the hyperfinite II_∞ factor R^∞ such that θ_n is outer, $\mathrm{Tr}\circ\theta_n$ is a finite multiple of the trace Tr and $\theta_n(R^\infty)'\cap R^\infty$ is atomic, for all n. Denote by $\mathcal{V}$ the set of all trace-preserving automorphisms θ of R^∞ such that any product $\theta_{i_0}\prod_{j=1}^{n}\theta\theta_{i_{2j-1}}\theta^{-1}\theta_{i_{2j}}$ is outer, for all $n\geqslant 1$, $i_1,i_2,...,i_{2n-1}\in\{1,2,3,...\}$ and $i_0,i_{2n}\in\{0,1,2,...\}$, where $\theta_0=\mathrm{id}_R$. Then $\mathcal{V}$ is a G_δ-dense subset of $\mathrm{Aut}(R^\infty)$.

The case $[R^\infty:\theta_n(R^\infty)]<\infty$, for all n, of (2) follows easily from [P3, Theorem 2.1], by arguing exactly as in the proof of Lemma A.2 above. The only change in that argument is the definition of the finite index subfactor $N\subset R$, which will again be a "diagonal" inclusion, but with $R\simeq N^t$ for some appropriate amplification of N, and N embedded into it by taking a partition $p_0,...,p_n\in\mathcal{P}(R)$ and defining $N\simeq p_0Rp$, $t=\tau(p_0)^{-1}$ and $N\hookrightarrow N^t=R$ by $x+\sum_{i=1}^{n}\theta_i(x)p_i$, $x\in p_0Rp_0$, where $\tau(p_i)/\tau(p_0)=d\mathrm{Tr}\circ\theta_i/d\mathrm{Tr}$, $\theta_i: p_0Rp_0\to p_iRp_i$ being "corners" of the endomorphisms $\theta_i: R^\infty\to R^\infty$. When the resulting subfactor is extremal, the rest of the argument is identical, while in case it is not extremal, then one replaces the symmetric enveloping algebra of $N\subset R$ by an appropriately defined enveloping algebra $\widetilde{M}$, containing N_∞ and satisfying appropriate commuting square properties.

Note that this result shows in particular that given any two subfactors of finite Jones index of the hyperfinite II_1 factor, $P\subset R$ and $Q\subset R$, there exists a subfactor $N\subset R$ having standard invariant "free product" of the standard invariants $\mathcal{G}_{P,R}$ and $\mathcal{G}_{Q,R}$, as considered in [BJ].

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