

A Selberg integral for the Lie algebra A_n

by

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To the memory of Atle Selberg

1. Introduction

1.1. \mathfrak{g} -Selberg integrals

In 1944 Selberg published the following remarkable multiple integral [24]. Let k be a positive integer, $t=(t_1, \dots, t_k)$, $dt=dt_1 \dots dt_k$ and

$$\Delta(t) = \prod_{1 \leq i < j \leq k} (t_i - t_j)$$

be the Vandermonde product.

THEOREM 1.1. (Selberg integral) *For $\alpha, \beta, \gamma \in \mathbb{C}$ such that*

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0 \quad \text{and} \quad \operatorname{Re}(\gamma) > -\min\left\{\frac{1}{k}, \frac{\operatorname{Re}(\alpha)}{k-1}, \frac{\operatorname{Re}(\beta)}{k-1}\right\},$$

there holds

$$\int_{[0,1]^k} |\Delta(t)|^{2\gamma} \prod_{i=1}^k t_i^{\alpha-1} (1-t_i)^{\beta-1} dt = \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma+1)}. \quad (1.1)$$

When $k=1$, the Selberg integral simplifies to the Euler beta integral [7]

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \quad (1.2)$$

which reduces to the standard definition of the *gamma function*

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad \operatorname{Re}(\alpha) > 0,$$

upon taking $(\beta, t) \mapsto (\zeta, t/\zeta)$ (with $\zeta \in \mathbb{R}$) and letting $\zeta \rightarrow \infty$.

At the time of its publication the Selberg integral was largely overlooked, but now, more than 60 years later, it is widely regarded as one of the most fundamental and important hypergeometric integrals. It has connections and applications to orthogonal polynomials, random matrices, finite reflection groups, hyperplane arrangements, Knizhnik–Zamolodchikov equations and more; see, e.g., [1], [3], [5], [6], [8], [15], [18], [29] and [30].

Because of the appearance of the Vandermonde product, the Selberg integral may be associated with the root system A_{k-1} . That such a viewpoint is useful is evidenced by Macdonald's famous ex-conjecture, which attaches a Selberg integral to any finite reflection group G [15]. To be precise, Macdonald conjectured a generalisation to G of the exponential limit of Theorem 1.1, known as Mehta's integral:

$$\frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} |\Delta(t)|^{2\gamma} e^{-(\sum_{i=1}^k t_i^2)/2} dt = \prod_{i=1}^k \frac{\Gamma(i\gamma+1)}{\Gamma(\gamma+1)};$$

see also [9], [15], [18], [21] and [22].

A different point of view—the one we wish to adopt in this paper—arises from the intimate connection between Knizhnik–Zamolodchikov (KZ) equations and hypergeometric integrals [23], [28], [29]. Let \mathfrak{g} be a simple Lie algebra of rank n , with simple roots $\bar{\alpha}_i$ and Chevalley generators e_i, f_i and $h_i, 1 \leq i \leq n$.⁽¹⁾ Let V_λ and V_μ be highest weight representations of \mathfrak{g} with highest weights λ and μ , and let $u = u(z, w)$ be a function with values in $V_\lambda \otimes V_\mu$ solving the KZ equations

$$\varkappa \frac{\partial u}{\partial z} = \frac{\Omega}{z-w} u \quad \text{and} \quad \varkappa \frac{\partial u}{\partial w} = \frac{\Omega}{w-z} u,$$

where Ω is the Casimir element. Solutions u with values in the space of singular vectors of weight $\lambda + \mu - \sum_{i=1}^n k_i \bar{\alpha}_i$ are expressible in terms of $k := k_1 + \dots + k_n$ dimensional integrals of hypergeometric type as follows [23]:

$$u(z, w) = \sum u_{I,J}(z, w) f^I v_\lambda \otimes f^J v_\mu$$

with

$$u_{I,J}(z, w) = \int_\gamma \Psi(z, w; t) \omega_{I,J}(z, w; t) dt.$$

⁽¹⁾ We use $\bar{\alpha}_i$ instead of the usual α_i to denote the simple roots to avoid a clash of notation with the exponents α_i in the A_n Selberg integral of Theorem 1.2.

In the above the sum is over all ordered multisets I and J with elements taken from $\{1, \dots, n\}$ such that their union contains the number i exactly k_i times, v_λ and v_μ are the highest weight vectors of V_λ and V_μ , $f^I v = (\prod_{i \in I} f_i)v$, $t = (t_1, \dots, t_k)$, $dt = dt_1 \dots dt_k$ and γ is a suitable integration cycle. The function $\omega_{I,J}$ is a rational function that will not concern us here and Ψ , known as the phase function, is defined as follows. The first k_1 integration variables are attached to the simple root $\bar{\alpha}_1$, the next k_2 integration variables are attached to the simple root $\bar{\alpha}_2$, and so on, such that $\bar{\alpha}_{t_j} := \bar{\alpha}_i$ if $k_1 + \dots + k_{i-1} < j \leq k_1 + \dots + k_i$. Then

$$\Psi(z, w; t) = (z-w)^{(\lambda, \mu)/\varkappa} \prod_{i=1}^k (t_i - z)^{-(\lambda, \bar{\alpha}_{t_i})/\varkappa} (t_i - w)^{-(\mu, \bar{\alpha}_{t_i})/\varkappa} \prod_{1 \leq i < j \leq k} (t_i - t_j)^{(\bar{\alpha}_{t_i}, \bar{\alpha}_{t_j})/\varkappa},$$

with (\cdot, \cdot) being the bilinear symmetric form on \mathfrak{h}^* (the space dual to the Cartan subalgebra \mathfrak{h}) normalised such that $(\theta, \theta) = 2$ for the maximal root θ .

In [20] Mukhin and Varchenko formulated a remarkable conjecture regarding the normalised phase function

$$\Psi(t) = \prod_{i=1}^k t_i^{-(\lambda, \bar{\alpha}_{t_i})/\varkappa} (1-t_i)^{-(\mu, \bar{\alpha}_{t_i})/\varkappa} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{(\bar{\alpha}_{t_i}, \bar{\alpha}_{t_j})/\varkappa}.$$

They proposed that if the space of singular vectors of weight $\lambda + \mu - \sum_{i=1}^n k_i \bar{\alpha}_i$ is 1-dimensional, then the integral

$$\int_{\Delta} \Psi(t) dt \tag{1.3}$$

(with $\Delta \subset [0, 1]^k$ being an appropriate integration domain not explicitly given) is expressible as a product of gamma functions. The original Selberg integral corresponds to the case $\mathfrak{g} = \mathfrak{sl}_2$ of the Mukhin–Varchenko conjecture.

In the following we restrict our attention to $\mathfrak{g} = \mathfrak{sl}_{n+1} = A_n$, with fundamental weights $\Lambda_1, \dots, \Lambda_n$; $(\Lambda_i, \bar{\alpha}_j) = \delta_{ij}$. If the weights of V_λ and V_μ are $\lambda = \sum_{i=1}^n \lambda_i \Lambda_i$ and $\mu = \sum_{i=1}^n \mu_i \Lambda_i$, and if we write $t = (t_1, \dots, t_k)$ as $t = (t^{(1)}, \dots, t^{(n)})$, with $t^{(s)} = (t_1^{(s)}, \dots, t_{k_s}^{(s)})$ the variables attached to the simple root $\bar{\alpha}_s$, then

$$\Psi(t) = \prod_{s=1}^n \left(|\Delta(t^{(s)})|^{2/\varkappa} \prod_{i=1}^{k_s} (t_i^{(s)})^{-\lambda_s/\varkappa} (1-t_i^{(s)})^{-\mu_s/\varkappa} \right) \prod_{s=1}^{n-1} |\Delta(t^{(s)}, t^{(s+1)})|^{-1/\varkappa},$$

where

$$\Delta(u, v) = \prod_{i=1}^{l(u)} \prod_{j=1}^{l(v)} (u_i - v_j)$$

for $u = (u_1, \dots, u_{l(u)})$ and $v = (v_1, \dots, v_{l(v)})$. In the case of \mathfrak{sl}_2 the phase function coincides with the integrand of the Selberg integral after identifying $\gamma = 1/\varkappa$, $\alpha = 1 - \lambda_1/\varkappa$ and $\beta = 1 - \mu_1/\varkappa$.

In [26] Tarasov and Varchenko dealt with the A_2 case of (1.3), obtaining a closed form evaluation for $\lambda=\lambda_2\Lambda_2$ and $\mu=\mu_1\Lambda_1+\mu_2\Lambda_2$. In the present paper we utilise the theory of Macdonald polynomials to extend this to A_n , and one of our main results is an explicit evaluation of (1.3) for $\lambda=\lambda_n\Lambda_n$ and $\mu=\sum_i\mu_i\Lambda_i$. If we write $\varkappa=1/\gamma$, $\lambda_i=(1-\alpha_i)/\gamma$ (so that $\alpha_1=\dots=\alpha_{n-1}=1$) and $\mu_i=(1-\beta_i)/\gamma$, and let $\Delta=C_\gamma^{k_1,\dots,k_n}[0,1]$ be the integration domain defined in (4.9) of §4, we may claim an evaluation of the (1.3) for $\mathfrak{g}=A_n$.

THEOREM 1.2. (A_n Selberg integral) *For a positive integer n let $0\leq k_1\leq k_2\leq\dots\leq k_n$ be integers and $k_0=k_{n+1}=0$. Let $\alpha,\beta_1,\dots,\beta_n,\gamma\in\mathbb{C}$ be such that*

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta_1) > 0, \dots, \operatorname{Re}(\beta_n) > 0, \quad -\min\left\{\frac{\operatorname{Re}(\alpha)}{k_n-1}, \frac{1}{k_n}\right\} < \operatorname{Re}(\gamma) < \frac{1}{k_n}$$

and

$$-\frac{\operatorname{Re}(\beta_s)}{k_s-k_{s-1}-1} < \operatorname{Re}(\gamma) < \frac{\operatorname{Re}(\beta_s+\dots+\beta_r)}{r-s}$$

for $1\leq s\leq r\leq n$. Then

$$\begin{aligned} &\int_{C_\gamma^{k_1,\dots,k_n}[0,1]} \prod_{s=1}^n \left(|\Delta(t^{(s)})|^{2\gamma} \prod_{i=1}^{k_s} (t_i^{(s)})^{\alpha_s-1} (1-t_i^{(s)})^{\beta_s-1} \right) \prod_{s=1}^{n-1} |\Delta(t^{(s)}, t^{(s+1)})|^{-\gamma} dt \\ &= \prod_{1\leq s\leq r\leq n} \prod_{i=1}^{k_s-k_{s-1}} \frac{\Gamma(\beta_s+\dots+\beta_r+(i+s-r-1)\gamma)}{\Gamma(\alpha_r+\beta_s+\dots+\beta_r+(i+s-r+k_r-k_{r+1}-2)\gamma)} \\ &\quad \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s+(i-k_{s+1}-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}, \end{aligned}$$

where $\alpha_1=\dots=\alpha_{n-1}=1$, $\alpha_n=\alpha$ and $dt=dt^{(1)}\dots dt^{(n)}$.

Remarks. (1) Whenever $A/0$ occurs in the conditions on $\alpha, \beta_1, \dots, \beta_n$ and γ , this is to be interpreted as $\pm\infty$ with the same sign as A . This ensures that the conditions are correct provided $k_s>k_{s-1}$, $1\leq s\leq n$. Only minor modifications are required if $k_s=k_{s-1}$ for some s . We also note that the condition $\operatorname{Re}(\gamma)<1/k_n$ comes from $\operatorname{Re}(\gamma)<\min\{1/k_s: 2\leq s\leq n\}$ and does not apply when $n=1$.

(2) For $k_1=\dots=k_{n-1}=0$ and $(k_n, \beta_n, t^{(n)})\rightarrow(k, \beta, t)$ the A_n Selberg integral simplifies to

$$\int_{C_\gamma^{0,\dots,0,k}[0,1]} |\Delta(t)|^{2\gamma} \prod_{i=1}^k t_i^{\alpha-1} (1-t_i)^{\beta-1} dt = \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma)}.$$

Since (see (4.9))

$$C_\gamma^{0,\dots,0,k}[0,1] = \{t\in\mathbb{R}^k : 0\leq t_k\leq t_{k-1}\leq\dots\leq t_1\leq 1\},$$

this is equivalent to the Selberg integral (1.1). Indeed, by the symmetry of the integrand, we may replace $C_\gamma^{0,\dots,0,k}[0, 1]$ by $[0, 1]^k$ provided the right-hand side is multiplied by $k!$. Absorbing this factor in the ratio of gamma functions yields (1.1). More generally, the integration domain $C_\gamma^{k_1,\dots,k_n}[0, 1]$ is such that $\Delta(t^{(s)}) \geq 0$, and the absolute value sign in $|\Delta(t^{(s)})|^{2\gamma}$ (but not in $|\Delta(t^{(s)}, t^{(s+1)})|^{-\gamma}$) may be omitted.

(3) Denoting the A_n Selberg integral by

$$I_{k_1,\dots,k_n}^{A_n}(\alpha; \beta_1, \dots, \beta_n; \gamma),$$

it readily follows that

$$I_{\underbrace{0,\dots,0}_{n-m}, l_1, \dots, l_m}^{A_n}(\alpha; \beta_1, \dots, \beta_n; \gamma) = I_{l_1, \dots, l_m}^{A_m}(\alpha; \beta_{n-m+1}, \dots, \beta_n; \gamma).$$

In particular, we have

$$C_\gamma^{\overbrace{0,\dots,0}^{n-m}, l_1, \dots, l_m}[0, 1] = C_\gamma^{l_1, \dots, l_m}[0, 1].$$

The case $k_1 = \dots = k_{n-1} = 0, k_n = k$ discussed in (2) is of course a special case of this more general reduction formula.

(4) By an appropriate change of integration variables (see §7 for details), it follows that

$$I_{1,1,k_3,\dots,k_n}^{A_n}(\alpha; \beta_1, \beta_2, \dots, \beta_n; \gamma) = I_{1,k_3,\dots,k_n}^{A_{n-1}}(\alpha; \beta_1 + \beta_2 - \gamma, \beta_3, \dots, \beta_n; \gamma) \frac{\Gamma(1-\gamma)\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma + 1)}. \tag{1.4}$$

By iteration, all but the last k_s equal to 1 may thus be eliminated.

(5) Upon taking $n=2, (k_1, k_2) \mapsto (k_2, k_1)$ and $(\beta_1, \beta_2) \mapsto (\beta_2, \beta_1)$, we obtain the \mathfrak{sl}_3 Selberg integral of Tarasov and Varchenko [26, Theorem 3.3]; see also [30].

(6) If we denote the set of positive roots of A_n by Φ_+ (i.e., $\Phi_+ = \{\bar{\alpha}_s + \dots + \bar{\alpha}_r : 1 \leq s \leq r \leq n\}$), then the product over $1 \leq s \leq r \leq n$ on the right-hand side of the A_n Selberg integral corresponds to the following product over Φ_+ :

$$\prod_{1 \leq s \leq r \leq n} g(\beta_s + \dots + \beta_r) = \prod_{\bar{\alpha} \in \Phi_+} g((\Lambda, \bar{\alpha})),$$

where $\Lambda = \beta_1 \Lambda_1 + \dots + \beta_n \Lambda_n$.

By replacing $(\beta_s, t^{(s)}) \mapsto (\zeta \beta_s, t^{(s)}/\zeta)$, with $\zeta \in \mathbb{R}$, and then letting ζ tend to infinity, we obtain the following exponential form of Theorem 1.2, with $C_\gamma^{k_1,\dots,k_n}[0, \infty]$ being the domain defined in (4.9) of §4.

COROLLARY 1.3. (First A_n exponential Selberg integral) *For a positive integer n let $0 \leq k_1 \leq k_2 \leq \dots \leq k_n$ be integers and $k_0 = k_{n+1} = 0$. Let $\alpha, \beta_1, \dots, \beta_n, \gamma \in \mathbb{C}$ be such that*

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta_1) > 0, \dots, \operatorname{Re}(\beta_n) > 0 \quad \text{and} \quad -\min\left\{\frac{\operatorname{Re}(\alpha)}{k_n-1}, \frac{1}{k_n}\right\} < \operatorname{Re}(\gamma) < \frac{1}{k_n}.$$

Then

$$\begin{aligned} \int_{C_\gamma^{k_1, \dots, k_n} [0, \infty]} \prod_{s=1}^n \left(|\Delta(t^{(s)})|^{2\gamma} \prod_{i=1}^{k_s} (t_i^{(s)})^{\alpha_s-1} e^{-\beta_s t_i^{(s)}} \right) \prod_{s=1}^{n-1} |\Delta(t^{(s)}, t^{(s+1)})|^{-\gamma} dt \\ = \prod_{1 \leq s \leq r \leq n} (\beta_s + \dots + \beta_r)^{-(\alpha_r + (k_r - k_{r+1} - 1)\gamma)(k_s - k_{s-1})} \\ \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s + (i - k_{s+1} - 1)\gamma) \Gamma(i\gamma)}{\Gamma(\gamma)}, \end{aligned}$$

where $\alpha_1 = \dots = \alpha_{n-1} = 1$ and $\alpha_n = \alpha$.

Replacing $t_i \mapsto 1 - t_i$ for all $1 \leq i \leq k$ in the Selberg integral (1.1) leads to an interchange of α and β . Consequently the classical Selberg integral has just a single exponential form. This (α, β) -symmetry is no longer present for $n > 1$, and replacing $t_i^{(s)} \mapsto 1 - t_i^{(s)}$ for all $1 \leq i \leq k_s$ and $1 \leq s \leq n$ followed by $(\alpha, t^{(s)}) \mapsto (\zeta, t^{(s)}/\zeta)$ (with $\zeta \in \mathbb{R}$) and then letting ζ tend to infinity, results in a second exponential form of Theorem 1.2. Below $\bar{C}_\gamma^{k_1, \dots, k_n} [0, \infty]$ is the integration domain defined in (5.8) of §5.

COROLLARY 1.4. (Second A_n exponential Selberg integral) *For a positive integer n let $0 \leq k_1 \leq k_2 \leq \dots \leq k_n$ be integers and $k_0 = k_{n+1} = 0$. Let $\beta_1, \dots, \beta_n, \gamma \in \mathbb{C}$ be such that*

$$\operatorname{Re}(\beta_1) > 0, \dots, \operatorname{Re}(\beta_n) > 0, \quad -\frac{1}{k_n} < \operatorname{Re}(\gamma) < \frac{1}{k_n}$$

and

$$-\frac{\operatorname{Re}(\beta_s)}{k_s - k_{s-1} - 1} < \operatorname{Re}(\gamma) < \frac{\operatorname{Re}(\beta_s + \dots + \beta_r)}{r - s}$$

for $1 \leq s \leq r \leq n$. Then

$$\begin{aligned} \int_{\bar{C}_\gamma^{k_1, \dots, k_n} [0, \infty]} e^{-\sum_{i=1}^{k_n} t_i^{(n)}} \prod_{s=1}^n \left(|\Delta(t^{(s)})|^{2\gamma} \prod_{i=1}^{k_s} (t_i^{(s)})^{\beta_s-1} \right) \prod_{s=1}^{n-1} |\Delta(t^{(s)}, t^{(s+1)})|^{-\gamma} dt \\ = \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \Gamma(\beta_s + \dots + \beta_r + (i + s - r - 1)\gamma) \\ \times \prod_{1 \leq s \leq r \leq n-1} \prod_{i=1}^{k_s - k_{s-1}} \frac{1}{\Gamma(1 + \beta_s + \dots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)} \\ \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \Gamma(1 + (i - k_{s+1} - 1)\gamma). \end{aligned}$$

1.2. Outline

In §2 we review some standard facts about Macdonald polynomials needed to prove an identity for the q, t -analogues of the Littlewood–Richardson coefficients (Theorem 2.1). In §3 we apply Theorem 2.1 to establish a new A_n q -binomial theorem for Macdonald polynomials (Theorem 3.2). In §4 we first utilise the $q=1$ case of this theorem to prove the exponential A_n Selberg integral of Corollary 1.3. Then, in §5, we exploit the full q -binomial theorem to obtain a multidimensional q -integral which yields Theorem 1.2 in the $q \rightarrow 1$ limit. In §6 we generalise the A_n Selberg integral by including a Jack polynomial in the integrand (Theorem 6.1). Finally, in §7, we give the full details of two special cases of Theorem 1.2, corresponding to $(k_1, \dots, k_{n-1}, k_n) = (1, \dots, 1, k)$ and $\gamma=0$, respectively.

2. Macdonald polynomials

Our main tool in the proof of Theorem 1.2 is the theory of symmetric functions, and in §2.1 and §2.2 we review some well-known facts from the theory. For a more comprehensive introduction we refer the reader to [14], [16] and [25].

2.1. Preliminaries

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \dots$, with finitely many λ_i unequal to zero. The length and weight of λ , denoted by $l(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero λ_i , respectively. As usual, we identify two partitions that differ only in their string of zeros, so that $(6, 3, 3, 1, 0, 0)$ and $(6, 3, 3, 1)$ represent the same partition. When $|\lambda| = N$ we say that λ is a partition of N , and the unique partition of zero is denoted by 0 . The multiplicity of the part i in the partition λ is denoted by $m_i = m_i(\lambda)$, and occasionally we will write $\lambda = (1^{m_1} 2^{m_2} \dots)$.

We identify a partition with its *Ferrers graph*, defined by the set of points in $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$, and further make the usual identification between Ferrers graphs and (Young) diagrams by replacing points by squares.

The conjugate λ' of λ is the partition obtained by reflecting the diagram of λ in the main diagonal, so that, in particular, $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$. The statistic $n(\lambda)$ is given by

$$n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i = \sum_{i \geq 1} \binom{\lambda'_i}{2}.$$

The dominance partial order on the set of partitions of N is defined by $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all $i \geq 1$. If $\lambda \geq \mu$ and $\lambda \neq \mu$ then $\lambda > \mu$.

If λ and μ are partitions then $\mu \subseteq \lambda$ if (the diagram of) μ is contained in (the diagram of) λ , i.e., $\mu_i \leq \lambda_i$ for all $i \geq 1$. If $\mu \subseteq \lambda$ then the skew-diagram $\lambda - \mu$ denotes the set-theoretic difference between λ and μ , i.e., those squares of λ not contained in μ .

Let $s = (i, j)$ be a square in the diagram of λ . Then $a(s)$, $a'(s)$, $l(s)$ and $l'(s)$ are the *arm-length*, *arm-colength*, *leg-length* and *leg-colength* of s , respectively, defined by

$$a(s) = \lambda_i - j, \quad a'(s) = j - 1, \tag{2.1a}$$

$$l(s) = \lambda'_j - i, \quad l'(s) = i - 1. \tag{2.1b}$$

This may be used to define the *generalised hook-length polynomials* [16, (VI.8.1)]

$$c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1}), \tag{2.2a}$$

$$c'_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)}), \tag{2.2b}$$

where the products are over all squares of λ . We further set

$$b_\lambda(q, t) = \frac{c_\lambda(q, t)}{c'_\lambda(q, t)}. \tag{2.3}$$

For a non-negative integer N the q -shifted factorial $(b; q)_N$ is defined as $(b; q)_0 = 1$ and

$$(b; q)_N = (1 - b)(1 - bq) \dots (1 - bq^{N-1}). \tag{2.4}$$

We also need the q -shifted factorial for negative (integer) values of N . This may be obtained from the above by

$$(b; q)_{-N} = \frac{1}{(bq^{-N}; q)_N}.$$

This implies in particular that $1/(q; q)_{-N} = 0$ for positive N .

The definition (2.4) may be extended to partitions by

$$(b; q, t)_\lambda = \prod_{s \in \lambda} (1 - bq^{a'(s)} t^{-l'(s)}) = \prod_{i=1}^{l(\lambda)} (bt^{1-i}; q)_{\lambda_i}.$$

With this notation, the polynomials (2.2) can be expressed as [12, Proposition 3.2]

$$c_\lambda(q, t) = (t^n; q, t)_\lambda \prod_{1 \leq i < j \leq n} \frac{(t^{j-i}; q)_{\lambda_i - \lambda_j}}{(t^{j-i+1}; q)_{\lambda_i - \lambda_j}}, \tag{2.5a}$$

$$c'_\lambda(q, t) = (qt^{n-1}; q, t)_\lambda \prod_{1 \leq i < j \leq n} \frac{(qt^{j-i-1}; q)_{\lambda_i - \lambda_j}}{(qt^{j-i}; q)_{\lambda_i - \lambda_j}}, \tag{2.5b}$$

where n is any integer such that $n \geq l(\lambda)$.

Finally, we introduce the usual condensed notation for q -shifted factorials as

$$(a_1, \dots, a_k; q)_N = (a_1; q)_N \dots (a_k; q)_N$$

and

$$(a_1, \dots, a_k; q, t)_\lambda = (a_1; q, t)_\lambda \dots (a_k; q, t)_\lambda.$$

2.2. Macdonald polynomials

Let \mathfrak{S}_n denote the symmetric group, and $\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ be the ring of symmetric polynomials in n independent variables.

Let $x = (x_1, \dots, x_n)$. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of at most n parts, the *monomial symmetric function* m_λ is defined as

$$m_\lambda(x) = \sum x^\alpha.$$

Here the sum is over all distinct permutations α of λ , and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. For $l(\lambda) > n$ we set $m_\lambda(x) = 0$. The monomial symmetric functions m_λ for $l(\lambda) \leq n$ form a \mathbb{Z} -basis of Λ_n .

For a non-negative integer r , the power sums p_r are given by $p_0 = 1$ and $p_r = m_{(r)}$ for $r > 1$. Hence

$$p_r(x) = \sum_{i=1}^n x_i^r.$$

More generally the power-sum products are defined as $p_\lambda(x) = p_{\lambda_1}(x) \dots p_{\lambda_n}(x)$.

Following Macdonald we define the scalar product $\langle \cdot, \cdot \rangle_{q,t}$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^n \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}},$$

with $z_\lambda = \prod_{i \geq 1} m_i! i^{m_i}$ and $m_i = m_i(\lambda)$. If we denote the ring of symmetric functions in n variables over the field $\mathbb{F} = \mathbb{Q}(q, t)$ of rational functions in q and t by $\Lambda_{n,\mathbb{F}}$, then the Macdonald polynomial $P_\lambda(x; q, t)$ is the unique symmetric polynomial in $\Lambda_{n,\mathbb{F}}$ such that [16, (VI.4.7)]

$$P_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu}(q, t) m_\mu(x) \tag{2.6}$$

and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{if } \lambda \neq \mu.$$

The Macdonald polynomials $P_\lambda(x; q, t)$ with $l(\lambda) \leq n$ form an \mathbb{F} -basis of $\Lambda_{n,\mathbb{F}}$. If $l(\lambda) > n$ then $P_\lambda(x; q, t) = 0$. From (2.6) it follows that $P_\lambda(x; q, t)$ for $l(\lambda) \leq n$ is homogeneous of degree $|\lambda|$:

$$P_\lambda(zx; q, t) = z^{|\lambda|} P_\lambda(x; q, t), \tag{2.7}$$

where z is a scalar.

For $f \in \Lambda_{n,\mathbb{F}}$ and a partition λ such that $l(\lambda) \leq n$, the evaluation homomorphism $u_\lambda^{(n)}: \Lambda_{n,\mathbb{F}} \rightarrow \mathbb{F}$ is defined as

$$u_\lambda^{(n)}(f) = f(q^{\lambda_1} t^{n-1}, q^{\lambda_2} t^{n-2}, \dots, q^{\lambda_n} t^0). \tag{2.8}$$

We extend this to $f \in \mathbb{F}(x_1, \dots, x_n)^{\mathfrak{S}_n}$ for those f for which the right-hand side of (2.8) is well defined. The principal specialisation formula for Macdonald polynomials corresponds to [16, Example VI.6.5]

$$u_0^{(n)}(P_\lambda) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{n-l'(s)}}{1 - q^{a(s)} t^{l(s)+1}} = t^{n(\lambda)} \frac{(t^n; q, t)_\lambda}{c_\lambda(q, t)}. \quad (2.9)$$

For more general evaluations we have the symmetry [16, (VI.6.6)]

$$u_\lambda^{(n)}(P_\mu) u_0^{(n)}(P_\lambda) = u_\mu^{(n)}(P_\lambda) u_0^{(n)}(P_\mu) \quad (2.10)$$

for $l(\lambda), l(\mu) \leq n$. It will be convenient to also define $u_{\lambda; z}^{(n)}$ as

$$u_{\lambda; z}^{(n)}(f) = f(zq^{\lambda_1} t^{n-1}, zq^{\lambda_2} t^{n-2}, \dots, zq^{\lambda_n} t^0). \quad (2.11)$$

For homogeneous functions of degree d we of course have

$$u_{\lambda; z}^{(n)}(f) = z^d u_\lambda^{(n)}(f). \quad (2.12)$$

Due to the stability $P_\lambda(x_1, \dots, x_n; q, t) = P_\lambda(x_1, \dots, x_n, 0; q, t)$ for $l(\lambda) \leq n$, we may extend the P_λ to an infinite alphabet, and in the remainder of this section we assume that x (and y) contain countably many variables. By abuse of terminology we will still refer to $P_\lambda(x; q, t)$ as a Macdonald polynomial, instead of a Macdonald function. Then the Cauchy identity for Macdonald polynomials is given by [16, (VI.4.13)]

$$\sum_\lambda b_\lambda(q, t) P_\lambda(x; q, t) P_\lambda(y; q, t) = \prod_{i, j \geq 1} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}, \quad (2.13)$$

with $b_\lambda(q, t)$ defined in (2.3).

The q, t -Littlewood–Richardson coefficients are defined as

$$P_\mu(x; q, t) P_\nu(x; q, t) = \sum_\lambda f_{\mu\nu}^\lambda(q, t) P_\lambda(x; q, t) \quad (2.14)$$

and trivially satisfy

$$f_{\mu\nu}^\lambda(q, t) = f_{\nu\mu}^\lambda(q, t)$$

and

$$f_{\mu\nu}^\lambda(q, t) = 0 \quad \text{unless } |\lambda| = |\mu| + |\nu|. \quad (2.15)$$

It can also be shown that [16, (VI.7.7)]

$$f_{\mu\nu}^\lambda(q, t) = 0 \quad \text{unless } \mu, \nu \subseteq \lambda. \quad (2.16)$$

The q, t -Littlewood–Richardson coefficients may be used to define the skew Macdonald polynomials

$$P_{\lambda/\mu}(x; q, t) = \sum_{\nu} f_{\mu\nu}^{\lambda}(q, t) P_{\nu}(x; q, t). \tag{2.17}$$

By (2.16), $P_{\lambda/\mu}(x; q, t) = 0$ unless $\mu \subseteq \lambda$ (in which case it is homogeneous of degree $|\lambda| - |\mu|$). Equivalent to (2.17) is

$$P_{\lambda}(x, y; q, t) = \sum_{\mu} P_{\lambda/\mu}(x; q, t) P_{\mu}(y; q, t). \tag{2.18}$$

Finally, we need the Kaneko–Macdonald definition of basic hypergeometric series with Macdonald polynomial argument [12], [17]:

$${}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, t; x \right] = \sum_{\lambda} t^{n(\lambda)} \frac{P_{\lambda}(x; q, t)}{c'_{\lambda}(q, t)} \frac{(a_1, \dots, a_{r+1}; q, t)_{\lambda}}{(b_1, \dots, b_r; q, t)_{\lambda}}. \tag{2.19}$$

When $x = (z)$ this reduces to the classical ${}_{r+1}\phi_r$ basic hypergeometric series [10]:

$${}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, t; (z) \right] = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_k}{(q, b_1, \dots, b_r; q)_k} z^k = {}_{r+1}\phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right].$$

The main result needed for ${}_{r+1}\Phi_r$ series is the q -binomial theorem [12, Theorem 3.5], [17, (2.2)] (see also [13, Theorem 3] and [19, Lemma 3.1]):

$${}_1\Phi_0 \left[\begin{matrix} a \\ - \end{matrix}; q, t; x \right] = \prod_{i \geq 1} \frac{(ax_i; q)_{\infty}}{(x_i; q)_{\infty}}. \tag{2.20}$$

Remark. In this paper we mostly view results such as (2.13) and (2.20) as formal identities. Later, when transforming formal power series to integrals, issues of convergence do become important. It is however not difficult to give necessary convergence conditions for each of the identities in this paper. For example, in (2.20), we may add $x = (x_1, \dots, x_n)$, $|q| < 1$ and $\max\{|x_1|, \dots, |x_n|\} < 1$, and view the ${}_1\Phi_0$ as a genuine hypergeometric function.

2.3. An identity for q, t -Littlewood–Richardson coefficients

A crucial role in our proof of the A_n Selberg integral of Theorem 1.2 is the following identity for the q, t -Littlewood–Richardson coefficients.

THEOREM 2.1. *Given two integers $0 \leq m \leq n$, let λ and μ be partitions such that $l(\lambda) \leq m$ and $l(\mu) \leq n$. Then*

$$\begin{aligned} & \sum_{\nu, \omega} t^{n(\nu) - |\omega|} f_{\omega\nu}^\lambda(q, t) u_0^{(n-m)}(P_{\mu/\omega}) \frac{(qt^{m-n-1}; q, t)_\nu}{c'_\nu(q, t)} \\ &= t^{n(\lambda) - m|\mu|} u_0^{(n)}(P_\mu) \frac{(qt^{m-1}; q, t)_\lambda}{c'_\lambda(q, t)} \prod_{i=1}^m \prod_{j=1}^n \frac{(qt^{j-i+m-n-1}; q)_{\lambda_i - \mu_j}}{(qt^{j-i+m-n}; q)_{\lambda_i - \mu_j}}. \end{aligned} \tag{2.21}$$

Since $f_{\omega\nu}^\lambda(q, t) = 0$ if $\omega \not\subseteq \lambda$ and $P_{\mu/\omega} = 0$ if $\omega \not\subseteq \mu$, we may add the restrictions $\omega \subseteq \lambda$ and $\omega \subseteq \mu$ to the sum over ω . We will in fact show that the summand on the left vanishes unless

$$\lambda_i \geq \mu_{i+n-m} \quad \text{for } 1 \leq i \leq m. \tag{2.22}$$

In other words, if μ^* is the partition formed by the last m parts of μ , i.e.,

$$\mu^* = (\mu_{n-m+1}, \dots, \mu_n),$$

then the summand vanishes unless $\mu^* \subseteq \lambda$. To see this, we recall from [16, (VI.7.13')] that

$$P_{\mu/\omega}(x_1, \dots, x_{n-m}; q, t) = \sum_T \psi_T(q, t) x^T,$$

where the sum is over all semistandard Young tableaux T of skew shape $\mu - \omega$ over the alphabet $\{1, \dots, n-m\}$; x^T is the monomial defined by T and $\psi_T \in \mathbb{F}$. For the shape $\mu - \omega$ to have an admissible filling it must have at most $n-m$ boxes in each of its columns. Hence $\omega_i \geq \mu_{i+n-m}$ for $1 \leq i \leq m$. Since we already established that the summand vanishes unless $\omega \subseteq \lambda$, a necessary condition for non-vanishing of the summand is thus given by (2.22). Since $1/(q; q)_{-N} = 0$ for every positive integer N , it is easily seen that also the double product on the right-hand side of (2.21) vanishes unless (2.22) holds.

Theorem 2.1 for arbitrary $0 \leq m \leq n$ corresponds to the $u=0$ case of a more general result established in [32, Theorem 4.1]. For $m=n$, so that

$$P_{\mu/\omega}(x_1, \dots, x_{n-m}; q, t) = \delta_{\mu\omega},$$

the theorem simplifies to [31, Proposition 3.2]. A proof of Theorem 2.1 is included below for the sake of completeness.

Proof of Theorem 2.1. Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ so that the Cauchy identity (2.13) becomes

$$\sum_{\eta} b_{\eta}(q, t) P_{\eta}(x; q, t) P_{\eta}(y; q, t) = \prod_{i=1}^m \prod_{j=1}^n \frac{(tx_i y_j; q)_{\infty}}{(x_i y_j; q)_{\infty}}.$$

Next we apply the homomorphisms $u_{\lambda; z}^{(m)}$ (acting on x) and $u_{\mu}^{(n)}$ (acting on y), and use the homogeneity (2.7) of the Macdonald polynomials. Hence

$$\sum_{\eta} z^{|\eta|} b_{\eta}(q, t) u_{\lambda}^{(m)}(P_{\eta}) u_{\mu}^{(n)}(P_{\eta}) = \prod_{i=1}^m \frac{(zt^{n+m-i}; q)_{\infty}}{(zt^{m-i}; q)_{\infty}} \prod_{i=1}^m \prod_{j=1}^n \frac{(zt^{n+m-i-j}; q)_{\lambda_i + \mu_j}}{(zt^{n+m-i-j+1}; q)_{\lambda_i + \mu_j}}. \tag{2.23}$$

The summand on the left vanishes unless $l(\eta) \leq \min\{n, m\}$. Assuming such an η , we may twice use the symmetry (2.10) to rewrite the left-hand side as

$$\text{LHS(2.23)} = \sum_{\eta} z^{|\eta|} b_{\eta}(q, t) \frac{u_{\eta}^{(m)}(P_{\lambda}) u_{\eta}^{(n)}(P_{\mu}) u_0^{(m)}(P_{\eta}) u_0^{(n)}(P_{\eta})}{u_0^{(m)}(P_{\lambda}) u_0^{(n)}(P_{\mu})}.$$

In the remainder we assume that $n \geq m$ and apply (2.18) as well as (2.7) to get

$$\begin{aligned} u_{\eta}^{(n)}(P_{\mu}) &= P_{\mu}(q^{\eta_1} t^{n-1}, \dots, q^{\eta_m} t^{n-m}, t^{n-m-1}, \dots, t, 1; q, t) \\ &= \sum_{\omega} P_{\omega}(q^{\eta_1} t^{n-1}, \dots, q^{\eta_m} t^{n-m}, q, t) u_0^{(n-m)}(P_{\mu/\omega}) \\ &= \sum_{\omega} t^{(n-m)|\omega|} u_{\eta}^{(m)}(P_{\omega}) u_0^{(n-m)}(P_{\mu/\omega}). \end{aligned}$$

Thus,

$$\text{LHS(2.23)} = \sum_{\eta, \omega} z^{|\eta|} t^{(n-m)|\omega|} b_{\eta}(q, t) \frac{u_0^{(n-m)}(P_{\mu/\omega}) u_{\eta}^{(m)}(P_{\lambda}) u_{\eta}^{(m)}(P_{\omega}) u_0^{(m)}(P_{\eta}) u_0^{(n)}(P_{\eta})}{u_0^{(m)}(P_{\lambda}) u_0^{(n)}(P_{\mu})}.$$

Next we use that

$$\begin{aligned} u_{\eta}^{(m)}(P_{\lambda}) u_{\eta}^{(m)}(P_{\omega}) &= u_{\eta}^{(m)}(P_{\lambda} P_{\omega}) \\ &= u_{\eta}^{(m)}\left(\sum_{\nu} f_{\omega\lambda}^{\nu} P_{\nu}\right) \quad (\text{by (2.14)}) \\ &= \sum_{\nu} f_{\omega\lambda}^{\nu} u_{\eta}^{(m)}(P_{\nu}) \end{aligned}$$

to rewrite this as

$$\text{LHS(2.23)} = \sum_{\eta, \omega, \nu} z^{|\eta|} t^{(n-m)|\omega|} f_{\omega\lambda}^{\nu}(q, t) b_{\eta}(q, t) \frac{u_0^{(n-m)}(P_{\mu/\omega}) u_{\eta}^{(m)}(P_{\nu}) u_0^{(m)}(P_{\eta}) u_0^{(n)}(P_{\eta})}{u_0^{(m)}(P_{\lambda}) u_0^{(n)}(P_{\mu})}.$$

By one more application of (2.10), this becomes

$$\text{LHS(2.23)} = \sum_{\eta, \omega, \nu} z^{|\eta|} t^{(n-m)|\omega|} f_{\omega\lambda}^{\nu}(q, t) b_{\eta}(q, t) \frac{u_0^{(n-m)}(P_{\mu/\omega}) u_{\nu}^{(m)}(P_{\eta}) u_0^{(m)}(P_{\nu}) u_0^{(n)}(P_{\eta})}{u_0^{(m)}(P_{\lambda}) u_0^{(n)}(P_{\mu})}.$$

The sum over η may now be evaluated as follows:

$$\begin{aligned} & \sum_{\eta} z^{|\eta|} b_{\eta}(q, t) u_0^{(n)}(P_{\eta}) u_{\nu}^{(m)}(P_{\eta}) \\ &= \sum_{\eta} t^{n(\eta)} \frac{(t^n; q, t)_{\eta}}{c'_{\eta}(q, t)} u_{\nu; z}^{(m)}(P_{\eta}) && \text{(by (2.3) and (2.9))} \\ &= u_{\nu; z}^{(m)} \left(\sum_{\eta} t^{n(\eta)} \frac{(t^n; q, t)_{\eta}}{c'_{\eta}(q, t)} P_{\eta}(x; q, t) \right) \\ &= u_{\nu; z}^{(m)} \left({}_1\Phi_0 \left[\begin{matrix} t^n \\ - \end{matrix}; q, t; x \right] \right) && \text{(by (2.19))} \\ &= u_{\nu; z}^{(m)} \left(\prod_{i=1}^m \frac{(x_i t^n; q)_{\infty}}{(x_i; q)_{\infty}} \right) && \text{(by (2.20))} \\ &= \prod_{i=1}^m \frac{(z q^{\nu_i} t^{n+m-i}; q)_{\infty}}{(z q^{\nu_i} t^{m-i}; q)_{\infty}} \\ &= \frac{(z t^{m-1}; q, t)_{\nu}}{(z t^{n+m-1}; q, t)_{\nu}} \prod_{i=1}^m \frac{(z t^{n+m-i}; q)_{\infty}}{(z t^{m-i}; q)_{\infty}}. \end{aligned}$$

We thus arrive at

$$\begin{aligned} \text{LHS(2.23)} &= \prod_{i=1}^m \frac{(z t^{n+m-i}; q)_{\infty}}{(z t^{m-i}; q)_{\infty}} \\ &\quad \times \sum_{\omega, \nu} t^{(n-m)|\omega|} f_{\omega\lambda}^{\nu}(q, t) \frac{u_0^{(n-m)}(P_{\mu/\omega}) u_0^{(m)}(P_{\nu})}{u_0^{(m)}(P_{\lambda}) u_0^{(n)}(P_{\mu})} \frac{(z t^{m-1}; q, t)_{\nu}}{(z t^{n+m-1}; q, t)_{\nu}}. \end{aligned}$$

Finally equating this with the right-hand side of (2.23) and replacing $z \mapsto z t^{1-n-m}$ yields

$$\begin{aligned} & \sum_{\omega, \nu} t^{(n-m)|\omega|} f_{\omega\lambda}^{\nu}(q, t) u_0^{(n-m)}(P_{\mu/\omega}) u_0^{(m)}(P_{\nu}) \frac{(z t^{-n}; q, t)_{\nu}}{(z; q, t)_{\nu}} \\ &= u_0^{(m)}(P_{\lambda}) u_0^{(n)}(P_{\mu}) \prod_{i=1}^m \prod_{j=1}^n \frac{(z t^{1-i-j}; q)_{\lambda_i + \mu_j}}{(z t^{2-i-j}; q)_{\lambda_i + \mu_j}}. \end{aligned} \tag{2.24}$$

Both sides of this identity trivially vanish if $l(\lambda) > m$. Furthermore, the summand on the left vanishes if $l(\nu) > m$. Hence we may without loss of generality assume in the following that $l(\lambda) \leq m$ and $l(\nu) \leq m$. (The latter of course refers to a restriction on the summation index.) We may also assume that the largest part of ν is bounded, since $f_{\omega\lambda}^{\nu} = 0$ if $|\omega| + |\lambda| \neq |\nu|$ and $P_{\mu/\omega} = 0$ if $\omega \not\subseteq \mu$. In particular, $\nu_1 \leq |\lambda| + |\mu|$.

The above considerations imply that $\lambda, \nu \subseteq (N^m)$ for sufficiently large N . Given such N , we can define the complements of λ and ν with respect to (N^m) . Denoting these partitions by $\hat{\lambda}$ and $\hat{\nu}$, we have $\hat{\lambda}_i = N - \lambda_{m+1-i}$ and $\hat{\nu}_i = N - \nu_{m+1-i}$ for $1 \leq i \leq m$.

We now replace λ and ν by $\hat{\lambda}$ and $\hat{\nu}$, respectively, in (2.24) and then eliminate the hats using [31, p. 263]

$$f_{\omega\hat{\lambda}}^{\hat{\nu}}(q, t) = t^{n(\nu)-n(\lambda)} f_{\omega\nu}^{\lambda}(q, t) \frac{(qt^{m-1}; q, t)_{\nu} c'_{\lambda}(q, t) u_0^{(m)}(P_{\lambda})}{(qt^{m-1}; q, t)_{\lambda} c'_{\nu}(q, t) u_0^{(m)}(P_{\nu})},$$

[4, (4.1)]

$$(a; q, t)_{\hat{\lambda}} = (-q/a)^{|\lambda|} t^{(m-1)|\lambda|-n(\lambda)} q^{n(\lambda')-N|\lambda|} \frac{(a; q, t)_{(Nm)}}{(q^{1-N}t^{m-1}/a; q, t)_{\lambda}}$$

and

$$u_0^{(m)}(P_{\hat{\lambda}}) = t^{\binom{m}{2}N+(1-m)|\lambda|} u_0^{(m)}(P_{\lambda}).$$

This last result follows from [4, (4.3)]

$$P_{\hat{\lambda}}(x; q, t) = (x_1 \dots x_m)^N P_{\lambda}(x^{-1}; q, t)$$

and the homogeneity (2.7). As a result, we end up with

$$\begin{aligned} \sum_{\omega, \nu} t^{n(\nu)-|\omega|} f_{\omega\nu}^{\lambda}(q, t) u_0^{(n-m)}(P_{\mu/\omega}) \frac{(qt^{m-1}, q^{1-N}t^{m-1}/z; q, t)_{\nu}}{c'_{\nu}(q, t)(q^{1-N}t^{n+m-1}/z; q, t)_{\nu}} \\ = t^{n(\lambda)-m|\mu|} u_0^{(n)}(P_{\mu}) \frac{(qt^{m-1}; q, t)_{\lambda}}{c'_{\lambda}(q, t)} \prod_{i=1}^m \prod_{j=1}^n \frac{(q^{1-N}t^{j-i+m-1}/z; q)_{\lambda_i-\mu_j}}{(q^{1-N}t^{j-i+m}/z; q)_{\lambda_i-\mu_j}}, \end{aligned}$$

where we have also used that

$$\frac{(a; q)_{N-k}}{(b; q)_{N-k}} = \frac{(a; q)_N}{(b; q)_N} \frac{(q^{1-N}/b; q)_k}{(q^{1-N}/a; q)_k} \left(\frac{b}{a}\right)^k$$

to rewrite the double product on the right.

Specialising $z \rightarrow q^{-N}t^n$ eliminates all references to N and completes the proof. \square

3. A_n basic hypergeometric series

In this section we will be working with n different sets of variables $x^{(1)}, \dots, x^{(n)}$, where

$$x^{(s)} = (x_1^{(s)}, \dots, x_{k_s}^{(s)})$$

are such that $k_1 \leq k_2 \leq \dots \leq k_n$.

Our main object of interest is the following generalisation of the Kaneko–Macdonald basic hypergeometric series (2.19).

Definition. (A_n basic hypergeometric series)

$$\begin{aligned}
 {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, t; x^{(1)}, \dots, x^{(n)} \right] \\
 = \sum_{\lambda^{(1)}, \dots, \lambda^{(n)}} \frac{(a_1, \dots, a_{r+1}; q, t)_{\lambda^{(n)}}}{(qt^{k_n-1}, b_1, \dots, b_r; q, t)_{\lambda^{(n)}}} \\
 \times \prod_{s=1}^n \left(t^{n(\lambda^{(s)})} \frac{(qt^{k_s-1}; q, t)_{\lambda^{(s)}}}{c'_{\lambda^{(s)}}(q, t)} P_{\lambda^{(s)}}(x^{(s)}; q, t) \right) \\
 \times \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}} \frac{(qt^{j-i+k_s-k_{s+1}-1}; q)_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}}{(qt^{j-i+k_s-k_{s+1}}; q)_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}}.
 \end{aligned} \tag{3.1}$$

Here the sum is over partitions $\lambda^{(s)}$ such that $l(\lambda^{(s)}) \leq k_s$ for $1 \leq s \leq n$ and

$$\lambda_i^{(s)} \geq \lambda_{i-k_s+k_{s+1}}^{(s+1)} \quad \text{for } 1 \leq i \leq k_s. \tag{3.2}$$

Remark. The sum over the partitions $\lambda^{(1)}, \dots, \lambda^{(n)}$ subject to (3.2) may alternatively be viewed as a sum over skew plane partitions of shape $\eta - \nu$ with $\eta = ((k_n)^n)$ being a partition of rectangular shape and $\nu = (k_n - k_1, k_n - k_2, \dots, k_n - k_{n-1})$.

The above definition simplifies to (2.19) when $n=1$, and to

$$\begin{aligned}
 {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, t; (z_1), (z_2), \dots, (z_n) \right] \\
 = \sum_{0 \leq j_n \leq \dots \leq j_1} \frac{(a_1, \dots, a_{r+1}; q)_{j_n}}{(q, b_1, \dots, b_r; q)_{j_n}} z_1^{j_1} \dots z_n^{j_n} \prod_{s=1}^{n-1} \frac{(q/t; q)_{j_s-j_{s+1}}}{(q; q)_{j_s-j_{s+1}}}
 \end{aligned}$$

when $k_1 = k_2 = \dots = k_n = 1$. Introducing new summation indices by $m_s = j_s - j_{s+1}$, $1 \leq s \leq n$ (where $j_{n+1} := 0$), this gives

$$\begin{aligned}
 {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, t; (z_1), (z_2), \dots, (z_n) \right] \\
 = {}_{r+1}\phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, z_1 \dots z_n \right] \prod_{s=1}^{n-1} {}_1\phi_0 \left[\begin{matrix} q/t \\ - \end{matrix} ; q, z_1 \dots z_s \right].
 \end{aligned}$$

Summing the ${}_1\phi_0$ series by the q -binomial theorem [10, (II.3)]

$${}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix} ; q, z \right] = \frac{(az; q)_\infty}{(z; q)_\infty},$$

we get

$$\begin{aligned}
 {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, t; (z_1), (z_2), \dots, (z_n) \right] \\
 = {}_{r+1}\phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, z_1 \dots z_n \right] \prod_{s=1}^{n-1} \frac{(qz_1 \dots z_s/t; q)_\infty}{(z_1 \dots z_s; q)_\infty}.
 \end{aligned} \tag{3.3}$$

Using Theorem 2.1 this may be generalised as follows.

THEOREM 3.1. Let $x^{(1)} = (x_1^{(1)}, \dots, x_{k_1}^{(1)})$ and

$$x^{(s)} = z_s(1, t, \dots, t^{k_s-1}) \quad \text{for } 2 \leq s \leq n.$$

Then,

$$\begin{aligned} {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, t; x^{(1)}, \dots, x^{(n)} \right] \\ = {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, t; \hat{x}^{(n)} \right] \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \frac{(q\hat{x}_i^{(s)} t^{k_s-k_{s+1}-1}; q)_\infty}{(\hat{x}_i^{(s)}; q)_\infty}. \end{aligned}$$

Here the $\hat{x}^{(s)}$ are recursively defined as $\hat{x}^{(1)} = x^{(1)}$ and

$$\hat{x}^{(s)} = z_s(t^{k_{s-1}-1}\hat{x}^{(s-1)}, t^{k_{s-1}}, t^{k_{s-1}+1}, \dots, t^{k_s-1}) \quad \text{for } 2 \leq s \leq n.$$

Taking $k_1 = k_2 = \dots = k_n = 1$ and $x^{(1)} = (z_1)$ (so that $\hat{x}^{(s)} = (z_1 \dots z_s)$), this reduces to (3.3).

Before presenting a proof, we will give several important consequences of Theorem 3.1.

THEOREM 3.2. (A_n q -binomial theorem) *With the same notation as in Theorem 3.1,*

$${}_1\Phi_0 \left[\begin{matrix} a \\ - \end{matrix} ; q, t; x^{(1)}, \dots, x^{(n)} \right] = \prod_{i=1}^{k_n} \frac{(a\hat{x}_i^{(n)}; q)_\infty}{(\hat{x}_i^{(n)}; q)_\infty} \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \frac{(q\hat{x}_i^{(s)} t^{k_s-k_{s+1}-1}; q)_\infty}{(\hat{x}_i^{(s)}; q)_\infty}.$$

Eliminating the hats from the double product on the right yields

$$\begin{aligned} {}_1\Phi_0 \left[\begin{matrix} a \\ - \end{matrix} ; q, t; x^{(1)}, \dots, x^{(n)} \right] &= \prod_{i=1}^{k_1} \left(\frac{(az_2 \dots z_n x_i^{(1)} t^{k_1+\dots+k_{n-1}-n+1}; q)_\infty}{(z_2 \dots z_n x_i^{(1)} t^{k_1+\dots+k_{n-1}-n+1}; q)_\infty} \right. \\ &\quad \times \left. \prod_{r=1}^{n-1} \frac{(qz_2 \dots z_r x_i^{(1)} t^{k_1+\dots+k_r-k_{r+1}-r}; q)_\infty}{(z_2 \dots z_r x_i^{(1)} t^{k_1+\dots+k_r-1-r+1}; q)_\infty} \right) \\ &\quad \times \prod_{s=2}^n \prod_{i=1}^{k_s-k_{s-1}} \frac{(az_s \dots z_n t^{i+s+k_{s-1}+\dots+k_{n-1}-n-1}; q)_\infty}{(z_s \dots z_n t^{i+s+k_{s-1}+\dots+k_{n-1}-n-1}; q)_\infty} \\ &\quad \times \prod_{2 \leq s \leq r \leq n-1} \prod_{i=1}^{k_s-k_{s-1}} \frac{(qz_s \dots z_r t^{i+s-r+k_{s-1}+\dots+k_r-k_{r+1}-2}; q)_\infty}{(z_s \dots z_r t^{i+s-r+k_{s-1}+\dots+k_{r-1}-1}; q)_\infty}. \end{aligned}$$

Proof. If we take Theorem 3.1 with $r=1$, the ${}_1\Phi_0$ on the right may be summed by (2.20), leading to the desired result. □

If we further specialise $x^{(1)} = z_1(1, t, \dots, t^{k_1})$ in Theorem 3.2, we obtain a more symmetric q -binomial theorem.

COROLLARY 3.3. Let $x^{(s)} = z_s(1, t, \dots, t^{k_s-1})$ for $1 \leq s \leq n$ and set $k_0 = 0$. Then

$$\begin{aligned}
 {}_1\Phi_0 \left[\begin{matrix} a \\ - \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] &= \prod_{s=1}^n \prod_{i=1}^{k_s - k_{s-1}} \frac{(az_s \dots z_n t^{i+s+k_{s-1}+\dots+k_{n-1}-n-1}; q)_\infty}{(z_s \dots z_n t^{i+s+k_{s-1}+\dots+k_{n-1}-n-1}; q)_\infty} \\
 &\times \prod_{1 \leq s \leq r \leq n-1} \prod_{i=1}^{k_s - k_{s-1}} \frac{(qz_s \dots z_r t^{i+s-r+k_{s-1}+\dots+k_r-k_{r+1}-2}; q)_\infty}{(z_s \dots z_r t^{i+s-r+k_{s-1}+\dots+k_{r-1}-1}; q)_\infty}.
 \end{aligned} \tag{3.4}$$

When $k_1 = \dots = k_n = k$ the above significantly simplifies to

$$\begin{aligned}
 {}_1\Phi_0 \left[\begin{matrix} a \\ - \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] \\
 = \prod_{i=1}^k \left(\frac{(az_1 \dots z_n t^{i-1+(n-1)(k-1)}; q)_\infty}{(z_1 \dots z_n t^{i-1+(n-1)(k-1)}; q)_\infty} \prod_{s=1}^{n-1} \frac{(qz_1 \dots z_s t^{i-2+(s-1)(k-1)}; q)_\infty}{(z_1 \dots z_s t^{i-1+(s-1)(k-1)}; q)_\infty} \right),
 \end{aligned}$$

where $x^{(s)} = z_s(1, t, \dots, t^{k-1})$ for $1 \leq s \leq n$.

Remark. It is again easily seen that Theorem 3.2 and Corollary 3.3 are true as functions of $x^{(1)}, z_2, \dots, z_n$ or z_1, z_2, \dots, z_n when $|q| < 1$ and

$$\max\{|x_1^{(1)}|, \dots, |x_{k_1}^{(1)}|, |z_2|, \dots, |z_n|\} < 1$$

or

$$\max\{|z_1|, \dots, |z_n|\} < 1.$$

Proof of Theorem 3.1. We abbreviate the sequences a_1, \dots, a_{r+1} and b_1, \dots, b_r by A and B , respectively, and assume that $n > 1$.

If we apply identity (2.21) to eliminate the double product $\prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}}$ in the definition (3.1) of the ${}_{r+1}\Phi_r$ series, we obtain

$$\begin{aligned}
 {}_{r+1}\Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] &= \sum_{\substack{\lambda^{(1)}, \dots, \lambda^{(n)} \\ \nu^{(1)}, \dots, \nu^{(n-1)} \\ \omega^{(1)}, \dots, \omega^{(n-1)}}} t^{n(\lambda^{(n)})} \frac{P_{\lambda^{(n)}}(x^{(n)}; q, t)}{c'_{\lambda^{(n)}}(q, t)} \frac{(A; q, t)_{\lambda^{(n)}}}{(B; q, t)_{\lambda^{(n)}}} \\
 &\times \prod_{s=1}^{n-1} \left(t^{n(\nu^{(s)})+k_s|\lambda^{(s+1)}|-|\omega^{(s)}|} f_{\omega^{(s)}\nu^{(s)}}^{\lambda^{(s)}}(q, t) P_{\lambda^{(s)}}(x^{(s)}; q, t) \right. \\
 &\quad \left. \times \frac{u_0^{(k_{s+1}-k_s)}(P_{\lambda^{(s+1)}/\omega^{(s)}})(qt^{k_s-k_{s+1}-1}; q, t)_{\nu^{(s)}}}{u_0^{(k_{s+1})}(P_{\lambda^{(s+1)}}) c'_{\nu^{(s)}}(q, t)} \right).
 \end{aligned} \tag{3.5}$$

The $\lambda^{(1)}$ -dependent part of the summand is given by

$$f_{\omega^{(1)}\nu^{(1)}}^{\lambda^{(1)}}(q, t) P_{\lambda^{(1)}}(x^{(1)}; q, t).$$

Hence the sum over $\lambda^{(1)}$ may be performed by (2.14) to yield

$$\begin{aligned}
 & {}_{r+1}\Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] \\
 &= \sum_{\substack{\lambda^{(2)}, \dots, \lambda^{(n)} \\ \nu^{(2)}, \dots, \nu^{(n-1)} \\ \omega^{(2)}, \dots, \omega^{(n-1)}}} t^{n(\lambda^{(n)})} \frac{P_{\lambda^{(n)}}(x^{(n)}; q, t)}{c'_{\lambda^{(n)}}(q, t)} \frac{(A; q, t)_{\lambda^{(n)}}}{(B; q, t)_{\lambda^{(n)}}} \frac{t^{k_1 |\lambda^{(2)}|}}{u_0^{(k_2)}(P_{\lambda^{(2)}})} \\
 &\quad \times \prod_{s=2}^{n-1} \left(t^{n(\nu^{(s)}) + k_s |\lambda^{(s+1)}| - |\omega^{(s)}|} f_{\omega^{(s)} \nu^{(s)}}^{\lambda^{(s)}}(q, t) P_{\lambda^{(s)}}(x^{(s)}; q, t) \right. \\
 &\quad \quad \left. \times \frac{u_0^{(k_{s+1} - k_s)}(P_{\lambda^{(s+1)}/\omega^{(s)}}) (qt^{k_s - k_{s+1} - 1}; q, t)_{\nu^{(s)}}}{u_0^{(k_{s+1})}(P_{\lambda^{(s+1)}}) c'_{\nu^{(s)}}(q, t)} \right) \\
 &\quad \times \sum_{\omega^{(1)}} t^{-|\omega^{(1)}|} u_0^{(k_2 - k_1)}(P_{\lambda^{(2)}/\omega^{(1)}}) P_{\omega^{(1)}}(x^{(1)}; q, t) \\
 &\quad \times \sum_{\nu^{(1)}} t^{n(\nu^{(1)})} \frac{(qt^{k_1 - k_2 - 1}; q, t)_{\nu^{(1)}}}{c'_{\nu^{(1)}}(q, t)} P_{\nu^{(1)}}(x^{(1)}; q, t).
 \end{aligned}$$

By (2.7) and (2.18) the sum over $\omega^{(1)}$ gives

$$P_{\lambda^{(2)}}(t^{-1}x^{(1)}, 1, \dots, t^{k_2 - k_1 - 1}; q, t),$$

and by (2.19) and (2.20) the sum over $\nu^{(1)}$ gives

$${}_1\Phi_0 \left[\begin{matrix} qt^{k_1 - k_2 - 1} \\ - \end{matrix}; q, t; x^{(1)} \right] = \prod_{i=1}^{k_1} \frac{(qx_i^{(1)} t^{k_1 - k_2 - 1}; q)_\infty}{(x_i^{(1)}; q)_\infty}.$$

Substituting these two results and once again using (2.7) to absorb the factor $t^{k_1 |\lambda^{(2)}|}$, we find that

$$\begin{aligned}
 & {}_{r+1}\Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] = \prod_{i=1}^{k_1} \frac{(qx_i^{(1)} t^{k_1 - k_2 - 1}; q)_\infty}{(x_i^{(1)}; q)_\infty} \\
 &\quad \times \sum_{\substack{\lambda^{(2)}, \dots, \lambda^{(n)} \\ \nu^{(2)}, \dots, \nu^{(n-1)} \\ \omega^{(2)}, \dots, \omega^{(n-1)}}} t^{n(\lambda^{(n)})} \frac{P_{\lambda^{(n)}}(x^{(n)}; q, t)}{c'_{\lambda^{(n)}}(q, t)} \frac{(A; q, t)_{\lambda^{(n)}}}{(B; q, t)_{\lambda^{(n)}}} \frac{P_{\lambda^{(2)}}(t^{k_1 - 1}x^{(1)}, t^{k_1}, \dots, t^{k_2 - 1}; q, t)}{u_0^{(k_2)}(P_{\lambda^{(2)}})} \\
 &\quad \times \prod_{s=2}^{n-1} \left(t^{n(\nu^{(s)}) + k_s |\lambda^{(s+1)}| - |\omega^{(s)}|} f_{\omega^{(s)} \nu^{(s)}}^{\lambda^{(s)}}(q, t) P_{\lambda^{(s)}}(x^{(s)}; q, t) \right. \\
 &\quad \quad \left. \times \frac{u_0^{(k_{s+1} - k_s)}(P_{\lambda^{(s+1)}/\omega^{(s)}}) (qt^{k_s - k_{s+1} - 1}; q, t)_{\nu^{(s)}}}{u_0^{(k_{s+1})}(P_{\lambda^{(s+1)}}) c'_{\nu^{(s)}}(q, t)} \right).
 \end{aligned}$$

Comparing this with (3.5) we see that up to the term

$$\frac{P_{\lambda^{(2)}}(t^{k_1-1}x^{(1)}, t^{k_1}, \dots, t^{k_2-1}; q, t)}{u_0^{(k_2)}(P_{\lambda^{(2)}})}, \tag{3.6}$$

we have effectively reduced n to $n-1$. The naive approach would be to apply $u_{0;t^1-k_1}^{(k_1)}$ acting on $x^{(1)}$. Then (3.6) collapses to 1 as desired, but

$$u_{0;t^1-k_1}^{(k_1)} \left(\prod_{i=1}^{k_1} \frac{(x_i^{(1)}t^{k_1-k_2-1}; q)_\infty}{(x_i^{(1)}; q)_\infty} \right) = \prod_{i=1}^{k_1} \frac{(t^{k_1-k_2-i}; q)_\infty}{(t^{1-i}; q)_\infty}$$

is not well defined. The correct approach is to apply $u_{0;z_2}^{(k_2)}$ acting on $x^{(2)}$, resulting in

$$\begin{aligned} u_{0;z_2}^{(k_2)} \left({}_{r+1}\Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n)} \right] \right) &= \prod_{i=1}^{k_1} \frac{(qx_i^{(1)}t^{k_1-k_2-1}; q)_\infty}{(x_i^{(1)}; q)_\infty} \\ &\times \sum_{\substack{\lambda^{(2)}, \dots, \lambda^{(n)} \\ \nu^{(2)}, \dots, \nu^{(n-1)} \\ \omega^{(2)}, \dots, \omega^{(n-1)}}} t^{n(\lambda^{(n)})} \frac{P_{\lambda^{(n)}}(x^{(n)}; q, t)}{c'_{\lambda^{(n)}}(q, t)} \frac{(A; q, t)_{\lambda^{(n)}}}{(B; q, t)_{\lambda^{(n)}}} \\ &\times \prod_{s=2}^{n-1} \left(t^{n(\nu^{(s)})+k_s|\lambda^{(s+1)}|-|\omega^{(s)}|} f_{\omega^{(s)}\nu^{(s)}}^{\lambda^{(s)}}(q, t) P_{\lambda^{(s)}}(\hat{x}^{(s)}; q, t) \right. \\ &\quad \left. \times \frac{u_0^{(k_{s+1}-k_s)}(P_{\lambda^{(s+1)}/\omega^{(s)}})(qt^{k_s-k_{s+1}-1}; q, t)_{\nu^{(s)}}}{u_0^{(k_{s+1})}(P_{\lambda^{(s+1)}}) c'_{\nu^{(s)}}(q, t)} \right), \end{aligned}$$

where $\hat{x}^{(2)} = z_2(t^{k_1-1}x^{(1)}, t^{k_1}, \dots, t^{k_2-1})$ and $\hat{x}^{(s)} = x^{(s)}$ for $3 \leq s \leq n-1$. Again, comparing this with (3.5), we have thus proved the following intermediate result.

LEMMA 3.4. *Assume that $n > 1$ and let*

$$x^{(2)} = z_2(1, t, \dots, t^{k_2-1}) \quad \text{and} \quad \hat{x}^{(2)} = z_2(t^{k_1-1}x^{(1)}, t^{k_1}, \dots, t^{k_2-1}).$$

Then,

$${}_{r+1}\Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] = {}_{r+1}\Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; \hat{x}^{(2)}, x^{(3)}, \dots, x^{(n)} \right] \prod_{i=1}^{k_1} \frac{(qx_i^{(1)}t^{k_1-k_2-1}; q)_\infty}{(x_i^{(1)}; q)_\infty}.$$

This is readily iterated, resulting in Theorem 3.1. □

4. Proof of the A_n exponential Selberg integral

Although Corollary 1.3 follows as a straightforward limit of the A_n Selberg integral, it is proved here directly from Corollary 3.3. The advantage of first dealing with the exponential integral instead of the full A_n Selberg integral is that it makes the introduction of $C_\gamma^{k_1, \dots, k_n}[a, b]$ slightly simpler. Throughout the proof we assume that $\gamma \neq 0$.

Applying $u_{0; z_1}^{(k_1)} \dots u_{0; z_n}^{(k_n)}$ (with $u_{0; z_s}^{(k_s)}$ acting on $x^{(s)}$) to (3.1), and using (2.5) and (2.9), we get

$$\begin{aligned} r_{+1} \Phi_r \left[\begin{matrix} A \\ B \end{matrix}; q, t; x^{(1)}, \dots, x^{(n)} \right] &= \sum_{\lambda^{(1)}, \dots, \lambda^{(n)}} \frac{(A; q, t)_{\lambda^{(n)}}}{(qt^{k_n-1}, B; q, t)_{\lambda^{(n)}}} \\ &\times \prod_{s=1}^n \left(t^{2n(\lambda^{(s)})} z_s^{|\lambda^{(s)}|} \prod_{1 \leq i < j \leq k_s} \frac{1 - t^{j-i} q^{\lambda_i^{(s)} - \lambda_j^{(s)}} (t^{j-i+1}; q)_{\lambda_i^{(s)} - \lambda_j^{(s)}}}{1 - t^{j-i} (qt^{j-i-1}; q)_{\lambda_i^{(s)} - \lambda_j^{(s)}}} \right) \\ &\times \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}} \frac{(qt^{j-i+k_s-k_{s+1}-1}; q)_{\lambda_i^{(s)} - \lambda_j^{(s+1)}}}{(qt^{j-i+k_s-k_{s+1}}; q)_{\lambda_i^{(s)} - \lambda_j^{(s+1)}}}, \end{aligned} \tag{4.1}$$

where $x^{(s)} = z_s(1, t, \dots, t^{k_s-1})$ for $1 \leq s \leq n$. Taking $r=0$ and $A=a_1=a$, this may be equated with the right-hand side of (3.4). Then, replacing $t \mapsto q^\gamma$ and $a \mapsto q^{\alpha+(k_n-1)\gamma}$, and letting $q \rightarrow 1^-$ using

$$(q^x; q)_k \rightarrow \frac{\Gamma(x+k)}{\Gamma(x)}$$

and

$$\frac{(q^x z; q)_\infty}{(q^y z; q)_\infty} \rightarrow (1-z)^{y-x}, \quad |z| < 1, \tag{4.2}$$

yields

$$\begin{aligned} &\sum_{\lambda^{(1)}, \dots, \lambda^{(n)}} \prod_{i=1}^{k_n} \frac{\Gamma(\alpha + \mu_i^{(n)})}{\Gamma(1 + \mu_i^{(n)})} \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}} \frac{\Gamma(1 - \gamma + \mu_i^{(s)} - \mu_j^{(s+1)})}{\Gamma(1 + \mu_i^{(s)} - \mu_j^{(s+1)})} \\ &\times \prod_{s=1}^n \left(z_s^{|\lambda^{(s)}|} \prod_{1 \leq i < j \leq k_s} \frac{(\mu_i^{(s)} - \mu_j^{(s)}) \Gamma(\gamma + \mu_i^{(s)} - \mu_j^{(s)})}{\Gamma(1 - \gamma + \mu_i^{(s)} - \mu_j^{(s)})} \right) \\ &= \prod_{1 \leq s \leq r \leq n} (1 - z_s \dots z_r)^{-(\alpha_r + (k_r - k_{r+1} - 1)\gamma)(k_s - k_{s-1})} \\ &\times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s + (i - k_{s+1} - 1)\gamma) \Gamma(i\gamma)}{\Gamma(\gamma)}. \end{aligned} \tag{4.3}$$

Here $\mu_i^{(s)} := \lambda_i^{(s)} + (k_s - i)\gamma$, $k_0 = k_{n+1} = 0$ and α_s is defined as in Theorem 1.2.

Next we would like to replace $z_s \mapsto e^{-\varepsilon\beta_s}$ and $\lambda_i^{(s)} \mapsto t_i^{(s)}/\varepsilon$ (with $\varepsilon > 0$), and take the $\varepsilon \rightarrow 0$ limit using Stirling's formula

$$\lim_{x \rightarrow \infty} x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1$$

to transform the above sum into an integral. There is however the complication that the difference $\lambda_i^{(s)} - \lambda_j^{(s+1)}$ is not necessarily non-negative. Let

$$u_{ij}^{(s)} = i - j - k_s + k_{s+1}. \tag{4.4}$$

Then, for $\lambda_i^{(s)} - \lambda_j^{(s+1)} > 0$,

$$\frac{\Gamma(1 - \gamma + \mu_i^{(s)} - \mu_j^{(s+1)})}{\Gamma(1 + \mu_i^{(s)} - \mu_j^{(s+1)})} \rightarrow \frac{\Gamma(1 - (u_{ij}^{(s)} + 1)\gamma + (t_i^{(s)} - t_j^{(s+1)})/\varepsilon)}{\Gamma(1 - u_{ij}^{(s)}\gamma + (t_i^{(s)} - t_j^{(s+1)})/\varepsilon)} \sim \varepsilon^\gamma (t_i^{(s)} - t_j^{(s+1)})^{-\gamma},$$

but for $\lambda_i^{(s)} - \lambda_j^{(s+1)} < 0$, by the Euler reflection formula,

$$\begin{aligned} \frac{\Gamma(1 - \gamma + \mu_i^{(s)} - \mu_j^{(s+1)})}{\Gamma(1 + \mu_i^{(s)} - \mu_j^{(s+1)})} &= \frac{\Gamma(-\mu_i^{(s)} + \mu_j^{(s+1)})}{\Gamma(\gamma - \mu_i^{(s)} + \mu_j^{(s+1)})} R_{ij}^{(s)}(\gamma) \\ &\rightarrow \frac{\Gamma(u_{ij}^{(s)}\gamma - (t_i^{(s)} - t_j^{(s+1)})/\varepsilon)}{\Gamma((u_{ij}^{(s)} + 1)\gamma - (t_i^{(s)} - t_j^{(s+1)})/\varepsilon)} R_{ij}^{(s)}(\gamma) \\ &\sim \varepsilon^\gamma (t_j^{(s+1)} - t_i^{(s)})^{-\gamma} R_{ij}^{(s)}(\gamma) \\ &= \varepsilon^\gamma |t_i^{(s)} - t_j^{(s+1)}|^{-\gamma} R_{ij}^{(s)}(\gamma), \end{aligned}$$

where

$$R_{ij}^{(s)}(\gamma) = \frac{\sin(\pi u_{ij}^{(s)}\gamma)}{\sin(\pi(u_{ij}^{(s)} + 1)\gamma)}. \tag{4.5}$$

Hence, depending on the relative order of $\lambda_i^{(s)}$ and $\lambda_j^{(s+1)}$, we may or may not pick up a ratio of sine functions, and for small ε the summand of (4.3) takes the form

$$\begin{aligned} &\varepsilon^{k_1 + \dots + k_n - \sum_{1 \leq s \leq r \leq n} (\alpha_r + (k_r - k_{r+1} - 1)\gamma)(k_s - k_{s-1})} \prod R_{ij}^{(s)}(\gamma) \\ &\times \prod_{i=1}^{k_n} (t_i^{(n)})^{\alpha-1} \prod_{s=1}^{n-1} |\Delta(t^{(s)}, t^{(s+1)})|^{-\gamma} \prod_{s=1}^n e^{-\beta_s \sum_{i=1}^{k_s} t_i^{(s)}} \Delta(t^{(s)})^{2\gamma}, \end{aligned}$$

where the first product is over all i, j and s such that $t_i^{(s)} < t_j^{(s+1)}$. From this it follows that we must first fix a complete ordering between the parts of $\lambda^{(s)}$ and $\lambda^{(s+1)}$. Any such ordering compatible with (3.2) may be described by a map [26]

$$M_s: \{1, \dots, k_s\} \longrightarrow \{1, \dots, k_{s+1}\} \tag{4.6a}$$

such that

$$M_s(i) \leq M_s(i+1) \tag{4.6b}$$

and

$$1 \leq M_s(i) \leq k_{s+1} - k_s + i, \tag{4.6c}$$

as follows:

$$\lambda_{M_s(i)}^{(s+1)} \leq \lambda_i^{(s)} \leq \lambda_{M_s(i)-1}^{(s+1)} \quad \text{for } 1 \leq i \leq k_s, \tag{4.7}$$

where $\lambda_0^{(s+1)} = \infty$. Note in particular that (4.6b) must hold so that (4.7) is compatible with the ordering among the parts of $\lambda^{(s)}$. Similarly (4.6c) must hold so that (4.7) is compatible with (3.2). A straightforward counting argument shows that there are exactly

$$\frac{k_{s+1} - k_s + 1}{k_{s+1} + 1} \binom{k_s + k_{s+1}}{k_s}$$

different maps.

Now define $D^{k_1, \dots, k_n}[a, b] \subseteq [a, b]^{k_1 + \dots + k_n}$ as the set of points

$$(\lambda^{(1)}, \dots, \lambda^{(n)}) = (\lambda_1^{(1)}, \dots, \lambda_{k_1}^{(1)}, \dots, \lambda_1^{(n)}, \dots, \lambda_{k_n}^{(n)})$$

satisfying

$$a \leq \lambda_{k_s}^{(s)} \leq \dots \leq \lambda_1^{(s)} \leq b \quad \text{for } 1 \leq s \leq n$$

and (3.2). Given admissible maps M_1, \dots, M_{n-1} we define

$$D_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}[a, b] \subseteq D^{k_1, \dots, k_n}[a, b]$$

by requiring that (4.7) holds for $1 \leq s \leq n-1$. As chains, we have

$$D^{k_1, \dots, k_n}[a, b] = \sum_{M_1, \dots, M_{n-1}} D_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}[a, b], \tag{4.8}$$

and summing over $\lambda^{(1)}, \dots, \lambda^{(n)}$ amounts to summing over the lattice points in

$$D^{k_1, \dots, k_n}[0, \infty].$$

Due to the decomposition (4.8), we know exactly which factors of the form $R_{ij}^{(s)}(\gamma)$ are picked up when we go from sum to integral. Indeed, from (4.7) we have that $\lambda_i^{(s)} \leq \lambda_j^{(s+1)}$ for $1 \leq j \leq M_s(i) - 1$. This gives rise to the factor

$$\prod_{j=1}^{M_s(i)-1} R_{ij}^{(s)}(\gamma) = \frac{\sin(\pi(i+k_{s+1}-k_s-M_s(i)+1)\gamma)}{\sin(\pi(i+k_{s+1}-k_s)\gamma)}.$$

Taking the product over i and s , this yields the total factor

$$F_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}(\gamma) = \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \frac{\sin(\pi(i+k_{s+1}-k_s-M_s(i)+1)\gamma)}{\sin(\pi(i+k_{s+1}-k_s)\gamma)}.$$

Hence, making the variable changes $z_s \mapsto e^{-\varepsilon\beta_s}$ and $\lambda_i^{(s)} \mapsto t_i^{(s)}/\varepsilon$, and letting ε tend to zero, we obtain Corollary 1.3, where the integration is over the chain

$$C_\gamma^{k_1, \dots, k_n}[a, b] = \sum_{M_1, \dots, M_{n-1}} F_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}(\gamma) D_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}[a, b]. \tag{4.9}$$

For $n=2$ this corresponds to the chain introduced by Tarasov and Varchenko in [26] (up to some trivial notational changes). Of course, to correctly interpret (4.9) in the context of Corollary 1.3 (and also Theorem 1.2) we have to replace $\lambda_i^{(s)}$ by $t_i^{(s)}$ in all of the above. In particular (4.7) becomes

$$t_{M_s(i)}^{(s+1)} \leq t_i^{(s)} \leq t_{M_s(i)-1}^{(s+1)} \quad \text{for } 1 \leq i \leq k_s,$$

and we integrate the $t_i^{(s)}$ such that

$$(t_1^{(1)}, \dots, t_{k_1}^{(1)}, \dots, t_1^{(n)}, \dots, t_{k_n}^{(n)}) \in D^{k_1, \dots, k_n}[a, b].$$

5. Proof of the A_n Selberg integral

Throughout this section we assume that $\gamma \neq 0$ and $0 < q < 1$, and use

$$(a; q)_z = \frac{(a; q)_\infty}{(aq^z; q)_\infty} \quad \text{for } z \in \mathbb{C}.$$

Let Γ_q be the q -gamma function [10, (I.35)]

$$\Gamma_q(x) = \frac{(q; q)_{x-1}}{(1-q)^{x-1}},$$

and define

$$\Delta_\gamma(x^{(s)}; q) = \prod_{1 \leq i < j \leq k_s} (x_j^{(s)})^{2\gamma} \left(1 - q^{(j-i)\gamma} \frac{x_i^{(s)}}{x_j^{(s)}}\right) \left(q^{1+(j-i-1)\gamma} \frac{x_i^{(s)}}{x_j^{(s)}}; q\right)_{2\gamma-1}$$

and

$$\Delta_\gamma(x^{(s)}, x^{(s+1)}; q) = \prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}} (x_j^{(s+1)})^{-\gamma} \left(q^{1-u_{ij}^{(s)}\gamma} \frac{x_i^{(s)}}{x_j^{(s+1)}}; q\right)_{-\gamma},$$

with $u_{ij}^{(s)}$ as in (4.4).

Using the above definitions as well as (4.1), Corollary 3.3 can be written in the form

$$\begin{aligned}
 & (1-q)^{k_1+\dots+k_n} \sum_{\lambda^{(1)}, \dots, \lambda^{(n)}} \prod_{i=1}^{k_n} (q^{1+(k_n-i)\gamma} x_i^{(n)}; q)_{\alpha-1} \prod_{s=1}^n \prod_{i=1}^{k_s} (x_i^{(s)})^{\beta_s} \\
 & \times \prod_{s=1}^{n-1} \Delta_\gamma(x^{(s)}, x^{(s+1)}; q) \prod_{s=1}^n \Delta_\gamma(x^{(s)}; q) \\
 & = \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s-k_{s-1}} \frac{\Gamma_q(\beta_s+\dots+\beta_r+(i+s-r-1)\gamma)}{\Gamma_q(\alpha_r+\beta_s+\dots+\beta_r+(i+s-r+k_r-k_{r+1}-2)\gamma)} \\
 & \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma_q(\alpha_s+(i-k_{s+1}-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\gamma)}.
 \end{aligned} \tag{5.1}$$

Here α_s is as defined in Theorem 1.2, $x_i^{(s)} := q^{\lambda_i^{(s)}}$ and (t, z_s, a) has been replaced by $(q^\gamma, q^{\beta_s-k_{s-1}\gamma}, q^{(k_n-1)\gamma+\alpha})$.

The reader familiar with Jackson or q -integrals will recognise (5.1) as a $(k_1+\dots+k_n)$ -dimensional such integral. The standard 1-dimensional Jackson integral is given by

$$\int_0^1 f(x) d_q x = (1-q) \sum_{i=0}^{\infty} f(q^i) q^i,$$

which simplifies (at least formally) to the Riemann integral

$$\int_0^1 f(x) dx$$

in the $q \rightarrow 1^-$ limit. Generalising this to arbitrary dimensions as

$$\int_{[0,1]^n} f(x) d_q x = (1-q)^n \sum_{i_1, \dots, i_n=0}^{\infty} f(q^{i_1}, \dots, q^{i_n}) q^{i_1+\dots+i_n},$$

where $x=(x_1, \dots, x_n)$ and $d_q x = d_q x_1 \dots d_q x_n$, (5.1) corresponds to the restricted q -integral

$$\begin{aligned}
 & \int_{\overline{D}^{k_1, \dots, k_n} [0,1]} \prod_{i=1}^{k_n} (q^{1+(k_n-i)\gamma} x_i^{(n)}; q)_{\alpha-1} \prod_{s=1}^n \prod_{i=1}^{k_s} (x_i^{(s)})^{\beta_s-1} \\
 & \times \prod_{s=1}^{n-1} \Delta_\gamma(x^{(s)}, x^{(s+1)}; q) \prod_{s=1}^n \Delta_\gamma(x^{(s)}; q) d_q x^{(1)} \dots d_q x^{(n)} \\
 & = \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s-k_{s-1}} \frac{\Gamma_q(\beta_s+\dots+\beta_r+(i+s-r-1)\gamma)}{\Gamma_q(\alpha_r+\beta_s+\dots+\beta_r+(i+s-r+k_r-k_{r+1}-2)\gamma)} \\
 & \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma_q(\alpha_s+(i-k_{s+1}-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\gamma)},
 \end{aligned} \tag{5.2}$$

where $\bar{D}^{k_1, \dots, k_n}[a, b] \subseteq [a, b]^{k_1 + \dots + k_n}$ is the set of all points

$$(x^{(1)}, \dots, x^{(n)}) = (x_1^{(1)}, \dots, x_{k_1}^{(1)}, \dots, x_1^{(n)}, \dots, x_{k_n}^{(n)})$$

such that

$$a \leq x_1^{(s)} \leq \dots \leq x_{k_s}^{(s)} \leq b \quad \text{for } 1 \leq s \leq n$$

and

$$x_i^{(s)} \leq x_{i-k_s+k_{s+1}}^{(s+1)} \quad \text{for } 1 \leq i \leq k_s \text{ and } 1 \leq s \leq n-1.$$

Assuming that

$$x_i^{(s)} < x_j^{(s)} \quad \text{for } 1 \leq i < j \leq k_s \tag{5.3a}$$

and

$$x_i^{(s)} < x_j^{(s+1)} \quad \text{for } 1 \leq i \leq k_s \text{ and } 1 \leq j \leq k_{s+1}, \tag{5.3b}$$

it follows from (4.2) that

$$\lim_{q \rightarrow 1^-} \Delta_\gamma(x^{(s)}; q) = \Delta(-x^{(s)})^{2\gamma}, \tag{5.4a}$$

$$\lim_{q \rightarrow 1^-} \Delta_\gamma(x^{(s)}, x^{(s+1)}; q) = \Delta(-x^{(s)}, -x^{(s+1)})^{-\gamma}. \tag{5.4b}$$

Hence, for (5.3) (and $x_i^{(n)} < 1$ for $1 \leq i \leq k_n$) the $q \rightarrow 1^-$ limit of the integrand of (5.2) is

$$\prod_{i=1}^{k_n} (1-x_i^{(n)})^{\alpha-1} \prod_{s=1}^n \left(\Delta(-x^{(s)})^{2\gamma} \prod_{i=1}^{k_s} (x_i^{(s)})^{\beta_s-1} \right) \prod_{s=1}^{n-1} \Delta(-x^{(s)}, -x^{(s+1)})^{-\gamma}. \tag{5.5}$$

The problem we are now facing is exactly the same as that of §4: the $x_i^{(s)}$ are not necessarily compatible with (5.3b). This forces us to also consider the limit of the factors making up $\Delta_\gamma(x^{(s)}, x^{(s+1)}; q)$ when $x_i^{(s)} > x_j^{(s+1)}$. In computing the limit (5.4b), we used that

$$\lim_{q \rightarrow 1^-} \left(q^{1-u_{ij}^{(s)}\gamma} \frac{x_i^{(s)}}{x_j^{(s+1)}}; q \right)_{-\gamma} = \left(1 - \frac{x_i^{(s)}}{x_j^{(s+1)}} \right)^{-\gamma},$$

but this is only correct for $x_i^{(s)} < x_j^{(s+1)}$. When $x_i^{(s)} > x_j^{(s+1)}$ we may use the q -reflection formula [27, (168a)]

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{2\sqrt{-1}q^{z/2}\theta_1(\sqrt{-1}\log q^{z/2}; q^{1/2})}{(1-q)\theta_1'(0; q^{1/2})}$$

(upon recalling that $x_i^{(s)} := q^{\lambda_i^{(s)}}$) to write

$$\left(q^{1-u_{ij}^{(s)}\gamma} \frac{x_i^{(s)}}{x_j^{(s+1)}}; q \right)_{-\gamma} = \frac{(q^{-1/2}x_j^{(s+1)}/x_i^{(s)})^\gamma \theta_1(\sqrt{-1}\log q^{u_{ij}^{(s)}\gamma/2}; q^{1/2})}{(q^{u_{ij}^{(s)}\gamma}x_j^{(s+1)}/x_i^{(s)})^\gamma \theta_1(\sqrt{-1}\log q^{(u_{ij}^{(s)}+1)\gamma/2}; q^{1/2})}.$$

Since

$$\frac{q^{u/2}\theta_1(\sqrt{-1} \log q^{u/2}; q^{1/2})}{q^{v/2}\theta_1(\sqrt{-1} \log q^{v/2}; q^{1/2})} = \frac{(1-q^u)(q^{1+u}; q)_\infty (q^{1-u}; q)_\infty}{(1-q^v)(q^{1+v}; q)_\infty (q^{1-v}; q)_\infty} \rightarrow \frac{u}{v} \prod_{n=1}^\infty \frac{1-u^2/n^2}{1-v^2/n^2} = \frac{\sin \pi u}{\sin \pi v},$$

as $q \rightarrow 1^-$, we find

$$\lim_{q \rightarrow 1^-} \left(q^{1-u_{ij}^{(s)}\gamma} \frac{x_i^{(s)}}{x_j^{(s+1)}}; q \right)_{-\gamma} = \left(\frac{x_i^{(s)}}{x_j^{(s+1)}} - 1 \right)^{-\gamma} R_{ij}^{(s)}(\gamma) = \left| 1 - \frac{x_i^{(s)}}{x_j^{(s+1)}} \right|^{-\gamma} R_{ij}^{(s)}(\gamma),$$

where $R_{ij}^{(s)}$ is given by (4.5). We therefore conclude that the $q \rightarrow 1^-$ limit of the integrand of (5.2) is given by (5.5) with

$$\Delta(-x^{(s)}, -x^{(s+1)})^{-\gamma} \rightarrow |\Delta(-x^{(s)}, -x^{(s+1)})|^{-\gamma} = |\Delta(x^{(s)}, x^{(s+1)})|^{-\gamma}$$

multiplied by a factor $R_{ij}^{(s)}(\gamma)$ for each $x_i^{(s)} > x_j^{(s+1)}$.

The rest of the proof simply follows §4. We introduce the maps M_s as in (4.6), with $M_s(i)$ such that (compare with (4.7))

$$x_{M_s(i)-1}^{(s+1)} \leq x_i^{(s)} \leq x_{M_s(i)}^{(s+1)} \quad \text{for } 1 \leq i \leq k_s, \tag{5.6}$$

where $x_0^{(s+1)} = 0$. Furthermore, we define $\bar{D}_{M_1, \dots, M_n}^{k_1, \dots, k_n}[a, b] \subseteq \bar{D}^{k_1 + \dots + k_n}[a, b]$ by requiring that (5.6) holds. Therefore,

$$\bar{D}^{k_1, \dots, k_n}[a, b] = \sum_{M_1, \dots, M_{n-1}} \bar{D}_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}[a, b] \tag{5.7}$$

as chains. Also, defining

$$\bar{C}_\gamma^{k_1, \dots, k_n}[a, b] = \sum_{M_1, \dots, M_{n-1}} F_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}(\gamma) \bar{D}_{M_1, \dots, M_{n-1}}^{k_1, \dots, k_n}[a, b], \tag{5.8}$$

we get

$$\begin{aligned} & \int_{\bar{C}_\gamma^{k_1, \dots, k_n}[0, 1]} \prod_{i=1}^{k_n} (1-x_i^{(n)})^{\alpha-1} \prod_{s=1}^n \left(|\Delta(x^{(s)})|^{2\gamma} \prod_{i=1}^{k_s} (x_i^{(s)})^{\beta_s-1} \right) \\ & \quad \times \prod_{s=1}^{n-1} |\Delta(x^{(s)}, x^{(s+1)})|^{-\gamma} dx^{(1)} \dots dx^{(n)} \\ & = \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma(\beta_s + \dots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma(\alpha_r + \beta_s + \dots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)} \\ & \quad \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s + (i - k_{s+1} - 1)\gamma) \Gamma(i\gamma)}{\Gamma(\gamma)} \end{aligned}$$

(where $|\Delta(x^{(s)})|^{2\gamma}$ may be replaced by $\Delta(-x^{(s)})^{2\gamma}$). Finally, making the variable changes $x_i^{(s)} = 1 - t_i^{(s)}$ for $1 \leq i \leq k_s$ and $1 \leq s \leq n$, so that $\bar{C}_\gamma^{k_1, \dots, k_n}[0, 1]$ is replaced by $C_\gamma^{k_1, \dots, k_n}[0, 1]$, completes the proof.

6. Further A_n integrals

Let $P_\lambda^{(\alpha)}$ be the *Jack polynomial*, obtained from the Macdonald polynomial P_λ as

$$P_\lambda^{(\alpha)}(x) = \lim_{t \rightarrow 1} P_\lambda(x; t^\alpha, t),$$

and let $(a)_N$ be the *Pochhammer symbol*

$$(a)_N = a(a+1) \dots (a+N-1).$$

Then it is an easy matter to generalise the previous derivation to yield an A_n integral involving the Jack polynomial.

THEOREM 6.1. *Let μ be a partition of at most k_1 parts. With the same conditions as in Theorem 1.2 we have*

$$\begin{aligned} & \int_{\bar{C}_\gamma^{k_1, \dots, k_n} [0,1]} P_\mu^{(1/\gamma)}(x^{(1)}) \prod_{s=1}^n \left(|\Delta(x^{(s)})|^{2\gamma} \prod_{i=1}^{k_s} (1-x_i^{(s)})^{\alpha_s-1} (x_i^{(s)})^{\beta_s-1} \right) \\ & \quad \times \prod_{s=1}^{n-1} |\Delta(x^{(s)}, x^{(s+1)})|^{-\gamma} dx^{(1)} \dots dx^{(n)} \\ & = \prod_{1 \leq i < j \leq k_1} \frac{((j-i+1)\gamma)_{\mu_i - \mu_j}}{((j-i)\gamma)_{\mu_i - \mu_j}} \\ & \quad \times \prod_{s=1}^n \prod_{i=1}^{k_1} \frac{(\beta_1 + \dots + \beta_s + (k_1 - s - i + 1)\gamma)_{\mu_i}}{(\alpha_s + \beta_1 + \dots + \beta_s + (k_1 + k_s - k_{s+1} - s - i)\gamma)_{\mu_i}} \\ & \quad \times \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma(\beta_s + \dots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma(\alpha_r + \beta_s + \dots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)} \\ & \quad \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s + (i - k_{s+1} - 1)\gamma) \Gamma(i\gamma)}{\Gamma(\gamma)}. \end{aligned}$$

When $n=1$ this simplifies to

$$\begin{aligned} & \frac{1}{k!} \int_{[0,1]^k} P_\mu^{(1/\gamma)}(x) |\Delta(x)|^{2\gamma} \prod_{i=1}^k x_i^{\alpha-1} (1-x_i)^{\beta-1} dx \\ & = \prod_{1 \leq i < j \leq k} \frac{\Gamma((j-i+1)\gamma + \mu_i - \mu_j)}{\Gamma((j-i)\gamma + \mu_i - \mu_j)} \\ & \quad \times \prod_{i=1}^k \frac{\Gamma(\alpha + (k-i)\gamma + \mu_i) \Gamma(\beta + (i-1)\gamma)}{\Gamma(\alpha + \beta + (2k-i-1)\gamma + \mu_i)}, \end{aligned} \tag{6.1}$$

where we have made the substitutions $(k_1, \alpha, \beta_1) \mapsto (k, \beta, \alpha)$ and have used the symmetry of the integrand to replace

$$\int_{\bar{C}_\gamma^k[0,1]} = \int_{0 \leq x_1 \leq x_2 \leq \dots \leq x_k \leq 1} \quad \text{by} \quad \frac{1}{k!} \int_{[0,1]^k}.$$

The integral (6.1) is due to Kadell [11, Theorem I] (see also [16, pp.385–386]). The special case $\mu=(1^r)$ of (6.1) corresponds to Aomoto’s integral [2], usually stated as

$$\begin{aligned} & \int_{[0,1]^k} |\Delta(x)|^{2\gamma} \prod_{i=1}^r x_i \prod_{i=1}^k x_i^{\alpha-1} (1-x_i)^{\beta-1} dx \\ &= \prod_{i=1}^r \frac{\alpha+(k-i)\gamma}{\alpha+\beta+(2k-i-1)\gamma} \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma+1)}, \end{aligned}$$

for $0 \leq r \leq k$. The equivalence of Aomoto’s integral and the $\mu=(1^r)$ case of (6.1) follows by symmetrising the integrand of the former, using that

$$\sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \dots x_{i_r} = e_r(x) = P_{(1^r)}^{(1/\gamma)}(x), \tag{6.2}$$

with e_r being the r th elementary symmetric function.

By taking $\mu=(1^r)$ in Theorem 6.1, we obtain the following A_n analogue of Aomoto’s integral:

$$\begin{aligned} & \int_{\bar{C}_\gamma^{k_1, \dots, k_n}[0,1]} e_r(x^{(1)}) \prod_{s=1}^n \left(|\Delta(x^{(s)})|^{2\gamma} \prod_{i=1}^{k_s} (1-x_i^{(s)})^{\alpha_s-1} (x_i^{(s)})^{\beta_s-1} \right) \\ & \quad \times \prod_{s=1}^{n-1} |\Delta(x^{(s)}, x^{(s+1)})|^{-\gamma} dx^{(1)} \dots dx^{(n)} \\ &= \binom{k_1}{r} \prod_{s=1}^n \prod_{i=1}^r \frac{\beta_1 + \dots + \beta_s + (k_1 - i - s + 1)\gamma}{\alpha_s + \beta_1 + \dots + \beta_s + (k_1 + k_s - k_{s+1} - i - s)\gamma} \\ & \quad \times \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma(\beta_s + \dots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma(\alpha_r + \beta_s + \dots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)} \\ & \quad \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s + (i - k_{s+1} - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}. \end{aligned}$$

Proof of Theorem 6.1. In proving Theorem 1.2, we have not taken advantage of the full A_n q -binomial theorem as stated in Theorem 3.2, relying on the less general Corollary 3.3 instead. In going from the former to the latter we have specialised $x^{(1)}$ to $z_1(1, t, \dots, t^{k_1-1})$, or, equivalently, applied to $u_{0; z_1}^{(k_1)}$ acting on $x^{(1)}$.

If, more generally, we apply $u_{\mu; z_1}^{(k_1)}$ instead of $u_{0; z_1}^{(k_1)}$, the factor $u_0^{(k_1)}(P_{\lambda^{(1)}})$ in the summand is replaced by $u_{\mu}^{(k_1)}(P_{\lambda^{(1)}})$. Then invoking (2.10) this leads to the term

$$u_0^{(k_1)}(P_{\lambda^{(1)}}) \frac{u_{\lambda^{(1)}}^{(k_1)}(P_{\mu})}{u_0^{(k_1)}(P_{\mu})}$$

instead of just $u_0^{(k_1)}(P_{\lambda^{(1)}})$.

Of course not just the summand of (3.4) will change by the above, and by applying $u_{\mu; z_1}^{(k_1)}$ instead of $u_{0; z_1}^{(k_1)}$, the right-hand side of (3.4) picks up the additional factor

$$\prod_{s=1}^n \frac{(z_1 \dots z_s t^{k_1 + \dots + k_{s-1} + k_1 - s}; q, t)_{\mu}}{(a_s z_1 \dots z_s t^{k_1 + \dots + k_s - k_{s+1} + k_1 - s - 1}; q, t)_{\mu}},$$

where $a_1 = \dots = a_{n-1} = q$ and $a_n = at^{1-k_n}$. Accordingly, the identity (5.1) generalises to

$$\begin{aligned} & (1-q)^{k_1 + \dots + k_n} \sum_{\lambda^{(1)}, \dots, \lambda^{(n)}} P_{\mu}(y; q, q^{\gamma}) \prod_{i=1}^{k_n} (q^{1+(k_n-i)\gamma} x_i^{(n)}; q)_{\alpha-1} \prod_{s=1}^n \prod_{i=1}^{k_s} (x_i^{(s)})_{\beta_s} \\ & \times \prod_{s=1}^{n-1} \Delta_{\gamma}(x^{(s)}, x^{(s+1)}; q) \prod_{s=1}^n \Delta_{\gamma}(x^{(s)}; q) \\ & = u_0^{(k_1)}(P_{\mu}(q, q^{\gamma})) \prod_{s=1}^n \frac{(q^{\beta_1 + \dots + \beta_s + (k_1 - s)\gamma}; q, q^{\gamma})_{\mu}}{(q^{\alpha_s + \beta_1 + \dots + \beta_s + (k_1 + k_s - k_{s+1} - s - 1)\gamma}; q, q^{\gamma})_{\mu}} \\ & \times \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma_q(\beta_s + \dots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma_q(\alpha_r + \beta_s + \dots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)} \\ & \times \prod_{s=1}^n \prod_{i=1}^{k_s} \frac{\Gamma_q(\alpha_s + (i - k_{s+1} - 1)\gamma) \Gamma_q(i\gamma)}{\Gamma_q(\gamma)}, \end{aligned}$$

where $y = (y_1, \dots, y_{k_1})$ and $y_i = x_i^{(1)} q^{(k_i - i)\gamma}$. The rest of the proof proceeds exactly as before. □

7. Two simple examples

To end this paper we present the fully worked-out examples of the A_n Selberg integral for

$$(k_1, \dots, k_{n-1}, k_n) = (1, \dots, 1, k)$$

and for $\gamma = 0$.

7.1. The case $(k_1, \dots, k_{n-1}, k_n) = (1, \dots, 1, k)$

In this case there is only one map M_s for $1 \leq s \leq n-2$, corresponding to the identity map $M_s(1) = 1$. For $s = n-1$, however, there are k different maps, given by $M_{n-1}(1) = a$ for $1 \leq a \leq k$.

If we relabel the integration variables $t_1^{(s)} \mapsto u_s$ for $1 \leq s \leq n-1$ and $t_i^{(n)} \mapsto t_i$ for $1 \leq i \leq k$, then the above implies the inequalities

$$O_a: \begin{cases} 0 \leq t_k \leq \dots \leq t_1 \leq 1, \\ 0 \leq u_{n-1} \leq \dots \leq u_1 \leq 1, \\ t_a \leq u_{n-1} \leq t_{a-1} \end{cases}$$

with $1 \leq a \leq k$ and $t_0 = 1$. As a result we obtain the following $(k+n-1)$ -dimensional integral:

$$\begin{aligned} & \sum_{a=1}^k \frac{\sin(\pi(k-a+1)\gamma)}{\sin(\pi k\gamma)} \int_{O_a} \prod_{i=1}^{n-1} (1-u_i)^{\beta_i-1} \prod_{i=1}^k t_i^{\alpha-1} (1-t_i)^{\beta_n-1} \prod_{1 \leq i < j \leq k} (t_i - t_j)^{2\gamma} \\ & \quad \times \prod_{i=1}^{n-2} (u_i - u_{i+1})^{-\gamma} \prod_{i=1}^{a-1} (t_i - u_{n-1})^{-\gamma} \prod_{i=a}^k (u_{n-1} - t_i)^{-\gamma} du dt \\ & = \Gamma(1-k\gamma)\Gamma^{n-2}(1-\gamma) \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \\ & \quad \times \prod_{i=1}^{k-1} \frac{\Gamma(\beta_n+(i-1)\gamma)}{\Gamma(\alpha+\beta_n+(i+k-2)\gamma)} \prod_{i=1}^n \frac{\Gamma(\beta_1+\dots+\beta_i+(1-i)\gamma)}{\Gamma(A_i+\beta_1+\dots+\beta_i-i\gamma)}, \end{aligned} \tag{7.1}$$

where $A_1 = \dots = A_{n-2} = 1$, $A_{n-1} = 1 - (k-1)\gamma$, $A_n = \alpha + k\gamma$,

$$du = du_1 \dots du_{n-1} \quad \text{and} \quad dt = dt_1 \dots dt_k.$$

In the notation of the introduction the above integral corresponds to

$$I_{\underbrace{1, \dots, 1, k}_{n-1}}^{A_n}(\alpha; \beta_1, \dots, \beta_n; \gamma)$$

and, according to the recurrence (1.4), all but one of the ones may be eliminated.

To see this, assume that $n \geq 3$ and replace the integration variable u_1 by v as

$$v = \frac{u_1 - u_2}{1 - u_2}.$$

Noting that $1 - u_1 = (1 - v)(1 - u_2)$ and $u_1 - u_2 = v(1 - u_2)$ the integral over v may be identified as Euler's beta integral (1.2) with $\alpha = 1 - \gamma$. Therefore

$$I_{\underbrace{1, \dots, 1, k}_{n-1}}^{A_n}(\alpha; \beta_1, \dots, \beta_n; \gamma) = I_{\underbrace{1, \dots, 1, k}_{n-2}}^{A_{n-1}}(\alpha; \beta_1 + \beta_2 - \gamma, \beta_3, \dots, \beta_n; \gamma) \frac{\Gamma(1-\gamma)\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma + 1)},$$

in accordance with (1.4). Iterating the recursion, it follows that

$$I_{\underbrace{1, \dots, 1, k}_{n-1}}^{A_n}(\alpha; \beta_1, \dots, \beta_n; \gamma) = I_{1, k}^{A_2}(\alpha; \beta_1 + \dots + \beta_n - (n-2)\gamma, \beta_n; \gamma) \\ \times \Gamma^{n-2}(1-\gamma) \prod_{i=1}^{n-2} \frac{\Gamma(\beta_1 + \dots + \beta_i + (1-i)\gamma)}{\Gamma(A_i + \beta_1 + \dots + \beta_i - i\gamma)},$$

and (7.1) reduces to its A_2 or $n=2$ case

$$\sum_{a=1}^k \frac{\sin(\pi(k-a+1)\gamma)}{\sin(\pi k\gamma)} \int_{O'_a} (1-u)^{\beta_1-1} \prod_{i=1}^k t_i^{\alpha-1} (1-t_i)^{\beta_2-1} \prod_{1 \leq i < j \leq k} (t_i - t_j)^{2\gamma} \\ \times \prod_{i=1}^{a-1} (t_i - u)^{-\gamma} \prod_{i=a}^k (u - t_i)^{-\gamma} du dt \\ = \frac{\Gamma(\beta_1)\Gamma(1-k\gamma)}{\Gamma(1+\beta_1-k\gamma)} \frac{\Gamma(\alpha+\beta_2+(2k-2)\gamma)}{\Gamma(\alpha+\beta_1+\beta_2+(k-2)\gamma)} \frac{\Gamma(\beta_1+\beta_2-\gamma)}{\Gamma(\beta_2+(k-1)\gamma)} \\ \times \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta_2+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha+\beta_2+(i+k-2)\gamma)\Gamma(\gamma)},$$

where

$$O'_a = \{(u, t) : 0 \leq t_k \leq \dots \leq t_a \leq u \leq t_{a-1} \leq \dots \leq t_1 \leq 1\}.$$

7.2. The case $\gamma=0$

When $\gamma=0$, Theorem 2.1 collapses to the integral

$$\int_{C_0^{k_1, \dots, k_n} [0,1]} \prod_{s=1}^n \prod_{i=1}^{k_s} (t_i^{(s)})^{\alpha_s-1} (1-t_i^{(s)})^{\beta_s-1} dt^{(1)} \dots dt^{(n)} \\ = \prod_{s=1}^n \frac{1}{k_s!} \left(\frac{\Gamma(\alpha)\Gamma(\beta_s + \dots + \beta_n)}{\Gamma(\alpha + \beta_s + \dots + \beta_n)} \right)^{k_s - k_{s-1}}$$

with $\alpha_1 = \dots = \alpha_{n-1} = 1$ and $\alpha_n = \alpha$. Because the $t_i^{(s)}$ in the integrand are completely decoupled, the problem of evaluating this integral is purely combinatorial. Introducing the partitions $\lambda^{(s)}$ for $1 \leq s \leq n-1$ as $\lambda^{(s)} = (M_s(k_s), \dots, M_s(1))$ so that $\lambda^{(s)}$ has exactly k_s parts and $\lambda_i^{(s)} \leq k_{s+1} - i + 1$, the $\gamma=0$ integral may also be stated more explicitly as

$$\sum_{\substack{\lambda^{(1)}, \dots, \lambda^{(n-1)} \\ 1 \leq \lambda_i^{(s)} \leq k_{s+1} - i + 1}} \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \frac{k_{s+1} - i - \lambda_i + 2}{k_{s+1} - i + 1} \int \prod_{s=1}^n \prod_{i=1}^{k_s} (t_i^{(s)})^{\alpha_s-1} (1-t_i^{(s)})^{\beta_s-1} dt^{(1)} \dots dt^{(n)} \\ = \prod_{s=1}^n \frac{1}{k_s!} \left(\frac{\Gamma(\alpha)\Gamma(\beta_s + \dots + \beta_n)}{\Gamma(\alpha + \beta_s + \dots + \beta_n)} \right)^{k_s - k_{s-1}},$$

where the integration domain is given by

$$\max\{t_{\lambda_{k_s-i+1}^{(s)}}^{(s+1)}, t_{i+1}^{(s)}\} \leq t_i^{(s)} \leq \min\{t_{i-1}^{(s)}, t_{\lambda_{k_s-i+1}^{(s)}-1}^{(s+1)}\} \tag{7.2a}$$

for $1 \leq s \leq n-1$ and $1 \leq i \leq k_s$ (with $t_0^{(s)}=1$ and $t_{k_s+1}^{(s)}=0$), and

$$t_{i+1}^{(n)} \leq t_i^{(n)} \leq t_{i-1}^{(n)} \tag{7.2b}$$

for $1 \leq i \leq k_n$.

Due to the factor

$$\prod_{s=1}^{n-1} \prod_{i=1}^{k_s} (k_{s+1} - i - \lambda_i + 2),$$

we may relax the condition $\lambda_i^{(s)} \leq k_{s+1} - i + 1$ to $\lambda_1^{(s)} \leq k_{s+1}$, so that the sum becomes

$$\sum_{\substack{\lambda^{(1), \dots, \lambda^{(n-1)}} \\ l(\lambda^{(s)})=k_s \\ \lambda_1^{(s)} \leq k_{s+1}}}$$

Since $\alpha_1 = \dots = \alpha_{n-1} = 1$, one may, with a bit of pain, successively integrate over the $t_i^{(s)}$ starting with $s=1$. We will not present the full details of this calculation here, but remark that the key to unravelling the combinatorics encoded in the inequalities (7.2) is given by

$$\sum_{\substack{\lambda \\ l(\lambda)=r \\ \lambda_1 \leq n}} \prod_{j=1}^n \frac{1}{m_j!} \prod_{i=1}^r (n - \lambda_i - i + 2) (x_{\lambda_i} - x_{\lambda_{i-1}}) = e_r(x), \tag{7.3}$$

with $x_0=0$, $x=(x_1, \dots, x_n)$, m_j being the multiplicity of the part j in λ and e_r being the r th elementary symmetric function (6.2). To establish (7.3), we note that when written in terms of the multiplicities m_j it becomes

$$\frac{1}{(n-r)!} \sum_{\substack{m_1, \dots, m_n \geq 0 \\ M_1=r}} \prod_{j=1}^n \frac{(x_j - x_{j-1})^{m_j} (n-j+1 - M_{j+1})}{m_j!} = e_r(x),$$

where $M_j = m_j + \dots + m_n$. Multiplying this by t^r and summing over r using the generating function for the e_r [16, (I.2.2)] yields

$$\sum_{m_1, \dots, m_n \geq 0} \frac{1}{(n-M_1)!} \prod_{j=1}^n \frac{[t(x_j - x_{j-1})]^{m_j} (n-j+1 - M_{j+1})}{m_j!} = \prod_{i=1}^n (1 + tx_i).$$

This is true for any x_0 provided we add the factor $(1+tx_0)^{n-M_1}$ to the summand:

$$\sum_{m_1, \dots, m_n \geq 0} \frac{(1+tx_0)^{n-M_1}}{(n-M_1)!} \prod_{j=1}^n \frac{[t(x_j-x_{j-1})]^{m_j} (n-j+1-M_{j+1})}{m_j!} = \prod_{i=1}^n (1+tx_i). \quad (7.4)$$

For $n=0$ this is obviously correct. If we denote the sum on the left by $L(x_0, \dots, x_n)$, then

$$\begin{aligned} L(x_0, \dots, x_n) &= \sum_{m_2, \dots, m_n \geq 0} \frac{(1+tx_0)^{n-M_2}}{(n-1-M_2)!} \prod_{j=2}^n \frac{[t(x_j-x_{j-1})]^{m_j} (n-j+1-M_{j+1})}{m_j!} \\ &\quad \times \sum_{m_1=0}^{n-M_2} \left(\frac{tx_1-tx_0}{1+tx_0} \right)^{m_1} \binom{n-M_2}{m_1} \\ &= \sum_{m_2, \dots, m_n \geq 0} \frac{(1+tx_1)^{n-M_2}}{(n-1-M_2)!} \prod_{j=2}^n \frac{[t(x_j-x_{j-1})]^{m_j} (n-j+1-M_{j+1})}{m_j!} \\ &= (1+tx_1)L(x_1, \dots, x_n). \end{aligned}$$

By induction, (7.4) is thus true for all non-negative integers n .

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