Acta Math., 207 (2011), 375–389
DOI: 10.1007/s11511-012-0072-8

Operator-Lipschitz functions in Schatten–von Neumann classes

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This paper resolves a number of problems in the perturbation theory of linear operators, linked with the 45-year-old conjecture of M. G. Kreĭn. In particular, we prove that every Lipschitz function is operator-Lipschitz in the Schatten–von Neumann ideals S^α, $1<\alpha<\infty$. Alternatively, for every $1<\alpha<\infty$, there is a constant $c_\alpha>0$ such that

$$\|f(a)-f(b)\|_\alpha \leqslant c_\alpha \|f\|_{\mathrm{Lip}\,1}\, \|a-b\|_\alpha,$$

where f is a Lipschitz function with

$$\|f\|_{\mathrm{Lip}\,1} := \sup_{\substack{\lambda,\mu\in\mathbb{R}\\ \lambda\neq\mu}} \left| \frac{f(\lambda)-f(\mu)}{\lambda-\mu} \right| < \infty,$$

$\|\cdot\|_\alpha$ is the norm in S^α, and a and b are self-adjoint linear operators such that $a-b\in S^\alpha$.

Denote by F_α the class of functions $f:\mathbb{R}\to\mathbb{C}$ such that

$$f(a)-f(b)\in S^\alpha$$

for any self-adjoint a and b such that $a-b\in S^\alpha$, and put

$$\|f\|_{F_\alpha} := \sup_{a,b} \frac{\|f(a)-f(b)\|_\alpha}{\|a-b\|_\alpha}.$$

In [10], M. G. Kreĭn conjectured that the condition $f'\in L^\infty$ is sufficient for $f\in F_1$. The fact that this conjecture does not hold was established by Yu. Farforovskaya in 1972, see [6]. Earlier, in 1967, she had also established that the analogue of Kreĭn's conjecture does not hold for the case $\alpha=\infty$, see [4] and [5]. Later, T. Kato [8] (resp. E. B. Davies, [2]) showed

The first author was partially supported by ARC.

that already the absolute value function $f(t)=|t|$ does not belong to F_∞ (resp. F_1). An alternative approach to the study of the class F_1 was developed by V. Peller in [13], where he proved that $B^1_{\infty 1}\subseteq F_1$ and every $f\in F_1$ belongs to B^1_{11} locally, where B^s_{pq} are Besov spaces; this approach also shows that the class F_1 is properly contained in the class of all Lipschitz functions, since $\mathrm{Lip}\,1\not\subseteq B^1_{11}$. However, the question whether the class F_α contains all Lipschitz functions when $1<\alpha<\infty$, $\alpha\neq 2$, remained open. For example, $f(t)=|t|$ belongs to F_α for all $1<\alpha<\infty$, see, e.g., [2], [3] and [9]. Another sufficient condition ensuring that $f\in F_\alpha$ for every $1<\alpha<\infty$, obtained in [2] and [12], requires that the derivative f' is bounded and has bounded total variation. On the other hand, addressing the problem of description of the classes F_α, $1<\alpha<\infty$, $\alpha\neq 2$, in her 1974 paper [7], Yu. Farforovskaya observes that "a slight change in the proof given in [6]" leads to the existence of a Lipschitz function f_α which does not belong to F_α for every $1<\alpha<2$, and further proceeds with an explicit "example" of f_α such that $f'_\alpha\in L^\infty$, but $f_\alpha\notin F_\alpha$, $2<\alpha<\infty$. Presenting the same problem in [14], V. Peller conjectured that $f\in F_\alpha$, $1\leqslant\alpha<2$, implies that the lacunary Fourier coefficients of f' satisfy

$$\{\hat{f}'(2^n)\}_{n\geqslant 0}\in\ell^\alpha.$$

Finally, shortly before this paper was written, F. Nazarov and V. Peller discovered that $f(a)-f(b)\in S^{1,\infty}$, provided f is Lipschitz, and a and b are self-adjoint linear operators such that $a-b$ is 1-dimensional, where $S^{1,\infty}$ is the weak trace ideal of compact operators. From here, they also show that

$$\|f(a)-f(b)\|_\Omega\leqslant c_0\|a-b\|_1$$

for every self-adjoint a and b such that $a-b\in S^1$, where S^Ω is dual to the Matsaev ideal S^ω (see [11] for details).

The main objective of the present paper is to show that in fact M. G. Kreĭn's conjecture holds for all $1<\alpha<\infty$, that is $f'\in L^\infty$ implies $\|f\|_{F_\alpha}<\infty$. Equivalently, F_α, $1<\alpha<\infty$, coincides with the class of all Lipschitz functions. In particular, this shows that Farforovskaya's remark concerning F_α, $1<\alpha<2$, and her result for F_α, $2<\alpha<\infty$, given in [7] do not hold and that the conjecture of V. Peller [14] does not hold either.

The main result of the paper is the following theorem whose proof is based on Theorem 2.

THEOREM 1. *Let f be a Lipschitz function and let $\|f\|_{\mathrm{Lip}\,1}\leqslant 1$. For every $1<\alpha<\infty$ there is a constant $c_\alpha>0$ such that*

$$\|f(a)-f(b)\|_\alpha\leqslant c_\alpha\|a-b\|_\alpha, \tag{1}$$

where a and b are self-adjoint (possibly unbounded) linear operators such that $a-b\in S^\alpha$.

The proof of Theorem 1 is given at the end of §2.

The symbol c_α shall denote a positive numerical constant which depends only on α, $1\leqslant\alpha\leqslant\infty$, and which may vary from line to line or even within a line.

1. Schur multipliers of divided differences

Although(1) the principal result of the paper is proved for the ideals of compact operators, in the present section we shall work in the setting of an arbitrary semifinite von Neumann algebra. This wider setting brings no additional difficulties to our considerations but allows a very succinct notation. Yet a reader unfamiliar with the theory of semifinite von Neumann algebras may think of the algebra of all bounded linear operators on ℓ^2 equipped with the standard trace instead of the couple (M,τ), and of the Schatten–von Neumann ideals S^α instead of the non-commutative spaces L^α.

Let M be a von Neumann algebra with normal semifinite faithful trace τ. Let L^α, $1\leqslant\alpha\leqslant\infty$, be the L^p-space with respect to the couple (M,τ) (see [15]).

Let $\{e_k\}_{k\in\mathbb{Z}}\subseteq M$ be a sequence of mutually orthogonal projections and let $f:\mathbb{R}\mapsto\mathbb{C}$ be a Lipschitz function. We shall study the following linear operator

$$Tx=\sum_{k,j\in\mathbb{Z}}\phi_{kj}e_kxe_j, \quad \text{where} \quad \phi_{kj}=\begin{cases}\dfrac{f(k)-f(j)}{k-j}, & \text{if } k\neq j,\\ 0, & \text{if } k=j.\end{cases} \tag{2}$$

In this section we keep the sequence $\{e_k\}_{k\in\mathbb{Z}}$, the function f and the operator T fixed.

THEOREM 2. *If* $\|f\|_{\mathrm{Lip}\,1}\leqslant 1$, *then the operator* T *is bounded on every space* L^α, $1<\alpha<\infty$.

Proof. Without loss of generality, we may assume that $f(0)=0$ and that f is real-valued.

Let us fix $x\in L^\alpha$ and $y\in L^{\alpha'}$, with $1<\alpha,\alpha'<\infty$ such that $1/\alpha+1/\alpha'=1$. We shall prove that

$$|\tau(yTx)|\leqslant c_\alpha\|x\|_\alpha\|y\|_{\alpha'}.$$

Recall that the triangular truncation is a bounded linear operator on L^α, $1<\alpha<\infty$ (see e.g. [3]). Thus, we may further assume that the operator x is upper-triangular and y is lower-triangular.(2) For every element $z\in M$, we set $z_{kj}:=e_kze_j$ for brevity. We can now

(1) The proof presented in this section was substantially simplified in comparison to the firstly circulated argument. A similar simplification was observed independently by Mikael de la Salle, [18].

(2) An element $x\in M$ is called *upper-triangular* (with respect to the sequence $\{e_k\}_{k\in\mathbb{Z}}$) if and only if $e_kxe_j=0$ for every $k>j$. It is called *lower-triangular* if and only if x^* is upper-triangular.

write

$$\tau(yTx)=\sum_{k<j}\tau(y_{jk}\phi_{kj}x_{kj}). \tag{3}$$

Let us show that we may also assume that the function f takes only integral values at integral points. Indeed, by setting $a_k=f(k)-f(k-1)$, we have

$$\phi_{kj}=\frac{1}{j-k}\sum_{k<m\leqslant j}a_m \quad \text{for } k<j.$$

Thus, we continue

$$\tau(yTx)=\frac{1}{j-k}\sum_{k<j}\tau(y_{jk}x_{kj})\sum_{k<m\leqslant j}a_m=\sum_{m\in\mathbb{Z}}a_m\sum_{k<m\leqslant j}\frac{\tau(y_{jk}x_{kj})}{j-k}.$$

Recall that we have to show that

$$|\tau(yTx)|\leqslant c_\alpha\|x\|_\alpha\|y\|_{\alpha'},$$

for every sequence $\{a_m\}_{m\in\mathbb{Z}}\in\ell^\infty$ with $\|\{a_m\}_{m\in\mathbb{Z}}\|_\infty\leqslant 1$. From this, it is clear that it is sufficient to take $a_m\in\{-1,0,1\}$ and thus, the function f takes only integral values at integral points, since

$$f(k)=f(k)-f(0)=\sum_{1\leqslant m\leqslant k}a_m.$$

We may also assume that the function f is non-decreasing (otherwise, we take the function $f_1(t)=f(t)+t$). Thus, from now on we assume that f takes integral values at integral points, f is non-decreasing and

$$0\leqslant f(k)-f(j)\leqslant 2(k-j) \quad \text{for } j\leqslant k, \quad j,k\in\mathbb{Z}.$$

According to Lemma 6, we have

$$\phi_{kj}=\int_{\mathbb{R}}g(s)(f(j)-f(k))^{is}(j-k)^{-is}\,ds \quad \text{for } k<j, \tag{4}$$

where $g:\mathbb{R}\mapsto\mathbb{C}$ is such that

$$\int_{\mathbb{R}}|s|^n|g(s)|\,ds<\infty \quad \text{for } n\geqslant 0. \tag{5}$$

With this in mind, we now see from (3) and (4) that

$$\tau(yTx)=\int_{\mathbb{R}}g(s)\tau(y_sx_s)\,ds,$$

where (as in Lemma 5)

$$y_s = \sum_{k<j} (f(j)-f(k))^{is} y_{jk} \quad \text{and} \quad x_s = \sum_{k<j} (j-k)^{-is} x_{kj}.$$

Now it follows from Lemma 5 that

$$|\tau(y_s x_s)| \leqslant c_\alpha (1+|s|)^2 \|x\|_\alpha \|y\|_{\alpha'},$$

and therefore, from (5),

$$|\tau(yTx)| \leqslant c_\alpha \|x\|_\alpha \|y\|_{\alpha'} \int_{\mathbb{R}} (1+|s|)^2 |g(s)|\, ds \leqslant c_\alpha \|x\|_\alpha \|y\|_\alpha. \qquad \square$$

Remark 3. The operator T in (2) can also be defined with respect to two families of orthogonal projections: if $\{e_k\}_{k\in\mathbb{Z}}$ and $\{f_j\}_{j\in\mathbb{Z}}$ are two families of orthogonal projections, then

$$\widetilde{T}x = \sum_{k,j\in\mathbb{Z}} \phi_{kj} e_k x f_j, \tag{6}$$

where ϕ_{kj} is defined as in (2). In this case, the operator $\widetilde{T}$ is also bounded on every L^α, $1<\alpha<\infty$, provided $\|f\|_{\mathrm{Lip}\,1} \leqslant 1$. Indeed, to see the latter it is sufficient to consider the operator T as in (2) with respect to the family of orthogonal projections

$$\{e_j \otimes e_{11} + f_j \otimes e_{22}\}_{j\in\mathbb{Z}}$$

in the tensor product von Neumann algebra $M \otimes \mathbb{M}_2$, where $\mathbb{M}_2$ is the algebra of 2×2 matrices, and apply Theorem 2.

To prove Lemma 5 used in the proof of Theorem 2, we firstly need the following result whose proof follows rather quickly from the vector-valued Marcinkiewicz multiplier theorem.

THEOREM 4. *Let $\lambda = \{\lambda(n)\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that*

$$\sup_{n\in\mathbb{Z}} |\lambda(n)| \leqslant 1.$$

If the total variation of λ over every dyadic interval $2^k \leqslant |n| \leqslant 2^{k+1}$, $k\geqslant 0$, does not exceed 1, *then the linear operator S defined by*

$$Sx = \sum_{k,j\in\mathbb{Z}} \lambda(f(j)-f(k)) e_k x e_j, \quad x \in L^\alpha,$$

is bounded on every L^α, $1<\alpha<\infty$, where $f\colon \mathbb{Z} \mapsto \mathbb{Z}$ is any non-decreasing integer-valued function.

Proof. It was proved in [1] that if X is a Banach space with the UMD (unconditionality for martingale differences) property (see [15] for the relevant definitions) and if $h \in L^2([0,1], X)$ (i.e. the space of all Bochner square integrable functions on $[0,1]$ with values in X), then the linear operator(3)

$$Mh(t) = \sum_{n \in \mathbb{Z}} \lambda(n) \hat{h}(n) e^{2\pi i n t}, \quad t \in [0,1],$$

is bounded, provided

$$\sup_{n \in \mathbb{Z}} |\lambda(n)| \leqslant 1$$

and the total variation of the sequence λ over every dyadic interval does not exceed 1.

Recall that L^α is a Banach space with the UMD property for every $1<\alpha<\infty$ (see e.g. [15]). Consider the function

$$h_x(t) = u_t^* x u_t = \sum_{k,j \in \mathbb{Z}} e^{2\pi i (f(j)-f(k))t} e_k x e_j, \quad x \in L^\alpha, \ t \in [0,1],$$

where the unitary u_t is defined by

$$u_t = \sum_{k \in \mathbb{Z}} e^{2\pi i f(k) t} e_k.$$

Observe that the nth Fourier coefficient of h_x is

$$\hat{h}_x(n) = \sum_{f(j)-f(k)=n} e_k x e_j, \quad n \in \mathbb{Z}. \tag{7}$$

Noting that the mapping $L^\alpha \ni x \mapsto h_x \in L^2([0,1], L^\alpha)$ is a complemented isometric embedding of L^α into $L^2([0,1], L^\alpha)$ and that, from (7),

$$M(h_x) = h_{Sx},$$

we see that the boundedness of S on L^α, $1<\alpha<\infty$, follows from the boundedness of M on $L^2([0,1], L^\alpha)$. □

(3) Here $\{\hat{h}(n)\}_{n \in \mathbb{Z}}$ is the sequence of the Fourier coefficients, i.e.,

$$\hat{h}(n) = \int_0^1 h(t) e^{-2\pi i n t}\, dt, \quad n \in \mathbb{Z}.$$

LEMMA 5. *If* $x \in L^\alpha$ *and if*

$$x_s = \sum_{k<j} (f(j)-f(k))^{is} e_k x e_j, \quad s \in \mathbb{R},$$

then, for every $1<\alpha<\infty$, *there is a constant* $c_\alpha>0$ *such that*

$$\|x_s\|_\alpha \leqslant c_\alpha (1+|s|) \|x\|_\alpha,$$

where $f: \mathbb{Z} \mapsto \mathbb{Z}$ *is any non-decreasing integer-valued function.*

Proof. Clearly, the lemma follows from Theorem 4 if we estimate the total variation of the sequence $\lambda=\{n^{is}\}_{n>0}$ over dyadic intervals. To this end, via the fundamental theorem of the calculus, we see that

$$|n^{is}-(n+1)^{is}| \leqslant \frac{|s|}{n}, \quad n \geqslant 1,$$

and thus immediately

$$\sum_{2^k \leqslant n \leqslant 2^{k+1}} |n^{is}-(n+1)^{is}| \leqslant |s|, \quad k \geqslant 0.$$

The lemma is proved. □

LEMMA 6. *There is a function* $g: \mathbb{R} \mapsto \mathbb{C}$ *such that*

$$\int_{\mathbb{R}} |s|^n |g(s)| \, ds < \infty, \quad n \geqslant 0,$$

and such that, for every $\lambda, \mu>0$ *with* $0 \leqslant \lambda/\mu \leqslant 2$,

$$\frac{\lambda}{\mu} = \int_{\mathbb{R}} g(s) \lambda^{is} \mu^{-is} \, ds.$$

Proof. Let us consider a C^∞-function f such that

(i) $f \geqslant 0$;

(ii) $f(t)=0$, if $t \geqslant 1+\log 2$;

(iii) $f(t)=e^t$, if $t \leqslant \log 2$.

Observe that f and all its derivatives are L^2 functions, i.e.,

$$\|f^{(n)}\|_2 < \infty, \quad n \geqslant 0.$$

If we now set $g(s)=\hat{f}(s)$, where $\hat{f}$ is the Fourier transform of f, then it is known (see [17, Lemma 7]) that

$$\int_{\mathbb{R}} |s|^n |g(s)| \, ds \leqslant c_0 \max\{\|f^{(n)}\|_2, \|f^{(n+1)}\|_2\} < \infty, \quad n \geqslant 0.$$

Furthermore, via the inverse Fourier transform, we also have

$$e^t = \int_{\mathbb{R}} g(s) e^{its} \, ds, \quad t \leqslant \log 2,$$

and substituting $t=\log \lambda/\mu$ yields the desired relation. □

2. Double operator integrals of divided differences.

Here we present the continuous versions of the transformation (6) and Theorem 2. The proof of the continuous case employs Theorem 2 (and Remark 3) and some approximation procedure.

Let M be a semifinite von Neumann algebra with a normal semifinite faithful trace τ, and let L^α be the corresponding L^p-space with respect to the couple (M,τ), $1\leqslant\alpha\leqslant\infty$. Let us first introduce a continuous version of the transformation (6).

Let us fix spectral measures $dE_\lambda, dF_\mu\in M$, $\lambda,\mu\in\mathbb{R}$. Recall that if $x,y\in L^2$, then the mapping

$$(\lambda,\mu)\longmapsto d\nu_{x,y}(\lambda,\mu):=\tau(y\,dE_\lambda\,x\,dF_\mu),\quad \lambda,\mu\in\mathbb{R},$$

is a complex-valued σ-additive measure on $\mathbb{R}^2$ with total variation not exceeding $\|x\|_2\|y\|_2$ (see [12]). If now $\phi:\mathbb{R}^2\mapsto\mathbb{C}$ is a bounded Borel function, then the latter implies that

$$(x,y)\longmapsto\int_{\mathbb{R}^2}\phi(\lambda,\mu)\,d\nu_{x,y}(\lambda,\mu),\quad x,y\in L^2,$$

is a continuous bilinear form, i.e. there is a bounded linear operator T_ϕ on L^2 such that

$$\tau(yT_\phi(x))=\int_{\mathbb{R}^2}\phi\,d\nu_{x,y},\quad x,y\in L^2. \tag{8}$$

The operator T_ϕ is a continuous version of (6), which we shall study in the present section.

We will say that the operator T_ϕ introduced above is bounded on L^α, for some $1\leqslant\alpha\leqslant\infty$, if and only if it admits a bounded extension from $L^\alpha\cap L^2$ to L^α. The latter extension, if it exists, is unique (see [12]).

The following theorem is the main objective of the present section.

THEOREM 7. *If* $\|f\|_{\mathrm{Lip}\,1}\leqslant1$, *then the operator* T_{ϕ_f} *is bounded on every space* L^α, $1<\alpha<\infty$, *where*

$$\phi_f(\lambda,\mu)=\begin{cases}\dfrac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \text{if } \lambda\neq\mu,\\ 0, & \text{if } \lambda=\mu,\end{cases}\qquad \lambda,\mu\in\mathbb{R}.$$

We need the following auxiliary lemma first.

LEMMA 8. (Duhamel's formula, [19]) *Let* A *and* B *be self-adjoint linear operators. If* $A-B$ *is bounded, then*

$$e^{irA}-e^{irB}=ir\int_0^1 e^{ir(1-t)A}(A-B)e^{irtB}\,dt,\quad r\in\mathbb{R}.$$

In particular,

$$\|e^{irA}-e^{irB}\|\leqslant|r|\,\|A-B\|.$$

Proof of Theorem 7. We observe that the function f may be assumed to be compactly supported. Indeed, assume that Theorem 7 has been proved for compactly supported functions. Fix a Lipschitz function $f:\mathbb{R}\to\mathbb{C}$, take a sequence of characteristic functions $\chi_n=\chi_{[-n,n]}$ and set

$$\phi_n(\lambda,\mu)=\chi_n(\lambda)\phi_f(\lambda,\mu)\chi_n(\mu).$$

Since Theorem 7 is assumed proved for compactly supported Lipschitz functions, the sequence of operators T_{ϕ_n} is uniformly bounded, and we also have that $\lim_{n\to\infty}\phi_n=\phi_f$ pointwise. This implies that T_{ϕ_f} is also bounded (see [12, Lemma 2.6]).

Let us next assume that $x\in L^2$ is *diagonal* with respect to the measures dE_λ and dF_μ, i.e.,

$$dE_\lambda x=x\,dF_\lambda,\ \ \lambda\in\mathbb{R}\quad\Longleftrightarrow\quad x=\int_{\mathbb{R}} dE_\lambda\,x\,dF_\lambda.$$

In this case,

$$d\nu_{x,y}(\lambda,\mu)=\delta(\lambda-\mu)\tau(dE_\lambda\,x\,dF_\mu y),\quad y\in L^2,$$

where δ is the Dirac function and therefore, from (8),

$$\begin{aligned}|\tau(yT_\phi(x))|&=\left|\tau\left(y\int_{\mathbb{R}}\phi(\lambda,\lambda)\,dE_\lambda\,x\,dF_\lambda\right)\right|\leqslant\|y\|_{\alpha'}\left\|\int_{\mathbb{R}}\phi(\lambda,\lambda)\,dE_\lambda x dF_\lambda\right\|_\alpha\\&\leqslant\|y\|_{\alpha'}\,\|\phi\|_\infty\left\|\int_{\mathbb{R}}dE_\lambda\,x\,dF_\lambda\right\|_\alpha=\|\phi\|_\infty\,\|y\|_{\alpha'}\,\|x\|_\alpha\end{aligned}$$

for every $x\in L^2\cap L^\alpha$ and $y\in L^2\cap L^{\alpha'}$, with x diagonal. In other words, the operator T_ϕ is trivially bounded on the diagonal part of L^α for every bounded Borel function ϕ. This indicates that the essential part of the proof is the boundedness on the off-diagonal part of L^α, i.e., for those $x\in L^\alpha$ such that

$$dE_\lambda\,x\,dF_\lambda=0,\ \ \lambda\in\mathbb{R}\quad\Longleftrightarrow\quad\int_{\mathbb{R}}dE_\lambda\,x\,dF_\lambda=0.$$

Note also that, if $x=A-B$, then trivially

$$dE^A_\lambda\,x\,dE^B_\lambda=0,\quad\lambda\in\mathbb{R}.$$

From now on, we assume that $x\in L^2$ is off-diagonal with respect to the measures dE_λ and dF_μ. The proof is a two-stage approximation. Let $\{G_n\}_{n\geqslant1}$ be the family of dilated Gaussian functions as in Lemma 9. If now $f_n=G_n*f$, then

$$\phi_n(\lambda,\mu):=\phi_{f_n}(\lambda,\mu)=\int_{\mathbb{R}}G_n(s)\phi_f(\lambda-s,\mu-s)\,ds,\quad\lambda\neq\mu.$$

In other words, ϕ_n is the convolution (with respect to the diagonal shift on $\mathbb{R}^2$) of G_n and ϕ. From Lemma 9, we see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |\phi_n - \phi_f| \, d\nu_{x,y} = 0, \quad x, y \in L^2.$$

This implies that

$$\lim_{n\to\infty} \tau(yT_{\phi_n}(x)) = \tau(yT_{\phi_f}(x)), \quad x, y \in L^2. \tag{9}$$

Consequently, in order to see that T_{ϕ_f} is bounded on L^α, we have to prove that the operators T_{ϕ_n} are all bounded on L^α, uniformly with respect to $n=1,2,\dots$.

Observe that f_n is a rapidly decreasing C^∞ function for every $n=1,2,\dots$. Thus, if $\hat{f}_n$ is the Fourier transform of f_n, then

$$c_{n,m} := \int_{\mathbb{R}} |s^m \hat{f}_n(s)| \, ds < \infty, \quad m = 0, 1, ..., \quad n = 1, 2, \dots. \tag{10}$$

Furthermore, if $h_n(s)$ is the Fourier transform of the derivative f_n', i.e, $h_n(s) = s\hat{f}_n(s)$, then h_n is integrable and, for $\lambda \neq \mu$,

$$\begin{aligned} \phi_n(\lambda, \mu) = \phi_{f_n}(\lambda, \mu) = \frac{f_n(\lambda) - f_n(\mu)}{\lambda - \mu} &= \int_0^1 f_n'((1-t)\lambda + t\mu) \, dt \\ = \int_{\mathbb{R}} h_n(s) \int_0^1 e^{is((1-t)\lambda + t\mu)} \, dt \, ds&. \end{aligned} \tag{11}$$

Directly checking the definition (8) again, the latter implies that the double operator integral operator T_{ϕ_n} (with respect to any spectral measures $dE_\lambda, dF_\mu \in M$, $\lambda, \mu \in \mathbb{R}$) has the form

$$T_{\phi_n}(x) = \int_{\mathbb{R}} h_n(s) \int_0^1 e^{is(1-t)A} x e^{istB} \, dt \, ds, \tag{12}$$

where x is off-diagonal,

$$A = \int_{\mathbb{R}} \lambda \, dE_\lambda \quad \text{and} \quad B = \int_{\mathbb{R}} \mu \, dF_\mu.$$

Let us now fix the spectral measures dE_λ and dF_μ and introduce the following discrete approximations $dE_{m,\lambda}$ and $dF_{m,\mu}$, $\lambda, \mu \in \mathbb{R}$, $m = 1, 2, ...,$

$$E_m(\Omega) = \sum_{\substack{j \in \mathbb{Z} \\ j/m \in \Omega}} e_j \quad \text{and} \quad F_m(\Omega) = \sum_{\substack{k \in \mathbb{Z} \\ k/m \in \Omega}} f_k,$$

where

$$e_j = E\left[\frac{j}{m}, \frac{j+1}{m}\right), \quad f_k = F\left[\frac{k}{m}, \frac{k+1}{m}\right), \quad j, k \in \mathbb{Z} \quad \text{and} \quad \Omega \subseteq \mathbb{R} \text{ is Borel}.$$

Let also $T_{n,m}$ be the double operator integral operator associated with the function ϕ_n and the spectral measures $dE_{m,\lambda}$ and $dF_{m,\mu}$. On one hand, the operator $T_{n,m}$ has the form

$$T_{n,m}(x) = \sum_{j,k\in\mathbb{Z}} \phi_n\left(\frac{j}{m}, \frac{k}{m}\right) e_j x f_k = \sum_{j,k\in\mathbb{Z}} \frac{mf_n(j/m)-mf_n(k/m)}{j-k} e_j x f_k,$$

which is the operator given in (6) with respect to the function $f_{n,m}(t)=mf_n(t/m)$. Since $f_{n,m}$ is Lipschitz and

$$\|f_{n,m}\|_{\mathrm{Lip}\,1} = \|f_n\|_{\mathrm{Lip}\,1} = \|G_n * f\|_{\mathrm{Lip}\,1} \leqslant 1,$$

it follows from Theorem 2 that the family of operators $T_{m,n}$ is bounded on L^α uniformly for $m,n=1,2,\dots$. On the other hand, let us set

$$A_m = \int_{\mathbb{R}} \lambda\, dE_{m,\lambda} \quad \text{and} \quad B_m = \int_{\mathbb{R}} \mu\, dF_{m,\mu}.$$

It follows immediately from the spectral theorem that

$$\|A - A_m\| \leqslant \frac{1}{m} \quad \text{and} \quad \|B - B_m\| \leqslant \frac{1}{m}.$$

Furthermore, observing the elementary representation

$$\begin{aligned} &e^{is(1-t)A} x e^{istB} - e^{is(1-t)A_m} x e^{isB_m} \\ &\qquad = e^{is(1-t)A} x (e^{isB} - e^{isB_m}) + (e^{is(1-t)A} - e^{is(1-t)A_m}) x e^{istB_m}, \end{aligned}$$

we now estimate, using Lemma 8,

$$\|e^{is(1-t)A} x e^{istB} - e^{is(1-t)A_m} x e^{-istB_m}\|_\alpha \leqslant \frac{2|s|}{m} \|x\|_\alpha.$$

Employing (12) for the operators T_{ϕ_n} and $T_{n,m}$, we have

$$T_{\phi_n}(x) - T_{n,m}(x) = \int_{\mathbb{R}} h_n(s) \int_0^1 (e^{is(1-t)A} x e^{istB} - e^{is(1-t)A_m} x e^{istB_m})\, dt\, ds.$$

Using the estimate above for the integrand, we finally see that

$$\|T_{\phi_n}(x) - T_{n,m}(x)\|_\alpha \leqslant \frac{2}{m} \|x\|_\alpha \int_{\mathbb{R}} |s h_n(s)|\, ds = \frac{2c_{n,2}}{m} \|x\|_\alpha,$$

since $h_n(s)=s\hat{f}(s)$ and using (10). In particular, we have

$$\lim_{m\to\infty} \|T_{\phi_n}(x) - T_{n,m}(x)\|_\alpha = 0.$$

We have already observed above that the family of operators $T_{n,m}$ is uniformly bounded on L^α. Together with the limit above, it means that the family T_{ϕ_n} is also uniformly bounded on L^α. Recalling (9) finally yields that T_{ϕ_f} is bounded on L^α. The theorem is proved. □

LEMMA 9. *Let $d\nu$ be a finite Borel measure on $\mathbb{R}^2$ and let $\phi:\mathbb{R}^2\to\mathbb{C}$ be an integrable function on $(\mathbb{R}^2, d\nu)$, i.e., $\phi\in L^1(\mathbb{R}^2, d\nu)$. If G_n is the family of dilated Gaussian functions, i.e.,*

$$G_n(t)=nG(nt),\ G(t)=(2\pi)^{-1/2}e^{-t^2/2},\quad t\in\mathbb{R},\ n=1,2,...,$$

and if

$$\phi_n(\lambda,\mu)=\int_{\mathbb{R}} G_n(s)\phi(\lambda-s,\mu-s)\,ds,$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}|\phi_n-\phi|\,d\nu=0. \tag{13}$$

Proof. Since the class of uniformly continuous functions on $\mathbb{R}^2$ is norm dense in $L^1(\mathbb{R}^2, d\nu)$, we may assume that ϕ is uniformly continuous on $\mathbb{R}^2$. For continuous ϕ, the limit (13) can be proved by the standard approximation identity argument. Indeed, we shall actually show the uniform convergence

$$\lim_{n\to\infty}\|\phi_n-\phi\|_\infty=0,$$

which, since $d\nu$ is finite, implies (13).

Observe first that

$$\int_{\mathbb{R}} G_n(s)\,ds=1,\quad n=1,2,\dots.$$

Consequently, we have

$$\phi_n(\lambda,\mu)-\phi(\lambda,\mu)=\int_{\mathbb{R}} G_n(s)\phi(\lambda-s,\mu-s)\,ds-\phi(\lambda,\mu)\int_{\mathbb{R}} G_n(s)\,ds=\int_{\mathbb{R}}\Phi_n(\lambda,\mu,s)\,ds,$$

where

$$\Phi_n(\lambda,\mu,s)=G_n(s)(\phi(\lambda-s,\mu-s)-\phi(\lambda,\mu)).$$

Fix $\varepsilon>0$. Recall that ϕ is uniformly continuous, which means that there is $\delta>0$ such that

$$\sup_{\lambda,\mu\in\mathbb{R}}|\phi(\lambda-s,\mu-s)-\phi(\lambda,\mu)|\leqslant\varepsilon,\quad \text{if } |s|<\delta.$$

Using the latter $\delta>0$, we split

$$\phi_n(\lambda,\mu)-\phi(\lambda,\mu)=\omega_0(\lambda,\mu)+\omega_\infty(\lambda,\mu),$$

where

$$\omega_0(\lambda,\mu)=\int_{|s|<\delta}\Phi_n(\lambda,\mu,s)\,ds\quad\text{and}\quad\omega_\infty(\lambda,\mu)=\int_{|s|\geqslant\delta}\Phi_n(\lambda,\mu,s)\,ds.$$

We estimate the latter two terms separately. For ω_0, using the uniform estimate above,

$$\sup_{\lambda,\mu\in\mathbb{R}} |\omega_0(\lambda,\mu)| \leqslant \int_{|s|<\delta} G_n(s) \sup_{\lambda,\mu\in\mathbb{R}} |\phi(\lambda-s,\mu-s)-\phi(\lambda,\mu)|\,ds \leqslant \varepsilon \int_{\mathbb{R}} G_n(s)\,ds \leqslant \varepsilon.$$

On the other hand, for ω_∞,

$$\sup_{\lambda,\mu\in\mathbb{R}} |\omega_\infty(\lambda,\mu)| \leqslant 2\int_{|s|\geqslant\delta} G_n(s) \sup_{\lambda,\mu} |\phi(\lambda,\mu)|\,ds = 2\|\phi\|_\infty \int_{|s|\geqslant\delta} G_n(s)\,ds.$$

Noting that for every fixed δ, we have

$$\lim_{n\to\infty} \int_{|s|\geqslant\delta} G_n(s)\,ds = 0,$$

we conclude that there is N_ε such that for every $n\geqslant N_\varepsilon$,

$$\sup_{\lambda,\mu\in\mathbb{R}} |\omega_\infty(\lambda,\mu)| \leqslant \varepsilon.$$

Combining the estimates above for ω_0 and ω_∞, we obtain that for every $\varepsilon>0$, there is N_ε such that, for every $n\geqslant N_\varepsilon$,

$$\|\phi_n-\phi\|_\infty \leqslant \|\omega_0\|_\infty + \|\omega_\infty\|_\infty \leqslant 2\varepsilon.$$

The lemma is proved. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Observe that it is sufficient to prove that there is a constant c_α such that for every self-adjoint operator u and every bounded operator v,

$$\|[f(u),v]\|_\alpha \leqslant c_\alpha \|[u,v]\|_\alpha. \tag{14}$$

Indeed, the estimate (1) immediately follows from the inequality above with

$$u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let T_{ϕ_f} be the double operator integral operator associated with the function ϕ_f and the spectral measure $dE=dF=dE^u$, where dE^u_λ is the spectral measure of u. It follows from Theorem 7 that the operator T_{ϕ_f} is bounded on every L^α, $1<\alpha<\infty$, and that its norm does not depend on the operator u. Combining this fact with [16, Theorem 5.3] yields the estimate (14). □

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Received May 20, 2009
Received in revised form October 15, 2009