Acta Math., 209 (2012), 265–322
DOI: 10.1007/s11511-012-0087-1

Almost sure multifractal spectrum for the tip of an SLE curve

by

Fredrik Johansson Viklund

Columbia University
New York, NY, U.S.A.

Gregory F. Lawler

University of Chicago
Chicago, IL, U.S.A.

1. Introduction

The chordal Schramm–Loewner evolution ($\mathrm{SLE}_\varkappa$) is a 1-parameter family of probability measures on curves $\gamma:[0,\infty)\to\overline{\mathbb{H}}$, where $\mathbb{H}$ denotes the complex upper half-plane. It was invented by Schramm [21] as a candidate for the scaling limit of 2-dimensional lattice models from statistical physics that satisfy conformal invariance and a Markovian property in the limit. Several lattice models have since been shown to have scaling limits that can be described by SLE. Examples include loop-erased random walk and the uniform spanning tree [14], the percolation exploration-process [22], and the FK-Ising model [23]. We refer the reader to [8], [10] and [24] for surveys and further references.

The properties of the SLE curves themselves has been the focus of much research since their introduction in [21]. For example, Rohde and Schramm [20] proved existence and Hölder continuity in the standard parametrization, and an upper bound on the Hausdorff dimension. Beffara [1] later proved the more difficult lower bound on the dimension. Lind [16] found the lower bound on the optimal Hölder exponent and the present authors [5] proved that this exponent is sharp.

In this paper we will be interested in the behavior at the tip $\gamma(t)$ of the growing SLE curve. Since the curves are fractals, one cannot make sense of derivatives. Instead, the natural approach is to consider the behavior of $|g_t'(z)|$ for z near $\gamma(t)$, where g_t is a uniformizing conformal map from the complement of the curve to the upper half-plane. For technical reasons it is often easier to consider $f_t=g_t^{-1}$ near V_t, the pre-image of the tip on the real-line. Our main goal will be to derive the almost sure *tip multifractal spectrum* for SLE. For a suitable interval of α, it is defined, roughly speaking, as the

Johansson Viklund acknowledges the support of the Simons Foundation and the Knut and Alice Wallenberg Foundation. Lawler is supported by National Science Foundation grant DMS-0907143.

dimension of the subset of the curve corresponding to t for which $y|f_t'(V_t+iy)|$ decays like y^α when $y\to 0^+$. We shall see that the tip multifractal spectrum is closely related to the *multifractal spectrum of harmonic measure* at the tip of the growing curve. As a function of α, this spectrum measures the size of the part of the curve that corresponds to t for which the harmonic measure of a ball of radius ε centered at the tip decays like ε^α as $\varepsilon\to 0^+$. We remark that both these spectra are independent of the particular parametrization of the curve.

The multifractal spectrum of harmonic measure has been studied extensively in the physics and mathematics literature. For example, in the case of the paths of Brownian motion, the spectrum is determined by the Brownian intersection exponents, see [11] and the references therein. In two dimensions these exponents were established by Lawler, Schramm and Werner in [11]–[13]. In the case of the SLE curves, Duplantier has predicted, using non-rigorous so-called quantum-gravity methods, a harmonic measure spectrum for the tip of an SLE curve, see [4, §7]. However, this spectrum is different from the ones we will work with, as it describes the local dimension of harmonic measure in a radial setup; it corresponds in some sense to the analog of our function $\varrho(\beta)$ for a radial SLE curve at the bulk point. (See §3 for the definition of ϱ.) Duplantier and Binder also used quantum gravity arguments to predict the spectrum of harmonic measure for the *bulk* of SLE, see [3]. Roughly speaking, this spectrum is defined as the dimension of the subset of the curve away from the tip where, for a given α, harmonic measure in a ball of radius ε decays like ε^α as $\varepsilon\to 0^+$. Beliaev and Smirnov [2] made a start to proving this result by establishing the *average integral means spectrum* for SLE. To get the almost sure multifractal spectrum from the average integral means spectrum, one can formally apply the so-called multifractal formalism [17] and find the bulk spectrum by taking a Legendre transform of the average integral means spectrum. This approach is believed to be valid for SLE, although it has not been proved in this case. Indeed, to the best of our knowledge the present paper is the first to establish almost sure multifractal spectra for the $\mathrm{SLE}_\varkappa$ family.

The starting point of our analysis is estimation of moments of the derivative of f_t using the reverse-time Loewner flow; this was started by Rohde and Schramm in [20] and extended in many places, e.g., [2], [5], [6], [9] and [16]. (This is the analogue of the average integral means spectrum result for our problem.) In order to get almost sure results, one needs second-moment estimates. The ideas for that appear in [9] and they were used in, e.g., [5]. These ideas are also important in understanding the so-called natural parametrization of SLE curves, see [15].

1.1. Multifractal spectra for the tip

We now proceed to discuss in greater detail the multifractal spectra that we will consider. To motivate our definitions, we will start out in a slightly different setting than the one we will work with in the rest of the paper.

Suppose that ζ is a boundary point of a simply connected domain D. We say that ζ is *accessible* (*in D by η*) if $\eta:[0,1]\to\mathbb{C}$ is a simple curve with $\eta(0)=\zeta$ and $\eta((0,1])\subset D$. If ζ is accessible by η, let h be a conformal transformation of D onto $\mathbb{C}\setminus(-\infty,0]$ with $h(\zeta)=0$. By $h(\zeta)=0$, we mean $h(\eta(0^+))=0$. We now specialize to the following situation: Let $\widetilde{\gamma}:(-\infty,\infty)\to\mathbb{C}$ be a simple curve with $\widetilde{\gamma}(t)\to\infty$ as $t\to\pm\infty$. For each t, we consider the "slit" plane $D_t=\mathbb{C}\setminus\widetilde{\gamma}((-\infty,t])$, which is a simply connected domain whose boundary contains $\widetilde{\gamma}(t)$ and ∞. The (*non-tangential*) *tip multifractal spectrum* which we describe in this subsection is one way to describe the behavior near $\widetilde{\gamma}(t)$ of the conformal map uniformizing D_t, for different values of t. Clearly, the boundary point $\widetilde{\gamma}(t)$ is accessible in D_t by the curve $\eta^{(t)}(s)=\widetilde{\gamma}(t+s)$.

Remark. For endpoints of slits like $\widetilde{\gamma}(t)$ in D_t, there is only one possible meaning for $h(\widetilde{\gamma}(t))=0$, but for general D a boundary point ζ might be approached from different directions that correspond to different values of $h(\zeta)$. Formally, this can be understood using prime ends (see, e.g., [19, Chapter 2]). In the case at hand, the curve η specifies a particular direction/prime end.

Let $D=D_t$, take $\zeta=\widetilde{\gamma}(t)$, and set $g(z)=i\sqrt{h(z)}$, where the branch of the square root is chosen so that $\sqrt{1}=1$. Then g is a conformal transformation of D onto the upper half plane $\mathbb{H}$ with $g(\zeta)=0$. The map g is only unique up to composition with a Möbius transformation, that is, if $\tilde{g}$ is another such map, then $\tilde{g}(z)=(T\circ g)(z)$, where T is a Möbius transformation of $\mathbb{H}$ fixing 0. Similarly, h is not unique.

Let $\eta^*(s)=g(\eta(s))$. Then $\eta^*:(0,1]\to\mathbb{H}$ is a curve with $\eta^*(0^+)=0$. If $\eta_1^*:(0,1]\to\mathbb{H}$ is another curve with $\eta_1^*(0^+)=0$, and $\eta_1(s)=g^{-1}(\eta_1^*(s))$, then $\eta_1(0^+)=\zeta$ and ζ is accessible by η_1. (This uses the fact that the curve η exists and that we are considering a domain slit by a curve.) We say that η^* satisfies a *weak cone condition* if there is a subpower function (see §2.1) ψ such that, for all $s>0$,

$$|\mathrm{Re}\,\eta^*(s)| \leqslant (\mathrm{Im}\,\eta^*(s))\,\psi\left(\frac{1}{\mathrm{Im}\,\eta^*(s)}\right),$$

and we say that η is *weakly non-tangential* if $g\circ\eta$ satisfies a weak cone condition. It is not difficult to see that this definition is independent of the choice of g. One example of a weakly non-tangential curve for D is

$$\eta(s)=g^{-1}(si), \quad 0<s\leqslant 1.$$

We will use this particular curve to define the tip multifractal spectrum but the definition will be the same for any weakly non-tangential curve.

Next, we let $f=g^{-1}$, so that f is a conformal transformation of $\mathbb{H}$ onto D. Since $f(is)=\eta(s)$, $s>0$, is a simple curve, the length of $\eta((0,s])$ is given by

$$v(f;s):=\int_0^s |f'(iy)|\,dy. \tag{1.1}$$

A sufficient condition for the existence of a limiting $\zeta=\eta(0^+)$ is that $v(f;0^+)=0$ which is equivalent to

$$v(f;t)<\infty, \quad t>0.$$

We can also use the plane slit by the negative real axis as uniformizing domain and write $f(w)=F(-w^2)$, where $F\colon\mathbb{C}\setminus(-\infty,0]\to D$ with $F(0)=\zeta$. Then $F^{-1}(\eta((0,s]))=[0,s^2]$. In particular, the length of $F^{-1}(\eta((0,s]))$ is s^2, and the length of $f^{-1}(\eta((0,s]))$ is s.

We say that the (*non-tangential*) *scaling exponent* at the boundary point ζ is θ if

$$v(f;s)\approx^* s^{2\theta}, \quad s\to 0^+.$$

In particular, if $D=\mathbb{C}\setminus(-\infty,t]$, then the scaling exponent at t equals 1. (Recall that $f\colon\mathbb{H}\to D$ and see (2.1) for the definition of $\approx^*$.) More generally, if γ is differentiable at t, then $\theta=1$ at t. Note that the Beurling estimates (see Lemma 2.6) imply that $\theta\leqslant 1$. (In fact, the same bound holds for a lim sup version of the definition of θ.) The scaling exponent is closely related to the behavior of $|f'(iy)|$ as $y\to 0^+$. Indeed, if $y|f'(iy)|\approx^* y^{1-\beta}$ for some $\beta<1$, then, as we will show in Proposition 2.7,

$$v(f;y)\approx^* y^{1-\beta}, \quad y\to 0^+,$$

so that

$$\theta=\tfrac{1}{2}(1-\beta).$$

Although the definition of $v(f;y)$ depends on the choice of the conformal map f, it is not hard to see that the scaling exponent θ is independent of the choice.

Returning to the curve $\tilde{\gamma}$, we can consider T_θ, the set of t such that the scaling exponent of D_t at $\tilde{\gamma}(t)$ equals θ. The tip multifractal spectrum can then be defined to be either of the following two functions:

$$\theta\longmapsto \dim_{\mathrm{H}}(T_\theta) \quad\text{and}\quad \theta\longmapsto\dim_{\mathrm{H}}[\tilde{\gamma}(T_\theta)],$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. The first function depends on the choice of the parametrization of $\tilde{\gamma}$ and the second is independent of the parametrization. One could also define lim inf and lim sup versions of this. The main goal of this paper is to compute the tip multifractal spectrum for the chordal SLE path. For technical convenience, we will use an alternative definition in terms of the behavior of $|f'(iy)|$ as $y\to 0^+$ and we will use β rather than θ as our variable.

Suppose now that $\gamma=\gamma(t)$ is a curve in $\overline{\mathbb{H}}$ with $\gamma(0^+)\in\mathbb{R}$. Let H_t be the unbounded connected component of $\mathbb{H}\setminus\gamma([0,t])$. One way to define the multifractal spectrum of harmonic measure at the tip is as the function

$$\alpha \longmapsto \dim_{\mathrm{H}}[\gamma(T_\alpha^{\mathrm{hm}})],$$

where T_α^{hm} is the set of t such that the normalized harmonic measure from infinity of a ball of radius $\varepsilon>0$ about the tip $\gamma(t)$ scales like ε^α as $\varepsilon\to 0^+$. We will both use this definition and a slightly different (non-equivalent) definition that is more closely related to the tip multifractal spectrum that we described above. See §2.3 for precise definitions.

1.2. Main results

Let $\hat f_t(z)=f_t(z+V_t)$, where $f_t\colon\mathbb{H}\to H_t$ is the chordal $\mathrm{SLE}_\varkappa$ Loewner chain. That is, f_t solves, for $t\geqslant 0$, the chordal Loewner partial differential equation

$$\partial_t f_t(z) = -f_t'(z)\frac{a}{z-V_t}, \quad f_0(z)=z,$$

where $a=2/\varkappa$ and V_t is standard Brownian motion. Further, for $-1\leqslant\beta\leqslant 1$, let

$$\varrho(\beta) = \frac{\varkappa}{8(\beta+1)}\left[\left(\frac{\varkappa+4}{\varkappa}\right)(\beta+1)-1\right]^2,$$

and set

$$\beta_\pm = -1+\frac{\varkappa}{12+\varkappa\mp 4\sqrt{8+\varkappa}}.$$

Define

$$\Theta_\beta = \{t\in(0,2] : y|\hat f_t'(iy)| \approx^* y^{1-\beta}\}.$$

See §2.1 for the definition of $\approx^*$.

THEOREM. (Tip multifractal spectrum) *Suppose that* $\varkappa>0$ *and* $\beta_-\leqslant\beta\leqslant\beta_+$. *For chordal* $\mathrm{SLE}_\varkappa$, *almost surely*,

$$\dim_{\mathrm{H}}(\Theta_\beta) = \frac{2-\varrho(\beta)}{2} \quad \textit{and} \quad \dim_{\mathrm{H}}[\gamma(\Theta_\beta)] = \frac{2-\varrho(\beta)}{1-\beta}.$$

See the precise statement in Theorem 3.1; we prove more than we state here.

Notice that we obtain Beffara's theorem on the dimension of the $\mathrm{SLE}_\varkappa$ curves [1] as a special case of Theorem 3.1.

Using the tip multifractal spectrum and some additional work, we can derive the almost sure spectrum for harmonic measure at the tip, see §2.3 for more details. Although we modify the definition of the spectrum somewhat, we prove in Theorem 3.2 the stronger almost sure version of the theorem. To state it, define

$$\alpha_\pm = \frac{1}{1-\beta_\pm},$$

where $\beta_\pm$ are as above. Let $\mathrm{hm}_t(\,\cdot\,)=\lim_{y\to\infty} y\,\mathrm{hm}(iy,\cdot,H_t)$ be the renormalized harmonic measure from infinity in H_t. For each $t\geqslant 0$, let $\gamma(t)=\lim_{y\to 0^+}\hat{f}_t(iy)$ be the tip at time t of the growing $\mathrm{SLE}_\varkappa$ curve and set

$$\Theta_\alpha^{\mathrm{hm}} = \{t\in(0,2] : \mathrm{hm}_t(E_{t,\varepsilon}) \approx^* \varepsilon^\alpha\},$$

where $E_{t,\varepsilon}$ is the part of ∂H_t that contains $\gamma(t)$ as the prime end corresponding to V_t and is separated from ∞ by $\partial\mathcal{B}(\gamma(t),\varepsilon)=\{z:|z-\gamma(t)|=\varepsilon\}$. We have the following result.

Theorem. (Multifractal spectrum for harmonic measure at the tip) *Suppose that* $\varkappa>0$ *and* $\alpha_-\leqslant\alpha\leqslant\alpha_+$. *For chordal* $\mathrm{SLE}_\varkappa$, *almost surely*,

$$\dim_{\mathrm{H}}[\gamma(\Theta_\alpha^{\mathrm{hm}})] = \alpha\left(1-\frac{4}{\varkappa}\right)+\frac{(4+\varkappa)^2}{8\varkappa}-\frac{\varkappa}{8}\left(\frac{\alpha^2}{2\alpha-1}\right). \tag{1.2}$$

In §6 we prove Theorem 3.3 which together with Theorem 3.2 and a Beurling estimate shows that the right-hand side of (1.2) gives the harmonic measure spectrum for a (one-sided) version which is closer to the usual definition, but for a smaller range of α.

1.3. Overview of the paper

Our paper is organized as follows. The next section discusses some preliminary facts. After setting some notation about asymptotics in §2.1, the deterministic Loewner equation is discussed in §2.2. Much in this subsection is standard but we have included this in order to phrase the results appropriately for our purposes. Also, we want to separate estimates that deal only with the Loewner equation itself from those that are particular to SLE. In this subsection, there are three kinds of results: those that hold for all conformal maps of $\mathbb{H}$ for which we use the letter h; those that hold for all solutions of the chordal Loewner equation for which we use g_t and $f_t=g_t^{-1}$; and towards the end facts about solutions of the Loewner equation for driving functions that are weakly Hölder-$\frac{1}{2}$. We also formally define the tip multifractal spectra in this section.

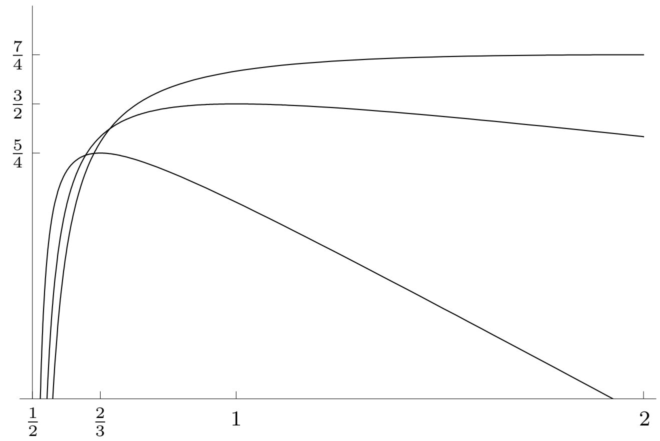

Figure 1. Multifractal spectrum of harmonic measure at the tip for $\mathrm{SLE}_\varkappa$, $\varkappa=2,4,6$. The maximum is the Hausdorff dimension of the curve.

The main theorem is not stated in full until §3, where the Schramm–Loewner evolution (SLE) is discussed. From here on a value of the SLE parameter $\varkappa$ is fixed and a large number of $\varkappa$-dependent parameters are defined. Although we do not discuss it directly, what we are doing is establishing the guess for the value of the multifractal spectrum in terms of the Legendre transform of a logarithmic moment generating function.

The basic proof of the main theorem can be found in §4. This section is relatively short because it relies on estimates on the moments of the derivative, some of which were established in [5] and [9]; the necessary additions are proved in §5. To make the paper self-contained we have also included an appendix that discusses a key result from [9]. §6 uses the *forward* Loewner flow to prove a result on the harmonic measure spectrum stated in §3.3. We warn the reader that some of the notation in §6 does not agree with the earlier sections and that the assumption $\varkappa<8$ is made there.

1.4. Acknowledgement

We would like to thank the referee for his/her careful reading and useful comments that helped us improve the quality of our paper.

2. Preliminaries

2.1. Notation

In order to avoid writing bulky expressions with ratios of logarithms, we will adopt the following notation.

We call a function $\psi\colon [0,\infty)\to(0,\infty)$ a (*positive*) *subpower function* if it is continuous, non-decreasing, and

$$\lim_{x\to\infty} x^{-u}\psi(x)=0$$

for all $u>0$.

If f and g are positive functions tending to zero with y, we write

$$f(y)\approx^* g(y), \quad y\to 0^+, \tag{2.1}$$

if there exists a subpower function ψ such that

$$\psi\left(\frac{1}{y}\right)^{-1} g(y)\leqslant f(y)\leqslant \psi\left(\frac{1}{y}\right) g(y), \quad y\to 0^+.$$

We write

$$f(y)\preccurlyeq g(y), \quad y\to 0^+,$$

if

$$\limsup_{y\to 0^+}\frac{\log g(y)}{\log f(y)}\leqslant 1,$$

and we write

$$f(y)\preccurlyeq_{\text{i.o.}} g(y), \quad y\to 0^+,$$

if

$$\liminf_{y\to 0^+}\frac{\log g(y)}{\log f(y)}\leqslant 1.$$

Here "i.o." stands for "infinitely often". Clearly $f(y)\preccurlyeq g(y)$ implies that $f(y)\preccurlyeq_{\text{i.o.}} g(y)$, but the converse is not true. Similarly we write $f(y)\succcurlyeq g(y)$ and $f(y)\succcurlyeq_{\text{i.o.}} g(y)$ for

$$\liminf_{y\to 0^+}\frac{\log g(y)}{\log f(y)}\geqslant 1 \quad\text{and}\quad \limsup_{y\to 0^+}\frac{\log g(y)}{\log f(y)}\geqslant 1,$$

respectively. We write $f(y)\approx g(y)$ if $f(y)\preccurlyeq g(y)$ and $f(y)\succcurlyeq g(y)$, that is, if

$$\lim_{y\to 0^+}\frac{\log g(y)}{\log f(y)}=1.$$

Note that, if $\beta>0$, then

$$f(y)\approx y^{\beta} \iff f(y)\approx^* y^{\beta}.$$

We will also use the notation for asymptotics for functions $f(n)$ and $g(n)$ as $n\to\infty$ along the positive integers. We summarize the notation in the following table:

Notation	Definition, as $y \to 0^+$
$f(y) \approx^* g(y)$	$\psi\left(\frac{1}{y}\right)^{-1} g(y) \leqslant f(y) \leqslant g(y)\psi\left(\frac{1}{y}\right)$
$f(y) \preccurlyeq g(y)$	$\limsup_{y\to 0^+} \frac{\log g(y)}{\log f(y)} \leqslant 1$
$f(y) \preccurlyeq_{\text{i.o.}} g(y)$	$\liminf_{y\to 0^+} \frac{\log g(y)}{\log f(y)} \leqslant 1$
$f(y) \succcurlyeq g(y)$	$\liminf_{y\to 0^+} \frac{\log g(y)}{\log f(y)} \geqslant 1$
$f(y) \succcurlyeq_{\text{i.o.}} g(y)$	$\limsup_{y\to 0^+} \frac{\log g(y)}{\log f(y)} \geqslant 1$

2.2. Chordal Loewner equation

In this section, we review some facts about conformal mappings and the chordal Loewner equation. See [8, Chapters 3 and 4] for proofs of theorems stated without proof here.

Let $\widetilde{\gamma}: (-\infty, \infty) \to \mathbb{C}$ be a simple curve as in the introduction. The chordal Loewner equation describes the evolution of $\widetilde{\gamma}((0,\infty))$ given $\widetilde{\gamma}((-\infty,0])$. Let $\tilde{g}$ be a conformal transformation of $\mathbb{C}\setminus\widetilde{\gamma}((-\infty,0])$ onto the upper half-plane $\mathbb{H}$ with $\tilde{g}(\gamma(0))=0$ and $\tilde{g}(\infty)=\infty$. In order to describe $\widetilde{\gamma}(t), t>0$, it suffices to describe

$$\gamma(t) := \tilde{g}(\widetilde{\gamma}(t)), \quad 0 \leqslant t < \infty,$$

and this is what the Loewner equation in $\mathbb{H}$ does. For the remainder of the paper, we will consider a curve γ in $\mathbb{H}$ as above. (In general, however, we will not assume it is simple.) The Riemann mapping theorem implies that there is a unique conformal transformation g_t of $\mathbb{H}\setminus\gamma((0,t])$ onto $\mathbb{H}$ with $g_t(z)=z+o(1)$ as $z\to\infty$. We can expand g_t at infinity,

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}),$$

where $a(t)$ by definition is the half-plane capacity of $\gamma((0,t])$. It is continuous and strictly increasing. We make the (slightly) stronger assumption that $a(t)\to\infty$ as $t\to\infty$. Then the chordal Loewner integral equation states that

$$g_t(z) = z + \int_0^t \frac{da(s)}{g_s(z) - V_s}, \quad t \leqslant T_z,$$

where $V_s=g_s(\gamma(s))$ and $T_z=\inf\{t:\operatorname{Im} g_t(z)=0\}=\inf\{t:g_t(z)-V_t=0\}$. It can be shown that $s\mapsto V_s$ is a continuous function and it is called the *Loewner driving function* (or *term*). It is convenient to choose a parametrization of γ such that $a(t)=at$ for some $a>0$, in which case we arrive at the Loewner differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z)-V_t}, \quad g_0(z)=z. \tag{2.2}$$

Let

$$f_t(z)=g_t^{-1}(z) \quad \text{and} \quad \hat{f}_t(z)=f_t(z+V_t)=g_t^{-1}(z+V_t).$$

By differentiating both sides of $f_t(g_t(z))=z$ with respect to t, we see that

$$\partial_t f_t(z) = -f_t'(z)\frac{a}{z-V_t}, \tag{2.3}$$

where we used the notation $f_t'(z)=\partial_z f_t(z)$. Since $g_t(\gamma(t))=V_t$, we get

$$\gamma(t)=f_t(V_t)=\lim_{y\to 0^+} f_t(V_t+iy)=\lim_{y\to 0^+}\hat{f}_t(iy). \tag{2.4}$$

We let

$$v_t(y)=v(\hat{f}_t;y)=\int_0^y |\hat{f}_t'(iu)|\,du.$$

Note that if g_t satisfies (2.2) and $g_t^*=g_{t/a}$, then

$$\partial_t g_t^*(z)=\frac{1}{g_t^*(z)-V_t^*}, \quad g_0^*(z)=z,$$

where $V_t^*=V_{t/a}$.

Conversely, we can start with a continuous function $t\mapsto V_t$ and $a>0$, and define a Loewner chain $(g_t, t\geqslant 0)$ by (2.2). We define $\gamma(t)$ by (2.4) provided that the limit exists. As mentioned above, if $v_t(y)<\infty$ for some $y>0$, then $v_t(0^+)=0$ and the limit in (2.4) exists. More work is needed to determine whether γ is a continuous function of t or not. We say that the family of conformal maps g_t is generated by a curve if γ, as defined by (2.4), exists and is a continuous function of t. We do not assume that the curve is simple. If H_t denotes the unbounded component of $\mathbb{H}\setminus\gamma((0,t])$, then g_t is the unique conformal transformation of H_t onto $\mathbb{H}$ satisfying

$$g_t(z)=z+\frac{at}{z}+O(|z|^{-2}), \quad z\to\infty.$$

LEMMA 2.1. *For every t and every $y>0$ with $v_t(y)<\infty$,*

$$\tfrac{1}{4}y|\hat{f}_t'(iy)|\leqslant|\gamma(t)-\hat{f}_t(iy)|\leqslant v_t(y). \tag{2.5}$$

Proof. The second estimate is immediate from the definition of $v_t(y)$ and the first inequality follows from the Koebe-$\frac{1}{4}$ theorem applied to $\hat{f}_t$ on the open disk of radius y about iy. □

LEMMA 2.2. *If f_t satisfies* (2.3) *and $z=x+iy\in\mathbb{H}$, then, for $s\geqslant 0$,*

$$e^{-5as/y^2}|f_t'(z)|\leqslant|f_{t+s}'(z)|\leqslant e^{5as/y^2}|f_t'(z)|. \tag{2.6}$$

In particular, if $s\leqslant y^2$, then

$$e^{-5a}|f_t'(z)|\leqslant|f_{t+s}'(z)|\leqslant e^{5a}|f_t'(z)|.$$

Proof. Without loss of generality, we may assume that $a=1$. Differentiating (2.3) with respect to z yields

$$\partial_t f_t'(z)=-f_t''(z)\frac{1}{z-V_t}+f_t'(z)\frac{1}{(z-V_t)^2}.$$

Note that $|z-V_t|\geqslant y$. Applying Bieberbach's theorem (the $n=2$ case of the Bieberbach conjecture) to the disk of radius y about z, we can see that

$$|f_t''(z)|\leqslant\frac{4}{y}|f_t'(z)|,$$

and hence

$$|\partial_t f_t'(z)|\leqslant\frac{5}{y^2}|f_t'(z)|,$$

which implies (2.6). □

The Koebe distortion and growth theorems are traditionally stated in terms of univalent functions defined on the unit disk (see, e.g., [19, Chapter 2]). We will use these theorems for univalent functions on $\mathbb{H}$, and the next proposition gives the appropriate results.

PROPOSITION 2.3. *Let $h:\mathbb{H}\to\mathbb{C}$ be a conformal transformation, $x\in\mathbb{R}$, $y>0$ and $r\geqslant 1$. Then*

$$(x^2+4)^{-2}|h'(iy)|\leqslant|h'(y(x+i))|\leqslant(x^2+4)^2|h'(iy)|, \tag{2.7}$$

$$|h(y(x+i))-h(iy)|\leqslant\tfrac{1}{2}(x^2+4)^{3/2}y|x|\,|h'(iy)|, \tag{2.8}$$

$$r^{-3}|h'(iy)|\leqslant|h'(iyr)|\leqslant r|h'(iy)|, \tag{2.9}$$

$$|h(iyr)-h(iy)|\leqslant\tfrac{1}{2}(r^2-1)y|h'(iy)|. \tag{2.10}$$

Proof. By scaling, we may assume that $y=1$. Let

$$G(z)=\frac{z-i}{z+i},\quad G'(z)=\frac{2i}{(z+i)^2},$$

which is a conformal transformation of $\mathbb{H}$ onto the unit disk $\mathbb{D}$ with $G(i)=0$ and $|G'(i)|=\frac{1}{2}$. We can write

$$h(z)=f(G(z)),\quad h'(z)=f'(G(z))G'(z),$$

where f is a univalent function on $\mathbb{D}$. The distortion theorem tells us that

$$|f'(w)|\leqslant\frac{1+|w|}{(1-|w|)^3}|f'(0)|,\quad |w|<1,$$

and the growth theorem states that

$$|f(w)-f(0)|\leqslant\frac{|w|}{(1-|w|)^2}|f'(0)|,\quad |w|<1.$$

As $|G'(i)|=\frac{1}{2}$, we get

$$|h'(z)|\leqslant\frac{2|G'(z)|(1+|G(z)|)}{(1-|G(z)|)^3}\,|h'(i)|, \tag{2.11}$$

and

$$|h(z)-h(i)|\leqslant\frac{2|G(z)|}{(1-|G(z)|)^2}|h'(i)|. \tag{2.12}$$

Since

$$|G(x+i)|=\frac{|x|}{\sqrt{x^2+4}}\quad\text{and}\quad |G'(x+i)|=\frac{2}{x^2+4},$$

we plug this into (2.11) and see that

$$|h'(x+1)|\leqslant\tfrac{1}{16}\big(\sqrt{x^2+4}+|x|\big)^4|h'(i)|\leqslant(x^2+4)^2|h'(i)|.$$

This gives the second inequality in (2.7) and the first follows easily by real translation. Plugging into (2.12) gives

$$|h(x+i)-h(x)|\leqslant\tfrac{2}{16}|x|\sqrt{x^2+4}\big(\sqrt{x^2+4}+x\big)^2|h'(i)|\leqslant\tfrac{1}{2}(x^2+4)^{3/2}|x|\,|h'(i)|.$$

Since $r\geqslant1$, we obtain

$$|G(ir)|=\frac{r-1}{1+r}=\left|G\left(\frac{i}{r}\right)\right|,\quad |G'(ir)|=\frac{2}{(1+r)^2}\quad\text{and}\quad\left|G'\left(\frac{i}{r}\right)\right|=\frac{2r^2}{(r+1)^2}.$$

Plugging this into (2.11) and (2.12) gives (2.9) and (2.10). □

COROLLARY 2.4. *If $h:\mathbb{H}\to\mathbb{C}$ is a conformal transformation, then, for every $y>0$,*

$$v(h;y)\geqslant\frac{y|h'(iy)|}{2},\tag{2.13}$$

$$\frac{2}{3}v(h;2^{-n})\leqslant\sum_{j=n}^{\infty}2^{-j}|h'(i2^{-j})|\leqslant\frac{8}{3}v(h;2^{-n}).$$

Proof. We write

$$v(h;2^{-n})=\sum_{j=n}^{\infty}\int_{2^{-j-1}}^{2^{-j}}|h'(iy)|\,dy.$$

Using (2.9) (which holds for $r>1$), we get

$$v(h;y)=\int_0^y|h'(is)|\,ds=y\int_0^1|h'(iry)|\,dr\geqslant y|h'(iy)|\int_0^1 r\,dr=\frac{y|h'(iy)|}{2},$$

$$\int_{r/2}^{r}|h'(iy)|\,dy\leqslant r|h_t'(ir)|\int_{1/2}^{1}\frac{ds}{s^3}=\frac{3r}{2}|h'(ir)|,$$

$$\int_{r/2}^{r}|h'(iy)|\,dy\geqslant r|h'(ir)|\int_{1/2}^{1}s\,ds=\frac{3r}{8}|h'(ir)|.$$ □

We define the following measure of the modulus of continuity of V_t:

$$\Delta(t,s)=\sup_{0\leqslant r\leqslant s^2}\sqrt{s^{-2}(V_{t+r}-V_t)^2+4}.$$

Note that $\Delta(t,s)\geqslant 2$, and it is of order 1 if

$$\sup_{0\leqslant r\leqslant s^2}|V_{t+r}-V_t|\approx s.$$

The definition of $\Delta(t,s)$ with the 4 has been chosen to make the statement of the next proposition cleaner.

PROPOSITION 2.5. *If $t\geqslant 0$ and $0\leqslant s\leqslant y^2$ with $v_t(y)+v_{t+s}(y)<\infty$, then*

$$|\gamma(t+r)-\gamma(t)|\leqslant v_t(y)+v_{t+s}(y)+e^{5a}|\hat f_t'(iy)|\Delta(t,y)^4y.\tag{2.14}$$

Proof. By the triangle inequality and (2.5), we have

$$\begin{aligned}|\gamma(t+r)-\gamma(t)|&\leqslant|\hat f_t(iy)-\gamma(t)|+|\hat f_{t+s}(iy)-\gamma(t+s)|+|\hat f_t(iy)-\hat f_{t+s}(iy)|\\&\leqslant v_t(y)+v_{t+s}(y)+|f_{t+s}(V_{t+s}+iy)-f_t(V_t+iy)|.\end{aligned}$$

Also,

$$\begin{aligned}|f_{t+s}(V_{t+s}+iy)-f_t(V_t+iy)|\\ \leqslant |f_{t+s}(V_{t+s}+iy)-f_{t+s}(V_t+iy)|+|f_{t+s}(V_t+iy)-f_t(V_t+iy)|.\end{aligned}$$

Using (2.8) and (2.6), we see that

$$\begin{aligned}|f_{t+s}(V_{t+s}+iy)-f_{t+s}(V_t+iy)| &\leqslant \tfrac{1}{2}|f'_{t+s}(V_t+iy)|\Delta(t,y)^4 y\\ &\leqslant \tfrac{1}{2}e^{5a}|f'_t(V_t+iy)|\Delta(t,y)^4 y\\ &= \tfrac{1}{2}e^{5a}|\hat f'_t(iy)|\Delta(t,y)^4 y.\end{aligned}$$

Also (2.6) and (2.3) imply that

$$|\partial_r f_{t+s}(V_t+iy)| \leqslant \frac{a}{y}e^{5ar/y^2}|\hat f'_t(iy)|, \tag{2.15}$$

and hence

$$\begin{aligned}|f_{t+s}(V_t+iy)-f_t(V_t+iy)| &\leqslant |f'_t(V_t+iy)|\int_0^{y^2}\frac{a}{y}e^{5au/y^2}\,du\\ &= \tfrac{1}{5}ye^{5a}|\hat f'_t(iy)| \qquad (2.16)\\ &< \tfrac{1}{2}ye^{5a}\Delta(t,y)^4|\hat f'_t(iy)|.\end{aligned}$$

□

LEMMA 2.6. *There exist $c>0$ such that, for $t\geqslant 0$ and $0<y\leqslant 1$,*

$$c\frac{y}{\sqrt{2at+1}} \leqslant |\hat f'_t(iy)| \leqslant \frac{\sqrt{2at+1}}{y}.$$

Proof. We may assume $a=1$, for otherwise we consider $g^*_t=g_{t/a}$. Let $w=\hat f_t(iy)$, that is, $g_t(w)=V_t+iy$, and let $Y_s=\operatorname{Im} g_s(w)$. The Loewner equation implies that $\partial_s(Y_s^2)\geqslant -2$ and hence $\operatorname{Im} w\leqslant\sqrt{2t+1}$. Similarly, $\operatorname{Im}\gamma(s)\leqslant\sqrt{2t}\leqslant\sqrt{2t+1}$ for $0\leqslant s\leqslant t$. The Loewner equation also implies that $\partial_s(Y_s/|g'_s(w)|)\leqslant 0$, which implies that

$$y|\hat f'_t(iy)| = \frac{Y_t}{|g'_t(w)|} \leqslant \frac{Y_0}{|g'_0(w)|} = \operatorname{Im} w \leqslant \sqrt{2t+1}.$$

This gives the second inequality.

For the first inequality, let $d=\operatorname{dist}(w,\gamma([0,t])\cup\mathbb{R})$. The Beurling estimate [8, Theorem 3.76] implies that there is a $c_*<\infty$ such that the probability that a Brownian motion starting at w goes distance $\sqrt{2t+1}$ without hitting $\gamma([0,t])\cup\mathbb{R}$ is bounded above by

$$c_*\left(\frac{d}{\sqrt{2t+1}}\right)^{1/2}.$$

By the gambler's ruin estimate, the probability that a Brownian motion in $\mathbb{H}$ starting at iy reaches $I_t := \{\widetilde{w} : \operatorname{Im} \widetilde{w} = 2\sqrt{2t+1}\,\}$ before hitting the real line equals $y(2\sqrt{2t+1}\,)^{-1}$. Since the imaginary part decreases in the forward Loewner flow, it follows from conformal invariance that the probability that a Brownian motion starting at w reaches I_t before hitting $\gamma([0,t]) \cup \mathbb{R}$ is at least $y(2\sqrt{2t+1}\,)^{-1}$. Therefore,

$$c_* \left(\frac{d}{\sqrt{2t+1}} \right)^{1/2} \geqslant \frac{y}{2\sqrt{2t+1}}.$$

The Koebe-$\frac{1}{4}$ theorem implies that $d \leqslant 4y|\hat{f}_t'(iy)|$, and plugging in we get

$$|\hat{f}_t'(iy)| \geqslant \frac{y}{16c_*^2\sqrt{2t+1}}. \qquad \square$$

PROPOSITION 2.7. *Let $h: \mathbb{H} \to \mathbb{C}$ be a conformal transformation and $v(h;y)$ be defined as in* (1.1). *Then, for every $\beta<1$, as $y \to 0^+$,*

$$\begin{aligned} y|h'(iy)| \preccurlyeq y^{1-\beta} \quad &\Longleftrightarrow \quad v(h;y) \preccurlyeq y^{1-\beta}, \\ y|h'(iy)| \succcurlyeq_{\text{i.o.}} y^{1-\beta} \quad &\Longleftrightarrow \quad v(h;y) \succcurlyeq_{\text{i.o.}} y^{1-\beta}, \\ y|h'(iy)| \approx^* y^{1-\beta} \quad &\Longleftrightarrow \quad v(h;y) \approx^* y^{1-\beta}. \end{aligned} \tag{2.17}$$

Proof. Using Corollary 2.4, all of the assertions follow easily except the fact that $v(h;y) \approx^* y^{1-\beta}$ implies $y|h'(iy)| \approx^* y^{1-\beta}$, which we will show here. Assume $v(h;y) \approx^* y^{1-\beta}$. By (2.13), we know that $y|h'(iy)| \preccurlyeq y^{1-\beta}$. For $0<\varepsilon<1-\beta$, let

$$u = u_\varepsilon = \frac{3\varepsilon}{1-\beta-\varepsilon},$$

and note that (2.9) implies, for y sufficiently small, that

$$\begin{aligned} y^{1-\beta+\varepsilon} \leqslant v(h;y) = v(h;y^{1+u}) + \int_{y^{1+u}}^{y} |h'(is)|\,ds \leqslant y^{1-\beta+2\varepsilon} + \int_{y^{1+u}}^{y} |h'(is)|\,ds \\ \leqslant y^{1-\beta+2\varepsilon} + y^{1-3u}|h'(iy)|. \end{aligned}$$

Hence, for all sufficiently small y,

$$y|h'(iy)| \geqslant \tfrac{1}{2} y^{1-\beta} y^{3u_\varepsilon+\varepsilon}.$$

Since $u_\varepsilon \to 0$ as $\varepsilon \to 0^+$, this gives $y|h'(iy)| \succcurlyeq y^{1-\beta}$. $\square$

Definition. For every $-1\leqslant\beta\leqslant 1$, let

$$\begin{aligned}
\overline{\Theta}_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \succcurlyeq_{\text{i.o.}} y^{1-\beta}\},\\
\Theta_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \approx^* y^{1-\beta}\},\\
\widetilde{\Theta}_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \preccurlyeq y^{1-\beta}\},\\
\underline{\Theta}_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \preccurlyeq_{\text{i.o.}} y^{1-\beta}\},\\
\underline{\Theta}^*_\beta &= \{t\in(0,2]: v_t(y) \preccurlyeq_{\text{i.o.}} y^{1-\beta}\},
\end{aligned}$$

where in each case the asymptotics are as $y\to 0^+$.

If $\beta\neq 1$, we can write these sets as the set of $t\in(0,2]$ such that

$$\begin{aligned}
\limsup_{y\to 0^+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} &\geqslant \beta,\\
\lim_{y\to 0^+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} &= \beta,\\
\limsup_{y\to 0^+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} &\leqslant \beta,\\
\liminf_{y\to 0^+} \frac{\log|\hat{f}_t'(iy)|}{\log(1/y)} &\leqslant \beta,\\
\liminf_{y\to 0^+} \frac{\log v_t(y)}{\log(1/y)} &\leqslant \beta-1,
\end{aligned}$$

respectively.

Using Lemma 2.6, we can see that for every $\beta>1$,

$$\overline{\Theta}_\beta = \Theta_\beta = \Theta_{-\beta} = \widetilde{\Theta}_{-\beta} = \underline{\Theta}_{-\beta} = \varnothing.$$

Note that (2.13) implies that $\underline{\Theta}^*_\beta\subset\underline{\Theta}_\beta$. By Proposition 2.7, we can also write

$$\Theta_\beta = \{t\in(0,2]: v_t(y)\approx^* y^{1-\beta} \text{ as } y\to 0^+\},$$

and similarly for $\overline{\Theta}_\beta$ and $\widetilde{\Theta}_\beta$. Also, $\overline{\Theta}_\beta\cup\widetilde{\Theta}_\beta=(0,2]$ and

$$\Theta_\beta \subset \overline{\Theta}_\beta\cap\widetilde{\Theta}_\beta\cap\underline{\Theta}^*_\beta.$$

Definition. The driving function V_t is *weakly Hölder-$\frac{1}{2}$* on $[0,2]$ if, for each $\alpha<\frac{1}{2}$, V_t is Hölder continuous of order α on $[0,2]$.

Two equivalent definitions are the following.

- If

$$\delta(s)=\sup\{|V_{t+s}-V_t|:0\leqslant t\leqslant t+s\leqslant 2\},$$

then

$$\psi(x)=\sup_{s\geqslant 1/x} s^{-1/2}\delta(s)$$

is a subpower function.

- There is a subpower function ψ such that for all $0\leqslant t\leqslant 2$ and $0\leqslant s\leqslant 1$,

$$\Delta(t,s)\leqslant\psi\left(\frac{1}{s}\right).$$

The next proposition shows that for weakly Hölder-$\frac{1}{2}$ functions V_t, it suffices to consider dyadic y and corresponding t in the definition of Θ_β, etc.

PROPOSITION 2.8. *Suppose that V_t is weakly Hölder-$\frac{1}{2}$ on $[0,2]$. For each $t\in[0,2]$ define*

$$t_n=t_n(t)=\frac{j-1}{2^{2n}},\quad \textit{if}\ \frac{j-1}{2^{2n}}\leqslant t<\frac{j}{2^{2n}}.$$

Then, for $-1\leqslant\beta\leqslant 1$, the following holds:

- *we have*

$$\begin{aligned}\Theta_\beta&=\{t\in(0,2]:2^{-n}|\hat f'_{t_n}(i2^{-n})|\approx^* 2^{-n(1-\beta)}\},\\ \overline{\Theta}_\beta&=\{t\in(0,2]:2^{-n}|\hat f'_{t_n}(i2^{-n})|\succcurlyeq_{\text{i.o.}} 2^{-n(1-\beta)}\},\\ \underline{\Theta}_\beta&=\{t\in(0,2]:2^{-n}|\hat f'_{t_n}(i2^{-n})|\preccurlyeq_{\text{i.o}} 2^{-n(1-\beta)}\},\end{aligned}$$

where the asymptotics are as $n\to\infty$ along the integers;

- *if $t\in\overline{\Theta}_\beta$, then*

$$v_t(y)\succcurlyeq_{\text{i.o.}} y^{1-\beta}\ \ \textit{and}\ \ |\gamma(t)-\hat f_t(iy)|\succcurlyeq_{\text{i.o.}} y^{1-\beta},\quad y\to 0^+;$$

- *if $t\in\tilde{\Theta}_\beta$, then*

$$v_t(y)\preccurlyeq y^{1-\beta}\ \ \textit{and}\ \ |\gamma(t)-\hat f_t(iy)|\preccurlyeq y^{1-\beta},\quad y\to 0^+;\tag{2.18}$$

- *if $t\in\Theta_\beta$, then*

$$v_t(y)\approx^* y^{1-\beta}\ \ \textit{and}\ \ |\gamma(t)-\hat f_t(iy)|\approx^* y^{1-\beta},\quad y\to 0^+.$$

Proof. Note that

$$|\hat{f}_t'(i2^{-n})| = |f_t'(V_t+i2^{-n})| \leqslant \Delta(t,2^{-n})^4 |f_t'(V_{t_n}+i2^{-n})| \leqslant e^{5a}\Delta(t,2^{-n})^4 |\hat{f}_{t_n}'(i2^{-n})|,$$

and similarly

$$|\hat{f}_t'(i2^{-n})| \geqslant e^{-5a}\Delta(t,2^{-n})^{-4} |\hat{f}_{t_n}'(i2^{-n})|.$$

Hence, if V_t is weakly Hölder-$\frac{1}{2}$, then there is a subpower function ψ such that, for all t and n,

$$\psi(2^n)^{-1} |\hat{f}_{t_n}'(i2^{-n})| \leqslant |\hat{f}_t'(i2^{-n})| \leqslant \psi(2^n) |\hat{f}_{t_n}'(i2^{-n})|.$$

This implies the first assertion. The remaining ones, which do not require V_t to be weakly Hölder-$\frac{1}{2}$, follow from (2.5). □

2.3. Harmonic measure at the tip

We will now discuss harmonic measure giving two non-equivalent definitions, one that is standard and one which is more directly related to the multifractal spectrum we have discussed.

In this subsection γ denotes a curve in $\mathbb{H}$ with one endpoint on the real line. We assume that the curve comes from a Loewner chain driven by a continuous function V_t, so it may have double points but it does not cross itself. Let H_t be the unbounded connected component of $\mathbb{H}\setminus\gamma([0,t])$. As before, we write $g_t: H_t \to \mathbb{H}$ for the normalized conformal mapping so that $\lim_{y\to 0^+} \hat{f}_t(iy) = \gamma(t)$, where $f_t = g_t^{-1}$ and $\hat{f}_t(z) = f_t(z+V_t)$. If the curve has double points, we are interpreting $\gamma(t)$ in terms of prime ends, and we then tacitly understand $\gamma(t)$ as the prime end corresponding to V_t.

If $z \in H_t$, then $\mathrm{hm}_{t,z}$ will denote the usual harmonic measure of $\mathbb{R}\cup\gamma((0,t])$ from z, that is, the hitting measure of Brownian motion starting at z stopped when it reaches ∂H_t. We let

$$\mathrm{hm}_t(U) = \lim_{y\to\infty} y\, \mathrm{hm}_{t,iy}(U),$$

which is the normalized harmonic measure from the boundary point at infinity. Note that for each $z \in H_t$, hm_t and $\mathrm{hm}_{t,z}$ are mutually absolutely continuous. Also, conformal invariance, the normalization at infinity, and the well-known Poisson formula in $\mathbb{H}$ together show that, for bounded U,

$$\mathrm{hm}_t(U) = \frac{1}{\pi} \mathrm{length}(g_t(U)).$$

Let

$$\tilde{\mu}(t,\varepsilon) = \mathrm{hm}_t[\overline{\mathcal{B}}(\gamma(t),\varepsilon)],$$

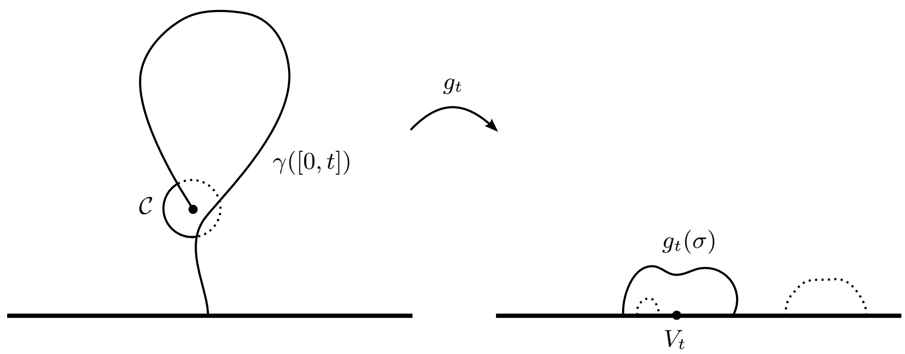

Figure 2. The image of $\partial\mathcal{B}(\gamma(t),\varepsilon)$ can have many components. The crosscut $g_t(\sigma)$ separates the interval $[x_-,x_+]\ni V_t$ from ∞ in $\mathbb{H}$ and the (normalized) harmonic measure $\mu(t,\varepsilon)$ equals $(x_+-x_-)/\pi$. By conformal invariance, $\mu(t,\varepsilon)$ equals the harmonic measure of the part of ∂H_t separated from ∞ by σ in H_t.

where $\mathcal{B}(z,\varepsilon)$ denotes the open disk of radius ε about z with closure $\overline{\mathcal{B}}(z,\varepsilon)$. For $\alpha>0$, define

$$\widetilde{\Theta}_\alpha^{\mathrm{hm}}=\{t\in(0,2]:\tilde{\mu}(t,\varepsilon)\approx^*\varepsilon^\alpha \text{ as } \varepsilon\to 0^+\}.$$

We define the multifractal spectrum of harmonic measure at the tip by

$$\alpha\longmapsto \dim_{\mathrm{H}}[\gamma(\widetilde{\Theta}_\alpha^{\mathrm{hm}})].$$

This multifractal spectrum can be hard to compute. One of the difficulties is that $\mathcal{B}(\gamma(t),\varepsilon)\cap H_t$ can contain many connected components whose images under g_t are far apart. We will give a different definition that is more directly related to the tip multifractal spectrum in this paper.

Fix $t>0$ and $\varepsilon>0$, and let $\mathcal{B}=\mathcal{B}(\gamma(t),\varepsilon)$. Let $O=O_{t,\varepsilon}$ denote the connected component of $\mathcal{B}\cap H_t$ that contains $\gamma(t)$ (considered as a prime end) on its boundary. Let $\mathcal{C}$ be the collection of connected components σ' of $\partial\mathcal{B}\cap H_t$ that is in ∂O and such that σ' separates (the prime end) $\gamma(t)$ from infinity in H_t, that is, every curve from $\gamma(t)$ to infinity in H_t passes through σ'. We let $\sigma=\sigma_\varepsilon$ be the unique member of $\mathcal{C}$ that separates all other elements of $\mathcal{C}$ from infinity in H_t. Let $E=E_{t,\varepsilon}$ be the part of ∂H_t separated from infinity by the crosscut σ. Note that E is connected. We will be interested in the decay rate of the harmonic measure of E as $\varepsilon\to0^+$. Let

$$x_-=x_{-,t,\varepsilon}<V_t<x_+=x_{+,t,\varepsilon}$$

denote the images of the endpoints of σ under g_t. (Since g_t maps onto the "nice" domain $\mathbb{H}$, these points always exist; see, e.g., [19, Chapter 2].) In other words, E is the preimage of the interval $[x_-,x_+]$ under g_t, and we define

$$\mu(t,\varepsilon)=\mathrm{hm}_t(E)=\frac{x_+-x_-}{\pi}.$$

It is not necessarily true that $E \subset \overline{\mathcal{B}}(\gamma(t),\varepsilon)$; see Figure 2. However, an estimate using the Beurling projection theorem shows that there is a $c<\infty$ such that

$$\mu\big(t, \tfrac{1}{2}\varepsilon\big) \leqslant c\tilde{\mu}(t,\varepsilon). \tag{2.19}$$

We define

$$\Theta_\alpha^{\mathrm{hm}} = \{t \in (0,2] : \mu(t,\varepsilon) \approx^* \varepsilon^\alpha \text{ as } \varepsilon \to 0^+\}.$$

The next lemma makes the connection with the tip multifractal spectrum.

Lemma 2.9. *If* $\frac{1}{2} \leqslant \alpha < \infty$, *then*

$$\Theta_\alpha^{\mathrm{hm}} = \Theta_{1-1/\alpha}.$$

Proof. We will prove that there exist $0<c_1,c_2<\infty$ such that for all $t \geqslant 0$ and all sufficiently small $\varepsilon>0$, one has

$$\mu(t, 2v_t(\varepsilon)) \geqslant c_1\varepsilon, \tag{2.20}$$

$$\mu(t,\varepsilon)|\hat{f}_t'(i\mu(t,\varepsilon))| \leqslant c_2\varepsilon. \tag{2.21}$$

The lemma follows immediately from these estimates combined with Proposition 2.7.

Let η_ε denote the line segment $(0, i\varepsilon]$. The harmonic measure from infinity of η_ε in $\mathbb{H}\setminus\eta_\varepsilon$ equals $c_1\varepsilon$ for a specific constant c_1, and hence by conformal invariance the harmonic measure from infinity of $\eta_\varepsilon^* := \hat{f}_t \circ \eta_\varepsilon$ in $H_t \setminus \eta_\varepsilon^*$ is also $c_1\varepsilon$. Since η_ε^* is a curve of length $v_t(\varepsilon)$ and one of its endpoints is $\gamma(t)$, the interior of η_ε^* is contained in $O_{t,v_t(\varepsilon)}$. From this and a Beurling estimate as in (2.19), we get (2.20).

It remains to prove (2.21). To this end, let $\sigma_\varepsilon = \sigma_{\varepsilon,t}$ be the open arc whose endpoints are mapped to $x_{-,\varepsilon} < x_{+,\varepsilon}$ as above. Let $\ell_\varepsilon = x_{+,\varepsilon} - x_{-,\varepsilon}$ and note that $\mu(t,\varepsilon) = \ell_\varepsilon/\pi$. Set $y_\varepsilon = \ell_\varepsilon$ and $z_\varepsilon = V_t + iy_\varepsilon$. By the distortion theorem it suffices to show that $y_\varepsilon|\hat{f}_t'(iy_\varepsilon)| \leqslant c\varepsilon$. Recall that $g_t(\sigma_\varepsilon)$ is a crosscut of $\mathbb{H}$ connecting $x_{-,\varepsilon}$ with $x_{+,\varepsilon}$. Since $y_\varepsilon = \ell_\varepsilon$, there is an absolute constant $c_2>0$ such that harmonic measure of $g_t(\sigma_\varepsilon)$ from z_ε in $\mathbb{H}\setminus g_t(\sigma_\varepsilon)$ is at least c_2. By conformal invariance, this is also true for the harmonic measure of σ_ε from $\hat{f}_t(iy_\varepsilon)$ in $H_t \setminus \sigma_\varepsilon$. By the distortion theorem and the Koebe-$\frac{1}{4}$ theorem, we know that $\operatorname{dist}(\hat{f}_t(iy_\varepsilon), \partial H_t) \asymp y_\varepsilon |\hat{f}_t'(iy_\varepsilon)|$. Note also that

$$\operatorname{dist}(\hat{f}_t(iy_\varepsilon), \partial H_t) \leqslant \operatorname{dist}(\hat{f}_t(iy_\varepsilon), \partial H_t \cup \sigma_\varepsilon) + 2\varepsilon,$$

since σ_ε is a crosscut of H_t of diameter at most 2ε. The needed estimate then comes from the Beurling estimate which implies that in any simply connected domain D, if $V \subset \partial D$, then

$$\mathrm{hm}_D(z,V) \leqslant c\sqrt{\frac{\operatorname{diam}(V)}{\operatorname{dist}(z,\partial D)}},$$

and this completes the proof. $\square$

3. Tip spectrum for SLE

Let $\varkappa>0$ and $a=2/\varkappa$. Then the chordal Schramm–Loewner evolution with parameter $\varkappa$ ($\mathrm{SLE}_\varkappa$) is the solution to the Loewner equation (2.2) with $a=2/\varkappa$, where V_t is a standard Brownian motion. It is well known that, with probability 1, V_t is weakly Hölder-$\frac{1}{2}$. Let

$$d=\min\left\{1+\tfrac{1}{8}\varkappa,2\right\}.$$

It was proved by Beffara [1] that d is the Hausdorff dimension of the path $\gamma([0,2])$. This will follow as a particular case of our main theorem, so we will not need to assume this result. However, it is convenient to use this notation.

3.1. Main theorem

Before stating the main theorem, we will define some special values of the parameter β. See §3.4 for more details. Let

$$\varrho(\beta)=\frac{\varkappa}{8(\beta+1)}\left[\left(\frac{\varkappa+4}{\varkappa}\right)(\beta+1)-1\right]^2, \tag{3.1}$$

and define

$$\hat{d}_\beta=\frac{2-\varrho(\beta)}{2}\quad\text{and}\quad d_\beta=\frac{2\hat{d}_\beta}{1-\beta}=\frac{2-\varrho(\beta)}{1-\beta}.$$

The maximum value of $\hat{d}_\beta$ equals 1 and is obtained at

$$\beta_\#:=\frac{\varkappa}{\varkappa+4}-1.$$

The maximum value of d_β equals d and is obtained at

$$\beta_*:=\frac{\varkappa}{\max\{4,\varkappa-4\}}-1.$$

We define $\beta_-\leqslant\beta_\#\leqslant\beta_*\leqslant\beta_+$ by $\varrho(\beta_-)=\varrho(\beta_+)=2$. A straightforward computation gives

$$\beta_+=-1+\frac{\varkappa}{12+\varkappa-4\sqrt{8+\varkappa}}, \tag{3.2}$$

$$\beta_-=-1+\frac{\varkappa}{12+\varkappa+4\sqrt{8+\varkappa}}<0. \tag{3.3}$$

Also $-1<\beta_-<\beta_+\leqslant 1$, with equality only for $\varkappa=8$.

Remark. The function $\beta_+(\varkappa)$ determines the optimal Hölder exponent for the $\mathrm{SLE}_\varkappa$ path in the capacity parametrization: with probability 1, the chordal $\mathrm{SLE}_\varkappa$ path away from the base is Hölder-α for $\alpha<\frac{1}{2}(1-\beta_+)$ and not Hölder-α for $\alpha>\frac{1}{2}(1-\beta_+)$. See [5, Theorem 1.1].

We recall from §2.1 that

$$\begin{aligned}
\Theta_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \approx^* y^{1-\beta}\},\\
\overline{\Theta}_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \succcurlyeq_{\text{i.o.}} y^{1-\beta}\},\\
\widetilde{\Theta}_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \preccurlyeq y^{1-\beta}\},\\
\underline{\Theta}_\beta &= \{t\in(0,2]: y|\hat{f}_t'(iy)| \preccurlyeq_{\text{i.o.}} y^{1-\beta}\},\\
\underline{\Theta}^*_\beta &= \{t\in(0,2]: v_t(y) \preccurlyeq_{\text{i.o.}} y^{1-\beta}\},
\end{aligned}$$

where the asymptotics are as $y\to0^+$. We can now state our main result.

THEOREM 3.1. *For chordal* $\mathrm{SLE}_\varkappa$, *if* $-1\leqslant\beta\leqslant1$, *the following facts hold with probability* 1:

- *If* $\beta_-\leqslant\beta\leqslant\beta_+$, *then*

$$\dim_{\mathrm{H}}(\Theta_\beta)=\hat{d}_\beta \quad \textit{and} \quad \dim_{\mathrm{H}}[\gamma(\Theta_\beta)]=d_\beta. \tag{3.4}$$

- *If* $\beta_\#\leqslant\beta\leqslant\beta_+$, *then*

$$\dim_{\mathrm{H}}(\overline{\Theta}_\beta)=\hat{d}_\beta. \tag{3.5}$$

- *If* $\beta_*\leqslant\beta\leqslant\beta_+$, *then*

$$\dim_{\mathrm{H}}[\gamma(\overline{\Theta}_\beta)]=d_\beta. \tag{3.6}$$

- *If* $\beta_-\leqslant\beta\leqslant\beta_\#$, *then*

$$\dim_{\mathrm{H}}(\underline{\Theta}_\beta)=\hat{d}_\beta. \tag{3.7}$$

- *If* $\beta_-\leqslant\beta\leqslant\beta_*$, *then*

$$\dim_{\mathrm{H}}[\gamma(\underline{\Theta}^*_\beta)]=d_\beta. \tag{3.8}$$

- *If* $\beta>\beta_+$, *then*

$$\overline{\Theta}_\beta=\varnothing.$$

- *If* $\beta<\beta_-$, *then*

$$\underline{\Theta}_\beta=\varnothing.$$

3.2. Remarks

- It follows from the theorem that, with probability 1, the results hold for a dense set of β. This implies that, with probability 1, (3.5)–(3.8) hold for all β. However, we have not shown whether or not for a particular realization there might be an exceptional β for which (3.4) does not hold.

• The restriction to $t\in(0,2]$ is only a convenience. By scaling we get a similar result for $t\in(0,\infty)$.

• The relationship $\dim_{\mathrm{H}}[\gamma(\Theta_\beta)]=2\dim_{\mathrm{H}}(\Theta_\beta)/(1-\beta)$ can be understood as follows. For s small, the image of the interval $[t,t+s^2]$ under $\hat{f}_t$ can be approximated by a set of diameter $s|\hat{f}_t'(is)|$ containing $\hat{f}_t(is)$. If $|\hat{f}_t'(is)|\approx s^{-\beta}$, then this set has diameter $s^{1-\beta}$. That is to say, intervals of length (diameter) s^2 in a covering of Θ_β are sent to sets of diameter $s^{1-\beta}$. Note that this is in contrast to complex Brownian motion where intervals of length s^2 are always sent to sets whose diameter is of order s.

• Since $\Theta_\beta\subset\overline{\Theta}_\beta\cap\widetilde{\Theta}_\beta\cap\underline{\Theta}^*_\beta$ and $\underline{\Theta}^*_\beta\subset\underline{\Theta}_\beta$, it suffices to prove the lower bounds for Θ_β in (3.4) and the upper bounds for $\overline{\Theta}_\beta$, $\underline{\Theta}_\beta$ and $\underline{\Theta}^*_\beta$ in (3.5)–(3.8). The upper bounds will be proved in §4.1 and the lower bounds in §4.2.

• To prove the upper bound (3.5) it suffices to show that, for each $s>0$,

$$\dim_{\mathrm{H}}(\overline{\Theta}_\beta\cap(s,2])\leqslant\hat{d}_\beta,$$

and similarly for (3.6)–(3.8). This is what we do in §4.1.

• Recall that $\underline{\Theta}^*_\beta\subset\underline{\Theta}_\beta$. It is open whether or not

$$\dim_{\mathrm{H}}[\gamma(\underline{\Theta}_\beta)]\leqslant d_\beta.$$

• Note that $(0,2]=\overline{\Theta}_{\beta_*}\cup\underline{\Theta}^*_{\beta_*}$. It follows that

$$\dim_{\mathrm{H}}(\gamma((0,2]))=d_{\beta_*}=d.$$

Hence, Beffara's theorem on the dimension of the path [1], [9] is a particular case of the theorem.

• The statements about the dimension of $\gamma(\Theta_\beta)$, $\gamma(\overline{\Theta}_\beta)$ and $\gamma(\underline{\Theta}^*_\beta)$ are independent of the parametrization of the curve.

• Using the Markov property for SLE it is not hard to show that, with probability 1, either Θ_β is dense in $(0,\infty)$ or it is empty. Also, $\dim_{\mathrm{H}}[\gamma(\Theta_\beta\cap[t_1,t_2])]$ is the same for all $0<t_1<t_2\leqslant2$. In particular, in order to prove the lower bound on dimension, it suffices to prove that, for all $\alpha<d_\beta$,

$$\mathbb{P}\{\dim_{\mathrm{H}}[\gamma(\Theta_\beta\cap[1,2])]\geqslant\alpha\}>0.$$

This is what we will do in §4.2. The proof proves the slightly stronger (for $\varkappa>4$) result

$$\mathbb{P}\{\dim_{\mathrm{H}}[\mathbb{H}\cap\gamma(\Theta_\beta\cap[1,2])]\geqslant\alpha\}>0.$$

• If $\varkappa=8$, we have $\beta_*=\beta_+=1$ and $\dim_{\mathrm{H}}[\gamma(\Theta_1)]=2$. This is related to the fact that this is the hardest case to establish the existence of the curve; the curve is almost surely not Hölder continuous (in the capacity parametrization) when $\varkappa=8$ [5]. For other values of $\varkappa$, we have $\beta_*<\beta_+<1$.

3.3. Multifractal spectrum of harmonic measure

Let $\Theta_\alpha^{\mathrm{hm}}$ be defined as in §2.3. Let

$$F_{\mathrm{tip}}(\alpha):=d_{1-1/\alpha}=\alpha\left(1-\frac{4}{\varkappa}\right)+\frac{(4+\varkappa)^2}{8\varkappa}-\frac{\varkappa}{8}\left(\frac{\alpha^2}{2\alpha-1}\right),$$

and let α_-, α_* and α_+ correspond to β_-, β_* and β_+, respectively, through the relation

$$\alpha=\frac{1}{1-\beta}.$$

Remark. We can compare the function F_{tip} with the conjectured almost sure bulk spectrum for $\mathrm{SLE}_\varkappa$ given by

$$F_{\mathrm{bulk}}(\alpha)=\alpha+\frac{(4+\varkappa)^2}{8\varkappa}-\frac{(4+\varkappa)^2}{8\varkappa}\left(\frac{\alpha^2}{2\alpha-1}\right).$$

THEOREM 3.2. *Suppose that* $\alpha_-\leqslant\alpha\leqslant\alpha_+$. *For chordal* $\mathrm{SLE}_\varkappa$, *with probability* 1,

$$\dim_{\mathrm{H}}[\gamma(\Theta_\alpha^{\mathrm{hm}})]=F_{\mathrm{tip}}(\alpha).$$

Proof. This is an immediate corollary of Theorem 3.1 and Lemma 2.9. □

Theorem 3.2 combined with (2.19) gives some information on $\widetilde{\Theta}_\alpha^{\mathrm{hm}}$. In §6, we will use the forward Loewner flow to give a proof of the following result.

THEOREM 3.3. *If* $0<\varkappa<8$ *and* $\frac{1}{2}\leqslant\alpha\leqslant\alpha_*$, *then, with probability* 1, *there exists a set* V *such that* $\dim_{\mathrm{H}}[\gamma(V)]\leqslant F_{\mathrm{tip}}(\alpha)$ *and for* $t\notin V$, $\gamma(t)\in\mathbb{H}$,

$$\tilde{\mu}(t,2^{-n})\preccurlyeq 2^{-n\alpha},\quad n\to\infty. \tag{3.9}$$

Let

$$\widetilde{T}_\alpha^{\mathrm{hm}}=\{t\in(0,2]:\tilde{\mu}(t,2^{-n})\succcurlyeq 2^{-\alpha n}\text{ and }\gamma(t)\in\mathbb{H}\}$$

and note that Theorem 3.3 combined with (2.19) and Theorem 3.2 implies that, for each $\alpha_-\leqslant\alpha<\alpha_*$, with probability 1,

$$\dim_{\mathrm{H}}[\gamma(\widetilde{T}_\alpha^{\mathrm{hm}})]=F_{\mathrm{tip}}(\alpha).$$

Indeed, it follows directly from (2.19) and Theorem 3.2 that the lower bound on the dimension holds with probability 1. To get the upper bound, notice that $\widetilde{T}_\alpha^{\mathrm{hm}}$ is contained in $\{t\in(0,2]:\tilde{\mu}(t,2^{-n})\succcurlyeq_{\mathrm{i.o.}}2^{-n\alpha}\}$, which, for those t such that $\gamma(t)\in\mathbb{H}$, in turn is contained in the set V from Theorem 3.3.

3.4. Parameters

In the statement of the main theorem, β and ϱ were the parameters used. However, in deriving the result, it is useful to consider a number of other parameters. Let

$$r_* = \min\left\{1, \frac{8}{\varkappa}\right\} \quad \text{and} \quad r_c = \frac{1}{2} + \frac{4}{\varkappa},$$

and note that

$$0 < r_* \leqslant r_c,$$

where the second inequality is strict unless $\varkappa=8$. Let $r<r_c$; we define λ, ζ, β and ϱ as functions of r.

Let

$$\lambda = \lambda(r) = r\left(1+\tfrac{1}{4}\varkappa\right) - \tfrac{1}{8}\varkappa r^2. \tag{3.10}$$

We write $\lambda_*=\lambda(r_*)$, and similarly for other parameters. As r increases from $-\infty$ to r_c, λ increases from $-\infty$ to

$$\lambda_c = 1 + \frac{3\varkappa}{32} + \frac{2}{\varkappa}.$$

Since the relationship is injective, we can write either $\lambda(r)$ or $r(\lambda)$. Solving the quadratic equation gives

$$r(\lambda) = \frac{4+\varkappa-\sqrt{(4+\varkappa)^2-8\lambda\varkappa}}{\varkappa}.$$

Also,

$$\lambda(0) = 0 \quad \text{and} \quad \lambda_* = d.$$

Let

$$\zeta = \zeta(r) = r - \tfrac{1}{8}\varkappa r^2 = \lambda(r) - \tfrac{1}{4}\varkappa r, \tag{3.11}$$

and note that

$$\zeta_* = 2-d.$$

We can write ζ as a function of λ,

$$\zeta(\lambda) = \lambda + \frac{\sqrt{(4+\varkappa)^2-8\lambda\varkappa}-4-\varkappa}{4}.$$

We now briefly discuss some results from [9] and [5]. The reverse-time $\mathrm{SLE}_\varkappa$ Loewner flow h_t (see §5.2 for definitions) has the property that, for fixed t, the distribution of $|h_t'(z)|$ is the same as that of $|\hat{f}_t'(z)|$. Let

$$Z_t = X_t + iY_t = h_t(i) - V_t.$$

Then, if $r\in\mathbb{R}$ and λ and ζ are defined as above, we have that

$$|h_t'(z)|^{\lambda} Y_t(z)^{\zeta}[\sin\arg Z_t(z)]^{-r}$$

is a martingale. Typically one expects $Y_t(i)\asymp\sqrt{t}$ and $\sin\arg Z_t(i)\asymp 1$. If this is true, then the martingale property would imply that

$$\mathbb{E}[|\hat{f}_{t^2}'(i)|^{\lambda}] = \mathbb{E}[|h_{t^2}'(i)|^{\lambda}] \asymp t^{-\zeta}.$$

It turns out that this argument can be carried out if $r<r_c$, and this is the starting point for determining the multifractal spectrum.

We define $\beta=\beta(r)$ by the relation

$$\frac{d\zeta}{d\lambda} = -\beta.$$

A straightforward calculation gives

$$\beta(r) = -1+\frac{\varkappa}{4+\varkappa-\varkappa r} \quad \text{and} \quad \varkappa r(\beta) = 4+\varkappa-\frac{\varkappa}{\beta+1}.$$

Note that β increases with r with

$$\beta(-\infty) = -1, \quad \beta(0) = -\frac{4}{4+\varkappa} = \beta_{\#}, \quad \beta(r_*) = \beta_* \quad \text{and} \quad \beta_c = 1,$$

where $\beta_{\#}$ and β_* are as defined in the previous section. Roughly speaking, $\mathbb{E}[|\hat{f}_{t^2}'(i)|^{\lambda}]$ is carried on an event on which $|\hat{f}_{t^2}'(i)|\approx t^{\beta}$ and

$$\mathbb{P}\{|\hat{f}_{t^2}'(i)| \approx t^{\beta}\} \approx t^{-(\zeta+\lambda\beta)}. \tag{3.12}$$

We emphasize that the relation between r, λ and β for $-\infty<r<r_c$ is bijective, and in order to specify the values of the parameters it suffices to give the value of any one of these. For example, we could choose β as the independent variable and write $r(\beta)$ and $\lambda(\beta)$. This is the natural approach when proving Theorem 3.1, but the formulas tend to be somewhat simpler if we choose r to be the independent variable.

From (3.12), it is natural to define

$$\varrho = \varrho(r) = \zeta(r)+\lambda(r)\beta(r) = \frac{\varkappa^2 r^2}{8(4+\varkappa-\varkappa r)}.$$

We can also write ϱ as a function of β and a computation gives (3.1). Note that

$$\frac{d\varrho}{d\beta} = \frac{d\zeta}{d\lambda}\frac{d\lambda}{d\beta}+\lambda+\beta\frac{d\lambda}{d\beta} = \lambda.$$

Let r_+ and r_- denote the two values of r for which $\zeta(r)+\lambda(r)\beta(r)=2$, with corresponding values $\beta_+=\beta(r_+)$ and $\beta_-=\beta(r_-)$. Then

$$r_\pm = \frac{4}{\varkappa}(-2\pm\sqrt{8+\varkappa}),$$

and β_+ and β_- are given as in (3.2) and (3.3). Note that, if $\varkappa\neq 8$, then $r_+<r_c$.

Define

$$d_\beta = \frac{2-\varrho(\beta)}{1-\beta} = \frac{2-(\zeta+\beta\lambda)}{1-\beta}.$$

Note that d_β is maximized at $\beta=\beta_*$ (interpreted as a limit for $\varkappa=8$) with $d_*=d$. We can also define d as a function of r, that is

$$d(r) = 1+\frac{\varkappa-\frac{1}{8}\varkappa^2r^2}{8+\varkappa-2\varkappa r}.$$

Straightforward differentiation shows that $d'(r)=0$ implies $r=1$ or $r=8/\varkappa$. Note that $1=8/\varkappa=r_+$ if $\varkappa=8$ and

$$\begin{aligned} 1<r_+<\frac{8}{\varkappa}, &\quad \text{if } \varkappa<8,\\ \frac{8}{\varkappa}<r_+<1, &\quad \text{if } \varkappa>8. \end{aligned}$$

From this we can see that $d(r)$ achieves its maximum on $(-\infty, r_+)$ at $r=r_*$; in fact, $d(\beta)$ increases for $\beta<\beta_*$ and decreases for $\beta_*<\beta<\beta_+$.

In order to match the notation of [9], let

$$q = r_c - r = \frac{1}{2}+\frac{4}{\varkappa}-r. \tag{3.13}$$

Obviously, $q>0$ if and only if $r<r_c$. For future reference, we note that

$$\frac{1-2q}{1+2q} = \beta. \tag{3.14}$$

4. Proof of the main theorem

In this section we will present the proof of the theorem relying on estimates about moments of derivatives of the map $\hat{f}$. The upper bounds are proved in §4.1, and the lower bound is proved in §4.2.

4.1. Upper bounds

In this subsection (and this subsection only) we write

$$\hat{f}_{j,n} = \hat{f}_{(j-1)2^{-2n}}.$$

For each $t \in [0,\infty)$, we associate a dyadic time by defining

$$t_n = t_n(t) = \frac{j-1}{2^{2n}}, \quad \text{if } \frac{j-1}{2^{2n}} \leqslant t < \frac{j}{2^{2n}}.$$

We fix s with $0 < s \leqslant 2$ and allow constants to depend on s.

The next theorem states the derivative estimates that we will use for the upper bounds; a proof can be found in [5].

THEOREM 4.1. ([5]) *If $r < r_c$, there exists $c < \infty$ such that, for all $t \geqslant 1$,*

$$\mathbb{E}[|\hat{f}'_{t^2}(i)|^\lambda] \leqslant ct^{-\zeta}. \tag{4.1}$$

COROLLARY 4.2. *If $\beta \geqslant \beta_\#$, there is a $c < \infty$ such that, if*

$$N_{n,\beta} = \sum_{s2^{2n} \leqslant j \leqslant 2^{2n+1}} 1_{\{|\hat{f}'_{j,n}(i2^{-n})| \geqslant 2^{n\beta}\}},$$

then

$$\mathbb{E}[N_{n,\beta}] \leqslant c2^{n(2-\varrho)}. \tag{4.2}$$

Proof. The range $\beta \geqslant \beta_\#$ corresponds to $\lambda \geqslant 0$. Hence, by Chebyshev's inequality,

$$\mathbb{P}\{|\hat{f}'_{j,n}(i2^{-n})| \geqslant 2^{n\beta}\} \leqslant 2^{-n\beta\lambda}\mathbb{E}[|\hat{f}'_{j,n}(i2^{-n})|^\lambda] = 2^{-n\beta\lambda}\mathbb{E}[|\hat{f}'_j(i)|^\lambda] \leqslant c2^{-n\beta\lambda}j^{-\zeta/2},$$

and hence

$$\begin{aligned}\mathbb{E}[N_{n,\beta}] &= \sum_{s2^{2n} \leqslant j \leqslant 2^{2n+1}} \mathbb{P}\{|\hat{f}'_{j,n}(2^{-n})| \geqslant 2^{n\beta}\} \\ &\leqslant 2^{-n\beta\lambda} \sum_{s2^{2n} \leqslant j \leqslant 2^{2n+1}} j^{-\zeta/2} \leqslant c2^{n(2-\beta\lambda-\zeta)} = c2^{n(2-\varrho)}.\end{aligned}$$

□

COROLLARY 4.3. *If $\beta < \beta_\#$, there is a $c < \infty$ such that, if*

$$N^*_{n,\beta} = \sum_{s2^{2n} \leqslant j \leqslant 2^{2n+1}} 1_{\{|\hat{f}'_{j,n}(i2^{-n})| \leqslant 2^{n\beta}\}},$$

then

$$\mathbb{E}[N^*_{n,\beta}] \leqslant c2^{n(2-\varrho)}. \tag{4.3}$$

Proof. This is proved in the same way using $\lambda<0$. □

The standard technique to find upper bounds for Hausdorff dimension uses an appropriate sequence of covers for a set. We will now describe the covers that we will use. Let

$$I(j,n)=\left[\frac{j-1}{2^{2n}},\frac{j}{2^{2n}}\right].$$

If $b,\bar{b}\in\mathbb{R}$ with $-1<b<\bar{b}<1$, let $\overline{\mathcal{B}}(j,n,\bar{b})$ be the closed disk in $\mathbb{C}$ of radius $2^{n(\bar{b}-1)}$ centered at $\hat{f}_{j,n}(i2^{-n})$, and let

$$I_n(s,b)=\bigcup_j I(j,n)\quad\text{and}\quad \mathcal{B}_n(s,b,\bar{b})=\bigcup_j \overline{\mathcal{B}}(j,n,\bar{b}),$$

where in each case the union is over $s2^{2n}\leqslant j\leqslant 2^{2n+1}$ with $|\hat{f}'_{j,n}(i2^{-n})|\geqslant 2^{nb}$. Let

$$I^m(s,b)=\bigcup_{n=m}^{\infty} I_n(s,b)\quad\text{and}\quad \mathcal{B}^m(s,b,\bar{b})=\bigcup_{n=m}^{\infty}\mathcal{B}_n(s,b,\bar{b}).$$

LEMMA 4.4. *If* $-1<b<\beta<b_1<\bar{b}<1$ *then, for each* m,

$$\overline{\Theta}_\beta\cap(s,2]\subset I^m(s,b)\quad\textit{and}\quad \gamma(\overline{\Theta}_\beta\cap\widetilde{\Theta}_{b_1}\cap(s,2])\subset\mathcal{B}^m(s,b,\bar{b}).$$

Proof. Suppose that $t\in\overline{\Theta}_\beta\cap(s,2]$. By Proposition 2.8, there exists a subsequence $n_j\to\infty$ such that

$$|f'_{t_{n_j}}(i2^{-n_j})|\geqslant 2^{n_j b}.$$

In other words, there is a sequence n_j such that $I(t_{n_j}2^{2n_j},n_j)\in I_{n_j}(s,b)$. This proves the first assertion.

If $t\in\overline{\Theta}_\beta\cap\widetilde{\Theta}_{b_1}\cap(s,2]$ and $b_1<u<\bar{b}$, then (2.18) shows that, for all sufficiently large n,

$$|\gamma(t)-\hat{f}_t(i2^{-n})|\leqslant 2^{(u-1)n}.$$

The triangle inequality gives

$$|\gamma(t)-\hat{f}_{t_n}(i2^{-n})|\leqslant|\gamma(t)-\hat{f}_t(i2^{-n})|+|\hat{f}_t(i2^{-n})-\hat{f}_{t_n}(i2^{-n})|,$$

and estimating as in (2.16), we have, for n sufficiently large,

$$|\hat{f}_t(i2^{-n})-\hat{f}_{t_n}(i2^{-n})|\leqslant 2^{(u-1)n}.$$

Hence, for n sufficiently large,

$$|\gamma(t)-\hat{f}_{t_n}(i2^{-n})|\leqslant 2^{n_j(\bar{b}-1)}.$$

This implies that, for all sufficiently large j,

$$\gamma(t)\in\mathcal{B}(t_{n_j}2^{2n_j},b,\bar{b}).$$ □

Proposition 4.5. *If* $\beta \geqslant \beta_{\#}$, *then, with probability* 1,

$$\dim_{\mathrm{H}}(\overline{\Theta}_\beta \cap (s,2]) \leqslant \hat{d}_\beta. \tag{4.4}$$

Moreover, if $\beta > \beta_+$, *then, with probability* 1,

$$\overline{\Theta}_\beta \cap (s,2] = \varnothing.$$

Proof. It suffices to consider $\beta_{\#} < \beta < 1$. Suppose that $\beta_{\#} < b < \beta < 1$. Using the cover from Lemma 4.4, we get

$$\mathcal{H}^\alpha(\overline{\Theta}_\beta \cap (s,2]) \leqslant \sum_{n=m}^{\infty} N_{n,b} 2^{-2\alpha n},$$

and hence (4.2) implies that

$$\mathbb{E}[\mathcal{H}^\alpha(\overline{\Theta}_\beta \cap (s,2])] \leqslant c \sum_{n=m}^{\infty} 2^{n(2-\varrho(b))} 2^{-2\alpha n}.$$

The sum goes to zero, provided $2\alpha > 2 - \varrho(b)$, and hence, with probability 1,

$$\mathcal{H}^\alpha(\overline{\Theta}_\beta \cap (s,2]) = 0, \quad \alpha > 1 - \tfrac{1}{2}\varrho(b).$$

Letting $b \to \beta$ gives (4.4).

For the second assertion, note that

$$\mathbb{P}\{\overline{\Theta}_\beta \cap (s,2] \neq \varnothing\} \leqslant \sum_{n=m}^{\infty} \mathbb{E}[N_{n,b}] \leqslant c \sum_{n=m}^{\infty} 2^{n(2-\varrho(b))}.$$

If $\beta > \beta_+$, then $\varrho(\beta) > 2$ and we can find $b < \beta$ with $\varrho(b) > 2$. □

Lemma 4.6. *If* $\beta_{\#} \leqslant \beta < b_1 < 1$, *then, with probability* 1,

$$\dim_{\mathrm{H}}[\gamma(\overline{\Theta}_\beta \cap \widetilde{\Theta}_{b_1} \cap (s,2])] \leqslant \frac{2-\varrho(\beta)}{1-b_1}. \tag{4.5}$$

Proof. Choose b and $\bar{b}$ with $\beta_{\#} < b < \beta < b_1 < \bar{b} < 1$. Using the cover from Lemma 4.4, we get

$$\mathcal{H}^\alpha[\gamma(\overline{\Theta}_\beta \cap \widetilde{\Theta}_{b_1} \cap (s,2])] \leqslant \sum_{n=m}^{\infty} N_{n,b} 2^{\alpha(\bar{b}-1)n},$$

and hence (4.2) implies that

$$\mathbb{E}[\mathcal{H}^\alpha(\gamma(\overline{\Theta}_\beta \cap \widetilde{\Theta}_{b_1} \cap (s,2]))] \leqslant c \sum_{n=m}^{\infty} 2^{n(2-\varrho(b))} 2^{\alpha(\bar{b}-1)n}.$$

The sum on the right goes to zero provided

$$\alpha > \frac{2-\varrho(b)}{1-\bar{b}}.$$

We now choose sequences of values for b and $\bar{b}$ that converge to β and b_1, respectively, to conclude (4.5). □

Proposition 4.7. *If $\beta_*\leqslant\beta\leqslant 1$, then, with probability* 1,

$$\dim_{\mathrm{H}}[\gamma(\overline{\Theta}_\beta\cap(s,2])]\leqslant d_\beta. \tag{4.6}$$

Proof. If $\beta=b_0<b_1<b_2<...<b_k<1$ with $b_k>b_+$, then

$$\overline{\Theta}_\beta=\bigcup_{j=1}^{k}(\overline{\Theta}_{b_{j-1}}\cap\widetilde{\Theta}_{b_j}).$$

Therefore, (4.5) implies that

$$\dim_{\mathrm{H}}[\gamma(\overline{\Theta}_\beta\cap(s,2])]\leqslant\max\left\{\frac{2-\varrho(b_{j-1})}{1-b_j}:j=1,...,k\right\}.$$

By taking finer and finer partitions and using the continuity of ϱ, we see that

$$\dim_{\mathrm{H}}[\gamma(\overline{\Theta}_\beta\cap(s,2])]\leqslant\sup_{b\geqslant\beta}\frac{2-\varrho(b)}{1-b}=d_\beta.$$

The last equality uses $\beta\geqslant\beta_*$ and the fact, which can easily be verified (see §3.4), that the function

$$b\longmapsto\frac{2-\varrho(b)}{1-b}$$

is decreasing on the interval $[\beta_*,\beta_+]$. □

For $\beta<\beta_\#$ we use a slightly different cover. Let $I(j,n)$ be as above and let

$$I_n^*(s,b)=\bigcup_j I(j,n)\quad\text{and}\quad\mathcal{B}_n^*(s,b,\bar b)=\bigcup_j\mathcal{B}(j,n,\bar b),$$

where in each case the union is over $s2^{2n}\leqslant j\leqslant 2^{2n+1}$ with $|\hat f'_{j,n}(i2^{-n})|\leqslant 2^{nb}$. Let

$$I_*^m(s,b)=\bigcup_{n=m}^{\infty}I_n^*(s,b)\quad\text{and}\quad\mathcal{B}_*^m(s,b,\bar b)=\bigcup_{n=m}^{\infty}\mathcal{B}_n^*(s,b,\bar b).$$

Lemma 4.8. *If $-1<\beta<b<\bar b$, then*

$$\underline{\Theta}_\beta\cap(s,2]\subset I_*^m(s,b)\quad\textit{and}\quad\gamma(\underline{\Theta}^*_\beta\cap(s,2])\subset\mathcal{B}_*^m(s,b,\bar b).$$

Proof. Suppose that $t\in\underline{\Theta}_\beta\cap(s,2]$. By Proposition 2.8, there exists a subsequence $n_j\to\infty$ such that

$$|\hat f'_{t_{n_j}}(i2^{-n_j})|\leqslant 2^{n_jb}. \tag{4.7}$$

In other words, there is a sequence n_j such that $I(t_{n_j}2^{2n_j}, n_j)\in I^*_{n_j}(s,b)$. This proves the first assertion.

Now suppose that $t\in\underline{\Theta}^*_\beta\cap(s,2]\subset\underline{\Theta}_\beta\cap(s,2]$. Then there exists a sequence n_j such that both (4.7) holds and

$$v_t(2^{-n_j})\leqslant 2^{n_j(b-1)}.$$

Using the triangle inequality as in Lemma 4.4, we see that

$$|\gamma(t)-\hat{f}_{t_{n_j}}(i2^{-n_j})|\leqslant v_t(2^{-n_j})+|\hat{f}_t(i2^{-n_j})-\hat{f}_{t_{n_j}}(i2^{-n_j})|,$$

and arguing as before we see that, for j sufficiently large,

$$|\gamma(t)-\hat{f}_{t_{n_j}}(i2^{-n_j})|\leqslant 2^{n_j(\bar{b}-1)}$$

and

$$\gamma(t)\in\mathcal{B}^*(t_{n_j}2^{2n},b,\bar{b}). \qquad \square$$

PROPOSITION 4.9. *If* $\beta<\beta_\#$, *then, with probability* 1,

$$\dim_{\mathrm{H}}(\underline{\Theta}_\beta\cap(s,2])\leqslant\hat{d}_\beta \quad \textit{and} \quad \dim_{\mathrm{H}}[\gamma(\underline{\Theta}^*_\beta\cap(s,2])]\leqslant d_\beta.$$

Moreover, if $\beta<\beta_-$, *then, with probability* 1,

$$\underline{\Theta}_\beta\cap(s,2]=\varnothing.$$

Proof. This is proved in the same way as Proposition 4.5 and Lemma 4.6 using (4.3). $\square$

4.2. Lower bound

In this subsection we prove the lower bound for the dimension in (3.4). We fix r such that $\varrho=\lambda\beta+\zeta<2$ and recall that $r<r_c$. As has been pointed out, it suffices to show that, with positive probability,

$$\dim_{\mathrm{H}}(\Theta_\beta\cap[1,2])\geqslant\frac{2-\varrho}{2} \quad \text{and} \quad \dim_{\mathrm{H}}[\gamma(\Theta_\beta\cap[1,2])]\geqslant\frac{2-\varrho}{1-\beta}. \tag{4.8}$$

We will use a standard technique of Frostman to show that, with positive probability, there exist non-trivial positive measures μ and ν, whose supports are contained in

$$\Theta_\beta\cap[1,2] \quad \text{and} \quad \gamma(\Theta_\beta\cap[1,2]),$$

respectively, such that

$$\mathcal{E}_\alpha(\mu)<\infty,\ \ \alpha<\frac{2-\varrho}{2},\quad \text{and}\quad \mathcal{E}_\alpha(\nu)<\infty,\ \ \alpha<\frac{2-\varrho}{1-\beta},$$

where

$$\mathcal{E}_\alpha(\mu)=\iint\frac{\mu(dx)\,\mu(dy)}{|x-y|^\alpha}$$

is the energy integral. It is well known that this implies (4.8); see, e.g., [18, Theorem 8.9]. For this subsection, we will adopt a different notation than in the previous subsection. We let

$$\hat{f}_{j,n}=\hat{f}_{1+(j-1)/n^2},\quad j=1,2,...,n^2.$$

We will be studying $|\hat{f}'_{j,n}(i/n)|$. The proof considers a subset of times in $\Theta_\beta\cap[1,2]$ that behave in some sense nicely. The hard work is Theorem 4.10 which will be proved in §5. This theorem discusses the existence of some events $E_{j,n}$ on which

$$|\hat{f}'_{j,n}(iy)|\approx y^{-\beta},\quad n^{-1}\leqslant y\leqslant 1.$$

The definition of the events ("good times") will be left for §5.

Theorem 4.10. *Let $\varrho=\lambda\beta+\zeta<2$. There exist $0<c_1,c_2<\infty$, a subpower function ψ and events*

$$E_{j,n},\quad n=1,2,...,\ \ j=1,...,n^2,$$

such that the following holds. Let $E(j,n)=1_{E_{j,n}}$ and

$$F(j,n)=n^{\zeta-2}\left|\hat{f}'_{j,n}\left(\frac{i}{n}\right)\right|^\lambda E(j,n).$$

- *If $1\leqslant j\leqslant n^2$, then on the event $E_{j,n}$,*

$$\psi\left(\frac{1}{y}\right)^{-1}y^{-\beta}\leqslant|\hat{f}'_{j,n}(iy)|\leqslant\psi\left(\frac{1}{y}\right)y^{-\beta},\quad n^{-1}\leqslant y\leqslant 1. \tag{4.9}$$

- *If $1\leqslant j\leqslant k\leqslant n^2$, then*

$$c_1n^{-2}\leqslant\mathbb{E}[F(j,n)]\leqslant n^{\zeta-2}\mathbb{E}\left[\left|\hat{f}'_{j,n}\left(\frac{i}{n}\right)\right|^\lambda\right]\leqslant\frac{c_2}{n^2}, \tag{4.10}$$

- *If $1\leqslant j\leqslant k\leqslant n^2$, then*

$$\mathbb{E}[F(j,n)F(k,n)]\leqslant n^{-4}\left(\frac{n^2}{k-j+1}\right)^{\varrho/2}\psi\left(\frac{n^2}{k-j+1}\right). \tag{4.11}$$

- *If $1\leqslant j<k\leqslant n^2$ and $E(j,n)E(k,n)=1$, then*

$$\left|\hat{f}_{j,n}\left(\frac{i}{n}\right)-\hat{f}_{k,n}\left(\frac{i}{n}\right)\right|^{-(2-\varrho)(1-\beta)}\leqslant\left(\frac{n^2}{k-j+1}\right)^{(2-\varrho)/2}\psi\left(\frac{n^2}{k-j+1}\right)^{d_\beta}. \tag{4.12}$$

Proof. This theorem combines Propositions 5.8 and 5.9 with Lemmas 5.5 and 5.7 proved in §5. □

PROPOSITION 4.11. *Under the assumptions of Theorem 4.10, with positive probability there exists $A\subset[1,2]$ such that, for $t\in A$,*

$$\tfrac{1}{4}y^{-\beta}\psi(1/y)^{-1}\leqslant|\hat{f}_t'(iy)|\leqslant 4y^{-\beta}\psi(1/y),\quad 0<y\leqslant 1,\tag{4.13}$$

and such that

$$\dim_{\mathrm{H}}(A)\geqslant\frac{2-\varrho}{2}\quad\textit{and}\quad\dim_{\mathrm{H}}[\gamma(A)]\geqslant\frac{2-\varrho}{1-\beta}.$$

Proof assuming Theorem 4.10. We use a now standard argument to show that with positive probability a "Frostman measure" of appropriate dimension can be put on the set of t satisfying (4.13). The proof is very similar to that of [9, Lemma 10.3], so we omit some of the details.

Let $\mu_{j,n}$ denote the random measure on $\mathbb{R}$ that is a multiple of Lebesgue measure on $I(j,n):=[1+(j-1)n^{-2},1+jn^{-2}]$, where the multiple is chosen to that $\|\mu_{j,n}\|=F(j,n)$. Let $\nu_{j,n}$ denote the random measure on $\mathbb{C}$ that is a multiple of Lebesgue measure on the disk of radius $\frac{1}{4}n^{\beta-1}\psi(n^2)^{-1}$ centered at $\hat{f}_{j,n}(i/n)$, where the constant is chosen so that $\|\nu_{j,n}\|=F(j,n)$. Let

$$\mu_n=\sum_{j=1}^{n^2}\mu_{j,n}\quad\text{and}\quad\nu_n=\sum_{j=1}^{n^2}\nu_{j,n}.$$

Note that

$$\|\mu_n\|=\|\nu_n\|=\sum_{j=1}^{n^2}F(j,n).$$

From (4.10) and (4.11), we see that

$$\mathbb{E}[\|\mu_n\|]\geqslant c_1\quad\text{and}\quad\mathbb{E}[\|\mu_n\|^2]\leqslant c_2.$$

Hence, by the Cauchy–Schwarz inequality, there is a constant $c>0$ such that

$$\mathbb{P}\{\|\mu_n\|>0\}\geqslant c>0$$

uniformly in n. In Lemma 4.12 below we show that there is $c_\alpha<\infty$ such that the energy integrals satisfy

$$\mathbb{E}[\mathcal{E}_\alpha(\mu_n)]\leqslant c_\alpha,\;\;\alpha<\frac{2-\varrho}{2},\quad\text{and}\quad\mathbb{E}[\mathcal{E}_\alpha(\nu_n)]\leqslant c_\alpha,\;\;\alpha<\frac{2-\varrho}{1-\beta}.$$

We let μ denote a subsequential limit of the μ_n which with positive probability we know is non-trivial and satisfies $\mathcal{E}_\alpha(\mu)<\infty$ for all $\alpha<\frac{1}{2}(2-\varrho)$. Hence,

$$\dim_{\mathrm{H}}(\operatorname{supp}\mu)\geqslant\tfrac{1}{2}(2-\varrho).$$

Similarly, let ν denote a subsequential limit of the ν_n which is non-zero with positive probability and satisfies

$$\dim_{\mathrm{H}}(\operatorname{supp}\nu)\geqslant\frac{2-\varrho}{1-\beta}.$$

We claim that every $t\in\operatorname{supp}\mu$ satisfies (4.13). Indeed, the construction shows that, if $t\in\operatorname{supp}\mu$, then there is a subsequence $n_k\to\infty$ and $j_k\in\{1,...,n_k^2\}$ such that $E(j_k,n_k)=1$ and

$$\lim_{k\to\infty}\frac{j_k-1}{n_k^2}=t. \tag{4.14}$$

Suppose that, for some $t\in[1,2]$ and $0<y\leqslant1$, we had

$$|\hat{f}_t'(iy)|\geqslant4y^{-\beta}\psi\left(\frac{1}{y}\right).$$

Continuity would imply that, for all s in a neighborhood of t,

$$|\hat{f}_s'(iy)|\geqslant2y^{-\beta}\psi\left(\frac{1}{y}\right).$$

This implies that there is no sequence (j_k,n_k) as above with $E(j_k,n_k)$ satisfying (4.14). A similar argument shows that there cannot exist $t\in\operatorname{supp}\mu$ and y with

$$|\hat{f}_t'(iy)|\leqslant\frac{1}{4}y^{-\beta}\psi\left(\frac{1}{y}\right)^{-1},$$

and this gives (4.13). Similarly, $\operatorname{supp}\nu$ is contained in $\gamma(A')$, where A' denotes the set of $t\in[1,2]$ satisfying (4.13). ☐

LEMMA 4.12. *Suppose that the assumptions of Theorem 4.10 hold and let μ_n and ν_n be the random measures constructed in the proof of Proposition 4.11. For every $\alpha<\frac{1}{2}(2-\varrho)$ and $\alpha'<(2-\varrho)/(1-\beta)$ there exist constants $c_\alpha<\infty$ and $c_{\alpha'}<\infty$ depending only on α and α', respectively, such that the energy integrals satisfy*

$$\mathbb{E}[\mathcal{E}_\alpha(\mu_n)]\leqslant c_\alpha\quad\textit{and}\quad\mathbb{E}[\mathcal{E}_{\alpha'}(\nu_n)]\leqslant c_{\alpha'}.$$

Proof. We will show the details for ν_n, the argument for μ_n being similar. For $j=1,...,n^2$, let

$$z_{j,n}=\hat{f}_{j,n}\left(\frac{i}{n}\right),\quad r_n=\frac{n^{\beta-1}}{4\psi(n^2)}\quad\text{and}\quad m_{j,n}=\frac{F(j,n)}{\pi r_n^2},$$

where ψ and F are as in Theorem 4.10. Recall that $\nu_n = \sum_{j=1}^{n^2} \nu_{j,n}$, where

$$d\nu_{j,n}(z) = m_{j,n} 1_{\{\mathcal{B}(z_{j,n}, r_n)\}}\, dA(z)$$

and $dA(z)$ is the area measure. Suppose that $\alpha < (2-\varrho)/(1-\beta)$. By definition,

$$\mathbb{E}[\mathcal{E}_\alpha(\nu_n)] = \sum_{j,k=1}^{n^2} \mathbb{E}\left[\iint \frac{d\nu_{j,n}(z)\, d\nu_{k,n}(w)}{|z-w|^\alpha}\right].$$

First consider the case where $j=k$:

$$\iint \frac{d\nu_{k,n}(z)\, d\nu_{k,n}(w)}{|z-w|^\alpha} = \frac{F(k,n)^2}{\pi^2 r_n^4} \iint_{\mathcal{B}(z_{k,n}, r_n)\times\mathcal{B}(z_{k,n}, r_n)} \frac{dA(z)\, dA(w)}{|z-w|^\alpha}.$$

One can check that there is a constant $c<\infty$ such that the right-hand side is bounded by $cF(k,n)^2 r_n^{-\alpha}$. Taking the expected value, using (4.11) and writing $\varepsilon = (2-\varrho)/(1-\beta)-\alpha$, we see that

$$\mathbb{E}\left[\frac{F(k,n)^2}{r_n^\alpha}\right] \leqslant c n^{-2-\varepsilon(1-\beta)} \psi_1(n^2),$$

where ψ_1 is a subpower function. Consequently, since $\beta<1$, we conclude that there is a constant $c_1<\infty$ depending only on α such that

$$\sum_{k=1}^{n^2} \mathbb{E}\left[\iint \frac{d\nu_{k,n}(z)\, d\nu_{k,n}(w)}{|z-w|^\alpha}\right] \leqslant c_1$$

uniformly in n. Now let $j<k$. We have

$$\iint \frac{d\nu_{j,n}(z)\, d\nu_{k,n}(z)}{|z-w|^\alpha} = \frac{F(j,n)F(k,n)}{\pi^2 r_n^4} \iint_{\mathcal{B}(z_{j,n}, r_n)\times\mathcal{B}(z_{k,n}, r_n)} \frac{dA(z)\, dA(w)}{|z-w|^\alpha}.$$

Since $j<k$, we can bound the last integral by a constant times

$$\frac{r_n^4}{|z_{j,n}-z_{k,n}|^\alpha}.$$

Using (4.11) and (4.12), this gives

$$\mathbb{E}\left[\iint \frac{d\nu_{j,n}(z)\, d\nu_{k,n}(z)}{|z-w|^\alpha}\right] \leqslant c\left(\frac{n^{-2}}{k-j+1}\right).$$

We conclude by noting that the sum over $1 \leqslant j<k \leqslant n^2$ of this last quantity is uniformly bounded in n. $\square$

5. Estimating the moments

In this section $r<r_c$ with corresponding values of ζ, β, ϱ and λ. All constants may depend on r. Let us give an overview of the section. We begin by discussing the reverse-time Loewner flow and how it relates to f_t. We then go on to define the "good times" which, roughly speaking, are T for which the reverse flow driven by $t\mapsto V_{T-t}-V_t$ behaves "as expected". (Here, V_t is a two-sided Brownian motion.) We make this precise in a number of lemmas that show how the derivative of the map and the flow of suitable points can be controlled on the event that T is "good". The needed correlation estimates can then be derived using moment bounds from [9] and [5]. However, to have the paper self-contained, we discuss some of the critical bounds in Appendix A.

5.1. Reverse Loewner flow

Here we state the basic lemma that relates the reverse Loewner flow to the forward flow for SLE. We will estimate the moments for h and $\tilde{h}$ rather than for $\hat{f}$.

If V_t is a continuous function, define g_t to be the solution to the forward-time (chordal) Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z)-V_t}, \quad g_0(z)=z. \tag{5.1}$$

Let $f_t(z)=g_t^{-1}(z)$ and $\hat{f}_t(z)=f_t(z+V_t)$. If U_t is another continuous function, let h_t be the solution to the reverse-time (chordal) Loewner equation

$$\partial_t h_t(z) = \frac{a}{U_t-h_t(z)}, \quad h_0(z)=z. \tag{5.2}$$

The next lemma relates the forward-time and reverse-time equations; although versions of this have appeared before, we give a short proof. We point out that this is a fact about the Loewner equation itself; no assumptions are made about the function V other than continuity.

Lemma 5.1. *Assume that V_t, $-\infty<t<\infty$, is a continuous function with $V_0=0$. For each $T\geqslant 0$, let*

$$U_{t,T} = V_{T-t}-V_T.$$

Let g_t, $0\leqslant t<\infty$, be the solution to the forward-time Loewner equation (5.1), and let f_t and $\hat{f}_t$ be as above. Let $h_{t,T}$, $0\leqslant t<\infty$, be the solution to the reverse-time Loewner equation (5.2) with $U_t=U_{t,T}$. Let

$$Z_{t,T}(z) = h_{t,T}(z)-U_{t,T}.$$

Then

$$h_{T,T}(z)=\hat{f}_T(z)-V_T=\hat{f}_T(z)+U_T,$$

and, if $0\leqslant S\leqslant T$ *and* $z,w\in\mathbb{H}$, *then*

$$h_{T,T}(z)=h_{S,S}(Z_{T-S,T}(z))+U_{T-S,T}, \tag{5.3}$$

$$\hat{f}_T(z)-\hat{f}_S(w)=h_{S,S}(Z_{T-S,T}(z))-h_{S,S}(w). \tag{5.4}$$

In particular,

$$h'_{T,T}(z)=h'_{S,S}(Z_{T-S,T}(z))h'_{T-S,T}(z).$$

Proof. Fix T and write

$$U_t:=U_{t,T} \quad \text{and} \quad h_t:=h_{t,T}.$$

For $0\leqslant S\leqslant T$, we have

$$U_{t,S}=V_{S-t}-V_S=U_{T-S+t}-U_{T-S}.$$

Let $u_t(z)=h_{T-t}(z)+V_T$. Then u_t satisfies

$$\partial_t u_t(z)=-\partial_t h_{T-t}(z)=\frac{a}{h_{T-t}(z)-(V_t-V_T)}=\frac{a}{u_t(z)-V_t},$$

with initial value $u_0(z)=h_T(z)+V_T$. Thus we see from (5.1) that $u_t(z)$ and $g_t(z)$ satisfy the same ordinary differential equation but with different initial values. However,

$$\tilde{u}_t(z):=g_t(h_T(z)+V_T)$$

and $u_t(z)$ does satisfy the same ordinary differential equation with the same initial conditions, and so it follows that

$$u_T(z)=g_T(h_T(z)+V_T).$$

On the other hand, as $u_T(z)=h_0(z)+V_T=z+V_T$, we get

$$z+V_T=g_T(h_T(z)+V_T),$$

and since $\hat{f}_T(z)=g_T^{-1}(z+V_T)$, we can write this as

$$h_T(z)=\hat{f}_T(z)-V_T=\hat{f}_T(z)+U_T.$$

For $0\leqslant s\leqslant T$, define $h_t^{(s)}$ by

$$h_{s+t}=h_t^{(s)}\circ h_s.$$

By (5.2), we see that

$$\partial_t[h_t^{(s)}(z)-U_s]=\frac{a}{U_{t+s}-h_t^{(s)}(z)}=\frac{a}{U_{t+s}-U_s-[h_t^{(s)}(z)-U_s]},$$
$$h_0^{(s)}(z)-U_s=z-U_s.$$

Therefore,

$$h_t^{(s)}(z)-U_s=h_{t,T-s}(z-U_s), \tag{5.5}$$

which implies that

$$h_{s+t}(z)=h_{t,T-s}(Z_{s,T}(z))+U_s. \tag{5.6}$$

Setting $s=T-S$ and $t=S$ gives (5.3), and setting $s=S$ and $t=T-S$ gives

$$\hat{f}_{T-S}(w)=h_{T-S}(w)-U_{T-S,T-S}=h_{T-S,T-S}(w)-(U_T-U_S),$$
$$\hat{f}_T(z)=h_T(z)-U_T=h_{T-S,T-S}(Z_{S,T}(z))+U_S-U_T.$$

Subtracting these equations gives (5.4). The final assertion follows from (5.3) and the chain rule. □

The preceding lemma holds for all continuous V_t. If V_t is a standard Brownian motion, then so is $U_{t,T}$ for each T. We get the following corollary.

Lemma 5.2. *Let $0<S<T$ and let g_t, $0\leqslant t\leqslant T$, be the solution to* (5.1), *where V_t is a standard Brownian motion. Let U_t be a standard Brownian motion and let h and $\tilde{h}$ be the solutions to*

$$\partial_t h_t(z)=\frac{a}{U_t-h_t(z)},\quad h_0(z)=z, \tag{5.7}$$
$$\partial_t \tilde{h}_t(z)=\frac{a}{\widetilde{U}_t-\tilde{h}_t(z)},\quad \tilde{h}_0(z)=z,$$

where $\widetilde{U}_t=U_{T-S+t}-U_{T-S}$. Then

$$h_{T-S+t}(z)=\tilde{h}_t(h_{T-S}(z)-U_{T-S})+U_{T-S}.$$

Moreover, the joint distribution of the functions

$$(\hat{f}'_S(w),\operatorname{Im}\hat{f}_S(w),\hat{f}'_T(z),\operatorname{Im}\hat{f}_T(z),\hat{f}_T(z)-\hat{f}_S(w))$$

is the same as the joint distribution of

$$(\tilde{h}'_S(w),\operatorname{Im}\tilde{h}_S(w),\tilde{h}'_S(Z)h'_{T-S}(z),\operatorname{Im}\tilde{h}_S(Z),\tilde{h}_S(Z)-\tilde{h}_S(w)),$$

where $Z=Z_{T-S}(z)=h_{T-S}(z)-U_{T-S}$.

5.2. Good times

Let $T>0$ and let $h_t=h_{t,T}$ be defined as in the proof of Lemma 5.1. More specifically, g_t is the solution to the forward-time Loewner equation (5.1) with a (two-sided) Brownian motion V_t as driving function, and h_t is the solution to the reverse-time Loewner equation (5.2) with $U_t=U_{t,T}=V_{T-t}-V_T$ as driving function. Let

$$Z_t(z)=X_t(z)+Y_t(z)=h_t(z)-U_t.$$

Recall from Lemma 5.1 that we have

$$h_{t+s}(z)=h_{t+s,T}(z)=h_{t,T-s}(Z_s(z))+U_s.$$

Note that

$$h'_{t+s}(z)=h'_{t,T-s}(Z_s(z))h'_s(z). \tag{5.8}$$

If ψ is a subpower function and $0<\delta\leqslant 1$, we let

$$\hat{\psi}_\delta(t)=\min\left\{\psi\left(\frac{t}{\delta}\right),\psi\left(\frac{1}{t}\right)\right\}=\begin{cases}\psi(t/\delta), & \text{if } t\leqslant\sqrt{\delta},\\ \psi(1/t), & \text{if } t>\sqrt{\delta}.\end{cases}$$

Note that, for every subpower function ψ and every $c<\infty$, there is an $M<\infty$ such that, for all $\delta>0$,

$$\hat{\psi}_\delta(t)\leqslant M, \quad \text{if } \delta\leqslant t\leqslant c\delta \text{ or } t\geqslant\frac{1}{c}. \tag{5.9}$$

Roughly speaking, $\hat{\psi}_\delta(t)$ is $O(1)$ for t comparable to δ or comparable to 1, but can be larger for other $\delta<t<1$.

Definition. We call a time T *ψ-good at δ* if the following conditions hold for $h_t=h_{t,T}$ with $\hat{\psi}=\hat{\psi}_\delta$ and $Z_t=X_t+iY_t=Z_t(\delta i)$:

$$Y_{t^2}\geqslant t\hat{\psi}(t)^{-1}, \qquad \delta\leqslant t\leqslant 2, \tag{5.10}$$

$$|X_{t^2}|\leqslant (t+\delta)\hat{\psi}(t), \qquad 0\leqslant t\leqslant 2, \tag{5.11}$$

$$\left(\frac{t}{\delta}\right)^{\beta}\psi(t)^{-1}\leqslant |h'_{t^2}(i\delta)|\leqslant\left(\frac{t}{\delta}\right)^{\beta}\psi(t), \qquad \delta\leqslant t\leqslant 2, \tag{5.12}$$

$$t^{-\beta}\psi(t)^{-1}\leqslant\frac{|h'_4(i\delta)|}{|h'_{t^2}(i\delta)|}=|h'_{4-t^2,T-t^2}(Z_{t^2})|\leqslant t^{-\beta}\psi(t), \qquad \delta\leqslant t\leqslant 2. \tag{5.13}$$

This definition depends on ψ and δ. Note that if T is ψ-good at δ and ϕ is a subpower function with $\psi\leqslant\phi$, then $\psi_\delta\leqslant\phi_\delta$ and T is ϕ-good at δ. In the remainder of this subsection, we derive some properties of ψ-good times. These will be used to estimate first and second moments for $|h'_{t^2}(\delta i)|^\lambda$ on the event that T is ψ-good at δ.

PROPOSITION 5.3. *For every subpower function ψ there is a subpower function ϕ such that, for all $\delta>0$, if T is ψ-good at δ, then*

$$|U_{t^2}| \leqslant t\phi\left(\frac{1}{t}\right), \quad \delta \leqslant t \leqslant 2. \tag{5.14}$$

Proof. Let $X_t = X_t(i\delta)$ and $Y_t = Y_t(i\delta)$. We let ϕ denote a subpower function, but it may change from line to line. From the Loewner equation, we know that

$$dX_t = -\frac{aX_t}{X_t^2+Y_t^2}\,dt - dU_t.$$

Hence,

$$|U_{t^2}| \leqslant |X_{t^2}| + a\int_0^{t^2} \frac{|X_s|\,ds}{X_s^2+Y_s^2}.$$

By (5.11), it suffices to show that

$$\int_0^{t^2} \frac{|X_s|\,ds}{X_s^2+Y_s^2} \leqslant \phi\left(\frac{1}{t}\right).$$

Using (5.10) and (5.11), we have

$$\frac{|X_s|}{X_s^2+Y_s^2} \leqslant \frac{\phi(1/s)}{\sqrt{s}},$$

and hence

$$\int_0^{t^2} \frac{|X_s|\,ds}{X_s^2+Y_s^2} \leqslant \int_0^{t^2} \frac{\phi(1/s)\,ds}{\sqrt{s}} = \int_{t^{-2}}^{\infty} \phi(x)x^{-3/2}\,dx = \tilde{\phi}\left(\frac{1}{t}\right)t,$$

where

$$\tilde{\phi}\left(\frac{1}{t}\right) = \frac{1}{t}\int_0^{t^2} \frac{\phi(1/s)\,ds}{\sqrt{s}}.$$

It is easy to check that $\tilde{\phi}$ is continuous and decays faster than x^{ε} for each ε. □

LEMMA 5.4. *If ψ is a subpower function, there is a $c>0$ such that, for every $0<\delta\leqslant 1$, if T is ψ-good at δ and $Y_t = Y_t(\delta i)$, then*

$$Y_{\delta^2} \geqslant (1+c)\delta. \tag{5.15}$$

Proof. Using (5.10) and (5.11), and the fact that Y_{t^2} increases with t, we see that there is a $c_1<\infty$ such that

$$|X_{t^2}| \leqslant c_1\delta \leqslant c_1 Y_{t^2}, \quad 0 \leqslant t \leqslant \delta.$$

The Loewner equation (5.2) implies that

$$\partial_s Y_s = \frac{aY_s}{X_s^2+Y_s^2} \geqslant \frac{a}{c_1^2+1}\,\frac{1}{Y_s}, \quad s \leqslant \delta^2,$$

from which (5.15) follows. □

LEMMA 5.5. *For every subpower function ψ, there is a c such that, if $0<\delta\leqslant 1$ and T is ψ-good at δ, then*

$$|h_{4-t^2,T-t^2}(Z_{t^2})-h_{4-t^2,T-t^2}(\delta i)|\geqslant c\hat{\psi}_\delta(t)^{-2}t^{1-\beta},\quad \delta\leqslant t\leqslant 2.$$

Proof. Using (5.10) and (5.15), we see that there is a $c_1>0$ such that if $\mathcal{B}$ denotes the open disk of radius $c_1t\hat{\psi}_\delta(t)^{-1}$ about Z_{t^2}, then $\delta i\notin\mathcal{B}$. Using (5.13) and the Koebe $\frac{1}{4}$-theorem, we see that $h_{4-t^2,T-t^2}(\mathcal{B})$ contains a disk of radius $\frac{1}{4}c_1\hat{\psi}_\delta(t)^{-2}t^{1-\beta}$ about $h_{4-t^2,T-t^2}(Z_{t^2})$. Since $h_{4-t^2,T-t^2}(\delta i)\notin\mathcal{B}$, the result follows. □

LEMMA 5.6. *For all subpower functions ψ and ϕ there is a subpower function $\bar{\psi}$ such that, if T is ψ-good at δ, then the following holds for all $\delta\leqslant t\leqslant 2$. Suppose that*

$$t\phi\left(\frac{1}{t}\right)^{-1}\leqslant y\leqslant t\phi\left(\frac{1}{t}\right)\quad\textit{and}\quad |x|\leqslant(t+\delta)\phi\left(\frac{1}{t}\right).$$

Then

$$t^{-\beta}\bar{\psi}\left(\frac{1}{t}\right)^{-1}\leqslant|h'_{4-t^2,T-t^2}(x+iy)|\leqslant t^{-\beta}\bar{\psi}\left(\frac{1}{t}\right).$$

Proof. By Proposition 2.3 and conditions (5.10) and (5.11),

$$\tilde{\psi}\left(\frac{1}{t}\right)^{-1}|h'_{4-t^2,T-t^2}(Z_{t^2})|\leqslant|h'_{4-t^2,T-t^2}(x+iy)|\leqslant\tilde{\psi}\left(\frac{1}{t}\right)|h'_{4-t^2,T-t^2}(Z_{t^2})|,$$

where $Z_{t^2}=Z_{t^2}(i\delta)$. The result then follows from (5.13). □

LEMMA 5.7. *For all subpower functions ψ and ϕ there is a subpower function $\bar{\psi}$ such that, if T is ψ-good at δ, then the following holds for $1\leqslant t\leqslant 2$. Let $w=x+iy$ with*

$$\delta\leqslant y\leqslant 1\quad\textit{and}\quad\left(\frac{x}{y}\right)^2+1\leqslant\phi\left(\frac{1}{y}\right).$$

Then,

$$y^{-\beta}\bar{\psi}_\delta\left(\frac{1}{y}\right)^{-1}\leqslant|h'_{t^2}(w)|\leqslant y^{-\beta}\bar{\psi}_\delta\left(\frac{1}{y}\right).$$

In particular,

$$y^{-\beta}\bar{\psi}_\delta\left(\frac{1}{y}\right)^{-1}\leqslant|h'_{t^2}(iy)|\leqslant y^{-\beta}\bar{\psi}_\delta\left(\frac{1}{y}\right).\tag{5.16}$$

Proof. We will do the case $t=2$; the argument is similar for $1\leqslant t\leqslant 2$. We let $\bar{\psi}$ denote a subpower function in this proof, but it may change from line to line. Since $(x/y)^2+1\leqslant\phi(1/y)$, we can see from Proposition 2.3 that

$$\bar{\psi}\left(\frac{1}{y}\right)^{-1}|h'_{t^2}(iy)|\leqslant|h'_{t^2}(w)|\leqslant\bar{\psi}\left(\frac{1}{y}\right)|h'_{t^2}(iy)|,$$

so we may assume that $x=0$. Let $s=y$. Using the Loewner equation (5.2), we can see that there is a $c<\infty$ such that

$$y \leqslant \operatorname{Im} h_{s^2}(iy) \leqslant cy, \quad |\operatorname{Re} h_{s^2}(iy)| \leqslant cy \quad \text{and} \quad |h'_{s^2}(iy)| \leqslant c.$$

The last estimate and (5.8) imply that

$$|h'_4(iy)| \asymp |h_{4-s^2,T-s^2}(Z_{s^2}(iy))|.$$

Using (5.14), we see that

$$|\operatorname{Re} Z_{s^2}(iy)| \leqslant y\left[c+\psi\left(\frac{1}{y}\right)\right].$$

By the previous result,

$$\bar{\psi}\left(\frac{1}{y}\right)^{-1} y^{-\beta} \leqslant |h'_{4-s^2,T-s^2}(Z_{s^2}(iy))| \leqslant \bar{\psi}\left(\frac{1}{y}\right) y^{-\beta}. \qquad \square$$

Definition. If n is a positive integer and $j=1,...,n^2$, we say that (j,n) is ψ-good if $T=1+(j-1)n^{-2}$ is ψ-good at n^{-1}. We let $E_{j,n}$ denote the event "(j,n) is ψ-good" and $E(j,n)$ denotes the indicator function of $E_{j,n}$.

It is important to note that on the event $E_{j,n}$, (5.16) implies that (4.9), the corresponding estimate for $|\hat{f}'|$, holds, with perhaps a different choice of subpower function ψ. Similarly, we note that Lemma 5.5 implies that (4.12) holds on the event $E_{j,n}$. The main estimate for the lower bound is the following.

PROPOSITION 5.8. *If $r<r_c$, there exist a subpower function ψ and $c>0$ such that, for all $n\geqslant 1$ and all $j=1,2,...,n^2$,*

$$\mathbb{E}\left[\left|\hat{f}'_{j,n}\left(\frac{i}{n}\right)\right|^{\lambda} E(j,n)\right] \geqslant cn^{-\zeta}. \tag{5.17}$$

Remark. For fixed n, the expectation in (5.17) is the same for all j.

Although the proof of Proposition 5.8 has essentially appeared in [9, Theorem 10.8], we have included Appendix A which discusses it. We point out that the assumption $r<r_c$ is crucial for the result.

5.3. Correlations

In this subsection we fix a subpower function ψ such that Proposition 5.8 holds. If n is a positive integer, we write j and k for positive integers satisfying $1\leqslant j<k\leqslant n^2$. We will

consider $E_{j,n} \cap E_{k,n}$ with indicator function $E(j,n)E(k,n)$. If j, k and n are fixed, we write

$$S = 1 + \frac{j-1}{n^2}, \quad T = 1 + \frac{k-1}{n^2}, \quad \tilde{h}_t = h_{t,S} \quad \text{and} \quad h_t = h_{t,T},$$

and recall that this means that $\tilde{h}_t$ and h_t are solutions to the reverse-time Loewner equation with $V_{S-t} - V_S$ and $V_{T-t} - V_T$ as driving functions, respectively.

PROPOSITION 5.9. *There is a subpower function ϕ such that, for all $1 \leqslant j < k \leqslant n^2$,*

$$\mathbb{E}\left[\left| \hat{f}'_T\left(\frac{i}{n}\right) \right|^\lambda \left| \hat{f}'_S\left(\frac{i}{n}\right) \right|^\lambda E(j,n)E(k,n) \right] \leqslant n^{-2\zeta} \left(\frac{n^2}{k-j} \right)^{(\lambda\beta+\zeta)/2} \phi\left(\frac{n^2}{k-j} \right).$$

Moreover, on the event $E_{j,n} \cap E_{k,n}$,

$$\left| \hat{f}_T\left(\frac{i}{n}\right) - \hat{f}_S\left(\frac{i}{n}\right) \right| \geqslant \left(\frac{k-j}{n^2} \right)^{(1-\beta)/2} \phi\left(\frac{n^2}{k-j} \right).$$

Proof. We write ϕ for a subpower function, but we let it vary from line to line; in the end we choose the maximum of all the subpower functions mentioned. Recall that $\hat{f}'_S(i/n) = \tilde{h}'_S(i/n)$, $\hat{f}'_T(i/n) = h'_T(i/n)$ and

$$\hat{f}_T\left(\frac{i}{n}\right) - \hat{f}_S\left(\frac{i}{n}\right) = \tilde{h}_S\left(Z_{T-S}\left(\frac{i}{n}\right) \right) - \tilde{h}_S\left(\frac{i}{n}\right).$$

The second assertion of the proposition follows immediately from Lemma 5.5, so we need only show the first one.

Since T is ψ-good at $1/n$, we know from (5.13) that

$$\left| h'_T\left(\frac{i}{n}\right) \right| \leqslant \left| h'_{T-S}\left(\frac{i}{n}\right) \right| \left(\frac{n^2}{k-j} \right)^{\beta/2} \phi\left(\frac{n^2}{k-j} \right).$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left[\left| \hat{f}'_T\left(\frac{i}{n}\right) \right|^\lambda \left| \hat{f}'_S\left(\frac{i}{n}\right) \right|^\lambda E(j,n)E(k,n) \right] \\ &\qquad \leqslant \left(\frac{n^2}{k-j} \right)^{\lambda\beta/2} \phi\left(\frac{n^2}{k-j} \right) \mathbb{E}\left[\left| h'_{T-S}\left(\frac{i}{n}\right) \right|^\lambda \left| \tilde{h}'_S\left(\frac{i}{n}\right) \right|^\lambda E(j,n)E(k,n) \right]. \end{aligned}$$

Note that $|h'_{T-S}(i/n)|^\lambda E(k,n)$ and $|\tilde{h}'_S(i/n)|^\lambda E(j,n)$ are independent random variables. Therefore,

$$\begin{aligned} &\mathbb{E}\left[\left| h'_{T-S}\left(\frac{i}{n}\right) \right|^\lambda \left| \tilde{h}'_S\left(\frac{i}{n}\right) \right|^\lambda E(j,n)E(k,n) \right] \\ &\qquad \leqslant \mathbb{E}\left[\left| \tilde{h}'_S\left(\frac{i}{n}\right) \right|^\lambda E(j,n) \right] \mathbb{E}\left[\left| h'_{T-S}\left(\frac{i}{n}\right) \right|^\lambda E(k,n) \right]. \end{aligned}$$

We now apply Theorem 4.1 to see that the right-hand side above is bounded above by

$$n^{-\zeta}(k-j)^{-\zeta/2} \phi\left(\frac{n^2}{k-j} \right) = n^{-2\zeta} \left(\frac{n^2}{k-j} \right)^{\zeta/2} \phi\left(\frac{n^2}{k-j} \right),$$

and this concludes the proof. ☐

6. Proof of Theorem 3.3

In this section we will use the *forward* Loewner flow to prove Theorem 3.3, which we now restate.

THEOREM 3.3. *If* $0<\varkappa<8$ *and* $\frac{1}{2}\leqslant\alpha\leqslant\alpha_*$, *then, with probability* 1, *there exists a set* V *such that* $\dim_{\mathrm{H}}[\gamma(V)]\leqslant F_{\mathrm{tip}}(\alpha)$ *and, for* $t\notin V$, $\gamma(t)\in\mathbb{H}$,

$$\tilde{\mu}(t,2^{-n})\preccurlyeq 2^{-n\alpha},\quad n\to\infty.$$

Throughout we will fix $\varkappa=2/a<8$. We will write u rather than α (to avoid having both α and a in formulas). To prove the theorem it suffices to show that for every bounded domain $D\subset\mathbb{H}$ bounded away from the real line, there is a set V_D with

$$\dim_{\mathrm{H}}[\gamma(V_D)]\leqslant F_{\mathrm{tip}}(u)$$

and such that (3.9) holds for $t\notin V_D$ with $\gamma(t)\in D$. We fix such a D and allow constants to depend on D. The basic strategy is typical for establishing upper bounds for multifractal spectra. We estimate a particular moment of $|g_\tau'(z)|$ for an appropriate stopping time, use Chebyshev's inequality to get an estimate on probabilities, and use this estimate to bound the dimension of a well-chosen covering.

We warn the reader again that some of the notation in this section is not consistent with that in other sections.

We parameterize $\mathrm{SLE}_\varkappa$ so that the conformal maps g_t satisfy

$$\partial_t g_t(z)=\frac{a}{g_t(z)-U_t},\quad g_0(z)=z, \tag{6.1}$$

where $U_t=-B_t$ is a standard Brownian motion. This is valid for $z\in\mathbb{C}\setminus\{0\}$ up to time $T_z\in(0,\infty]$. We let H_t be the unbounded component of $\mathbb{H}\setminus\gamma((0,t])$.

6.1. Preliminaries

Let

$$Z_t=Z_t(z)=X_t+iY_t=g_t(z)-U_t.$$

If $z\in\mathbb{H}$, let

$$\Delta_t=|g_t'(z)|,\quad \Upsilon_t=\frac{Y_t}{|g_t'(z)|},\quad \Theta_t=\arg Z_t\quad\text{and}\quad S_t=\sin\Theta_t.$$

Note that Υ_t equals $\frac{1}{2}$ times the conformal radius of H_t about z (or we can think of it as the conformal radius normalized so that the conformal radius of $\mathbb{H}$ about i equals 1). The Koebe-$\frac{1}{4}$ theorem implies that

$$\tfrac{1}{2}\Upsilon_t\leqslant \operatorname{dist}(z,\partial H_t)\leqslant 2\Upsilon_t. \tag{6.2}$$

Straightforward computations using (6.1) show that, for $z\in\mathbb{H}$,

$$\partial_t \Delta_t = \Delta_t \frac{a(Y_t^2 - X_t^2)}{|Z_t|^4} \quad \text{and} \quad \partial_t \Upsilon_t = -\Upsilon_t \frac{2aY_t^2}{|Z_t|^4}.$$

There exist $0<c_1<c_2<\infty$ such that, for $z\in D$,

$$c_1 \leqslant \Upsilon_0 \leqslant c_2 \quad \text{and} \quad c_1 \leqslant S_0 \leqslant 1. \tag{6.3}$$

Let, for $u\geqslant\frac{1}{2}$,

$$r = r(u) = \frac{1}{2} - 2a - \frac{1}{4u-2}, \tag{6.4}$$

and let

$$\lambda = \lambda_r = \frac{r^2}{2a} + r\left(1 - \frac{1}{2a}\right) \quad \text{and} \quad \xi = \xi_r = \frac{r^2}{4a} = \frac{\lambda}{2} - \frac{r}{2}\left(1 - \frac{1}{2a}\right). \tag{6.5}$$

Note that r increases with u. Define

$$\hat{u}_c = \frac{1}{2} + \frac{1}{8a-2}.$$

Note that $\hat{u}_c<\alpha_*=2a/(4a-1)$. If $u<\hat{u}_c$, since $a>\frac{1}{4}$,

$$r < r(\hat{u}_c) = 1 - 4a < \min\left\{\tfrac{1}{2} - 2a, 2 - 3a\right\} \quad \text{and} \quad -\lambda < r < 0.$$

The following is a straightforward Itô's formula calculation that we omit.

Proposition 6.1. *Let $r\in\mathbb{R}$ and let λ and ξ be as in (6.5). If $z\in\mathbb{H}$ and*

$$M_t = M_t(z) = |Z_t|^r Y_t^{\xi} \Delta_t^{\lambda} = S_t^{-r} \Upsilon_t^{\xi+r} \Delta_t^{\lambda+r},$$

then M_t is a local martingale satisfying

$$dM_t = M_t \frac{rX_t}{|Z_t|^2}\, dB_t.$$

Let $\mathcal{D}_n$ denote the set of dyadic rationals in $\mathbb{C}$,

$$z = \frac{j}{2^n} + i\frac{k}{2^n}, \quad j, k \in \mathbb{Z}.$$

Note that if $w\in\mathbb{C}$, then there exists $z\in\mathcal{D}_n$ with $|z-w|\leqslant 2^{-n}$, and hence

$$\mathcal{B}(w, 2^{-n}) \subset \mathcal{B}(z, 2^{-n+1}).$$

6.2. Basic strategy

Let

$$\tau_{n,z} = \inf\{s : \Upsilon_s(z) \leqslant 2^{-n+3}\}.$$

We will only consider n so large that $2^{-n+4} \leqslant c_1$, where c_1 is the constant in (6.3). Note that $\mathbb{P}\{\tau_{n,z} = \infty\} > 0$. If $\tau_{n,z} < \infty$, (6.2) implies that

$$2^{-n+2} \leqslant \operatorname{dist}(z, \partial H_{\tau_{n,z}}) = \operatorname{dist}(z, \gamma((0, \tau_{n,z}])) \leqslant 2^{-n+4}.$$

In particular, if $|w-z| \leqslant 2^{-n}$, then $\operatorname{dist}(w, \partial H_{\tau_{n,z}}) \geqslant 2^{-n+1}$.

Recall that we defined the normalized harmonic measure by

$$\operatorname{hm}_t(V) = \lim_{y\to\infty} y \operatorname{hm}(iy, V, H_t)$$

and $\tilde{\mu}(t,\varepsilon) = \operatorname{hm}_t[\bar{\mathcal{B}}(\gamma(t),\varepsilon)]$. Similarly, we define

$$\widehat{\operatorname{hm}}_t(V) := \lim_{y\to\infty} y \operatorname{hm}(iy, \partial V, H_t \setminus V),$$

and note that $\operatorname{hm}_t(V) \leqslant \widehat{\operatorname{hm}}_t(V)$. Set $\operatorname{hm}_{n,z} = \operatorname{hm}_{\tau_{n,z}}$, and similarly for $\widehat{\operatorname{hm}}$. If

$$|z-\gamma(t)| \leqslant 2^{-n}$$

then $\tau_{n,z} \leqslant t$, and hence, by monotonicity of harmonic measure, we have

$$\widehat{\operatorname{hm}}_{n,z}[\mathcal{B}(z, 2^{-n+1})] \geqslant \operatorname{hm}_t[\mathcal{B}(z, 2^{-n+1})] \geqslant \tilde{\mu}(t, 2^{-n}).$$

Let $\mathcal{D}_n(D)$ denote the set of $z \in \mathcal{D}_n$ such that $\operatorname{dist}(z, D) \leqslant 2^{-n}$ and

$$A_m^u = A_m^u(D) = \bigcup_{n=m}^{\infty} \bigcup_z \mathcal{B}(z, 2^{-n+1}),$$

where the inner union is over all $z \in \mathcal{D}_n(D)$ satisfying

$$\widehat{\operatorname{hm}}_{n,z}[\mathcal{B}(z, 2^{-n+1})] \geqslant 2^{-nu}. \tag{6.6}$$

Then, if $\gamma(t) \in D \setminus A_m^u$, for all sufficiently large n,

$$\tilde{\mu}(t, 2^{-n}) \leqslant 2^{-nu}.$$

Hence, for each m, A_m^u is a cover of $D \cap V_u$, where V_u is the set of $\gamma(t)$ that do not satisfy (3.9). Let $N_n = N_{n,u}(D)$ be the cardinality of the set of $z \in \mathcal{D}_n(D)$ satisfying (6.6). Then, for all s,

$$\mathcal{H}^s(D \cap V_u) \leqslant \lim_{m\to\infty} \mathcal{H}^s(A_m^u) \leqslant c \lim_{m\to\infty} \sum_{n=m}^{\infty} N_n 2^{-ns}.$$

The following proposition follows immediately.

Proposition 6.2. *Let $u, s>0$ and suppose that*

$$N_{n,u}(D) \preccurlyeq 2^{ns}, \quad n \to \infty.$$

Then

$$\dim_{\mathrm{H}}(D \cap V_u) \leqslant s.$$

Proof. The argument above shows that $\mathcal{H}^{s'}(D\cap V_u)=0$ for all $s'>s$. □

In order to show that for SLE

$$\dim_{\mathrm{H}}(D \cap V_u) \leqslant s$$

with probability 1, it suffices to show that

$$\mathbb{E}[N_{n,u}(D)] \preccurlyeq 2^{ns}, \quad n \to \infty.$$

Indeed, this relation and the Borel–Cantelli lemma imply that, with probability 1, $N_{n,u}(D) \leqslant 2^{ns'}$ for all $s'>s$. Note that

$$\mathbb{E}[N_{n,u}(D)] \leqslant c_D 2^{2n} \sup_{\mathrm{dist}(z,D)\leqslant 2^{-n}} \mathbb{P}\{\tau_{n,z}<\infty; \widehat{\mathrm{hm}}_{n,z}[\mathcal{B}(z,2^{-n+1})] \geqslant 2^{-nu}\}.$$

Notice that conformal invariance of harmonic measure and distortion estimates imply that, on the event $\tau_{n,z}<\infty$,

$$\widehat{\mathrm{hm}}_{n,z}[\mathcal{B}(z,2^{-n+1})] \asymp 2^{-n}|g'_{\tau_{n,z}}(z)|.$$

Indeed, $g_{\tau_{n,z}}(\mathcal{B}(z,2^{-n+1}))$ is a connected set whose diameter is comparable to

$$2^{-n}|g'_{\tau_{n,z}}(z)|,$$

and whose distance from the real axis is also comparable to

$$2^{-n}|g'_{\tau_{n,z}}(z)|.$$

Hence, there exists $c<\infty$ such that

$$\mathbb{E}[N_{n,u}(D)] \leqslant c_D 2^{2n} \sup_{\mathrm{dist}(z,D)\leqslant 2^{-n}} \mathbb{P}\{\tau_{n,z}<\infty; |g'_{\tau_{n,z}}(z)| \geqslant c2^{-n(u-1)}\}.$$

In the remainder of this section we will show that there exists $c=c_D<\infty$ such that, for all sufficiently large n and all z with $\mathrm{dist}(z,D)\leqslant 2^{-n}$,

$$\mathbb{P}\{\tau_{n,z}<\infty; |g'_{\tau_{n,z}}(z)| \geqslant c2^{-n(u-1)}\} \leqslant 2^{-n\varrho(u)}, \tag{6.7}$$

where

$$\varrho(u) = \left(\frac{1}{8a}+2a-1\right)\left(u-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{8a}\right)+\frac{1}{32a\left(u-\frac{1}{2}\right)}.$$

Then from the above arguments we know that, with probability 1,

$$\dim_{\mathrm{H}}(D\cap V_u) \leqslant 2-\varrho(u) = F_{\mathrm{tip}}(u).$$

The second equality is a straightforward calculation. The remainder of this section is devoted to establishing (6.7).

6.3. Weighting by the martingale

The local martingale M_t is not a martingale because it "blows up" on the event of measure zero that z is on the path $\gamma((0,\infty))$. However, if we choose stopping times τ such as $\tau_{n,z}$ which prevent the path from getting too close to z, then the stopped process $M_{t\wedge\tau}$ is a martingale. Let $\mathbb{P}^*$ and $\mathbb{E}^*$ denote probabilities and expectations with respect to the measure obtained by weighting by (the stopped martingale) M. The Girsanov theorem implies that

$$dB_t = \frac{rX_t}{|Z_t|^2}\,dt + dW_t, \quad 0 \leqslant t < \tau,$$

where W_t is a standard Brownian motion with respect to the measure $\mathbb{P}^*$. In particular,

$$d\Theta_t = \frac{(1-2a-r)X_tY_t}{|Z_t|^4}\,dt - \frac{Y_t}{|Z_t|^2}\,dW_t.$$

It is useful to use a "radial" parametrization $\sigma(t)$. We write $\widehat{Z}_t = Z_{\sigma(t)}$, $\widehat{X}_t = X_{\sigma(t)}$, etc. The radial parametrization is defined by

$$\widehat{\Upsilon}_t := \Upsilon_{\sigma(t)} = e^{-2at}.$$

Note that

$$-2a\widehat{\Upsilon}_t = \partial_t\widehat{\Upsilon}_t = -2a\widehat{\Upsilon}_t\frac{\widehat{Y}_t^2}{|\widehat{Z}_t|^4}\partial_t\sigma(t),$$

which implies that

$$\partial_t\sigma(t) = \frac{|\widehat{Z}_t|^4}{\widehat{Y}_t^2}.$$

Note also that

$$d\widehat{\Theta}_t = (1-2a)\cot\widehat{\Theta}_t\,dt + d\widehat{B}_t,$$

and the local martingale M_t satisfies

$$d\widehat{M}_t = -r\widehat{M}_t\cot\widehat{\Theta}_t\,d\widehat{B}_t.$$

Moreover, we have that

$$d\widehat{\Theta}_t = (1-2a-r)\cot\widehat{\Theta}_t\,dt + d\widehat{W}_t. \tag{6.8}$$

In the above, $\widehat{B}_t$ and $\widehat{W}_t$ are standard Brownian motions with respect to $\mathbb{P}$ and $\mathbb{P}^*$, respectively. Since $1-2a-r>\frac{1}{2}$, we compare with a Bessel process to see that in the measure $\mathbb{P}^*$, $\widehat{\Theta}_t$ never reaches $\{0,\pi\}$; see [8, Chapter 1]. It follows that $\widehat{M}_t$ is actually a martingale. Also, the invariant probability density for the stochastic differential equation (6.8) equals

$$f(\theta) = c\sin^{2(1-2a-r)}\theta.$$

Since $r<1-4a<3-4a$, it follows that $\sin^r\theta$ is integrable with respect to $f(\theta)\,d\theta$. The important fact for us is that there is a c such that, if $\widehat{\Theta}_t$ satisfies (6.8) with $\sin\widehat{\Theta}_0 \geqslant c_1$, then, for all $t>0$,

$$\mathbb{E}^*[\widehat{S}_t^r] \leqslant c. \tag{6.9}$$

Let

$$\tau_s = \inf\{t : \Upsilon_t = e^{-2as}\}.$$

For $r<\frac{1}{2}-2a$, we have for all s,

$$\mathbb{P}^*\{\tau_s<\infty\} = 1.$$

Then, using (6.9), we have

$$\begin{aligned}\mathbb{E}[|g'_{\tau_s}(z)|^{\lambda+r}; \tau_s<\infty] &= \mathbb{E}[M_{\tau_s} S^r_{\tau_s} \Upsilon^{-\xi-r}_{\tau_s}; \tau_s<\infty] = e^{2as(\xi+r)}\mathbb{E}[M_{\tau_s} S^r_{\tau_s}; \tau_s<\infty] \\ &= e^{2as(\xi+r)} M_0(z)\mathbb{E}^*[\widehat{S}_s^r] \leqslant ce^{2as(\xi+r)}.\end{aligned}$$

Since $\lambda+r>0$, if $\varepsilon=e^{-2as}$, then

$$\mathbb{P}\{\tau_s<\infty; |g'_{\tau_s}(z)| \geqslant \varepsilon^{u-1}\} \leqslant \varepsilon^{-(u-1)(\lambda+r)}\mathbb{E}[|g'_{\tau_s}(z)|^{\lambda+r}; \tau_s<\infty] \leqslant c\varepsilon^{\varrho(u)},$$

where

$$\varrho(u) = -(u-1)(\lambda+r)-(r+\xi).$$

Doing the algebra, we get

$$\varrho(u) = \left(\frac{1}{8a}+2a-1\right)\left(u-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{8a}\right)+\frac{1}{32a\left(u-\frac{1}{2}\right)}.$$

This proves (6.7) which concludes the proof of Theorem 3.3.

Appendix A. Proof of Proposition 5.8

In this appendix we will discuss the proof of the lower bound for the moments of the derivative. We shall use the notation from §5. Our goal is to prove the following result.

PROPOSITION *If $r<r_c$, there exist a subpower function ψ and $c>0$ such that, for all $n\geqslant 1$ and all $j=1,2,...,n^2$,*

$$\mathbb{E}\left[\left|\hat{f}'_{j,n}\left(\frac{i}{n}\right)\right|^\lambda E(j,n)\right]\geqslant cn^{-\zeta},$$

where $E(j,n)$ is the indicator of the event that $T=1+(j-1)n^{-2}$ is ψ-good at $1/n$.

Since the proof has essentially appeared in [9], we shall be rather brief and cite several results. Let h_t be the reverse-time SLE$_\varkappa$ flow. Scaling implies that the distribution of $(|h'_{t^2}(\delta i)|, t\in[\delta,2])$ is the same as that of $(|h'_{(t/\delta)^2}(i)|, t\in[\delta,2])$, so it will be sufficient to study $\mathbb{E}[|h'_s(i)|^\lambda]$ for $1\leqslant s\leqslant t$, as $t\to\infty$. We fix $a>0$ and $r<r_c$ and let all constants in the appendix depend on these parameters.

A.1. Setup

Let V_t be the driving Brownian motion for h and define

$$Z_t=h_t(i)-V_t,\quad X_t=\operatorname{Re} Z_t,\quad Y_t=\operatorname{Im} Z_t\quad\text{and}\quad K_t=\frac{X_t}{Y_t}.$$

The Loewner equation implies that we can write

$$|h'_t(i)|^\lambda=\exp\left[a\lambda\int_0^t\left(1-\frac{2}{1+K_t^2}\right)\frac{dt}{|Z_t|^2}\right]. \tag{A.1}$$

Itô's formula shows that, if λ, ζ and r are as in §3, then

$$M_t=|h'_t(i)|^\lambda Y_t^\zeta(1+K_t^2)^{r/2}$$

is a martingale; see [9, Proposition 2.1]. Note that the Loewner equation implies that $d\log Y_t/dt=a/|Z_t|^2$, and so the right-hand side of (A.1) suggests that we perform a time-change $t\mapsto\sigma(t)$ so that $\log Y_{\sigma(t)}$ grows linearly. Set

$$\sigma(t)=\inf\{s:\log Y_s=at\},$$

and let $\widehat{X}_t=X_{\sigma(t)}$, $\hat{h}'_t=h'_{\sigma(t)}$, etc. Differentiating both sides of $Y_{\sigma(t)}=e^{2at}$ shows that

$$\sigma(t)=\int_0^t e^{2at}(1+\widehat{K}_t^2)\,dt.$$

The equation for $\widehat{K}$ is

$$d\widehat{K}_t=-2a\widehat{K}_t\,dt+\sqrt{1+\widehat{K}_t^2}\,d\widehat{B}_t, \tag{A.2}$$

where $\widehat{B}$ is standard Brownian motion and a time change of V:

$$\widehat{B}_t = \int_0^{\sigma(t)} \frac{dV_t}{|Z_t|}.$$

It is now useful to consider a change of variable. Recall that

$$r_c = 2a + \frac{1}{2} = \frac{4}{\varkappa} + \frac{1}{2}$$

and let

$$d\hat{J}_t = -r_c \tanh \hat{J}_t\, dt + d\widehat{B}_t. \tag{A.3}$$

Then Itô's formula implies that $\widehat{K}_t = \sinh \hat{J}_t$ satisfies the stochastic differential equation (A.2). Define

$$\hat{L}_t = t - \int_0^t \frac{2\,dt}{1+\widehat{K}_t^2} = t - \int_0^t \frac{2\,dt}{\cosh^2 \hat{J}_t}, \tag{A.4}$$

so that $|\hat{h}_t'(i)| = e^{a\hat{L}_t}$. One can check that

$$N_t = M_{\sigma(t)} = e^{a\lambda \hat{L}_t} e^{a\zeta t} (1+\widehat{K}_t^2)^{r/2} = e^{a\lambda \hat{L}_t} e^{a\zeta t} \cosh^r \hat{J}_t$$

is again a martingale and

$$dN_t = r(\tanh \hat{J}_t) N_t\, d\widehat{B}_t;$$

see [9, Proposition 6.2].

A.2. Change of measure

The martingale property implies that

$$\mathbb{E}[e^{a\lambda \hat{L}_t} \cosh^r \hat{J}_t] = e^{-a\zeta t}.$$

The idea is to show that this integral is, roughly speaking, supported on an event on which $\cosh^r \hat{J}_t \asymp 1$ and $\hat{L}_t = \beta t + O(\sqrt{t}\,)$, where $\beta = \beta(r)$ is as in §5. We shall use Girsanov's theorem to change measure and compute probabilities in the new measure obtained by weighting by N. For an event E measurable with respect to the σ-algebra generated by $\widehat{B}_s$, $0 \leqslant s \leqslant t$, define the measure $\mathbb{P}_*(E) = \mathbb{E}[1_E N_t]$; note that $N_0 = 1$. By Girsanov's theorem, if W_t is standard Brownian motion under $\mathbb{P}_*$, then

$$d\hat{J}_t = -q \tanh \hat{J}_t\, dt + dW_t, \tag{A.5}$$

where $q := r_c - r$. The key observation is that $\hat{J}$ satisfying the last equation is a positive recurrent process whenever $q = r_c - r > 0$, and therefore has an invariant density.

LEMMA A.1. *Let $q>0$, assume that $\hat{J}_t$ satisfies the stochastic differential equation* (A.5) *under the measure $\mathbb{P}_*$ and let $\hat{L}_t$ be defined by* (A.4). *Then the following holds*:

(i) *The process $\hat{J}_t$ is positive recurrent with invariant density*

$$v_q(x) = \frac{C_q}{\cosh^{2q} x}, \quad -\infty < x < \infty,$$

where $C_q = \Gamma\left(q+\frac{1}{2}\right)/\Gamma\left(\frac{1}{2}\right)\Gamma(q)$. Moreover,

$$\int_{-\infty}^{\infty}\left(1-\frac{2}{\cosh^2 x}\right)v_q(x)\,dx = \frac{1-2q}{1+2q} = \beta.$$

(ii) *There exists a constant $c<\infty$ depending only on q such that, if $\hat{J}_0=0$, $k\geqslant 0$ and $u\geqslant 1$, then*

$$\mathbb{P}_*\{\text{there exists } t\in[k,k+1] \text{ with } \cosh \hat{J}_t \geqslant u\} \leqslant cu^{-2q}.$$

(iii) *There exists $c<\infty$ such that, for $0\leqslant s\leqslant t$,*

$$\mathbb{E}_*\left[\exp\left(\frac{(2q+1)|\hat{L}_t-\hat{L}_s-\beta(t-s)|}{\sqrt{t-s}}\right)\right] \leqslant c.$$

Proof. By the stochastic differential equation for $\hat{J}_t$, we have that the invariant density v solves the equation

$$\tfrac{1}{2}v''(x) - ((-q\tanh x)v(x))' = 0;$$

see, e.g., [7, §15.5]. A solution is given by $v=v_q$, and one can check that this is an invariant density for $\hat{J}_t$. Next, we note that we get an upper bound for the probability in (ii) by considering the same probability but with $\hat{J}_0$ having the law of the invariant distribution. Indeed, this follows by coupling the two processes. Let $x\geqslant 1$, suppose that $\hat{J}_0$ has the law of the invariant distribution and, for $k\geqslant 0$, set

$$Y = Y_{k,x} = \int_k^{k+2} 1_{\{|\hat{J}_t|\geqslant x-1\}}\,dt.$$

Then,

$$\mathbb{E}_*[Y] = 2\int_{|y|\geqslant x-1} v_q(y)\,dy \leqslant \frac{c}{\cosh^{2q} x}.$$

Using the strong Markov property and the fact that the drift $\tanh \hat{J}_t$ is bounded when $\hat{J}_t$ is close to x (and the fact that $\hat{J}_t$ has continuous paths) it follows that there exists a $\delta>0$ such that

$$\mathbb{E}_*[Y \mid \text{there exists } t\in[k,k+1] \text{ with } |\hat{J}_t|\geqslant x] \geqslant \delta,$$

and consequently

$$\mathbb{P}_*\{\text{there exists } t\in[k,k+1] \text{ with } |\hat{J}_t|\geqslant x\}\leqslant \delta^{-1}\mathbb{E}_*[Y].$$

Using the estimate on $\mathbb{E}_*[Y]$, this completes the proof of (ii).

We now turn to (iii). Since $|\hat{L}_t-\hat{L}_s-\beta(t-s)|\leqslant 2(t-s)$, it is enough to prove the bound for $t-s$ sufficiently large. We shall assume that $4/\sqrt{t-s}\leqslant q$. For $r\in\mathbb{R}$, let

$$\delta=r\left(\tfrac{1}{2}q+\tfrac{1}{4}\right)-\tfrac{1}{4}r^2 \quad\text{and}\quad \eta=r\left(\tfrac{1}{2}q-\tfrac{1}{4}\right)-\tfrac{1}{4}r^2.$$

Then Itô's formula shows that the process

$$O_t=e^{\delta(\hat{L}_t-\beta t)}e^{(\eta+\delta\beta)t}\cosh^r \hat{J}_t$$

is a martingale and so, for $s<t$,

$$\mathbb{E}_*[e^{\delta(\hat{L}_t-\hat{L}_t-\beta(t-s))}e^{(\eta+\delta\beta)(t-s)}\cosh^r \hat{J}_t]=\mathbb{E}_*[\cosh^r \hat{J}_s]. \tag{A.6}$$

It follows from (ii) that there is a constant $c<\infty$ such that $\mathbb{E}_*[\cosh^r \hat{J}_s]<c$. We combine this with (A.6) with the choice $r=4/\sqrt{t-s}$ which was assumed to be at most q. We can therefore Taylor expand δ and η to obtain

$$\mathbb{E}_*\left[\exp\left(\pm b\frac{\hat{L}_t-\hat{L}_s-\beta(t-s)}{\sqrt{t-s}}\right)(\cosh \hat{J}_t)^{\pm 4/\sqrt{t-s}}\right]\leqslant c, \tag{A.7}$$

where $b:=2q+1$. From this it follows directly that

$$\mathbb{E}_*\left[\exp\left(b\frac{\hat{L}_t-\hat{L}_s-\beta(t-s)}{\sqrt{t-s}}\right)\right]\leqslant c.$$

The remaining case, when b is replaced by $-b$, can be easily verified using (A.7) and condition (ii). □

The next lemma defines the event corresponding to a "good time" (for the time-changed process) in terms of $\hat{J}$ and $\hat{L}$. The proof is a consequence of the estimates in Lemma A.1 and the Chebyshev inequality.

Lemma A.2. ([9], Proposition 7.3) *For each $u,t>0$, let $E_{t,u}$ be the event that the following estimates hold for $0\leqslant s\leqslant t$:*

$$|\hat{J}_s|\leqslant u\log\min\{s+2,t-s+2\},$$
$$|\hat{L}_s-\beta s|\leqslant u\sqrt{s}\log(s+2),$$
$$|\hat{L}_t-\hat{L}_s-\beta(t-s)|\leqslant u\sqrt{t-s}\log(t-s+2).$$

Then,

$$\lim_{u\to\infty}\inf_{t>0}\mathbb{P}_*(E_{t,u})=1.$$

It remains to transfer these estimates to the processes in their original time-parametrizations. Using the lemmas proved in §5, Proposition 5.8 follows from the next result.

Proposition A.3. *There exists a subpower function* ψ *such that if*

$$\psi_{t,s}=\psi(\min\{e^{as},e^{a(t-s)}\}),\quad t\geqslant 0,$$

and E_t *is the event that the following estimates hold for* $0\leqslant s\leqslant t$:

$$\begin{gathered}Y_{e^{2as}}\geqslant e^{as}\psi_{t,s}^{-1},\\ |X_{e^{2as}}|\leqslant e^{as}\psi_{t,s},\\ \psi(e^{as})^{-1}e^{as\beta}\leqslant |h'_{e^{2as}}(i)|\leqslant e^{as\beta}\psi(e^{as}),\\ \psi(e^{a(t-s)})^{-1}e^{a(t-s)\beta}\leqslant \frac{|h'_{e^{2at}}(i)|}{|h'_{e^{2as}}(i)|}\leqslant e^{a(t-s)\beta}\psi(e^{a(t-s)}),\end{gathered}$$

then

$$\mathbb{E}[|h'_{e^{2at}}(i)|^{\lambda}1_{E_t}]\asymp e^{-a\zeta t}.$$

Proof. We shall use ψ to denote a subpower function that may change from expression to expression. In the end we choose the maximum of those that we have used. First note that Lemma A.2 implies that we can find a $u_*<\infty$ such that $\inf_{t>0}\mathbb{P}_*(E_{u_*,t})\geqslant\frac{1}{2}$. Let $E_t=E_{u_*,t}$ and write $E(t)$ for the indicator of E_t. Note that on E_t we have that $1\leqslant\cosh\hat{J}_t\leqslant c_*$, where c_* is a constant depending only on u_*; in the sequel we will allow constants (and subpower functions) to depend on u_*. Consequently,

$$\mathbb{E}[|h'_{\sigma(t)}(i)|^{\lambda}E(t)]\asymp e^{-a\zeta t}. \tag{A.8}$$

It remains to relate $\sigma(t)$ to the original time parametrization. When $s<t$, $\cosh\hat{J}_s$ is not uniformly bounded on E_t, but if $F(s,t)=2+\min\{s,t-s\}$, then taking exponentials gives

$$1\leqslant\cosh\hat{J}_s\leqslant cF(s,t)^{u_*},\quad 0\leqslant s\leqslant t,$$

on E_t. Hence, there is a constant $c_*<\infty$ such that, on E_t,

$$\frac{e^{2as}-1}{2a}\leqslant\int_0^s e^{2as}\cosh^2\hat{J}_s\,ds=\sigma(s)\leqslant c_*F(s,t)^{2u_*}e^{2as},\quad 0\leqslant s\leqslant t, \tag{A.9}$$

where the first inequality is immediate. We wish to use this to show that the bounds on $|h'_{\sigma(s)}(i)|$ from Lemma A.2 also hold for $|h'_{e^{2as}}(i)|$ with suitable subpower corrections. It is enough to consider s so large that $\sigma(s)\geqslant e^{2as}/4a$. Indeed, $|h'_{e^{2as}}(i)|\asymp 1$ for s bounded by a constant. In terms of Y, (A.9) implies that there is a constant $c<\infty$ such that

$$e^{as-c\log F(s,t)}\leqslant Y_{e^{2as}}\leqslant ce^{as},\quad 0\leqslant s\leqslant t. \tag{A.10}$$

Write

$$\sigma_s = \sigma_{s,t} = \sigma(s - c\log F(s,t)),$$

where c is as in (A.10) and, for a given subpower function ψ, set

$$\psi_{t,s} = \psi(\min\{e^{as}, e^{a(t-s)}\}).$$

Then we see that there is a subpower function ψ and $r \in [0, e^{2as}\psi_{t,s}]$ such that

$$e^{2as} = \sigma_s + r.$$

Next, note that

$$|h'_{e^{2as}}(i)| = |(h_r^{\sigma_s})'(Z_{\sigma_s})|\,|h'_{\sigma_s}(i)|,$$

where $r \mapsto h_r^{\sigma_s}$ solves the reverse-time Loewner equation driven by V_{σ_s+r} as in §5. Since $Y_{\sigma_s} \geqslant e^{as}\psi_{t,s}^{-1}$ and $r \leqslant e^{2as}\psi_{t,s}$, it follows that

$$\psi_{t,s}^{-1} \leqslant \frac{|h'_{e^{2as}}(i)|}{|h'_{\sigma_s}(i)|} \leqslant \psi_{t,s}, \tag{A.11}$$

and we can use Lemma A.2 to conclude that, on E_t,

$$\psi(e^{as})^{-1} e^{as\beta} \leqslant |h'_{e^{2as}}(i)| \leqslant e^{as\beta}\psi(e^{as}), \quad 0 \leqslant s \leqslant t, \tag{A.12}$$

and

$$\psi(e^{a(t-s)})^{-1} e^{a(t-s)\beta} \leqslant \frac{|h'_{e^{2at}}(i)|}{|h'_{e^{2as}}(i)|} \leqslant e^{a(t-s)\beta}\psi(e^{a(t-s)}), \quad 0 \leqslant s \leqslant t. \tag{A.13}$$

A similar argument shows that

$$\frac{|h'_{\sigma(t)}(i)|}{|h'_{\sigma_t}(i)|} \asymp 1,$$

which combined with (A.11) and (A.8) shows that

$$\mathbb{E}[|h'_{e^{2at}}(i)|^{\lambda} E(t)] \asymp e^{-a\zeta t}.$$

Finally, recall that $\cosh^2 \hat{J}_s = 1 + e^{-2as} X^2_{\sigma(s)}$, and so it follows that

$$X^2_{e^{2as}} \leqslant c e^{2as}\psi_{t,s}, \quad 0 \leqslant s \leqslant t.$$

This completes the proof. □

References

[1] Beffara, V., The dimension of the SLE curves. *Ann. Probab.*, 36 (2008), 1421–1452.

[2] Beliaev, D. & Smirnov, S., Harmonic measure and SLE. *Comm. Math. Phys.*, 290 (2009), 577–595.

[3] Binder, I. & Duplantier, B., Harmonic measure and winding of conformally invariant curves. *Phys. Rev. Lett.*, 89 (2002), 264101.

[4] Duplantier, B., Conformal fractal geometry & boundary quantum gravity, in *Fractal Geometry and Applications: a Jubilee of Benoît Mandelbrot*, Part 2, Proc. Sympos. Pure Math., 72, pp. 365–482. Amer. Math. Soc., Providence, RI, 2004.

[5] Johansson Viklund, F. & Lawler, G. F., Optimal Hölder exponent for the SLE path. *Duke Math. J.*, 159 (2011), 351–383.

[6] Kang, N.-G., Boundary behavior of SLE. *J. Amer. Math. Soc.*, 20 (2007), 185–210.

[7] Karlin, S. & Taylor, H. M., *A Second Course in Stochastic Processes*. Academic Press, New York, 1981.

[8] Lawler, G. F., *Conformally Invariant Processes in the Plane*. Mathematical Surveys and Monographs, 114. Amer. Math. Soc., Providence, RI, 2005.

[9] — Multifractal analysis of the reverse flow for the Schramm–Loewner evolution, in *Fractal Geometry and Stochastics* IV, Progr. Probab., 61, pp. 73–107. Birkhäuser, Basel, 2009.

[10] — Schramm–Loewner evolution (SLE), in *Statistical Mechanics*, IAS/Park City Math. Ser., 16, pp. 231–295. Amer. Math. Soc., Providence, RI, 2009.

[11] Lawler, G. F., Schramm, O. & Werner, W., Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187 (2001), 237–273.

[12] — Values of Brownian intersection exponents. II. Plane exponents. *Acta Math.*, 187 (2001), 275–308.

[13] — Values of Brownian intersection exponents. III. Two-sided exponents. *Ann. Inst. H. Poincaré Probab. Statist.*, 38 (2002), 109–123.

[14] — Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32 (2004), 939–995.

[15] Lawler, G. F. & Sheffield, S., A natural parametrization for the Schramm–Loewner evolution. *Ann. Probab.*, 39 (2011), 1896–1937.

[16] Lind, J. R., Hölder regularity of the SLE trace. *Trans. Amer. Math. Soc.*, 360 (2008), 3557–3578.

[17] Makarov, N. G., Fine structure of harmonic measure. *Algebra i Analiz*, 10 (1998), 1–62 (Russian); English translation in *St. Petersburg Math. J.*, 10 (1999), 217–268.

[18] Mattila, P., *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[19] Pommerenke, C., *Boundary Behaviour of Conformal Maps*. Grundlehren der Mathematischen Wissenschaften, 299. Springer, Berlin–Heidelberg, 1992.

[20] Rohde, S. & Schramm, O., Basic properties of SLE. *Ann. of Math.*, 161 (2005), 883–924.

[21] Schramm, O., Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118 (2000), 221–288.

[22] Smirnov, S., Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333 (2001), 239–244.

[23] — Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. of Math.*, 172 (2010), 1435–1467.

[24] Werner, W., Random planar curves and Schramm–Loewner evolutions, in *Lectures on Probability Theory and Statistics*, Lecture Notes in Math., 1840, pp. 107–195. Springer, Berlin–Heidelberg, 2004.

Fredrik Johansson Viklund
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
U.S.A.
fjv@math.columbia.edu

Gregory F. Lawler
Department of Mathematics
and Department of Statistics
University of Chicago
5734 S. University Avenue
Chicago, IL 60637
U.S.A.
lawler@math.uchicago.edu

Received November 12, 2010
Received in revised form April 23, 2012